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A GELFAND REPRESENTATION THEORY FOR C^*-ALGEBRAS

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Recent work by the author which was independently duplicated in part by Giles and Kummer has made it possible to generalize the Gelfand representation theorem for abelian C-*algebras to the non-abelian case. Let A be a C-algebra with unit. If A is abelian, it can be identified with the algebra of all continuous complex-valued functions on its maximal ideal space (with the hull-kernel topology). A less precise way of looking at this result would be to say that an abelian A is completely recoverable from the set of maximal ideals and a certain structure thereon (in this case, a topology). If we use the latter description as the basis for a theory applicable to non-abelian A, we find immediately that two changes are necessary. The set of maximal ideals is replaced by the set of maximal left ideals, and secondly, the structure defined thereon will not be a topology, though it will have many similar properties when viewed correctly. This paper shows how the C^*-algebra is recovered from the maximal left ideals (with structure).

I. Preliminaries. Consider the W^*-algebra A^{**}, the second Banach space dual of A [9, p. 236]. There exists a central projection $z \in A^{**}$ which is the supremum of all the minimal projections in A^{**} [3, p. 278]. Set $M = zA^{**}$. The minimal projections of M are in one to one correspondence with the maximal left ideals of A [3, p. 280 and 9, p. 48], so that we can define a structure on this set of minimal projections instead of directly on the maximal left ideals. Naturally the first thing we "build" is the algebra M. We then single out a class L of projections in M as the q-open projections as follows. First note that we can consider $A \subset M$ since $A \subset A^{**}$ and $A \rightarrow zA$ is a *-isomorphism [9, p. 39]. (Also we can view M as the direct sum of irreducible representations of A, one from each equivalence class.) A projection p in M is q-open if there exists a closed left ideal I of A such that the weak* closure $\bar{I}$ of I in M is of the form Mp. The q-open projections are analogous to the open sets of a topology.

If A were abelian, M would be the algebra of all bounded complex function on its maximal ideal space K. The q-open projections would be characteristic functions of open sets of K for the hull-kernel topology. A self-adjoint operator b in M actually lies in $A(\subset M)$ if and only if the spectral projections of b corresponding to open sets of real numbers are q-open projections in the above sense.

This is a restatement of Gelfand's theorem since a function is continuous if and only if its inverse images of open sets are open.

We may now state an identical theorem for the non-abelian case. The proof follows immediately from the addendum to [4] and Theorem II. 17 of [3].

THEOREM I.1. *A self-adjoint operator $b \in M$ lies in $A(\subset M)$ if and only if each spectral projection of b which corresponds to an open subset of the real numbers is also a q-open projection.*

This theorem says that we may reconstruct A from its set of maximal left ideals together with the above defined structure. As a corollary we note that if two algebras A_1 and A_2 have "isomorphic structures" then they are isomorphic.

COROLLARY I.2. *Let A_1 and A_2 be C^*-algebras with $M_i = z_i A_i^{**}$ and L_i the q-open projections in M_i $(i = 1, 2)$. If there exists a $*$-isomorphism $\varphi \colon M_1 \rightarrow M_2$ which maps L_1 onto L_2, then $\varphi \mid A_1$ is an isomorphism of A_1 onto A_2.*

This paper extends these results to C^*-algebras without unit with appropriate modifications suggested by the abelian case. A number of other "topological" results are proved, and counter-examples are given to close off several tempting avenues of approach.

To complete our terminology, we shall assume from now on that A is a C^*-algebra which may not have a unit. The above discussion still applies to get $z \in A^{**}$ and we set $M = zA^{**}$. Identify A and $zA \subset M$ and call M the pure state q-space of A. (The terminology is lifted from [11].) We have already defined q-open projections in M, and their complements (in M) are called q-closed. $\tilde{A}$ will denote the algebra A with unit adjoined as in [9, p. 7]. Note that A is a closed two-sided ideal in $\tilde{A}$ of co-dimension one. Thus $\tilde{A}^* \cong A^* \oplus \{\lambda f_\infty\}$, where f_∞ is the unique pure state of $\tilde{A}$ which vanishes on A. Also the pure state q-space $\tilde{M}$ of $\tilde{A}$ is: $M \cong M \oplus \{\lambda 1_\infty\}$, with $f_\infty(1_\infty) = 1$. In view of Theorem II.17 of [3] all the properties of open or closed projections in A^{**} (as considered in [3 and 4]) carry over immediately to corresponding properties of q-open or q-closed projections in M.

