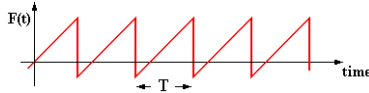


Periodic Waveforms

Only periodic functions of time will be considered in this subject. Non-periodic functions include noise in a system as well as modulated waveforms. A periodic sawtooth waveform is illustrated below, showing the period of the waveform. When calculations, such as the average value of the waveform, are made later, it may be advantageous to redefine a new zero for the time axis.



If the period is T seconds, then the frequency of the waveform is given by the number of times that this period occurs in a second, i.e.

$$\text{frequency, } f = 1/T \text{ Hz (hertz)}$$

A waveform which has only a constant value is a special case where the period is infinity and the frequency is 0 Hz.

All periodic waveforms, e.g. voltage, current or power waveforms, may be analyzed in terms of their Fourier series,

$$\text{i.e. } f(t) = a_0 + \sum_{n=1}^{\infty} \{ a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t) \}$$

$$\text{or } f(t) = a_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t + \phi_n)$$

d.c. term harmonic fundamental frequency
(i.e. n=1)

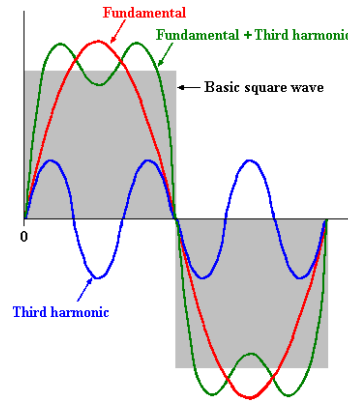
The fundamental frequency, where n = 1, has a period of T seconds, this being the period of the composite waveform.

An example of a square wave

What is a square wave? If the waveform represents the voltage as a function of time, then there will be two distinct voltage levels with equal time intervals at each voltage level. This is sometimes referred to as a waveform with a mark-space ratio of unity.

One complete period of the square wave is shown on the right in gray. The fundamental sine wave frequency which is the closest approximation to this square wave is shown in red. Now, taking the difference between the original square wave and the fundamental sine wave component leaves a result which is best approximated by a third harmonic component, shown in blue. Thus the sum of the fundamental and third harmonic components, shown in green is a closer approximation to the original square wave. Adding further odd harmonic components will improve the agreement with the original square wave.

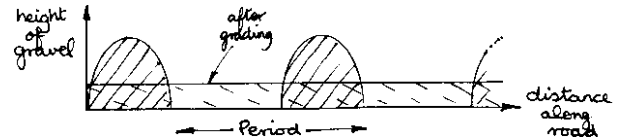
Any periodic waveform may be described in terms of a harmonic series.



The Average Value of a Waveform

All periodic functions have an average (or mean) value, even if it is zero. In electrical terms, this average value is also known as the d.c. level of the waveform.

So, what is the average value of a waveform? Consider the situation where equal loads of gravel have been placed at equal intervals along the bottom of a trench. A section of this situation is illustrated below.



On this diagram, the following points may be observed, either in terms of the gravel as a function of position or as a voltage waveform as a function of time:

- the period of the waveform;

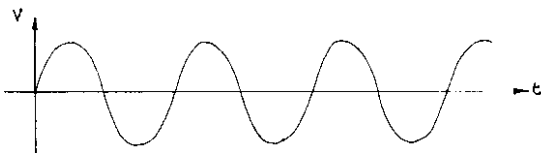
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2

- the average height of the gravel after it is smoothed out over the length of the trench;
- or the average voltage after it is smoothed out over time;

When presented mathematically, it is seen that the area under the periodic function must equal the area under the average value over the same time. This will be used shortly.

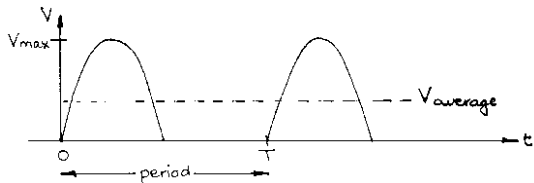
The Sine Wave



The average value of this waveform is zero.

If this waveform represents, say, the current waveform being applied to a battery, then there would be no net charge into the battery.

The Half-Wave Rectified Waveform (using diodes)



This waveform has an average value which is non-zero. In order to calculate the average value, it is noted that the area under the periodic function over a complete period must equal the area under the averaged curve. Thus

$$T \times V_{\text{average}} = \int_0^T V(t) dt$$

$$\text{i.e. } V_{\text{average}} = \frac{1}{T} \int_0^T V(t) dt$$

For the half-wave rectified waveform

$$V_{\text{average}} = \frac{1}{T} \int_0^{T/2} V_{\text{max}} \sin(\omega t) dt$$

