Regularity for C*-algebras and the Toms–Winter conjecture

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Parts of this talk concern joint work with:

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The Toms–Winter conjecture

Definition

An **Elliott algebra** is a simple separable amenable C*-algebra.

Conjecture (Toms–Winter, $\sim$2008)

If A is an Elliott algebra, then the following are equivalent:

(i) A has finite nuclear dimension;
(ii) A is $\mathcal{Z}$-stable (where $\mathcal{Z}$ is the Jiang–Su algebra);
(iii) A has strict comparison of positive elements.

Strict comparison of positive elements is a property of the Cuntz semigroup (an algebraic invariant); in practice, it is the easiest property to verify.

Partition of unity
(topology)

↙ ↘

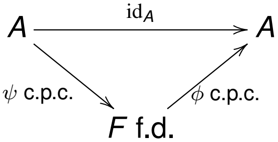
Noncommutative POU
= c.p. approx. prop.

Lebesgue covering
dimension

↘ ↙

Nuclear dimension

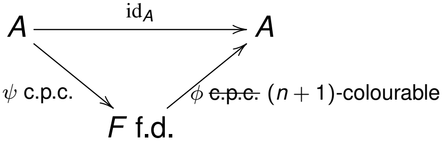
Completely positive approximation property:



commuting in point-$\|.\|$, i.e., $\|\phi(\psi(a)) - a\|$ small on a finite subset.

Nuclear dimension

Nuclear dimension at most n (Kirchberg–Winter '04, Winter–Zacharias '10):



commuting in point-$\|.\|$, i.e., $\|\phi(\psi(a)) - a\|$ small on a finite subset.

$(n+1)$-colourable: $F = F_0 \oplus \cdots \oplus F_n$ such that $\phi|_{F_i}$ is c.p.c. and orthogonality-preserving (a.k.a. order zero).

Eg. $\dim_{nuc} C(X) = \dim X$.

Nuclear dimension: some properties

Finite nuclear dimension is preserved by:

- quotients;

- hereditary subalgebras;

- extensions;

- tensor products;

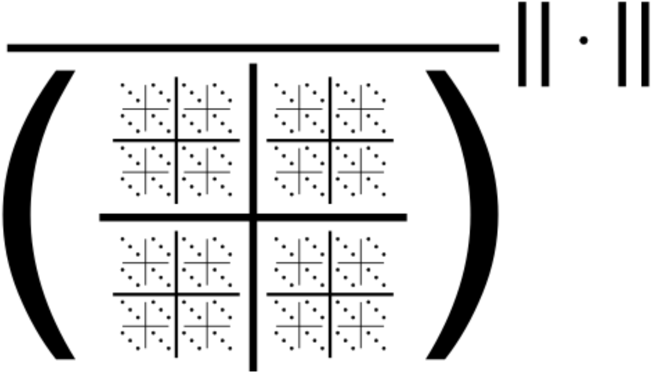
- ~~inductive limits.~~ if $\dim_{nuc}(\varinjlim A_k) \leq \sup \dim_{nuc}(A_k)$ (this was a mistake).

Eg. $\dim_{nuc} \mathcal{O}_n = 1$ (Winter–Zacharias '10)

$\dim_{nuc} A = 0$ if and only if A is AF.

The Jiang–Su algebra

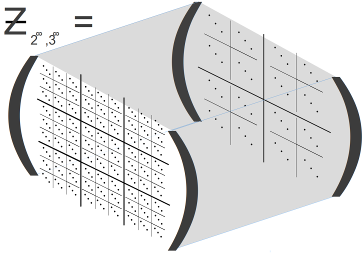
Recall: a UHF algebra is an inductive limit of matrix algebras



M_{2^∞}

$$M_{k^\infty} \cong M_{k^\infty} \otimes M_{k^\infty} \cong M_{k^\infty}^{\otimes\infty}.$$

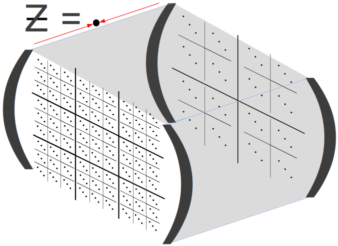
The Jiang–Su algebra



$$\mathcal{Z}_{2^\infty,3^\infty} := \{ f \in C([0,1], M_{2^\infty} \otimes M_{3^\infty}) \mid f(0) \in 1_{M_{2^\infty}} \otimes M_{3^\infty}, \; f(1) \in M_{2^\infty} \otimes 1_{M_{3^\infty}} \}.$$

This has no nontrivial projections.

The Jiang–Su algebra

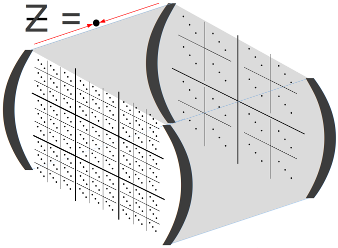


The Jiang–Su algebra is

$$\mathcal{Z} := \varinjlim(\mathcal{Z}_{2^\infty,3^\infty}, \alpha),$$

where $\alpha : \mathcal{Z}_{2^\infty,3^\infty} \to \mathcal{Z}_{2^\infty,3^\infty}$ is a trace-collapsing unital *-homomorphism.