II. The problem of compactness. Although the notion of compactness is vaguely introduced in [3], it is clear that a theory which claims to generalize locally compact Hausdorff spaces should generalize the notion of a compact set.

DEFINITION II.1. A projection $p \in M$ is *q-compact* if p is q-closed and there exists $b \in A^+$ $(= \{a \in A\colon a \geqq 0\})$ with $bp = p$.

There are a number of conditions equivalent to compactness for a set in a locally compact Hausdorff space. It would be desirable to show that many of them can be extended to equivalent conditions for q-compactness. The most desirable such condition would be:

Conjecture II.2. A regular [10, p. 408] projection $p \in M$ is compact if for every family $\{p_\alpha\}$ of q-closed projections such that the family $\{p_\alpha \wedge p\}$ has the finite intersection property, then $p \wedge \bigwedge_\alpha p_\alpha \neq 0$.

We shall prove this for certain p in Theorem II.6. The conjecture is false without the assumption of regularity (see Example IV.5).

LEMMA II.3. *Suppose B is a C^*-algebra, $b \in B^+$, $p \in B^{**}$ a projection and $\{a_\alpha\} \subset B$ an increasing net of positive elements with $\| b^{1/2} - b^{1/2}a_\alpha \| \underset{\alpha}{\rightarrow} 0$. If $b \geqq p$ (considering $B \subset B^{**}$), then $\| p - a_\alpha p \| \underset{\alpha}{\rightarrow} 0$.*

Proof. Since $\| b^{1/2} - b^{1/2}a_\alpha \| \underset{\alpha}{\rightarrow} 0$, clearly $\| (1 - a_\alpha)\, b\, (1 - a_\alpha) \| \underset{\alpha}{\rightarrow} 0$. Since $(1 - a_\alpha)\, b\, (1 - a_\alpha) \geqq (1 - a_\alpha)\, p\, (1 - a_\alpha)$, we get

$$\| (1 - a_\alpha)\, p\, (1 - a_\alpha) \| = \| (1 - a_\alpha) p \|^2 = \| p - a_\alpha p \|^2 \longrightarrow 0 .$$

LEMMA II.4. *If p is q-closed for A and we consider $\tilde{A}$ and $\tilde{M}$ as above with $M \subset \tilde{M}$ (hence $p \in \tilde{M}$) and there exists $b \in A^+$ with $b \geqq p$, then p is q-closed in $\tilde{M}$.*

Proof. Let $K = (p\ A^*\ p)^+$. Then K is $\sigma(A^*, A)$ closed by [3, II.2]. If K is not $\sigma(\tilde{A}^*, \tilde{A})$ closed, then there is a net $\{f_\alpha\} \subset K$ with $\| f_\alpha \| \subset K$ with $\| f_\alpha \| = 1$ and $f_\alpha \underset{\alpha}{\rightarrow} f$, $\sigma(\tilde{A}^*, \tilde{A})$, for some $f \in \tilde{A}^*$ with $\| f \| = f(1) = 1$. Since $\tilde{A}^* = A^* \oplus \{\lambda f_\infty\}$, we get $f = f_0 + \lambda f_\infty$ where $f_0 \in A^{*+}$ and $\lambda \geqq 0$. For any $c \in A$ with $c \geqq p$,

$$f_0(c) = f(c) = \lim_\alpha f_\alpha(c) \geqq \overline{\lim_\alpha} f_\alpha(p) = 1$$

since each $f_\alpha \in K$.

