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ON COMPOSITE $\boldsymbol{n}$ FOR WHICH $\varphi(n) \mid \boldsymbol{n} \boldsymbol{- 1}$. II Carl Pomerance

## ON COMPOSITE $n$ FOR WHICH $\varphi(n) \mid n-1$, II

## Carl Pomerance

The problem of whether there exists a composite $n$ for which $\varphi(n) \mid n-1$ ( $\varphi$ is Euler's function) was first posed by D. H. Lehmer in 1932 and still remains unsolved. In this paper we prove that the number of such $n$ not exceeding $x$ is $O\left(x^{1 / 2}(\log x)^{3 / 4}\right)$. We also prove that any such $n$ with precisely $K$ distinct prime factors is necessarily less than $K^{2 K}$. There are appropriate generalizations of these results to integers $n$ for which $\varphi(n) \mid n-a, a$ an arbitrary integer.

1. Introduction. In 1932, D. H. Lehmer [6] asked if there are any composite integers $n$ for which $\varphi(n) \mid n-1$, $\varphi$ being Euler's function. The answer to this question is still not known. Lieuwens [7] has shown that any such $n$ is divisible by at least 11 distinct primes; Kishore [5] has recently announced the analogous result for 13 primes.

If $S$ is any set of positive integers, denote by $N(S, x)$ the number of members of $S$ which do not exceed $x$. Let $L$ denote the set of composite $n$ for which $\varphi(n) \mid n-1$. Although Erdös was not specifically considering the problem of estimating $N(L, x)$, as a corollary of his paper [2], we have

$$
N(L, x)=O(x \exp (-c \log x \log \log \log x / \log \log x))
$$

for some $c>0$. In [11] we proved

$$
N(L, x)=O\left(x^{2 / 3}(\log \log x)^{1 / 3}\right)
$$

One result of this paper is

$$
\begin{equation*}
N(L, x)=O\left(x^{1 / 2}(\log x)^{3 / 4}\right) \tag{1.1}
\end{equation*}
$$

There is still clearly a wide gap between the possibility $L=\varnothing$ and (1.1), for the latter does not even establish that the members of $L$ are as scarce as squares! Note that we conjectured in [11] that for every $\varepsilon>0$,

$$
N(L, x)=O\left(x^{\varepsilon}\right)
$$

Important in proving (1.1) is the consideration for $n \in L$ of the distribution in the interval $[0, \log n]$ of the numbers $\log d$ for $d \mid n$. We show that these numbers do not leave any large gaps, in that any reasonable subinterval will contain some $\log d$.

We also prove another result of independent interest about the set $L$ : if $n \in L$ and $n$ is divisible by precisely $K$ distinct primes,
then

$$
\begin{equation*}
n<K^{2 K} \tag{1.2}
\end{equation*}
$$

This result is similar to a result of Borho [1] dealing with amicable numbers.

We establish results analogous to (1.1) and (1.2) for other sets of positive integers analogous to $L$. Recalling notation from [10], [11], we let

$$
F(a)=\{n: n \equiv \alpha(\bmod \varphi(n))\}
$$

for each integer $a$. From Sierpiński [12, p. 232], we have

$$
\begin{equation*}
F(0)=\{1\} \cup\left\{2^{i} \cdot 3^{j}: i>0, j \geqq 0\right\} . \tag{1.3}
\end{equation*}
$$

We have seen in [10] that $F(0)$ plays a special role for the sets $F(a)$. Indeed, if $a \notin F(0)$, then $F(a)$ has no member of the form $p a$ with $p$ prime, $p \nmid a$. However, if $a \in F(0)$, then every such number $p a$ is in $F(\alpha)$. Hence we are naturally led to consider the subsets

$$
F^{\prime \prime}(a)=\{n \in F(a): n \neq p a \text { for } p \text { prime, } p \nmid a\} .
$$

Note that $F^{\prime \prime}(1)=L \cup\{1\}$. We shall prove

$$
\begin{equation*}
N\left(F^{\prime}(a), x\right)=O\left(x^{1 / 2}(\log x)^{3 / 4}\right) \tag{1.4}
\end{equation*}
$$

for every integer $a$, where the implied constant depends on $a$. Note that (1.3) implies $N(F(0), x)=O\left((\log x)^{2}\right)$, so that (1.4) is true for $a=0$. However other results we prove will not be true for $a=0$. Throughout the remainder of this paper, $a$ will represent a nonzero integer.