Change the variable of integration

3

$$V_{\text{average}} = \frac{1}{\omega T} \int_0^{\pi} V_{\text{max}} \sin(\omega t) d(\omega t)$$

$$= \frac{1}{2\pi} \int_0^{\pi} V_{\text{max}} \sin(\omega t) d(\omega t)$$

(1/2π) (1/2π) (1/2π)
time period time for waveform

NOTE:

$$\text{When } t = T/2,$$

$$\omega t = \pi$$

$$\text{Likewise } \omega T = 2\pi$$

$$(\omega \cdot \frac{T}{\omega} = 2\pi)$$

$$(\omega = 2\pi f)$$

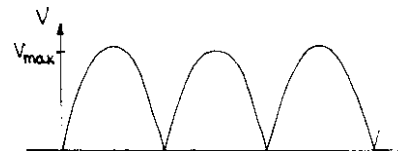
Note that between pi and 2pi the contribution from the complete waveform to the average value is zero, i.e. the sine function is only integrated over the time where it is present. Thus

$$V_{\text{average}} = \frac{V_{\text{max}}}{2\pi} [-\cos(\omega t)]_0^{\pi}$$

$$= \frac{V_{\text{max}}}{2\pi} [-(-1) + 1] = \frac{V_{\text{max}}}{\pi}$$

$$\text{Hence } V_{\text{average}} = 0.318 V_{\text{max}}$$

The Full-Wave Rectified Waveform



By inspection, the average value has doubled, giving

$$V_{\text{average}} = 0.637 \times V_{\text{max}}$$

This may be found either as

$$V_{\text{average}} = \frac{1}{2\pi} \left\{ \int_0^{\pi} V_{\text{max}} \sin(\omega t) d(\omega t) - \int_{\pi}^{2\pi} V_{\text{max}} \sin(\omega t) d(\omega t) \right\}$$

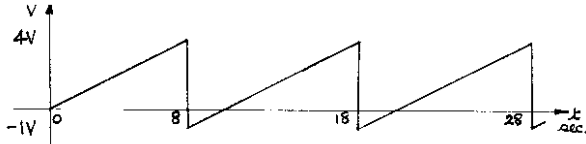
or, noting that the period of the waveform is now pi,

$$V_{\text{average}} = \frac{1}{\pi} \int_0^{\pi} V_{\text{max}} \sin(\omega t) d(\omega t)$$

4

Example

Calculate the average value for the following waveform:



The period of this waveform is 10 seconds, e.g. from 8 to 18 seconds. In that range, 8 ≤ t ≤ 18,

$$V(t) = 0.5t - 5$$

Therefore

$$V_{\text{average}} = \frac{1}{10} \int_8^{18} (0.5t - 5) dt$$

$$= \frac{1}{10} \left[0.25t^2 - 5t \right]_8^{18}$$

i.e. $V_{\text{average}} = \frac{1}{10} [81 - 90 - (16 - 40)] = 1.5$

Note that life may be easier if a new time scale is defined, where t' = 0 at t = 8 seconds, leading to

$$V_{\text{average}} = \frac{1}{10} \int_0^{10} (0.5t' - 1) dt'$$

Checking the Result

It is always wise, where possible, to check the result. Generally an approximate answer may be obtained for an average value by inspection of the waveform. Here, it is seen that

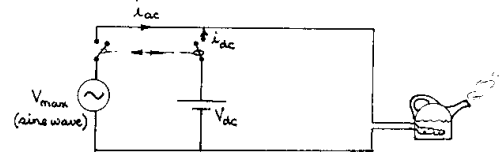
$$V_{\text{average}} = \frac{4 + (-1)}{2}$$

$$= 1.5 \text{ volts}$$

The R.M.S. Value of a Waveform

The r.m.s. value for a voltage or current waveform is its effective value from the point of view of power dissipation in a resistive circuit. The letters R, M, S standing

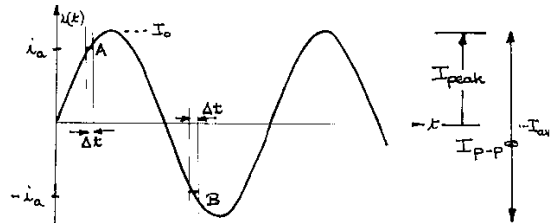
for root mean square describe the way in which the effective value of the waveform is derived. Consider the following situation:



The circuit connects two voltage sources, one at a time, to a kettle which has a resistance of R ohms. We wish to derive an equivalence between the sine wave and constant voltage sources for equal power transfers to the kettle.

Why can an a.c. voltage or current deliver power?

It is not the average current (which is zero for a sine wave source) which determines its power delivering capability. Consider the detail of the waveform which is illustrated below.

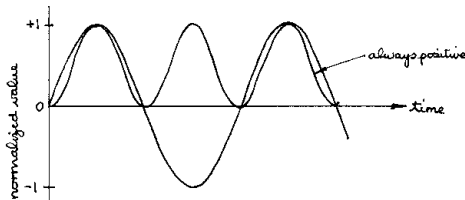


At A, the instantaneous power = $i_a^2 R$ and is non-zero.

At B, the instantaneous power = $(-i_a)^2 R = i_a^2 R$ and is also positive.

Since $i = I_0 \sin(\omega t)$, the instantaneous power

$$p(t) = I_0^2 R \sin^2(\omega t)$$



The power waveform $\propto \sin^2 \omega t$

where $\sin^2 \omega t = \frac{1}{2}(1 - \cos(2\omega t))$ show that this is logically correct.