Now if $1 \in A$, then A^* is $\sigma(\tilde{A}^*, \tilde{A})$ closed in $\tilde{A}^*$, so the conclusion of this lemma is immediate. If $1 \notin A$, let $\{a_\gamma\} \subset A^+$ be an increasing approximate unit. Then, by Lemma II.3, $\{a_\gamma\}$ is an approximate unit for p also. Thus given $\varepsilon > 0$ there exists $c \in A$ with $c \geqq p$ and $\| c \| \leqq 1 + \varepsilon$ by Theorem 1.2 of [2]. Hence $f_0(c) \geqq 1$ by the above. Since $\varepsilon > 0$ was arbitrary, $\| f_0 \| = 1$, so $\lambda = 0$, since

$$\| f \| = \| f_0 \| + | \lambda | = 1 .$$

Thus $f \in A^*$. Since $\{f_\alpha\} \subset K$, K is $\sigma(A^*, A)$ closed, and $f_\alpha \underset{\alpha}{\rightarrow} f$ in the $\sigma(\tilde{A}^*, \tilde{A})$ topology, we see that $f \in K$, so K is $\sigma(\tilde{A}^*, \tilde{A})$ closed.

THEOREM II.5. *If p is q-closed and there exists $b \in A$ with $b \geqq p$, then p is q-compact.*

Proof. Since p is q-closed for $\tilde{A}$ by Lemma II.4, there exist $\{b_\alpha\} \subset \tilde{A}$, $b_\alpha = a_\alpha + \lambda_\alpha 1$ with $a_\alpha \in A$, $1 \geqq b_\alpha \geqq p$ and $b_\alpha \downarrow p$ in M [5, proof of Prop. 1]. Thus each b_α (and hence a_α) commutes with p. Since $f_\infty(b_\alpha) \underset{\alpha}{\rightarrow} 0$, there exists α_0 with $f_\infty(b_{\alpha_0}) < 1/2$. Thus $\lambda_{\alpha_0} < 1/2$ since $f_\infty(a_{\alpha_0}) = 0$. Let $g(t)$ be a continuous function which has $g(t) = 1$ for $t \geqq 1/2$, $g(0) = 0$, $0 \leqq g(t) \leqq 1$ for all t. Then $g(a_{\alpha_0}) \geqq p$. (Since a_{α_0}, b_{α_0} and p all commute, we may view them as functions on a common locally compact space; this makes the assertion clear.) Since $g(a_{\alpha_0}) \in A$, the theorem follows.

The construction in the proof of last theorem will not work for all projections p in M having only the property that $p \leqq b \in A$, even though it easily works whenever p is central.

THEOREM II.6. *Suppose $1 \in A$ and A is separable. Then Conjecture* II. 2 *holds for central projections $p \in M$.*

Proof. Suppose p satisfies the intersection condition of Conjecture II.2. We need only show p is q-closed since $1 \in A$. If it is not q-closed, let $\bar{p}$ be its closure [3, II. 11] and let $q \leqq \bar{p} - p$ be a minimal projection. As in [1] there exists a strictly positive element a_0 in $\{a \in A: aq = qa = 0\} = I$, so we let p_n be the spectral projection of a_0 corresponding to the interval $[0, 1/n]$. Since $\bigwedge_n p_n \wedge p = 0$, there is some n_0 with $p_{n_0} p = 0$ by hypothesis. Since p is central, the spectral projection x of a_0 corresponding to $[1/n_0, \infty)$ is q-closed and $x \geqq p$. This contradicts $xq = 0$ and $q \leqq \bar{p}$.

THEOREM II.7. *If p is q-compact, then p satisfies the intersection condition of Conjecture* II.2.

Proof. Since p is also q-closed in $\tilde{M}$ by Lemma II.4, the theorem follows from [3, II.10] for if $\{p_\alpha\}$ are q-closed in M, then their q-closures $\{\bar{p}_\alpha\}$ in $\tilde{M}$ have no larger M component. (Recall that $\tilde{M} = M \oplus \{\lambda 1_\infty\}$ with $1_\infty M = \{0\}$.) Thus if $p \wedge \bigwedge_{\alpha \in J} \bar{p} \neq 0$ for all finite sets J, $p \wedge \bigwedge_\alpha \bar{p}_\alpha \neq 0$, so $p \wedge \bigwedge_\alpha p_\alpha \neq 0$, since $p \wedge \bar{p}_\alpha = p \wedge p_\alpha$.

Next we move in a different direction for a characterization of

M. If A were an abelian C^*-algebra of functions containing the constants and separating the points of the topological space Ω, then A consists of all continuous functions on Ω if and only if Ω is compact. Following [11] we define a q-space to be an atomic W^*-algebra. If M_1 is a q-space and $A \subset M_1$, is a weak* dense C^*-subalgebra with $1 \in A$, we can define a q-open projection in M_1 as a sup of range projections of elements of A. Naturally q-closed projections are complements of q-open projections. If $M_1 = M$, the two definitions coincide.