We also prove that if $n \in F^{\prime}(a)$ and $n$ is divisible by precisely $K$ distinct primes, then

$$
n<\max \left\{16|a|^{3},|a| \cdot K^{2 K}\right\} .
$$

Certain results of Norton [9] (see Suryanarayana [13]) enable us to state our theorems in a sharper form than could be done otherwise. The results of Meijer [8] might yield further improvements.

We wish to thank the referee who carefully read the paper and made several helpful suggestions.
2. Preliminary results. If $n$ is an integer at least 2, denote by $\omega(n)$ the number of distinct prime factors of $n, P(n)$ the largest prime factor of $n$, and $p(n)$ the least prime factor of $n$.

In our work with the sets $F^{\prime}(\alpha)$ it will be convenient to isolate the square free members. Note that every member of $F^{\prime \prime}(1)$ is
square free. Let

$$
F^{\prime \prime}(a)=\left\{n \in F^{\prime \prime}(a): n \text { is square free }\right\} .
$$

Lemma 1. $N\left(F^{\prime \prime}(\alpha), x\right) \leqq 4 a^{2}+\sum_{d \mid a} N\left(F^{\prime \prime}(\alpha / d), x / d\right)$.
Proof. Let $n \in F^{\prime}(a), 4 a^{2}<n \leqq x$. If $n=p a$ for some prime $p$, then $p \mid a$, so $n \leqq \alpha^{2}$. Hence $n \neq p a$ for every prime $p$. Let $m$ be the maximal square free divisor of $n$ and let $d=n / m$. Then every prime factor of $d$ also divides $m$. Hence $\varphi(m)=\varphi(n) / d$, so that $d \mid \alpha$ and $m \in F(a / d)$. Since $m \neq p a / d$ for every prime $p$, we have $m \in F^{\prime \prime}(\alpha / d)$.

Hence all we need verify is that if $n_{1}, n_{2} \in F^{\prime}(a)$ with maximal square free divisors $m_{1}, m_{2}$, and if $n_{1}, n_{2}>4 a^{2}$, then $m_{1}=m_{2}$ implies $n_{1}=n_{2}$. Now for any $n$ we have

$$
\begin{equation*}
\varphi(n)>\sqrt{n} / 2 \tag{2.1}
\end{equation*}
$$

(Sierpiński [12, p. 230]). Suppose $m_{1}=m_{2}$. Then $n_{1}$ and $n_{2}$ have the same set of prime factors. This implies $n_{1} / \varphi\left(n_{1}\right)=n_{2} / \varphi\left(n_{2}\right)$. Let $k_{i}=\left(n_{i}-\alpha\right) / \varphi\left(n_{i}\right)$ for $i=1,2$. Then

$$
k_{1}+a / \varphi\left(n_{1}\right)=k_{2}+a / \varphi\left(n_{2}\right) .
$$

From (2.1) and the assumption $n_{1}, n_{2}>4 a^{2}$, we have $0<|a| \varphi\left(n_{i}\right) \mid<1$ for $i=1,2$. But $k_{1}, k_{2}$ are integers and $a / \varphi\left(n_{1}\right), a / \varphi\left(n_{2}\right)$ have the same sign, so

$$
a / \varphi\left(n_{1}\right)=a / \varphi\left(n_{2}\right) .
$$

But $n_{1} / \varphi\left(n_{1}\right)=n_{2} / \varphi\left(n_{2}\right)$, so $n_{1}=n_{2}$, which was to be proved.
Lemma 2. If $n \geqq 16 a^{2}, n \in F^{\prime \prime}(a)$, then
(i) $k \doteq(n-a) / \varphi(n)$ is a positive integer at least 2 ;
(ii) if $m \mid n, m \neq n$, then $m / \varphi(m)<k$;
(iii) there is a prime $q>P(n)$ with $n q / \varphi(n q)>k$;
(iv) $\omega(n) \geqq 3$.

