Newtonian fluid

$$\sigma = -\mu \frac{dv}{dy}$$

$$\pi r^2 (P_2 - P_1) = (2\pi r L)\sigma$$

$$\Delta P \, x \, \pi r^2 = \sigma (2\pi r L)$$

$$\sigma = \Delta P \frac{(\pi r^2)}{2\pi r L} = \frac{\Delta P r}{2L}$$

$$v(r) = \frac{\Delta P R^2}{4\mu L}\left[1 - \left(\frac{r}{R}\right)^2\right]$$

$$\mu = \frac{\pi \Delta P R^4}{8 L V^0}$$

Definition of a Newtonian Fluid

$$\frac{F}{A} = \sigma_{yx} = \mu\left(-\frac{du}{dy}\right) = \mu\gamma_{yx}$$

For Newtonian behaviour (1) σ is proportional to γ and a plot passes through the origin; and (2) by definition the constant of proportionality,

Newtonian

$$\sigma = -\mu \frac{dv}{dy}$$

$$\left.\begin{aligned} \pi r^2 (P_2 - P_1) &= (2\pi r L)\sigma \\ \Delta P \, x \, \pi r^2 &= \sigma(2\pi r L) \end{aligned}\right\} \quad \sigma = \Delta P \frac{(\pi r^2)}{2\pi r L} = \frac{\Delta P r}{2L}$$

$$v(r) = \frac{\Delta P R^2}{4\mu L}\left[1 - \left(\frac{r}{R}\right)^2\right]$$

$$\mu = \frac{\pi \Delta P R^4}{8 L V^0}$$

Newtonian

From

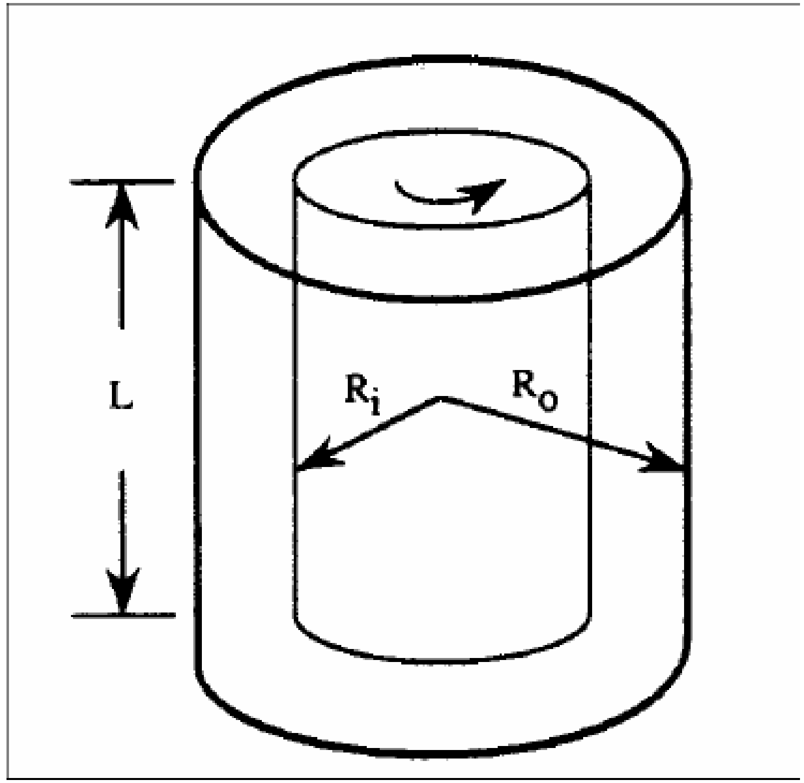
$$\sigma = \Delta P \frac{(\pi r^2)}{2\pi r L} = \frac{\Delta \mathrm{Pr}}{2\mathrm{L}} = \mu \frac{du}{dy}$$

and

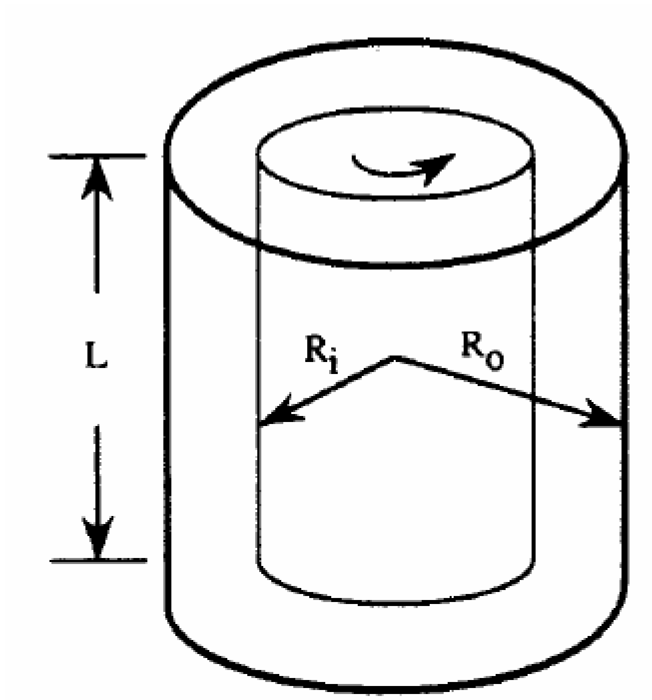
$$\mu = \frac{\pi \Delta P R^4}{8 L V^0} \qquad \therefore \Delta \mathrm{P} = \frac{8 \mu L Q}{\pi R^4}$$

$$\therefore \quad \gamma = 8\mathrm{v}/\mathrm{D}$$

Rotational Viscometer



Schematic illustration of coaxial-cylinder rotational viscometer.



$$\Omega = 2\pi r^2 L\sigma$$

$$\gamma = r\frac{d\omega}{dr}$$

$$\frac{\Omega}{2\pi L r^2} = -\mu\left[\frac{r d\omega}{dr}\right]$$

$$\int_0^{\omega_i} d\omega = -\frac{\Omega}{2\pi\ \mu L}\int_{R_0}^{R_i} r^{-3}\,dr$$

$$\omega_i = \frac{\Omega}{4\pi\mu L}\left[\frac{1}{R_i^2} - \frac{1}{R_0^2}\right]$$

$$\omega_i = 2\pi N$$

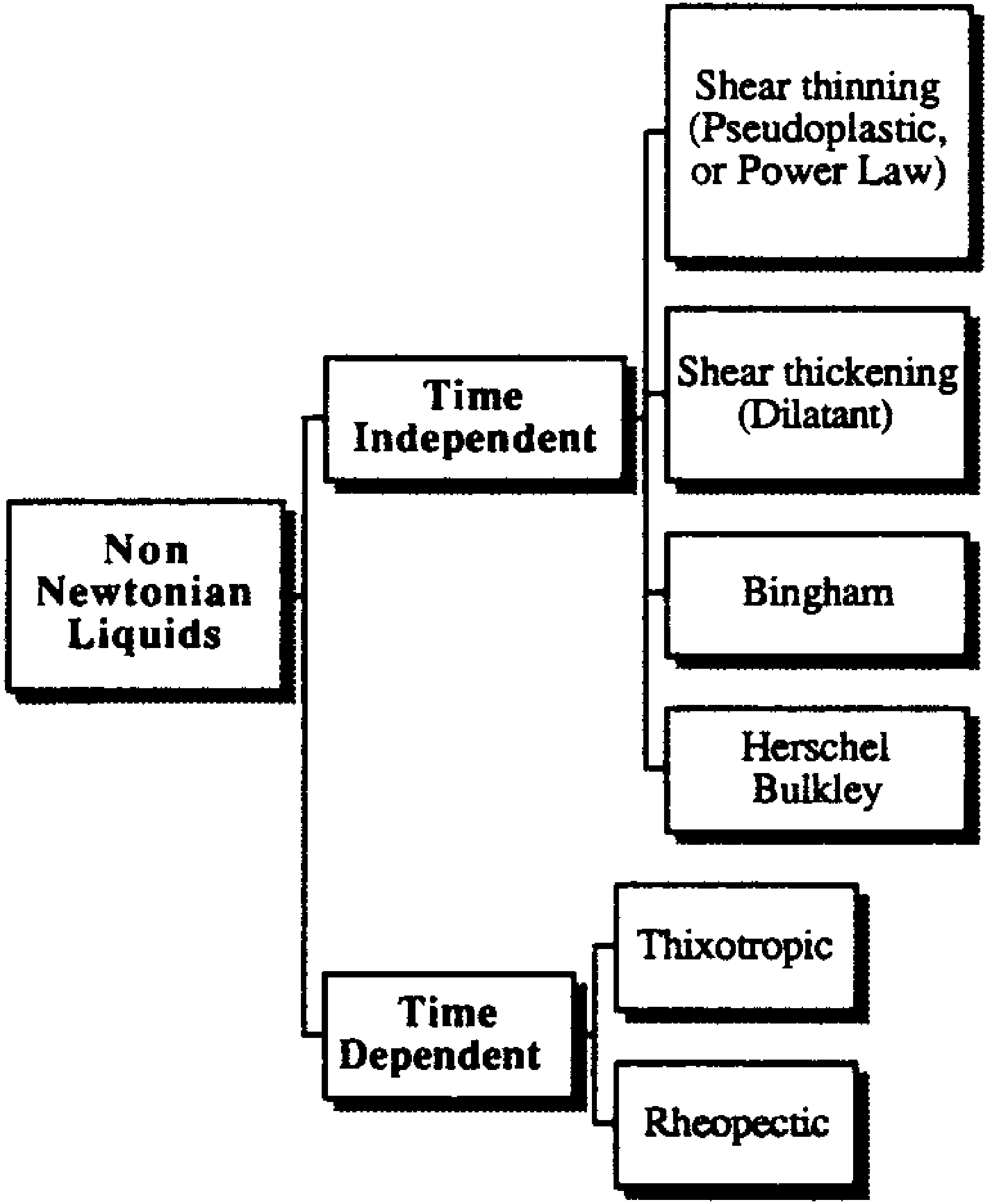
$$\mu = \frac{\Omega}{8\pi^2 NL}\left[\frac{1}{R_i^2} - \frac{1}{R_0^2}\right]$$