THEOREM II.8. *If A is separable and $A \subset M_1$ as above, then there is an A-preserving *-isomorphism between M_1 and M if and only if the q-closed projections of M_1 satisfy the intersection condition of Conjecture* II.2.

Proof. If M_1 is *-isomorphic to M under an A-preserving map the verification is routine. Now suppose the q-closed projections of M_1 satisfy the intersection condition. If every pure state of A extends to a normal state of M_1, there is a natural isomorphism between M_1 and M which preserves A because of the definition of M as a subset of A^{**}. Thus let f be a pure state of A with no normal extension to M_1. Let $\{a_j\} \subset A$ be an increasing positive abelian [1] approximate unit for $\{a \in A: f(a^*a + aa^*) = 0\}$. Then let p_{jn} be the spectral projection of a_α corresponding to the interval $(1/n, \infty)$. Cleary $\bigvee_{j,n} P_{jn} = 1$ in M_1, for if not, then $(1 - \bigvee_{j,n} P_{jn})$ would be one-dimensional, hence f could be extended to a normal functional on M_1 with support $(1 - \bigvee_{j,n} p_{jn})$. But $\{(1 - p_{jn}\}\}$ is a decreasing net of closed projections in M_1 with $\bigwedge_{j,n}(1 - \vee p_{jn}) = 0$. Thus $(1 - p_{jn}) = 0$ for some j and n. Hence a_j is invertible, so $f = 0$, a contradiction.

III. The Gelfand representation.

LEMMA III.1. *If p is q-closed, p_1 is q-compact, and $p_1 p = 0$, then there exists $a \in A^+$ with $\|a\| = 1$, $ap = 0$, and $ap_1 = p_1$.*

Proof. Set $A_1 = \{a \in A: ap = pa = 0\}$. Consider $\widetilde{A}_1 \subset \widetilde{A}$. By Lemma II.4, p_1 is q-closed for $\widetilde{A}$. Thus the unit ball of $p_1 A^* p_1 = p_1 \widetilde{A}_1^* p_1$ $= p_1 \widetilde{A}_1^* p_1$ is compact for the $\sigma(\widetilde{A}^*, \widetilde{A})$ topology, hence also for the weaker $\sigma(\widetilde{A}_1^*, \widetilde{A}_1)$. Thus $p_1 \widetilde{A}_1^* p_1$ is $\sigma(\widetilde{A}_1^*, \widetilde{A})$ closed, so p_1 is q-closed for $\widetilde{A}_1$ [3, II.2]. Now by [4, I.1] there exists $a \in \widetilde{A}_1^+$ with $\|a\| = 1$, $ap_1 = p_1$ and $ap_2 = 0$, where p_2 is the one dimensional projection in $\widetilde{M}_1$ which supports the pure state f_∞ which vanishes on A_1. Since $ap_2 = 0$, $a \in A_1$, so $ap = 0$.

This last Lemma generalizes Urysohn's Lemma. We now define an analog for a continuous function.

DEFINITION III.2. A self-adjoint operator $b \in M$ is *q-continuous* if each spectral projection of b corresponding to an open subset of the spectrum of b is also q-open.

Now we can state our best Gelfand representation theorem.

THEOREM III.3. *The self-adjoint elements of A are exactly those q-continuous elements b of M such that the spectral projections of b corresponding to closed subsets of the spectrum of b which don't contain 0 are q-compact* (i.e., b "vanishes at ∞").

Proof. Consider $A \subset \tilde{A}$, $M \subset \tilde{M}$. If $b \in \tilde{A}$, then $b \in A$, since $b \subset M$. But if p is the spectral projection of b corresponding to an open subset U of the spectrum of b, we consider two cases. First if $0 \notin U$, then $p \in M$, hence p is q-open since it is q-open for A by hypothesis. Secondly if $0 \in U$, then the complement of U is closed and doesn't contain 0, thus the spectral projection corresponding to it is q-compact for A, hence q-closed for $\tilde{A}$ by Lemma II.4. Thus b is q-continuous for $\tilde{A}$ and Theorem I.1 applies.