Proof. (i) First we note that $n$ is composite. Indeed if $n=p$, a prime, then the condition $p \in F^{\prime \prime}(a)$ implies $p-1 \mid a-1$ and $a \neq 1$. Then $p \leqq|a|+2<16 a^{2}$, a contradiction.

Now $n=k \varphi(n)+a$, so if $k \leqq 0$, then $n \leqq a$. Suppose $k=1$. Since $n$ is composite, $n$ has a divisor $d$ with $\sqrt{n} \leqq d<n$. Then $\varphi(n) \leqq n-d \leqq n-\sqrt{n}$. Then

$$
a=n-\varphi(n) \geqq \sqrt{n} \geqq 4|a|,
$$

a contradiction.
(ii) It is sufficient to prove (ii) for $m=n / p$ where $p=P(n)$. From (2.1) and the assumption $n \geqq 16 a^{2}$, we have $|a / \varphi(n)|<1 / 2$. Hence from the equation $n / \varphi(n)=k+a / \varphi(n)$ and (i) we have

$$
\begin{equation*}
(3 / 4) k \leqq k-1 / 2<n / \varphi(n)<k+1 / 2 . \tag{2.2}
\end{equation*}
$$

Then $m / \varphi(m)<k+1 / 2<2 k$, so

$$
\begin{equation*}
k \varphi(m)>m / 2 . \tag{2.3}
\end{equation*}
$$

Now

$$
\begin{equation*}
a=n-k \varphi(n)=m p-k \varphi(m p)=p(m-k \varphi(m))+k \varphi(m) . \tag{2.4}
\end{equation*}
$$

If $m=k \varphi(m)$, then (2.4) implies $a=k \varphi(m)$, so that $\alpha=m$ and $n \notin F^{\prime \prime}(\mathrm{a})$. Hence $m \neq k \varphi(m)$. If $m>k \varphi(m)$, then (2.3), (2.4) imply

$$
a \geqq p+k \varphi(m)>p+m / 2 \geqq(2 p m)^{1 / 2}>n^{1 / 2} \geqq 4|a|,
$$

a contradiction. Hence $m<k \varphi(m)$.
(iii) If $a>0$, clearly any prime $q>P(n)$ will do. Hence assume $a<0$. We first prove

$$
\begin{equation*}
P(n)<n / 2|a| . \tag{2.5}
\end{equation*}
$$

Indeed from (2.2) we have (with $m=n / P(n)$ )

$$
\frac{3}{4} k<\frac{n}{\varphi(n)}=\frac{m}{\varphi(m)} \cdot \frac{P(n)}{P(n)-1} \leqq \frac{2 m}{\varphi(m)} .
$$

Then from (ii) and (2.4) we have

$$
P(n)=(\alpha-k \varphi(m)) /(m-k \varphi(m)) \leqq|a|+k \varphi(m)<|a|+(8 / 3) m .
$$

If (2.5) fails, we have $m=n / P(n) \leqq 2|a|$, and it follows that $P(n)<(19 / 3)|a|$ and $n=m P(n)<16 \alpha^{2}$, a contradiction.

By Chebyshev's theorem there is a prime $q$ with $n / 2|a|<q<n /|a|$, and by (2.5), $q>P(n)$. Also

$$
\begin{aligned}
\frac{n q}{\varphi(n q)} & >\frac{n}{\varphi(n)} \cdot \frac{n /|\alpha|}{n /|\alpha|-1}=\frac{n^{2}}{\varphi(n)(n+a)} \\
& =\frac{k n^{2}}{(n-a)(n+\alpha)}>k,
\end{aligned}
$$

since $n^{2}>n^{2}-a^{2}>0$.
(iv) We noted in the proof of (i) that $\omega(n) \geqq 2$. Suppose $\omega(n)=2$. Let $n=p q$ with $p<q$. Let $r$ be a prime with $r>q$ and $p q r / \phi(p q r)>k \geqq 2$ (using (i) and (iii)). Since (2/1)(3/2)(5/4) $<4$, we have $k=2$ or 3 .

If $k=3$, then since $(2 / 1)(5 / 4)(7 / 6)<3$, we have $n=p q=6<16 a^{2}$.