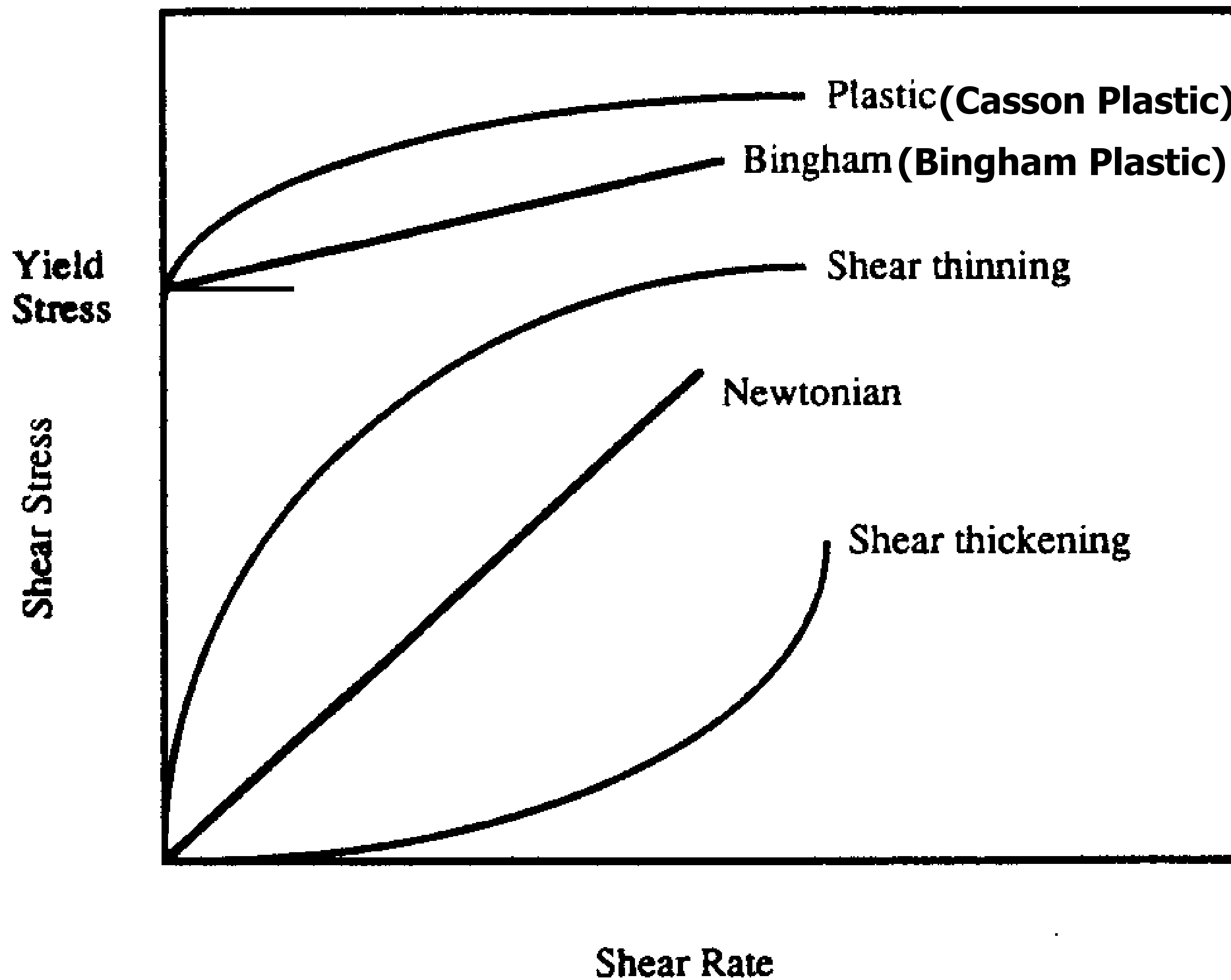
$$\mu = \frac{\Omega}{8\pi^2 NLR_i^2}$$

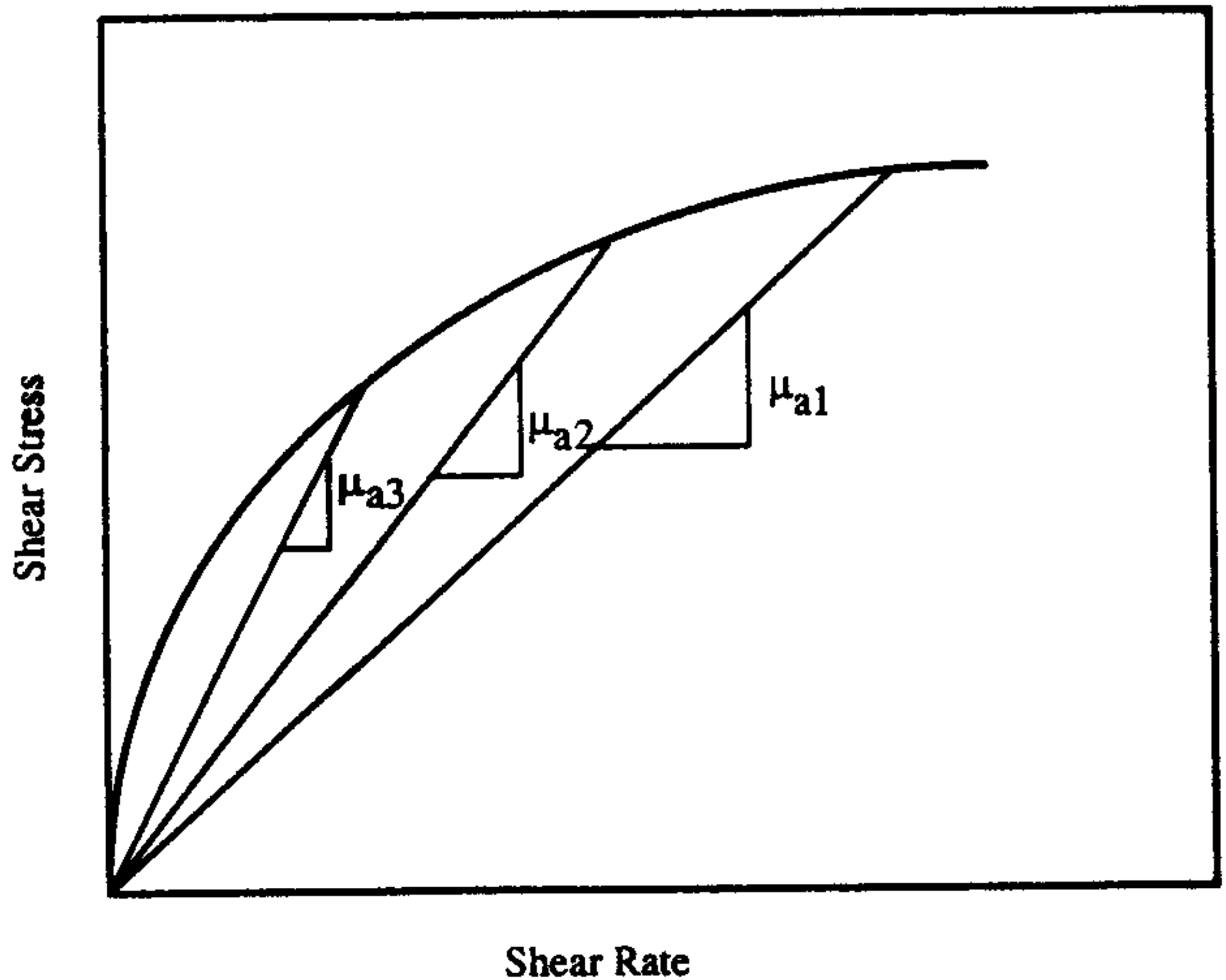
Non-Newtonian Fluids

Flow Characteristic of Non-Newtonian Fluid

- Fluids in which shear stress is not directly proportional to deformation rate are non-Newtonian flow: toothpaste and Lucite paint


Non Newtonian Liquids
Time Independent
Shear thinning (Pseudoplastic, or Power Law)
Shear thickening (Dilatant)
Bingham
Herschel Bulkley
Time Dependent
Thixotropic
Rheopectic


Plastic (Casson Plastic)
Bingham (Bingham Plastic)
Yield Stress
Shear thinning
Newtonian
Shear Stress
Shear thickening
Shear Rate



Viscosity changes with shear rate. Apparent viscosity (μ_a or η) is always defined by the relationship between shear stress and shear rate.

Model Fitting - Shear Stress vs. Shear Rate

Summary of Viscosity Models

Newtonian	$\tau = \eta \dot{\gamma}$
Pseudoplastic	$\tau = K \dot{\gamma}^{n} \quad (n < 1)$
Dilatant	$\tau = K \dot{\gamma}^{n} \quad (n > 1)$
Bingham	$\tau = \tau_y + \eta \dot{\gamma}^{n}$
Casson	$\tau^{1/2} = \tau_0^{1/2} + \eta_c^{1/2} \dot{\gamma}^{1/2}$
Herschel-Bulkley	$\tau = \tau_y + K \dot{\gamma}^{n}$

τ or σ = shear stress, γ^{o} = shear rate, μ_a or η = apparent viscosity

m or K or K'= consistency index, n or n'= flow behavior index

Herschel-Bulkley model (Herschel and Bulkley , 1926)

$$\sigma = m\left[\frac{du}{dy}\right]^{n} + \sigma_0$$

Values of coefficients in Herschel-Bulkley fluid model

Fluid	m	n	σ_0	Typical examples
Herschel-Bulkley	>0	$0<n<\infty$	>0	Minced fish paste, raisin paste
Newtonian	>0	1	0	Water,fruit juice, honey, milk, vegetable oil
Shear-thinning (pseudoplastic)	>0	$0<n<1$	0	Applesauce, banana puree, orange juice concentrate
Shear-thickening	>0	$1<n<\infty$	0	Some types of honey, 40 % raw corn starch solution
Bingham Plastic	>0	1	>0	Toothpaste, tomato paste

Non-Newtonian Fluid Behaviour

The flow curve (shear stress vs. shear rate) is either non-linear, or does pass through the origin, or both. Three classes can be distinguished.

(1) Fluids for which the rate of shear at any point is determined only by the value of the shear stress at that point at that instant; these fluids are variously known as “time independent”, “purely viscous”, “inelastic”, or “Generalised Newtonian Fluids” (GNF).

(2) More complex fluids for which the relation between shear stress and shear rate depends, in addition, on the duration of shearing and their kinematic history; they are called “time-dependent fluids”.

(3) Substances exhibiting characteristics of both ideal fluids and elastic solids and showing partial elastic recovery, after deformation; these are characterised as “visco-elastic” fluids.