For the abelian case it is well-known that if B is a C^*-algebra of continuous bounded functiohs on a locally compact Hausdorff space Ω such that the smallest topology on Ω making all $b \in B$ continuous agrees with the given topology, then B contains all continuous functions vanishing at ∞ on Ω. A similar result is true in general.

THEOREM III.4. *Let A_1 be a C^*-subalgebra of M such that the q-open projections for A_1 in M are the same as the q-open projections for A. Then $A_1 \supset A$ and $A_1 = A$ if $1 \in A$.*

Proof. Let $A_2 = A \cap A_1$. If p is q-open for A, then $p = \bigvee_\alpha p_\alpha$ where p_α is q-open with q-compact closure. For each α, p_α is also A_1 open, so there exists a net $\{a_\alpha^\gamma\} \subset A_1$ with $0 \leqq a_\alpha^\gamma \uparrow_\gamma p_\alpha$. By hypothesis each $a \in A_1$ is q-continuous, and since p_α has compact closure, Theorem III.3 applies to give $\{a_\alpha^\gamma\} \subset A$, hence in A_2. Thus p is A_2 open. We now apply Theorem III.3 of [3] and get $A_2 = A$. (Theorem III.3 of [3] is stated for algebras with unit, but considering $\tilde{A}_2$ and $\tilde{A}$ we get the result.)

Now if $1 \in A$, Theorem I.1 gives that $A_1 \subset A$, so $A_1 = A$.

Recall that one way of constructing the double centralizer $M(A)$ of A is to let $M(A)$ be the idealizer of A in A^{**}, i.e.,

$$M(A) = \{b \in A^{**}: bA + Ab \subset A\} .$$

We first prove a lemma bringing $M(A)$ into M.

LEMMA III.5. *The mapping $b \to bz$ is a $*$-isomorphism of $M(A)$ into M.*

Proof. Suppose $b \geqq 0$ in $M(A)$ and $zb = 0$. Then let $a \in A$ with $0 < a \leqq b$. Then $za = 0$ since $za \leqq zb = 0$. This means $a = 0$, a contradiction.

From now on consider $M(A)$ as a subalgebra of M. A tempting conjecture would be;

Conjecture III.6. The self-adjoint elements of $M(A)$ are exactly the q-continuous elements of M.

Our next result is one half of the conjecture.

THEOREM III.7. *Every self-adjoint element of $M(A)$ is q-continuous.*

Proof. Let $\{a_\alpha\} \subset A$ be a positive increasing approximate unit for A. Let $b \in M(A)$ be self-adjoint and let U be an open subset of the spectrum of b with p the spectral projection of b corresponding to U. Let $\{b_n\}$ be a sequence of continuous functions of b with $0 \leqq b_n \uparrow p$. Then $\{b_n^{1/2} a_\alpha b_n^{1/2}\}$ is a net in A which is $\leqq p$ and converges to p. Thus p is q-open for A.

In [7] Dixmier introduces the ideal center of a C^*-algebra which is a C^*-subalgebra of $M(A)$ containing A. Dixmier constructs it in A^{**} but Lemma III.5 assures us the idea carries over to M as well. We can characterize it in the obvious way.

COROLLARY III.8. *The ideal center of A consists of exactly those central elements of M which are q-continuous.*

Proof. We need to show that if d is central in M and p-continuous and $a \in A$, then $da \in A$. Clearly we need only consider $d, a \geqq 0$ and $\|d\| = \|a\| = 1$. For $\lambda > 0$, the spectral projection p of (da) corresponding to the interval $[\lambda, \infty)$ is less than or equal to the spectral projection of a corresponding to $[\lambda, \infty)$ which is q-compact since $a \in A$. By III.3 we need only show ad is q-continuous.

To show that (ad) is q-continuous, let (α, β) be an open interval and consider a and d as real functions on $\sigma(ad)$ (the spectrum of ad). Then let $t_0 \in K = \{t: a(t)d(t) \in (\alpha, \beta)\}$. For sufficiently small ε and δ we have $U \cap V \subset K$, where $U = \{t: a(t_0) - \varepsilon < t < a(t_0) + \varepsilon\}$ and $V = \{t; d(t_0) - \delta < t < d(t_0) + \delta\}$. Since K is a union of open sets of the form $U \cap V$, the spectral projection p of ad in M corresponding to K is a union of projections corresponding to sets of the form $U \cap V$. But for any U and V as above, the spectral projections of (ad) corresponding to U and V are both q-open and they commute. Hence their intersection corresponds to $U \cap V$ and it is q-open [3, II.7]. Thus p is a union of q-open projections, hence it is q-open [3, II.5].