Suppose $k=2$. Since $(5 / 4)(7 / 6)(11 / 10)<2$, we have $p=2$ or 3 . By (ii), $p / \varphi(p)<2$, so $p=3$. Since $(3 / 2)(7 / 6)(11 / 10)<2$, we have $q=5$. That is, $n=p q=15<16 \alpha^{2}$.

Lemma 3. Suppose $k$, $n$ are natural numbers with $n$ square free and $n / \varphi(n)>k$. If $m \mid n$ and $m / \varphi(m)<k$, then

$$
p(n / m)<\omega(n / m) \cdot(m+1) .
$$

Proof. Let $r=\omega(n / m), p=p(n / m)$. Then

$$
k<\frac{n}{\varphi(n)} \leqq \frac{m}{\varphi(m)} \cdot\left(\frac{p}{p-1}\right)^{r},
$$

so that

$$
m / k \varphi(m)>(1-1 / p)^{r} \geqq 1-r / p .
$$

Hence

$$
p<\frac{r k \varphi(m)}{k \varphi(m)-m}=r\left(1+\frac{m}{k \varphi(m)-m}\right) \leqq r(m+1) .
$$

3. Members of $F^{\prime \prime}(a)$ with $K$ prime factors.

Theorem 1. Suppose $n \geqq 16 a^{2}, n \in F^{\prime \prime}(\alpha), K=\omega(n)$. Let the prime factorization of $n$ be $p_{1} p_{2} \cdots p_{K}$ where $p_{1}>p_{2}>\cdots>p_{K}$. Then for $1 \leqq i \leqq K$, we have

$$
p_{i}<(i+1)\left(1+\prod_{j=i+1}^{K} p_{j}\right) .
$$

Proof. Let $m=\prod_{j=i+1}^{K} p_{j}$. By (iii) of Lemma 2 there is a prime $q>p_{1}$ with $n q / \varphi(n q)>k$. By (ii) of Lemma $2, m / \varphi(m)<k$. Since $p_{i}=p(n q / m)$ and $i+1=\omega(n q / m)$, Lemma 3 completes the proof.

Theorem 2. Suppose $n \geqq 16 a^{2}, n \in F^{\prime \prime}(\alpha), K=\omega(n)$. Then there is a positive constant $\beta$ independent of the choice of $a, n$ such that

$$
\begin{equation*}
p(n)<\beta K^{1 / 2}(\log K)^{1 / 2} \tag{3.1}
\end{equation*}
$$

In addition, if $K \geqq 4$, then $p(n) \leqq K-1$.
Proof. Let $p=p(n)$. Since there is a prime $q>P(n)$ with $n q / \varphi(n q)>k \geqq 2$ ((i) and (iii) of Lemma 2), it follows from Norton [9, Theorem 4] that there is an absolute constant $\beta_{1}>0$ with

$$
K+1=\omega(n q)>\beta_{1} p^{2} / \log p .
$$

By Theorem 1, $\log p<\log (2(K+1))<\beta_{2} \log K$ for some $\beta_{2}>0$ ((iv) of Lemma 2). Hence there is an absolute constant $\beta>0$ such that $p^{2}<\beta^{2} K \log K$, which proves (3.1).

Now assume $K \geqq 4$. Then $p \leqq K-1$ if $p=2$ or 3 . From $n q / \varphi(n q)>2$, we have $K+1 \geqq 7$ if $p=5$, so $p \leqq K-1$ in this case too. If $p \geqq 7$ we similarly get $K+1 \geqq 15$, so that using a result of Grün [3], we have

$$
p<(2 / 3)(K+1)+2<K-1
$$

Theorem 3. If $n \in F^{\prime \prime}(a), K=\omega(n)$, then

$$
n<\max \left\{16 a^{2}, K^{2^{K}}\right\}
$$

Proof. Assume $n \geqq 16 a^{2}$. By (iv) of Lemma 2 we have $K \geqq 3$. If $K=3$, we can show as follows that $n \leqq 435<3^{2^{3}}$. Write $n=p q r$ where $p<q<r$ are primes. By Lemma 2 there is a prime $s>r$ such that

$$
\begin{gather*}
p q r s / \varphi(p q r s)>k \geqq 2,  \tag{3.2}\\
p q / \varphi(p q)<k . \tag{3.3}
\end{gather*}
$$