Time-Independent Fluid Behaviour

1. Shear thinning or pseudoplastic fluids

Viscosity decrease with shear stress. Over a limited range of shear-rate (or stress) log (t) vs. log (g) is approximately a straight line of negative slope. Hence

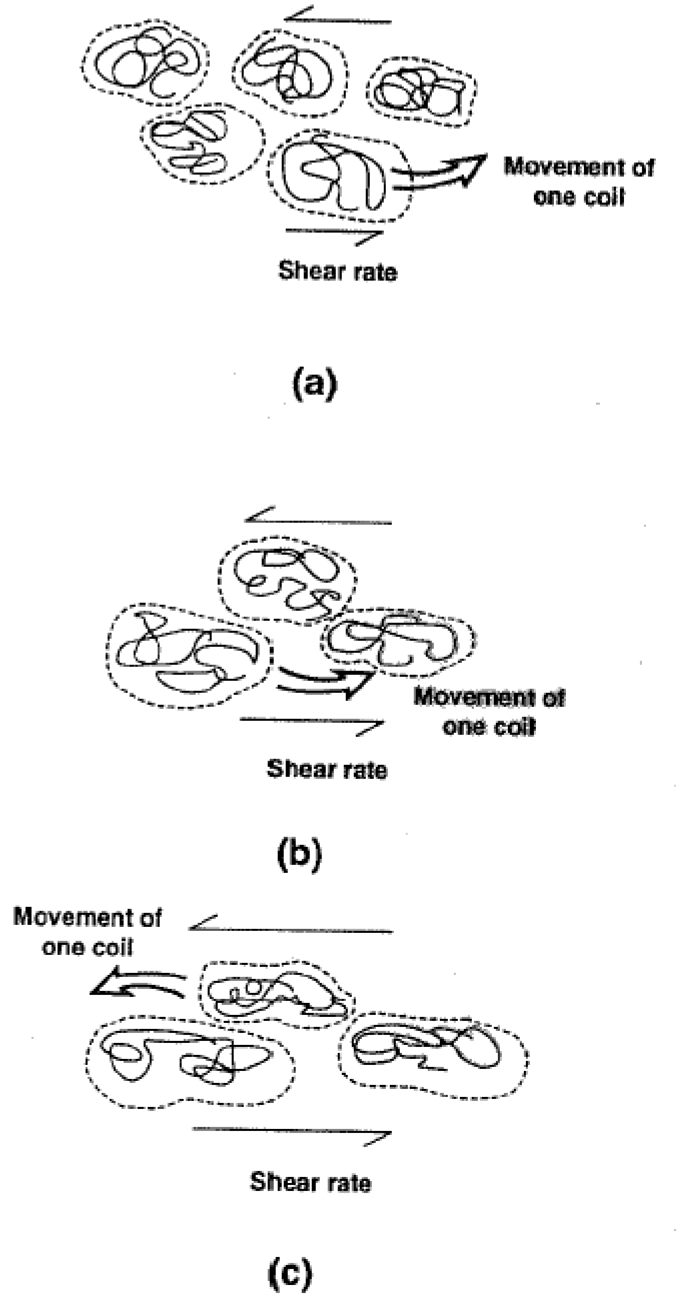
$\tau_{yx} = m(\gamma_{yx})^n$ (*) where m = fluid consistency coefficient

n = flow behaviour index

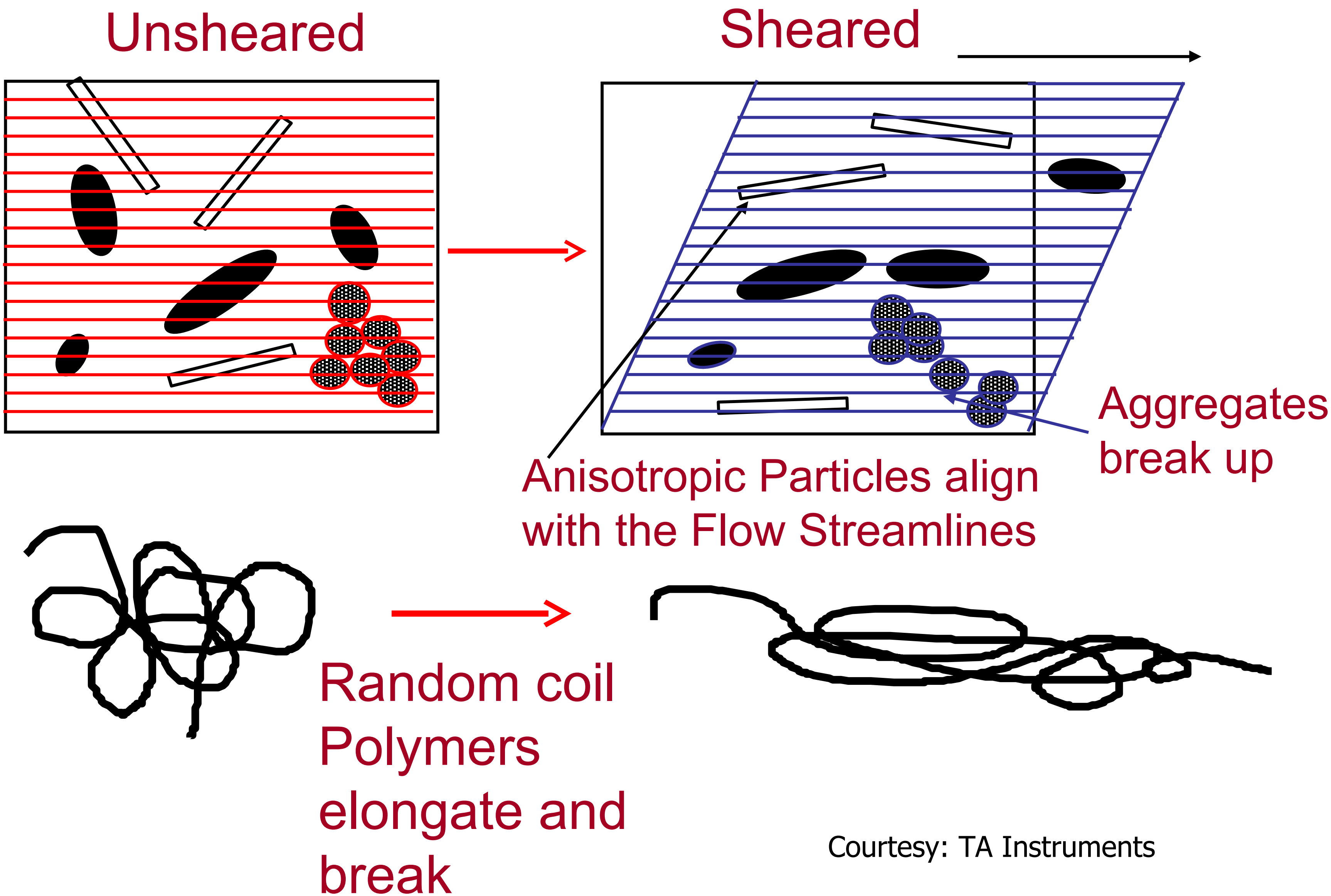
Re-arrange Eq. (*) to obtain an expression for apparent viscosity μ_{app} $(= \tau_{yx}/\gamma_{yx})$

Pseudoplastics

Flow of **pseudoplastics** is consistent with the random coil model of polymer solutions and melts. At low stress, flow occurs by **random coils** moving past each other w/o coil deformation. At moderate stress, the coils are deformed and slip past each other more easily. At high stress, the coils are distorted as much as possible and offer low resistance to flow. **Entanglements** between chains and the **reptation** model also are consistent with the observed viscosity changes.



Why Shear Thinning occurs



Courtesy: TA Instruments

Shear Thinning Behavior

- Shear thinning behavior is often a result of:
  - Orientation of non-spherical particles in the direction of flow. An example of this phenomenon is the pumping of fiber slurries.
  - Orientation of polymer chains in the direction of flow and breaking of polymer chains during flow. An example is polymer melt extrusion
  - Deformation of spherical droplets to elliptical droplets in an emulsion. An industrial application where this phenomenon can occur is in the production of low fat margarine.
  - Breaking of particle aggregates in suspensions. An example would be stirring paint.

Courtesy: TA Instruments

2. Viscoplastic Fluid Behaviour

Viscoplastic fluids behave as if they have a **yield stress (τ_0).** Until τ_0 is exceeded they do not appear to flow. A **Bingham plastic** fluid has a constant plastic viscosity

$$\tau_{yx} = \tau_0^B + \mu_B \gamma_{yx} \qquad \text{for} \quad |\tau_{yx}| > |\tau_0^B|$$

$$\gamma_{yx} = 0 \qquad \text{for} \quad |\tau_{yx}| < |\tau_0^B|$$

Often the two model parameters τ_0^B and μ_B are treated as curve fitting constants, even when there is no true yield stress.

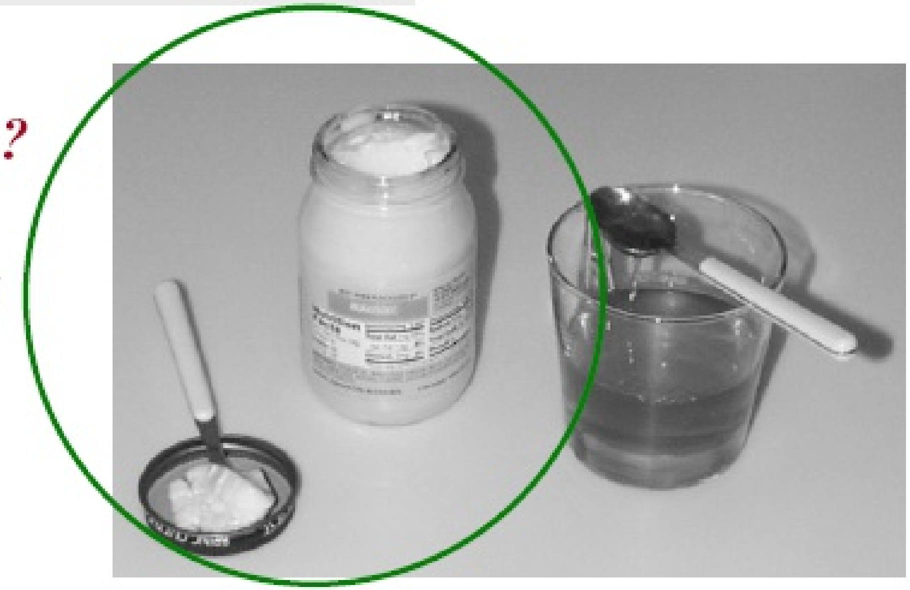
3. Shear-thickening or Dilatant Fluid Behaviour

Eq. (*) is applicable with $n>1$.