IV. Assorted results and examples. One interesting question is: What are all the different C^*-algebras which have a factor for their pure state p-space? If M is countably decomposable, then the question was answered in [13] where it was shown that the C^*-algebra must consist of exactly the compact operators in M (i.e., the C^*-algebra generated by the minimal projections). We can slightly extend this result.

THEOREM IV.1. *Suppose M is a factor. Then A consists of exactly the compact operators in M if any q-open projection p is countably decomposable.*

Proof. Let $A_0 = \{a \in A: ap = pa = a\}$. Then the pure state q-space M_0 of A_0 is pMp. By [13] A_0 consists of the compact operators in pMp. Thus A contains all the compact operators in M by [9, p.85]. But if A is strictly larger than the compact operators, then they form an ideal in A, so A has at least two inequivalent irreducible representations. This contradicts the assumption that M is a factor.

Next is a theorem of the Stone-Weierstrass type.

THEOREM IV.2. *Let $B \subset A$ be a C^*-subalgebra which separates the pure states of A and* 0. *If pBp is norm closed in M for each q-closed projection p for A, then $B = A$.*

Proof. By [3, III.2] M is also the pure state q-space for B. Let p_1 be the B-closure of p in M (i.e., the smallest projection $\geqq p$ which is q-closed for B). If $p_1 > p$, then there is a minimal projection p_2 in M with $p_2 \leqq p_1 - p$. Let $\{b_\alpha\} \subset B$ with $1 \geqq b_\alpha \downarrow_\alpha p_2$ in M. Then

$\| p_1 b_\alpha p_1 \| = 1$ for all α, but $\| p b_\alpha p \| \underset{\alpha}{\rightarrow} 0$ since p is q-closed. By [3, II.12] the map $B \rightarrow p_1 B p_1$ has closed range, and by hypothesis the map $p_1 B p_1 \xrightarrow{\varphi} pBp$ has closed range also. But since p_1 is the q-closure of p for B, the map φ is $1-1$. Thus φ^{-1} is continuous by the closed graph theorem, and this contradicts $\| p_1 b_\alpha p_1 \| = 1$, $\| p b_\alpha p \| \underset{\alpha}{\rightarrow} 0$.

The most difficult aspect of the q-theory is the existence of non-regular projections, even in the best of circumstances [4, I.2]. The next result shows that some interesting projections are regular.

PROPOSITION IV.3. *If p' is finite-dimensional, then p is regular.*

Proof. Let p_1 be the q-closure of p. Then p_1' is finite dimensional, so p_1' is q-closed [3, II.8]. Hence p_1 is q-open and q-closed, so $p_1' \in A$ by [3, II.18]. By considering $p_1 A p_1$, we can assume $p_1 = 1$. Let $b \in A$ with $\| b \| = 1$ and suppose $\| bp \| < 1$. This would be the case if p were not regular. Since $\| b^* b \| = 1$ and $\| b^* b p \| < 1$, we can assume $b > 0$. Let p_2 be the spectral projection of b corresponding to the open interval (δ, ∞), where $\| bp \| < \delta < 1$. Then p_2 is q-open and $p_2 \neq 0$, so $p_2 \wedge p \neq 0$ as follows. If $p_2 \wedge p = 0$, then $p_2' \vee p' = 1$. Since p' is finite dimensional, this implies that p_2 is finite dimensional. But then $p_2 \in A$, so we can get a minimal projection $p_3 \in A$ with $p_3 \leqq p'$. This contradicts $\bar{p} = 1$. Now if g is a pure state of A with $g(p_2 \wedge p) = 1$, then

$$g(bp) = g(b) = g(p_2 b p_2) \geqq g(\delta p_2) = \delta .$$

This contradicts the definition of δ.

The next proposition and example show how badly behaved non-regular projections can be and how reasonable regular projections are.