We proceed as with the proof of (iv) of Lemma 2. Say $k \geqq 3$. Then (3.2) implies $k=3, p=2, q \leqq 5$ or $k=4, n=p q r=30$. In the former case, (3.3) implies $q=5$, so (3.2) implies $n=p q r=70$. Now say $k=2$. Then (3.2), (3.3) imply $p=3$. Then (3.2) implies $q=5, \quad r \leqq 29 \quad$ (so $\quad n \leqq 3 \cdot 5 \cdot 29=435$ ) or $\quad q=7, \quad r \leqq 13 \quad$ (so $n \leqq 3 \cdot 7 \cdot 13=273$ ).

Assume $K \geqq 4$. Let the prime factorization of $n$ be $p_{1} p_{2} \cdots p_{K}$ where $p_{1}>p_{2}>\cdots>p_{K}$. By Theorem 2,

$$
p_{K}+1 \leqq K
$$

By Theorem 1, $p_{K-1}<K\left(p_{K}+1\right) \leqq K^{2}$. Hence

$$
p_{K-1} p_{K}+1<K^{3} .
$$

Again by Theorem 1, $p_{K-2}<(K-1)\left(p_{K-1} p_{K}+1\right)$, so that

$$
p_{K-2} p_{K-1} p_{K}+1<p_{K-2}\left(p_{K-1} p_{K}+1\right)<(K-1) K^{6}<K^{7} .
$$

Continuing in this fashion we get

$$
n=p_{1} p_{2} \cdots p_{K}<K^{2 K_{-1}}<K^{2^{K}}
$$

Theorem 4. If $n \in F^{\prime}(\alpha), K=\omega(n)$, then

$$
n<\max \left\{16|a|^{3},|a| \cdot K^{2^{K}}\right\}
$$

Proof. Assume $n \geqq 16|a|^{3}$. Following the proof of Lemma 1,
we find a positive integer $d$ with $d \mid(n, a)$ and $n / d \in F^{\prime \prime}(a / d)$. Then $n / d \geqq 16 a^{2}$, so Theorem 3 implies $n / d<K^{2 K}$. Hence

$$
n<d \cdot K^{2 K} \leqq|a| \cdot K^{2 K} .
$$

4. A combinatorial lemma.

LEMMA 4. Suppose $\delta \geqq 0, a_{1} \geqq a_{2} \geqq \cdots \geqq a_{t}>0, \quad B_{i}=\sum_{j=i}^{t} a_{j}$ for $1 \leqq i \leqq t$, and

$$
\begin{equation*}
a_{i} \leqq \delta+B_{i+1} \tag{4.1}
\end{equation*}
$$

for $1 \leqq i \leqq t-1$. Then given any $y$ with $0 \leqq y<B_{1}$, there is a subset $S$ of $\{1,2, \cdots, t\}$ with

$$
y-\delta-a_{t}<\sum_{i \in S} a_{i} \leqq y
$$

Proof. We may assume $y \geqq \delta+a_{t}$ for otherwise take $S=\varnothing$. We have

$$
\begin{equation*}
B_{1}>y, \quad B_{t} \leqq y \tag{4.2a,b}
\end{equation*}
$$

Let $s(0)=0$. Say we have either constructed a set $S$ as called for or we have inductively found an integer sequence $s(0)<s(1)<\cdots<s(i-1)<t$ where $i \geqq 1$ and

$$
\begin{equation*}
\sum_{j=1}^{i-1} a_{s(j)}+B_{s(i-1)+1}>y \tag{4.3a}
\end{equation*}
$$

Let $s(i)$ be maximal with

$$
\sum_{j=1}^{i-1} a_{s(j)}+B_{s(i)}>y .
$$

By (4.3a), (4.3b), $s(i)$ exists and $s(i-1)<s(i)<t$. Then since $a_{s(i)}+B_{s(i)+1}=B_{s(i)}$, we have

$$
\begin{equation*}
\sum_{j=1}^{i} a_{s(j)}+B_{s(i)+1}>y \tag{4.4a}
\end{equation*}
$$