Viscosity increases with shear stress. Dilatant: shear thickening fluids that contain suspended solids. Solids can become close packed under shear.

Other Inelastic Fluids

What about mayonnaise?



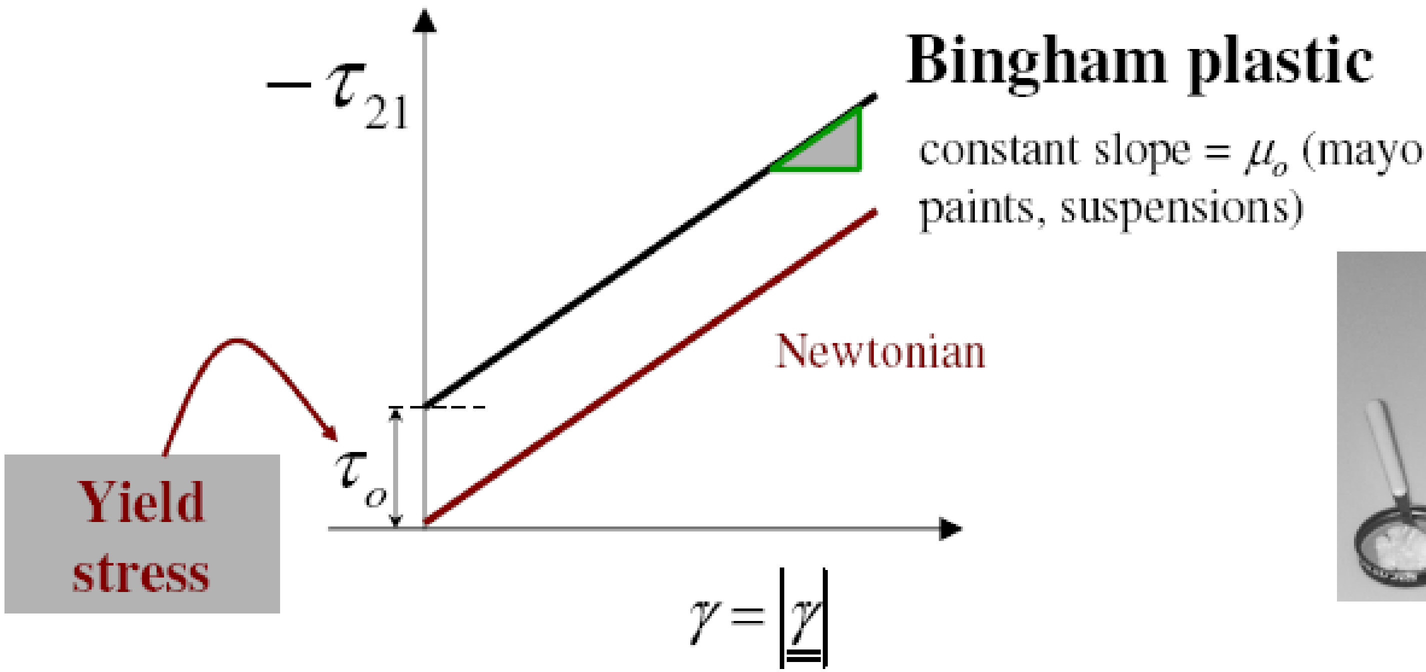
Mayonnaise and many other like fluids (paint, ketchup, most suspensions, asphalt) is able to sustain a **yield stress.**

Once the fluid begins to deform under an imposed stress, the viscosity may either be constant or may shear-thin. This type of steady shear viscosity behavior can be modeled with a GNF.

Source: Faith A. Morrison, Michigan Tech U.

Non-Newtonian Fluids

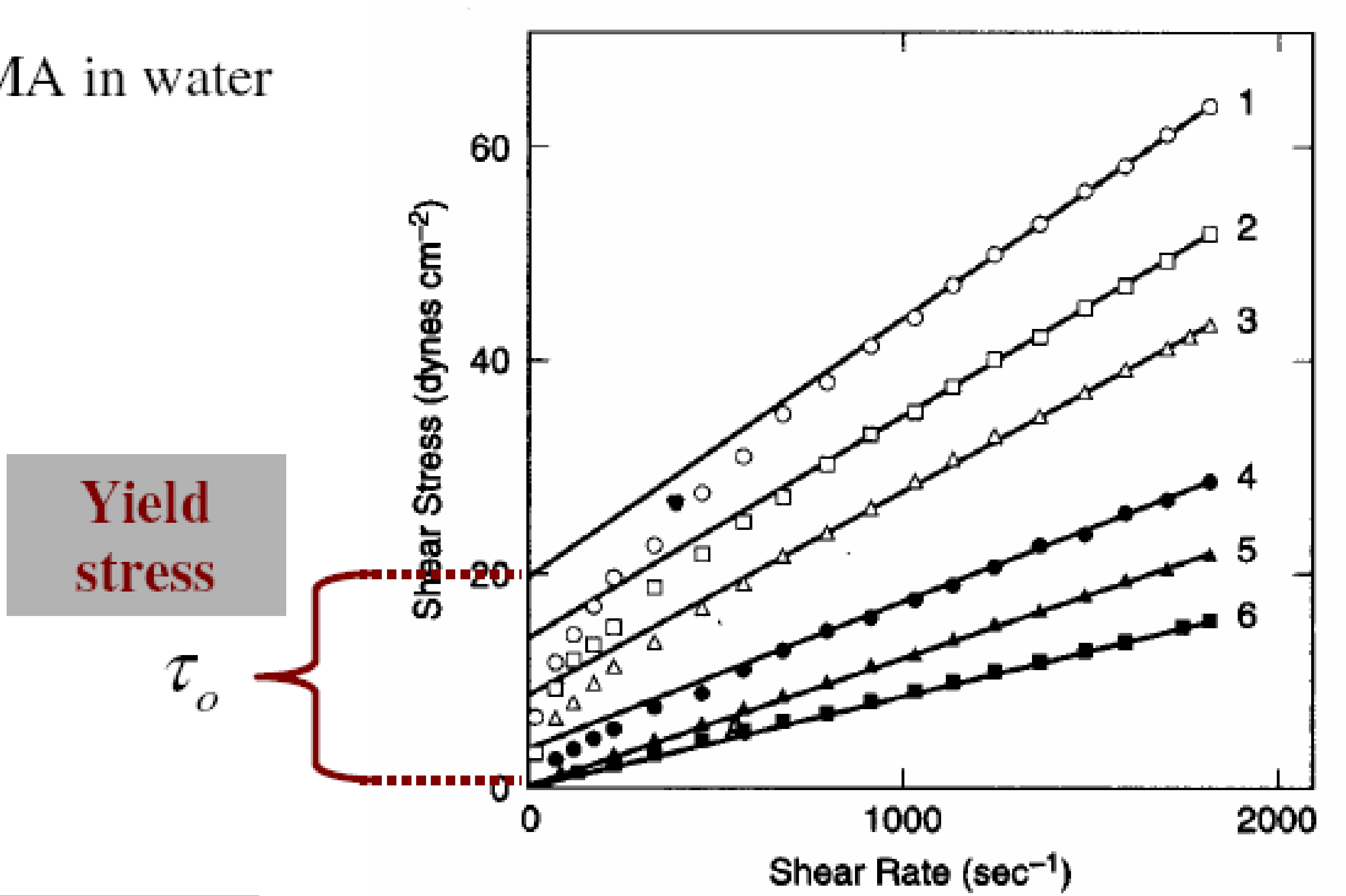
For some fluids, no flow occurs when moderate stresses are applied.



Source: Faith A. Morrison, Michigan Tech U.

Bingham plastic

PMMA in water



Friend and Hunter, 1971; dispersions of PMMA in water at various ζ-potentials; From Larson, p353.

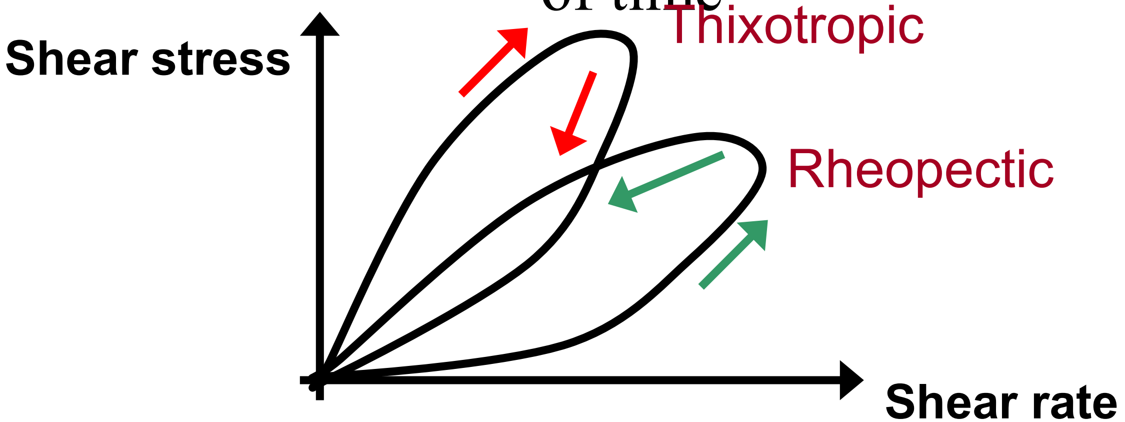
Source: Faith A. Morrison, Michigan Tech U.