PROPOSITION IV.4. *If $p \in M$ is regular, f a pure state of A, $b \in A$ with $b \geqq p$ and $f(b) = 0$, then $f(\bar{p}) = 0$* ($\bar{p}$ = closure of p).

Proof. Let $\{a_\alpha\}$ be an increasing positive approximate unit for $\{a \in A: f(a^* a + a a^*) = 0\}$. By Lemma II.3 and by [2, I.2] we can get $\{b_n\} \subset A$ with $b_n \geqq p$, $\| b_n \| \leqq 1 + 1/n$, $f(b_n) = 0$. Let p_1 be the support projection of f. If $f(\bar{p}) \neq 0$, then there exists a pure state g of A with $g(\bar{p}) = 1$ and $g(p_1) \neq 0$. By regularity and [10, 6.1] there exists a net $\{g_\gamma\}$ of states of A with $g_\gamma \underset{\gamma}{\rightarrow} g$, $\sigma(A^*, A)$, and $g_\gamma(p) = 1$ for all γ. Let b_0 be a limit point of $\{b_n\}$ for the weak* topology of M, clearly $\| b_0 \| \leqq 1$. Since $g_\gamma(b_n) \geqq g_\gamma(p) = 1$ for all γ

and all n, then $g(b_n) \geqq 1$ for all n. Hence $g(b_0) \geqq 1$. But $||b_0 + p_1|| = 1$ since $b_n p_1 = 0$ for all n implies $b_0 p_1 = 0$ (and similarly $p_1 b_0 = 0$). Hence $g(b_0 + p_1) \geqq 1 + g(p_1) > 1$, contradicting the asssumption that $|| g || = 1$.

EXAMPLE IV.5. Let us work in the direct sum $\sum_{n=1}^{\infty} \oplus B(H_n)$ of matrix algebras where dimension $H_n = 2$ for all n. Set

$$a = \sum_{n=1}^{\infty} \begin{pmatrix} 1/n & 0 \\ 0 & 0 \end{pmatrix}, \quad p = \sum_{n=1}^{\infty} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad q = \sum_{n=1}^{\infty} \begin{pmatrix} 1 - \gamma_n & (\gamma_n - \gamma_n^2)^{1/2} \\ (\gamma_n - \gamma_n^2)^{1/2} & \gamma_n \end{pmatrix}$$

where $\{\gamma_n\}_{n=1}^{\infty}$ is an enumeration of the rationals between 0 and 1 which contains each rational an infinite number of times. Set $b = p + q$ and let A be the C^*-algebra generated by a and b. Let p_1 be the range projection of a in M.

Conclusions from the example. (1) $b \geqq p_1$ but there is no $d \in A^+$ with $dp_1 = p_1$ (c.*f*., [12] page 11, line 11). (2) If f is the pure state at ∞ for A, then $f(b) = 0$ but $f(\bar{p}_1) \neq 0$, so p_1 is nonregular by Proposition IV.4. (3) Let p_2 be the support projection of f. Then $p_1 + p_2$ satisfies the intersection condition of Conjecture II.2, but $p_1 + p_2$ is not q-closed.

If φ: $A_1 \to A_2$ is a *-homomorphism of A_1 onto A_2, we may easily extend it to a normal *-homomorphism of M_1 onto M_2. However if φ is not onto, this extension may not be possible. The natural representation of the continuous function on the interval $[0, 1]$ into the algebra of all bounded operators on $L^2\,[0, 1]$ by $\varphi(f)h = fh$ has no such extension (the proof was communicated to me by R. Giles). In order to place q-theory into a category theory setting, one must restrict the class of allowable "*morphisms*" between two C^*-algebras. The following restriction is empty in the abelian case.

PROPOSITION IV.6. *A *-homomorphism φ taking the C^*-algebra A_1 into the C^*-algebra A_2 has a normal extension $\tilde{\varphi}$: $M_1 \to M_2$* (necessarily unique) *if and only if φ is continuous for the topologies generated by the seminorms $|| a ||_f = f(a^*a)$ for all pure states f of A_1* (or A_2 for the topology on A_2).

Proof. It $\tilde{\varphi}$ exists, the continuity is automatic for $\tilde{\varphi}$, hence for φ. The converse follows immediately from [14, p. 3 of appendix].

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