Note that $\sum_{j=1}^{i-1} a_{s(j)}+B_{s(i)+1} \leqq y$. Then we may assume

$$
\begin{equation*}
\sum_{j=1}^{i-1} a_{s(j)}+B_{s(i)+1} \leqq y-\delta-a_{t} \tag{4.5}
\end{equation*}
$$

for otherwise we may take

$$
S=\{s(1), s(2), \cdots, s(i-1), s(i)+1, s(i)+2, \cdots, t\}
$$

Then from (4.5) and from (4.1) applied to $a_{s^{(i)}}$, we have

$$
\begin{aligned}
\sum_{j=1}^{2} a_{s(j)}+B_{t} & =\sum_{j=1}^{i} a_{s(j)}+a_{t} \\
& \leqq \sum_{j=1}^{2-1} a_{s(j)}+\delta+B_{s(i)+1}+a_{t} \leqq y ;
\end{aligned}
$$

that is,

$$
\begin{equation*}
\sum_{j=1}^{i} a_{s(j)}+B_{t} \leqq y . \tag{4.4b}
\end{equation*}
$$

Since there is not an infinite increasing sequence of positive integers all less than $t$, this process must terminate with the construction of a suitable set $S$.
5. Estimates for $N\left(F^{\prime}(\alpha), x\right)$.

Theorem 5. For every $a, N\left(F^{\prime}(a), x\right)=O\left(x^{1 / 2}(\log x)^{3 / 4}\right)$, where the implied constant depends on a.

Proof. In view of Lemma 1, it will be sufficient to prove for every $a$ that $N\left(F^{\prime \prime}(\alpha), x\right)=O\left(x^{1 / 2}(\log x)^{3 / 4}\right)$, where the implied constant depends on $a$. We record for future reference: there are positive constants $\alpha, \gamma$ with

$$
\begin{gather*}
n / \varphi(n)<\alpha \log \log n, \quad n \geqq 3  \tag{5.1}\\
\omega(n)<\gamma \log n / \log \log n, \quad n \geqq 3 . \tag{5.2}
\end{gather*}
$$

(Hardy and Wright [4, pp. 353-355].)
Let $n \in F^{\prime \prime}(\alpha), 16 a^{2} \leqq n \leqq x, K=\omega(n)$. Let the prime factorization of $n$ be $p_{1} p_{2} \cdots p_{K}$ where $p_{1}>p_{2}>\cdots>p_{K}$. We may assume $n>x^{1 / 2}(\log x)^{3 / 4}$. Theorem 1 implies

$$
\log p_{i}<\log (2 K)+\sum_{j=\imath+1}^{K} \log p_{j}, \quad 1 \leqq i \leqq K-1
$$

We apply Lemma 4 with

$$
\delta=\log (2 K), \quad t=K, \quad a_{i}=\log p_{i}, \quad y=\frac{1}{2} \log x+\frac{3}{4} \log \log x .
$$

Hence there is an integer $m$ with $m \mid n$ and

$$
y-\delta-\log p_{K}<\log m \leqq y .
$$

Then

$$
x^{1 / 2}(\log x)^{3 / 4} / 2 K p_{K}<m \leqq x^{1 / 2}(\log x)^{3 / 4} .
$$

By (3.1), (5.2), we have

$$
\begin{aligned}
2 K p_{K} & <2 \beta K^{3 / 2}(\log K)^{1 / 2} \\
& <2 \beta(\gamma \log x / \log \log x)^{3 / 2}(\log (\gamma \log x / \log \log x))^{1 / 2} \\
& <\gamma^{\prime}(\log x)^{3 / 2}(\log \log x)^{-1}
\end{aligned}
$$

for some $\gamma^{\prime}>0$. Hence

$$
\begin{aligned}
f(x) & \doteq\left(1 / \gamma^{\prime}\right) x^{1 / 2}(\log x)^{-3 / 4} \log \log x<m \\
& \leqq x^{1 / 2}(\log x)^{3 / 4} \doteq g(x) .
\end{aligned}
$$