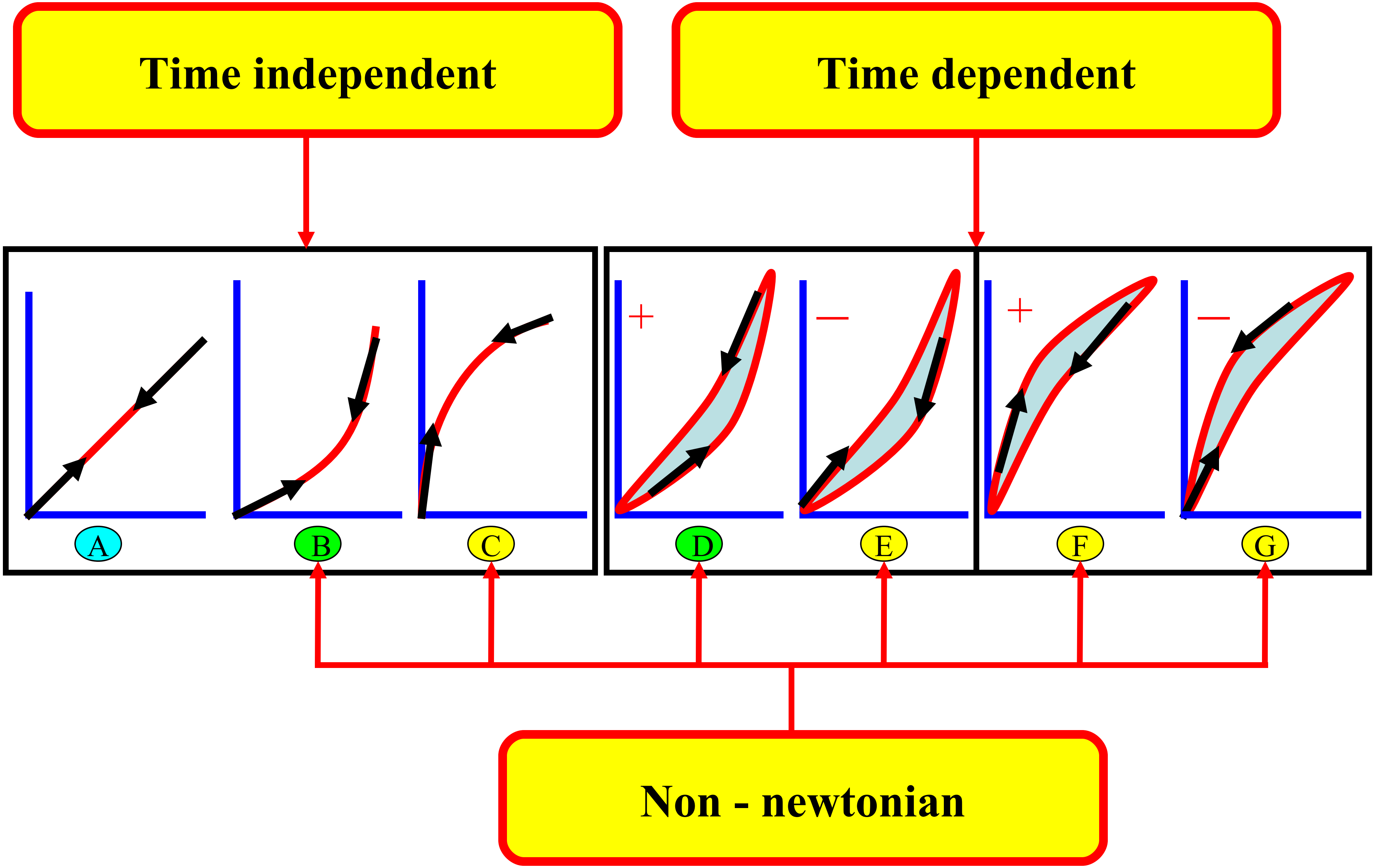
Time-dependent Fluid Behaviour

The response time of the material may be longer than response time of the measurement system, so the viscosity will change with time. Apparent viscosity depends not only on the rate of shear but on the "time for which fluid has been subject to shearing".

Thixotropic : Material structure breaks down as shearing action continues : e.g. gelatin, cream, shortening, salad dressing.

Rheopectic : Structure build up as shearing continues (not common in food : e.g. highly concentrated starch solution over long periods of time



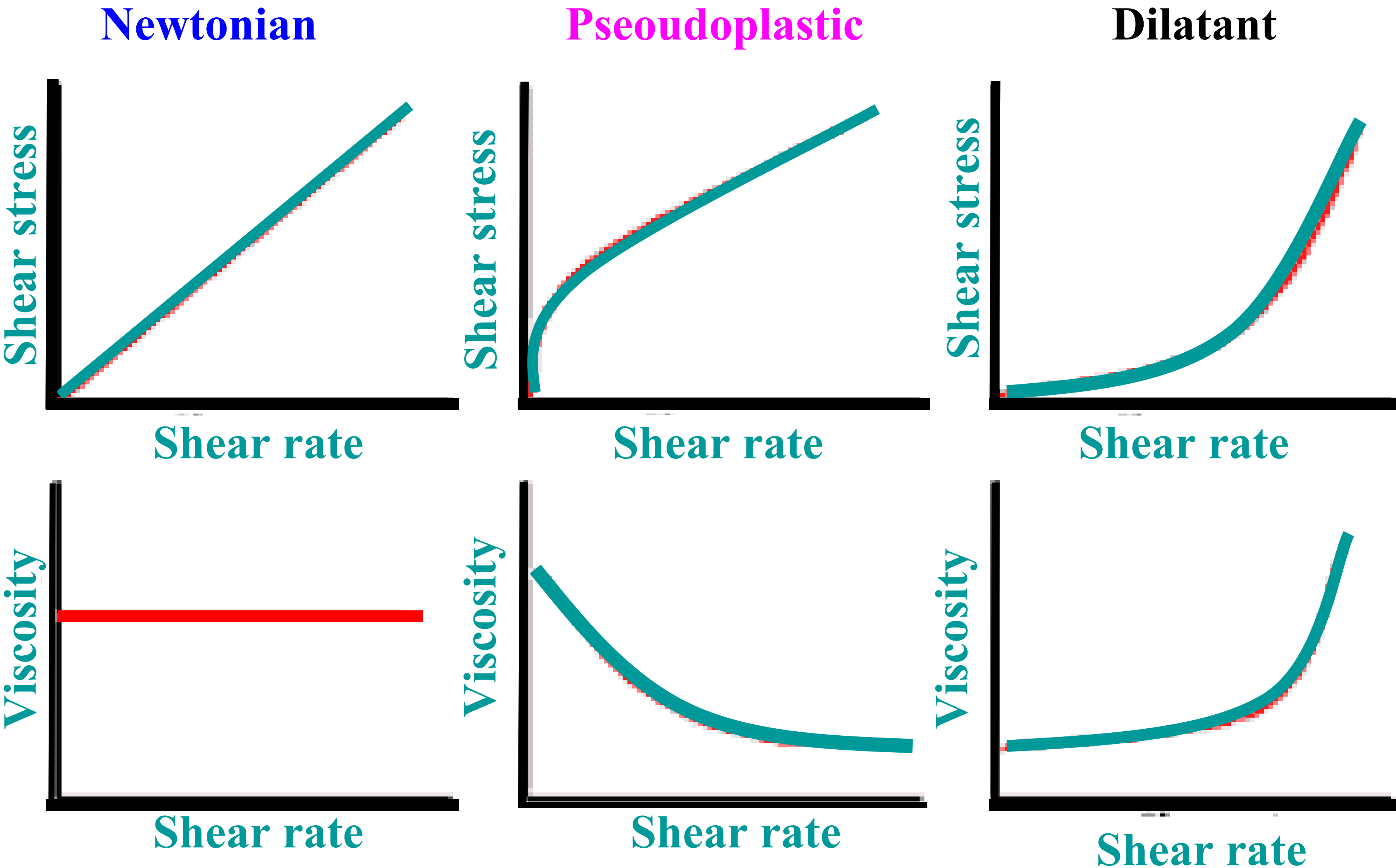


Rheological curves of Time - Independent and Time – Dependent Liquids

Visco-elastic Fluid Behaviour

A visco-elastic fluid displays both elastic and viscous properties.

A true visco-elastic fluid gives time dependent behaviour.



Common flow behaviours

Examples

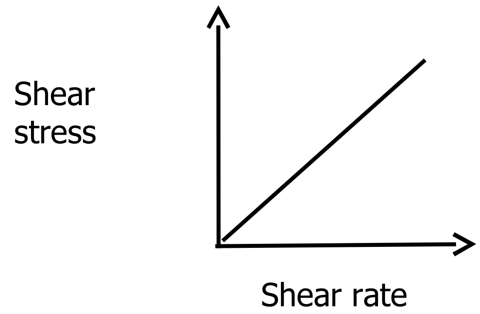
Newtonian flow occurs for simple fluids, such as water, petrol, and vegetable oil.

The Non-Newtonian flow behaviour of many microstructured products can offer real advantages. For example, paint should be easy to spread, so it should have a low apparent viscosity at the high shear caused by the paintbrush. At the same time, the paint should stick to the wall after its brushed on, so it should have a high apparent viscosity after it is applied. Many cleaning fluids and furniture waxes should have similar properties.

Examples

The causes of Non-Newtonian flow depend on the colloid chemistry of the particular product. In the case of water-based latex paint, the shear-thinning is the result of the breakage of hydrogen bonds between the surfactants used to stabilise the latex. For many cleaners, the shear thinning behaviour results from disruptions of liquid crystals formed within the products. It is the forces produced by these chemistries that are responsible for the unusual and attractive properties of these microstructured products.

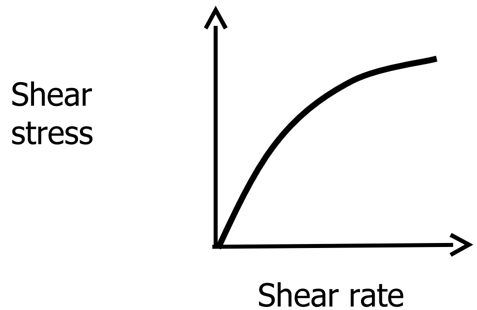
Newtonian Foods



Examples:

- Water
- Milk
- Vegetable oils
- Fruit juices
- Sugar and salt solutions

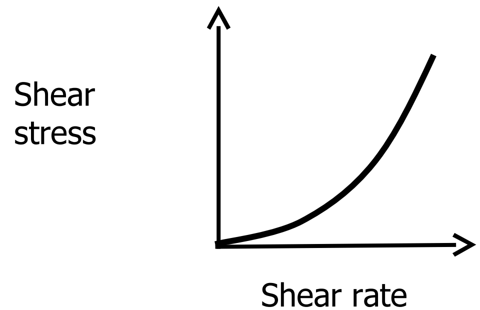
Pseudoplastic (Shear thinning) Foods



Examples:

- Applesauce
- Banana puree
- Orange juice concentrate
- Oyster sauce
- CMC solution

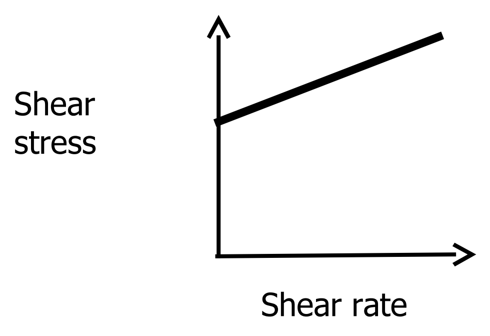
Dilatant (Shear thickening) Foods



Examples:

- Liquid Chocolate
- 40% Corn starch solution

Bingham Plastic Foods



Examples:

- Tooth paste
- Tomato paste

Non-Newtonian Fluids

Newtonian Fluid

$$\tau_{rz} = -\mu \frac{du_z}{dr}$$

Non-Newtonian Fluid

$$\tau_{rz} = -\eta \frac{du_z}{dr}$$

η is the apparent viscosity and is not constant for non-Newtonian fluids.