The Jiang–Su algebra



$$\mathcal{Z} := \varinjlim(\mathcal{Z}_{2^\infty,3^\infty}, \alpha).$$

$\mathcal{Z}$ is simple.

$K_0(\mathcal{Z}) = \mathbb{Z}$; $K_1(\mathcal{Z}) = 0$.

$\mathcal{Z}$ has unique trace.

$\mathcal{Z}$ is also strongly self-absorbing.

$\mathcal{Z} \cong \mathcal{Z}^{\otimes\infty}$.

$\mathcal{Z}$-stability

A C*-algebra A is **$\mathcal{Z}$-stable** if $A \cong A \otimes \mathcal{Z}$.

Theorem

If A is separable and unital, then it is $\mathcal{Z}$-stable if and only if $\mathcal{Z}$ embeds into

$$A_\infty \cap A',$$

where $A_\infty := c_b(\mathbb{N}, A)/c_0(\mathbb{N}, A)$.

Trivial observation: for any B, the C*-algebra $B \otimes \mathcal{Z}$ is $\mathcal{Z}$-stable.

$\mathcal{Z}$-stabilization is a way to tame a wild C*-algebra.

$\mathcal{Z}$-stability: some properties

$\mathcal{Z}$-stability is preserved by:

- quotients;

- hereditary subalgebras;

- extensions;

- tensor products;

- inductive limits.

Just like finite nuclear dimension.

Origins of the Toms–Winter conjecture: classification

Conjecture (Elliott, '90s)

Elliott algebras are classified by K-theory paired with traces.

Disproven by examples of Villadsen ('98), refined by Rørdam ('03), Toms ('08).

Villadsen's C*-algebras have "high topological dimension" (in some vague sense).

Classification results apply to C*-algebras of "low topological dimension", eg., purely infinite C*-algebras, AH algebras of slow dimension growth.

The Toms–Winter conjecture is an attempt to make "low topological dimension" less vague, more robust.

Origins of the Toms–Winter conjecture: classification

Classification can be used to prove (ii) $\Rightarrow$ (i) in many cases:

Theorem (Kirchberg ∼'94, Phillips '00)

Purely infinite Elliott algebras in the UCT-class satisfy the Elliott conjecture.

It follows that if A is an infinite Elliott algebra, in the UCT class, and is $\mathcal{Z}$-stable, then

$$A = \varinjlim A_n,$$

where A_n is a direct sum of $C(\mathbb{T}) \otimes M_k \otimes \mathcal{O}_m$'s.

Hence $\dim_{nuc}(A) < \infty$ (in fact ≤ 5).

Origins of the Toms–Winter conjecture: classification

Classification can be used to prove (ii) $\Rightarrow$ (i) in many cases:

Theorem (Gong '02, Elliott-Gong-Li '07, Lin '11)

Simple $\mathcal{Z}$-stable AH algebras satisfy the Elliott conjecture.

It follows that if A is a $\mathcal{Z}$-stable AH algebra then

$$A = \varinjlim A_n,$$

where A_n is a direct sum of $C(X) \otimes M_k$'s where $\dim X \leq 3$.

Hence, $\mathrm{dim}_{nuc} A < \infty$ (in fact, ≤ 3).

Origins of the Toms–Winter conjecture: classification

Classification can be used to prove (ii) $\Rightarrow$ (i) in many cases:

Similarly, Gong-Lin-Niu classification (arXiv '15) shows that if A is a $\mathcal{Z}$-stable Elliott algebra that is "rationally generalized tracial rank one" and in the UCT-class, then $\dim_{nuc}(A) \leq 2$.

Finite nuclear dimension implies $\mathcal{Z}$-stability

Theorem (Winter '10 & '12, T '14)

If A is simple and separable and $\dim_{nuc} A < \infty$ then $A \cong A \otimes \mathcal{Z}$.

$\mathcal{Z}$-stability implies finite nuclear dimension

It is desirable to establish that $\mathcal{Z}$-stability implies finite nuclear dimension without using classification, because:

- Classification requires strong hypotheses (UCT, simplicity, tracial approximation, . . .);

- Classification arguments are lengthy (Gong: 208 pages; Elliott-Gong-Li: 72 pages; Gong-Lin-Niu: 271 pages);

- Finite nuclear dimension is a useful hypothesis for classification (eg. Winter, arXiv ’13).

$\mathcal{Z}$-stability implies finite nuclear dimension

"Von Neumann algebraic" approach

If A is a $\mathcal{Z}$-stable unital Elliott algebra then it has finite nuclear dimension provided:

- A is infinite (Matui-Sato '14);

- A has unique trace and is quasidiagonal (Matui-Sato '14);

- A has unique trace (Sato-White-Winter, arXiv '14);

- the extreme boundary of $T(A)$ is compact (Brown-Bosa-Sato-T-White-Winter arXiv '15).

$\mathcal{Z}$-stability implies finite nuclear dimension

Subhomogeneous algebra approach

$A \otimes \mathcal{Z}$ has finite nuclear dimension provided:

- *A* is a commutative C*-algebra (T-Winter '14) (hence also if *A* is AH);

- *A* is a subhomogeneous C*-algebra (Elliott-Niu-Santiago-T arXiv '15) (hence also if *A* is ASH).

Using this fact, Elliott-Gong-Lin-Niu showed that simple $\mathcal{Z}$-stable ASH algebras are classifiable.