For each integer $m$ in the above interval we now count the number of choices for $n \in F^{\prime \prime}(a)$ with $n \leqq x$ and $m \mid n$. Since $\varphi(m) \mid \varphi(n)$ for such $n$, we have

$$
n \equiv 0(\bmod m), \quad n \equiv a(\bmod \varphi(m)),
$$

so by the generalized Chinese remainder theorem, there are at most $1+x /[m, \varphi(m)]$ choices for such $n$ (here [, ] denotes least common multiple). Now ( $m, \varphi(m)) \mid(n, \varphi(n))$ and $(n, \varphi(n)) \mid a$. Hence for each $m$, there are at most (using (5.1))

$$
\begin{aligned}
1+x /[m, \varphi(m)] & =1+x(m, \varphi(m)) / m \varphi(m) \\
& \leqq 1+|a| x / m \varphi(m)<1+|a| \alpha x \log \log x / m^{2}
\end{aligned}
$$

choices for $n \in F^{\prime \prime}(a)$ with $n \leqq x$ and $m \mid n$.
Hence we have

$$
\begin{aligned}
N\left(F^{\prime \prime}(a), x\right) & \leqq 16 a^{2}+x^{1 / 2}(\log x)^{3 / 4}+\sum_{f(x)<m \leq g(x)}\left(1+|a| \alpha x \log \log x / m^{2}\right) \\
& =O\left(x^{1 / 2}(\log x)^{3 / 4}\right)+O\left(x \log \log x \sum_{f(x<m} 1 / m^{2}\right) \\
& =O\left(x^{1 / 2}(\log x)^{3 / 4}\right)+O(x \log \log x / f(x)) \\
& =O\left(x^{1 / 2}(\log x)^{3 / 4}\right) .
\end{aligned}
$$

Remark. Both the referee and D. Suryanarayana kindly suggest the use of a fact due to Landau,

$$
\sum_{m>y} 1 / m \varphi(m)=O(1 / y),
$$

in the proof of Theorem 5, rather than (5.1). This enables us to get the slightly stronger estimate

$$
\begin{equation*}
N\left(F^{\prime}(\alpha), x\right)=O\left(x^{1 / 2}(\log x)^{3 / 4}(\log \log x)^{-1 / 2}\right) \tag{5.3}
\end{equation*}
$$

where the implied constant depends on $a$. In addition we note that if those $n \leqq x$ for which $p(n) \leqq(\log x)^{1 / 4}$ are treated separately from the remaining choices for $n$, then an extra factor of $1 / \log \log x$ may be introduced on the right of (5.3). It is conceivable that further
improvements are possible, even in the exponent on $\log x$ (perhaps by considering a sharper version of Lemma 4 where the constant $\delta$ is replaced by a variable $\delta_{i}$ which is usually small). It would seem to take a completely new idea however to lower the exponent on $x$.

## References

1. W. Borho, Eine Schranke für befreundete Zahlen mit gegebener Teileranzahl, Math. Nachr., 63 (1974), 297-301.
2. P. Erdös, On pseudoprimes and Carmichael numbers, Publ. Math. Debrecen, 4 (1956), 201-206.
3. O. Grün, Über ungerade vollkommene Zahlen, Math. Z., 55 (1952), 353-354.
4. G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers (Fourth Edition), Oxford, 1960.
5. M. Kishore, On the equation $k \varphi(M)=M-1$, Not. Amer. Math. Soc., 22 (1975), A-501-A-502.
6. D. H. Lehmer, On Euler's totient function, Bull. Amer. Math. Soc., 38 (1932), 745-757.
7. E. Lieuwens, Do there exist composite numbers $M$ for which $k \varphi(M)=M-1$ holds? Nieuw Arch. Wisk., (3), 18 (1970), 165-169.
8. H. G. Meijer, Sets of primes with intermediate density, Math. Scand., 34 (1974), 37-43.
9. K. K. Norton, Remarks on the number of factors of an odd perfect number, Acta Arith., 6 (1961), 365-374.
10. C. Pomerance, On the congruences $\sigma(n) \equiv a(\bmod n)$ and $n \equiv a(\bmod \varphi(n))$, Acta Arith., 26 (1975), 265-272.
11. C. Pomerance, On composite $n$ for which $\varphi(n) \mid n-1$, Acta Arith., 28 (1976), 387-389.
12. W. Sierpiński, Elementary Theory of Numbers, Warsaw, 1964.
13. D. Suryanarayana, On odd perfect numbers, Math. Student, 41 (1973), 153-154.

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