η - Apparent Viscosity

The shear rate dependence of η categorizes non-Newtonian fluids into several types.

Power Law Fluids:

- Pseudoplastic – η (viscosity) decreases as shear rate increases (shear rate thinning)
- Dilatant – η (viscosity) increases as shear rate increases (shear rate thickening)

Bingham Plastics:

- η depends on a critical shear stress (τ_0) and then becomes constant

Modeling Power Law Fluids

Oswald - de Waele

$$\tau_{rz} = K\left(-\frac{du_z}{dr}\right)^n = \left[K\left(\frac{du_z}{dr}\right)^{n-1}\right]\left(-\frac{du_z}{dr}\right)$$

(boxed term: μ_{eff})

where:
K = flow consistency index
n = flow behavior index

Note: Most non-Newtonian fluids are pseudoplastic n<1.

Modeling Bingham Plastics

Yield stress

$$\tau_{rz} = -\mu_{\infty} \frac{du_z}{dr} \pm \tau_0$$

$$\left|\tau_{rz}\right| \geq \tau_0$$

Frictional Losses Non-Newtonian Fluids

Recall:

$$h_f = 2f\frac{L}{D}\frac{\overline{V}^2}{g}$$

Applies to any type of fluid under any flow conditions

Power Law Fluid

$$\tau_{rz} = K\left(-\frac{du_z}{dr}\right)^n$$

$$\frac{du_z}{dr} = -\left(-\frac{1}{2}\frac{\Delta p}{KL}\right)^{1/n} r^{1/n}$$

Boundary Condition

$$r = R \qquad u_z = 0$$

Velocity Profile of Power Law Fluid Circular Conduit

Upon Integration and Applying Boundary Condition
We can derive the expression for u(r)

$$u_z = \left(-\frac{1}{2}\frac{\Delta p}{KL}\right)^{1/n}\left(\frac{n}{n+1}\right)\left[R^{\frac{n+1}{n}} - r^{\frac{n+1}{n}}\right]$$

Power Law Results (Laminar Flow)

$$\Delta p = -\frac{2^{n+2}\left(\frac{3n+1}{n}\right)^{n} LK\overline{V}^{\,n}}{D^{n+1}}$$

↑ Hagen-Poiseuille (laminar Flow) for Power Law Fluid ↑

Recall

$$f = -\frac{1}{4}\left(\frac{D}{L}\right)\left(\frac{2}{\overline{V}^{\,2}}\right)\frac{\Delta p}{\rho}$$

Laminar Friction Factor
Power Law Fluid

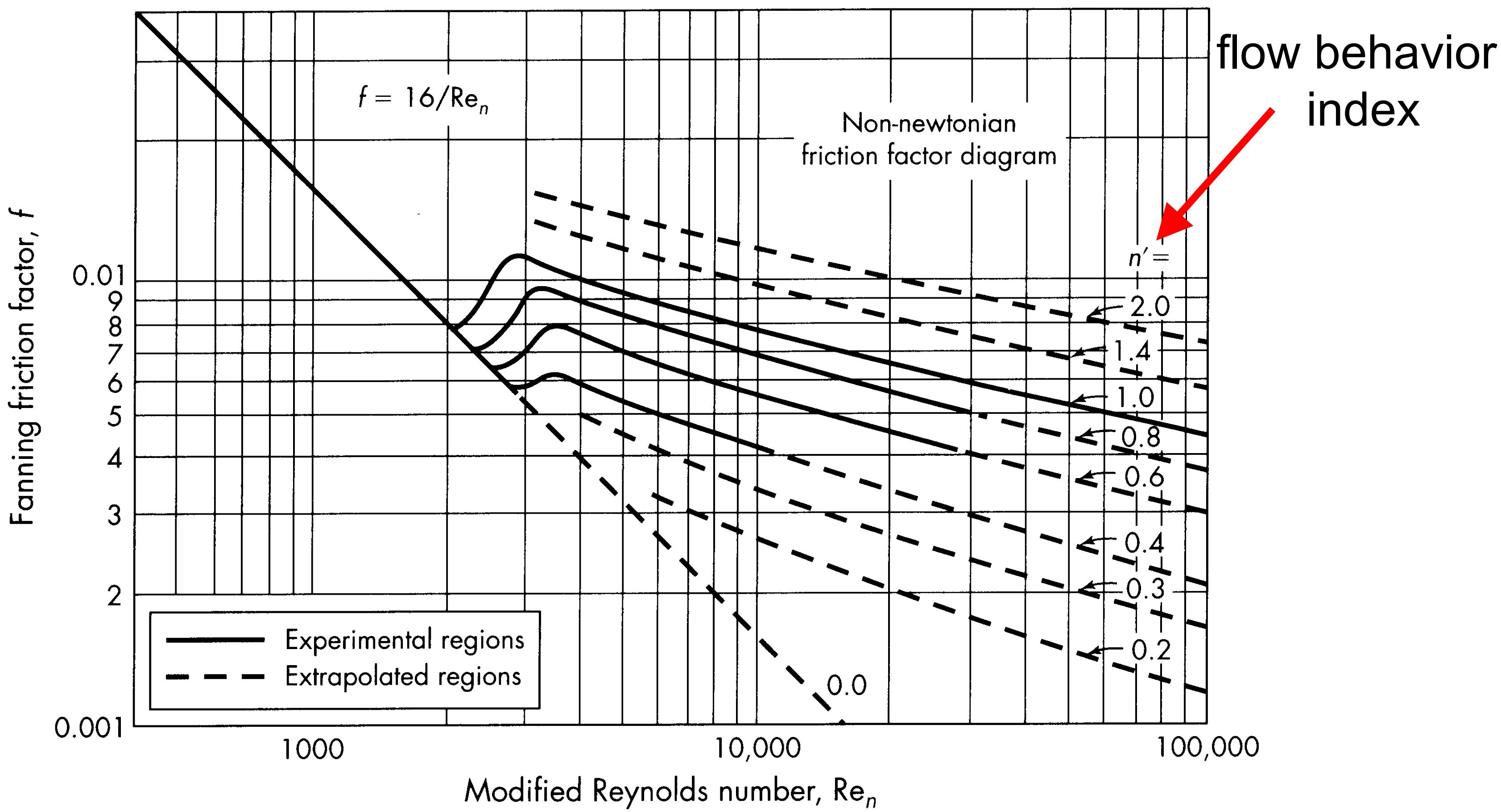
$$f = \frac{2^{n+1}\left(\frac{3n+1}{n}\right)^n K}{\overline{V}^{2-n} D^n \rho}$$

$$f = \frac{16}{Re_{PL}}$$

Define a Power Law Reynolds Number or Generalized Reynolds number (GRe)

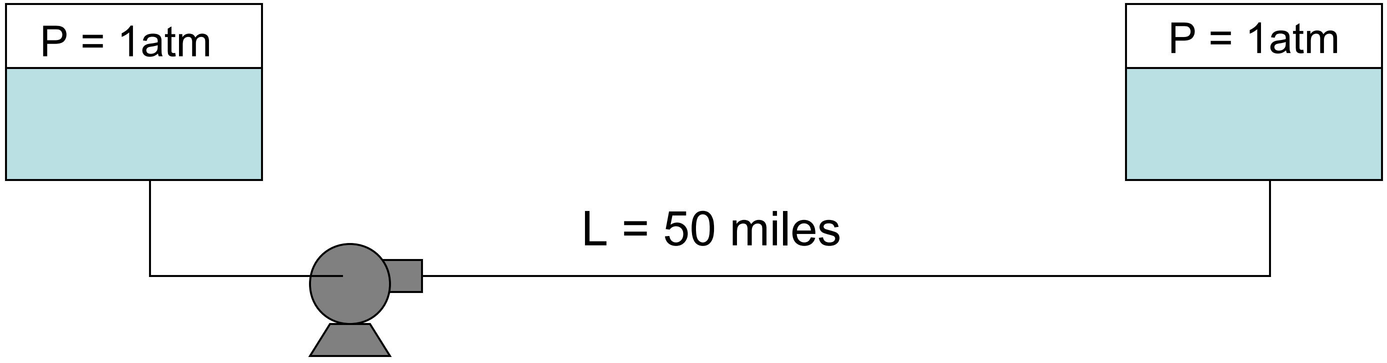
$$Re_{PL} = 2^{3-n}\left(\frac{n}{3n+1}\right)^n \frac{\overline{V}^{2-n} D^n \rho}{K}$$

Turbulent Flow



Power Law Fluid Example

A coal slurry is to be transported by horizontal pipeline. It has been determined that the slurry may be described by the power law model with a flow index of 0.4, an apparent viscosity of 50 cP at a shear rate of 100 /s, and a density of 90 lb/ft3. What horsepower would be required to pump the slurry at a rate of 900 GPM through an 8 in. Schedule 40 pipe that is 50 miles long ?



$$K'\left(\frac{\partial V}{\partial r}\right)^{n'} = \mu_{app}\left(\frac{\partial V}{\partial r}\right)$$

$$K' = 50cP\left(\frac{100}{s}\right)^{1-0.4} = 0.792\,\frac{kg}{m\ s^{1.6}}$$

$$\tilde{V} = \left(\frac{900gal}{\min}\right)*\left(\frac{1ft3}{7.48gal}\right)*\left(\frac{1\min}{60s}\right)*\left(\frac{1}{0.3474\,ft2}\right)*\left(\frac{m}{3.281\,ft}\right) = 1.759\,\frac{m}{s}$$

$$RE_N = 2^{(3-0.4)}\left(\frac{0.4}{3*(0.4)+1}\right)^{0.4}\left[\frac{(0.202\ m)^{0.4}\left(1442\,\frac{kg}{m^3}\right)\left(1.759\,\frac{m}{s}\right)^{1.6}}{0.792\,\frac{kg}{m\ s^{1.6}}}\right] = 7273$$

$$W_p = \frac{\Delta P}{\rho} + \frac{\Delta \alpha V^2}{2g_c} + \frac{g\Delta Z}{g_c} + h_f$$

$$W_p = h_f = 4f\left(\frac{L}{D}\right)\frac{V^2}{2}$$

$$f = 0.0048 \quad Fig\ 5.11$$

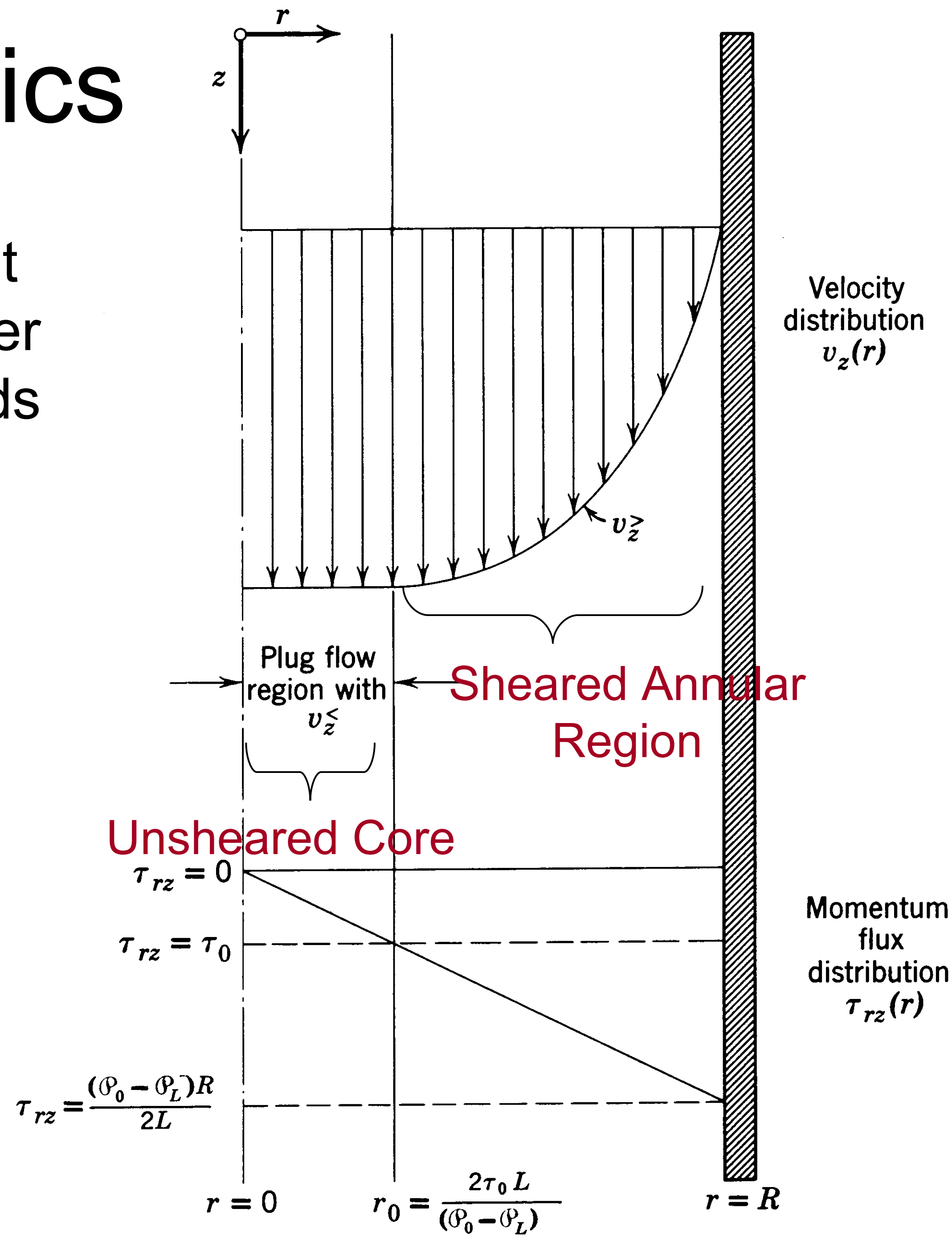
$$W_p = h_f = 4(0.0048)\left(\frac{80460m}{0.202m}\right)\frac{\left(1.760\frac{m}{s}\right)^2}{2} = 11{,}845\frac{m^2}{s^2}$$

$$\dot{m} = 1.759\frac{m}{s} * \left(0.0323\ m^2\right) * \left(1442\frac{kg}{m^3}\right) = 81.9\frac{kg}{s}$$

$$Power = \frac{81.9\frac{kg}{s}\left(11{,}845\frac{m^2}{s^2}\right)}{1000} = 970.1\ kW = 1300\ Hp$$

Bingham Plastics

Bingham plastics exhibit Newtonian behavior after the shear stress exceeds τ_o. For flow in circular conduits Bingham plastics behave in an interesting fashion.



Bingham Plastics

Unsheared Core

$$r \le r_c \qquad u_z = u_c = \frac{\tau_0}{2\mu_\infty r_c}\left(R - r_c\right)^2$$

Sheared Annular Region

$$r > r_c \qquad u_z = \frac{\left(R - r\right)}{\mu_\infty}\left[\frac{\tau_{rz}}{2}\left(1 + \frac{r}{R}\right) - \tau_0\right]$$

Laminar Bingham Plastic Flow

$$f = \frac{16}{\mathrm{Re}_{BP}}\left[1 + \frac{He}{6\,\mathrm{Re}_{BP}} - \frac{He^4}{3 f^3 \left(\mathrm{Re}_{BP}\right)^7}\right]$$ (Non-linear)

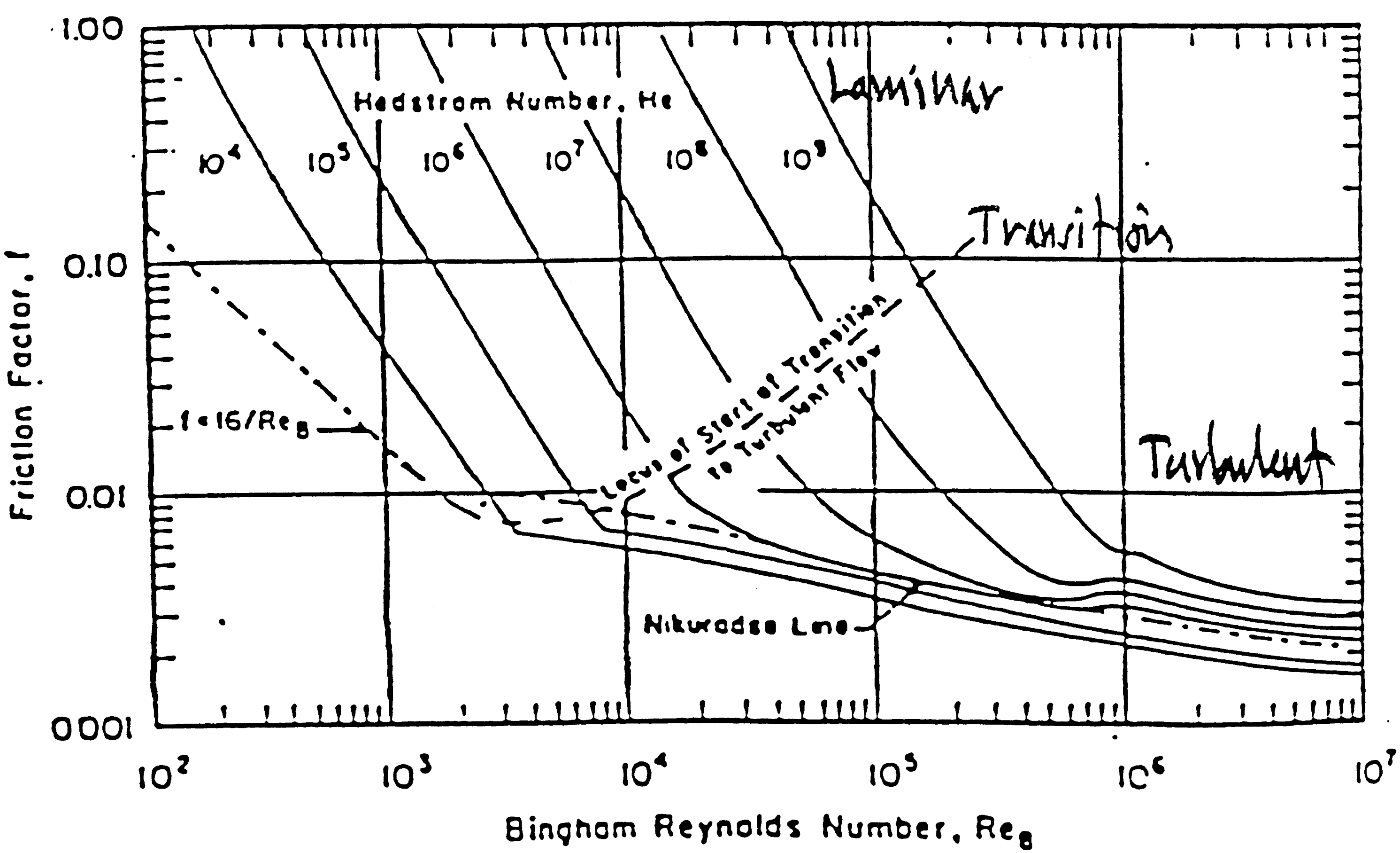
$$He = \frac{D^2 \rho \tau_0}{\mu_\infty^2}$$ Hedstrom Number

$$\mathrm{Re}_{BP} = \frac{D \rho \overline{V}}{\mu_\infty}$$

Turbulent Bingham Plastic Flow

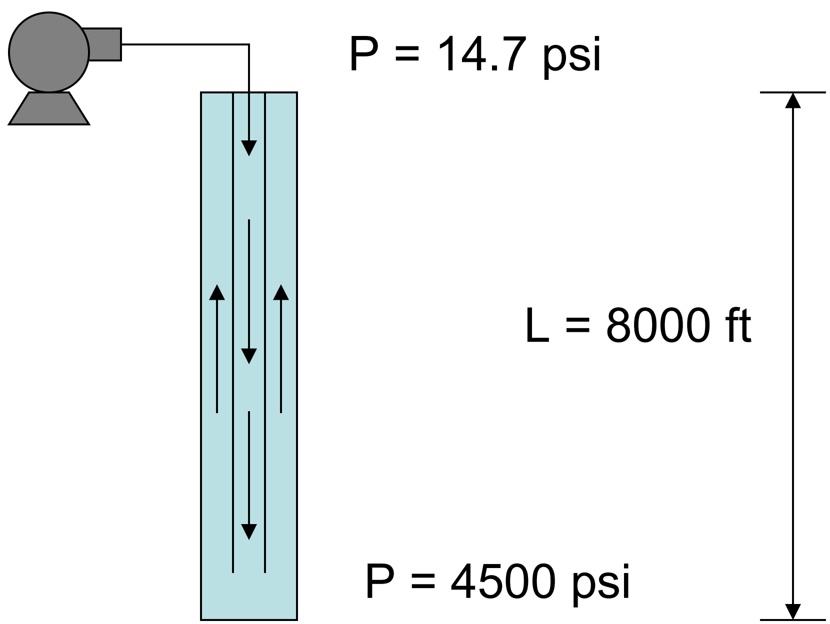
$$f = 10^{a}\,\mathrm{Re}_{BP}^{-0.193}$$

$$a = -1.378\left(1 + 0.146e^{-2.9x10^{-5}\,He}\right)$$



Bingham Plastic Example

Drilling mud has to be pumped down into an oil well that is 8000 ft deep. The mud is to be pumped at a rate of 50 GPM to the bottom of the well and back to the surface through a pipe having an effective diameter of 4 in. The pressure at the bottom of the well is 4500 psi. What pump head is required to do this ? The drilling mud has the properties of a Bingham plastic with a yield stress of 100 dyn/cm^2, a limiting (plastic) viscosity of 35 cP, and a density of 1.2 g/cm3.



$$D = \frac{4}{12}\,ft = 0.3333\ ft \qquad Area = 0.0873\ ft^2$$

$$V = 50\,\frac{gal}{\min} * \left(\frac{\min}{60s}\right) * \left(\frac{ft^3}{7.48gal}\right) * \left(\frac{1}{0.0873\,ft^2}\right) = 1.276\,\frac{ft}{s}$$

$$\rho = 1.2 * 62.4\,\frac{lb_m}{ft^3} = 74.88\,\frac{lb_m}{ft^3}$$

$$\mu = 35\ cP * \left(\frac{6.7197 \times 10 - 4\,\frac{lb_m}{ft\ s}}{cP}\right) = 0.0235\,\frac{lb_m}{ft\ s}$$

$$N_{RE} = \frac{0.3333\,ft * \left(1.276\,\frac{ft}{s}\right) * \left(74.88\,\frac{lb_m}{ft3}\right)}{0.0235\,\frac{lb_m}{ft\ s}} = 1355$$

$$\tau_o = 100\,\frac{dyn}{cm^2} = 100\,\frac{g}{s^2 cm}$$

$$N_{HE} = \frac{\left(4in\left(\frac{2.54\ cm}{in}\right)\right)^2 * \left(1.2\frac{g}{cm^3}\right) * \left(\frac{100g}{s^2 cm}\right)}{\left(0.35\frac{g}{cms}\right)^2} = 1.01\times 10^5$$

$$f = 0.14$$

$$W_p = \frac{\Delta P}{\rho} + \frac{\Delta \alpha V^2}{2g_c} + \frac{g\Delta Z}{g_c} + h_f$$

$$W_p = \frac{(4500 - 14.7)\frac{lb_f}{in^2}\left(\frac{144\ in^2}{ft^2}\right)}{74.88\frac{lb_m}{ft^3}} - 8000\frac{ft\ lb_f}{lb_m} + \frac{4*0.14*(8000\,ft)}{0.3333\,ft}\left(\frac{\left(1.276\frac{ft}{s}\right)^2}{2*\left(\frac{32.2\,ft\ lbm}{lb_f s^2}\right)}\right)$$

$$Wp = (8626 - 8000 + 339) = 965\frac{ft\ lb_f}{lb_m}$$

Viscometers

In order to get meaningful (universal) values for the viscosity, we need to use geometries that give the viscosity as a scalar invariant of the shear stress or shear rate. Generalized Newtonian models are good for these steady flows: tubular, axial annular, tangential annular, helical annular, parallel plates, rotating disks and cone-and-plate flows. Capillary, Couette and cone-and-plate viscometers are common.

Table 8.3 Shear Rate, Shear Stress and Viscosity
for Several Viscometers in steady flow
Ω = rotation speed, T = torque

Viscometer	Shear Rate, $\dot{\gamma}$	Shear Stress, $\tau\gamma$	Viscosity, η	Comment
Narrow Gap Concentric Cylinder $r_1/r_o > .97$	$\frac{r_o\Omega}{r_o - r}$	$\frac{T}{2\pi r_o^2 L}$	$\frac{T(r_o - r_1)}{2\pi r_o^3 \Omega_1 L}$	
Wide-Gap Concentric Cylinder	$\frac{2\Omega}{n(1-b^{2/n})}$	$\frac{T}{2\pi r_1^2 L}$	$\frac{T_n(1-b^{2/n})}{4\pi r_1^2 L \Omega_1}$	n refers to power law exponent
Rotating Cylinder in Large Volume	$\frac{2\Omega_1}{n}$	$\frac{T}{2\pi r_1^2 L}$	$\frac{T_n}{4\pi r_1^2 \Omega_1 L}$	.1 to 10 s^{-1}
Cone and Plate Viscometer	$\frac{\Omega_1}{\theta_0}$	$\frac{3T}{2\pi a^3}$	$\frac{3T\theta_0}{2\pi a^3 \Omega_1}$	
Parallel Plate Viscometer	$\frac{a\Omega_1}{n}$	$\frac{3T}{2\pi a^3(1+\frac{n}{3})}$	$\frac{3TL}{2\pi a^4 \Omega_1 (1+\frac{n}{3})}$	
Capillary Viscometer	$\frac{4Q}{\pi a^3}\left[\frac{3}{4}+\frac{1}{4}\frac{d\ln Q}{d\ln \tau_w}\right]$	$\frac{a}{2}\ \frac{dP}{dL}$ a = radius	$\frac{\pi a^4\, dP/dL}{8Q\left[\frac{3}{4}+\frac{1}{4}\frac{d\ln Q}{d\ln \tau_w}\right]}$	Q=flow

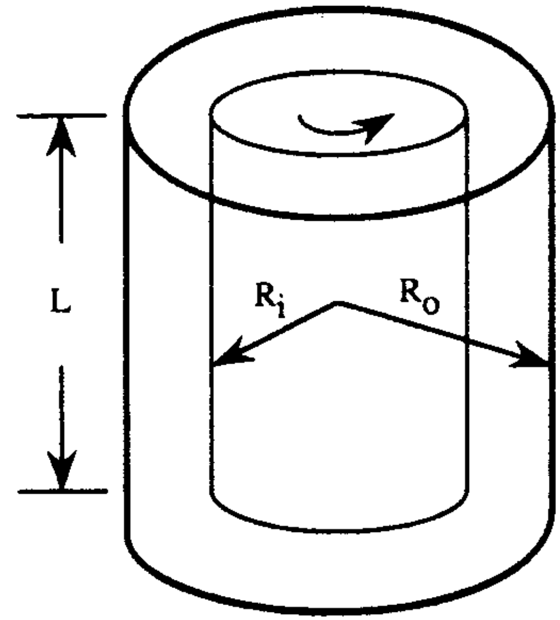
Non-newtonian fluid

- from

$$\Omega = 2\pi r^2 L\sigma$$

$$\gamma = r\frac{d\omega}{dr}$$

$$\frac{\Omega}{2\pi L r^2} = -\mu\left[\frac{r d\omega}{dr}\right]^n$$



Integrate from r = Ro→ Ri and ω = 0→ω_i

Non-newtonian fluid

η or μ_a

γ^o

- obtain

$$\tau = -K\left[\frac{4\pi N}{n}\right]^n = -K.n^{-n}\left[4\pi N\right]^{n-1}\left[4\pi N\right]$$

$$\eta = -K.n^{-n}\left[4\pi N\right]^{n-1}$$

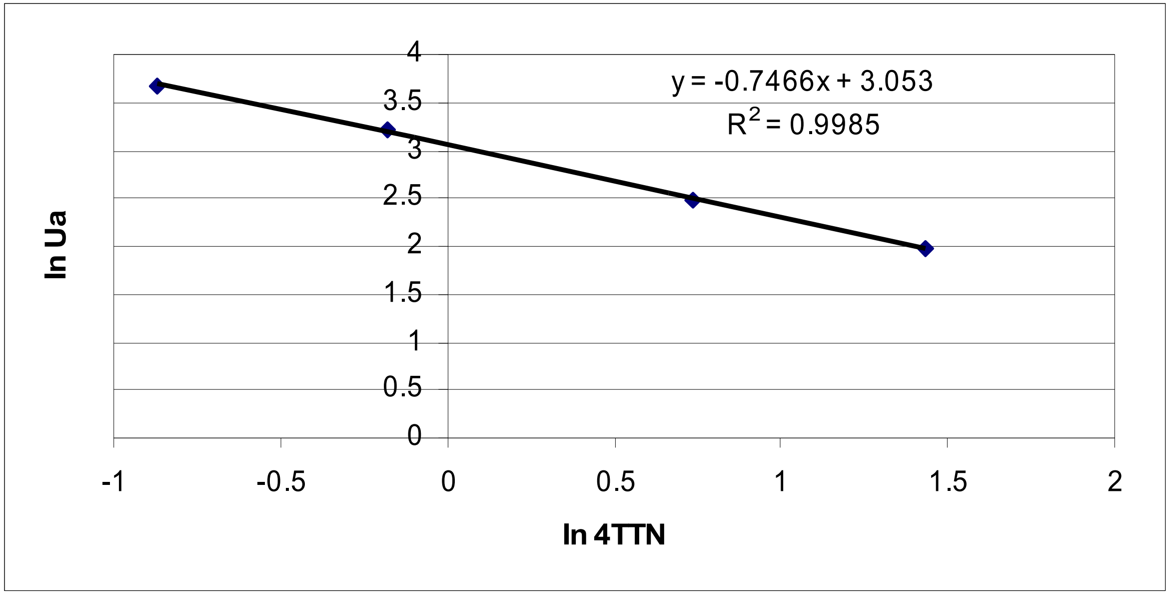
$$\ln\eta = \ln K - n\ln n + (n-1)\ln\left[4\pi N\right]$$

Linear : y = y-intercept + slope (x)

Example

Given the apparent viscosity data of a fluid in a rotational viscometer, determine the rheological parameters **K and n**

Speed (rpm)	Apparent viscosity (cP)	Apparent Viscosity (Pa s)	ln μ_A	N' (rps)	ln(4πN')
2	39500	39.5	3.6763	0.0333	-0.8702
4	25200	25.2	3.2268	0.0667	-0.1770
10	12060	12.1	2.4899	0.1667	0.7393
20	7230	7.2	1.9782	0.3333	1.4324



n = 1.7466

K = 56.09

(shear thinning)