Instantaneous power \Rightarrow average power
For equal average powers for the a.c. and d.c. sources,

$$\frac{1}{T} \int_0^T i^2(t) R dt = \frac{1}{T} \int_0^T I_{dc}^2 R dt$$

$$= I_{dc}^2 R \quad \text{cancel } R.$$

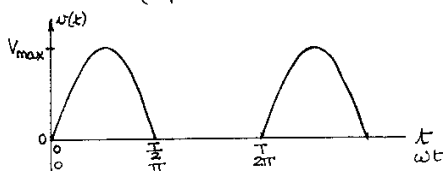
We can therefore express a current for the time-varying waveform that will produce the same power as a d.c. current of the same magnitude. This current is the effective or r.m.s. current,

i.e. $I_{\text{r.m.s.}} = \sqrt{\frac{1}{T} \int_0^T i^2 dt}$

root mean square

Thus i) take the square of the quantity, e.g.

$$\{I_{\text{peak}} \sin(\omega t)\}^2$$



ii) find the average: $\frac{1}{T} \int_0^T I_{\text{peak}}^2 \sin^2(\omega t) dt$

or $\frac{1}{2\pi} \int_0^{2\pi} I_{\text{peak}}^2 \sin^2(\omega t) d(\omega t)$

iii) take the square root:

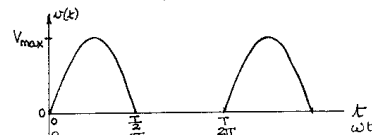
$$I_{\text{rms}} = \sqrt{\frac{I_{\text{peak}}^2}{2\pi} \int_0^{2\pi} \sin^2(\omega t) d(\omega t)}$$

NOTE: peak \equiv max

The mains electricity voltage is 240V r.m.s.

In a.c. circuit theory, clearly state the convention that you are using to express voltages and currents.

Example: The half-wave rectified voltage waveform.



$$V_{\text{rms}} = \sqrt{\frac{V_{\text{max}}^2}{2\pi} \left\{ \int_0^{\pi} \sin^2 \omega t d(\omega t) + \int_{\pi}^{2\pi} 0 d(\omega t) \right\}}$$

$$= V_{\text{max}} \sqrt{\frac{1}{2\pi} \left\{ \frac{1 - \cos(2\omega t)}{2} \right\} d(\omega t)}$$

$$= V_{\text{max}} \sqrt{\frac{1}{4\pi} \left[(\omega t) - \frac{1}{2} \sin(2\omega t) \right]_0^{\pi}}$$

$$\therefore V_{rms} = V_{max} \sqrt{\frac{1}{4\pi} (\pi - 0 - (0 - 0))}$$

giving

$$V_{rms} = \frac{V_{max}}{2}$$

Question: What is V_{rms} for full wave rectification?

Ans: $\sqrt{2}$ (half wave), i.e. $\frac{V_{max}}{\sqrt{2}} = 0.707 V_{max}$

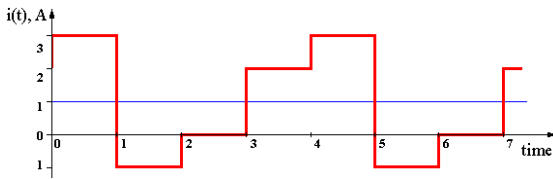
Question: What is V_{rms} for a sine wave without rectification?

Question: What is the peak value for the mains supply?

Ans: $\sqrt{2} \times 240 = 339.4$ volts

An Example of Average and R.M.S. Calculations

Calculate the average and r.m.s. values for the following waveform:



The Average Value

$$I_{dc} = \frac{1}{4} \left\{ \int_0^1 3 dt + \int_1^2 (-1) dt + \int_2^3 0 dt + \int_3^4 2 dt \right\}$$

Therefore

9

$$I_{dc} = \frac{1}{4} \{ 3 + (-1) + 0 + 2 \} = 1 \text{ A}$$

This is illustrated on the figure above.

The Effective or R.M.S. Value

$$I_{r.m.s.} = \sqrt{\frac{1}{4} \left\{ \int_0^1 3^2 dt + \int_1^2 (-1)^2 dt + \int_2^3 0^2 dt + \int_3^4 2^2 dt \right\}}$$

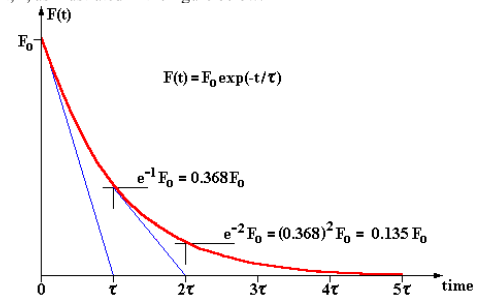
$$= \sqrt{\frac{1}{4} (9 + 1 + 0 + 4)}$$

$$I_{r.m.s.} = 1.87 \text{ A}$$

Exponentially Decaying Waveforms

The exponential waveform is common in transient analysis, i.e. during switching operations, for systems which include an energy storage element such as a capacitor or inductor.

For a stable system, the exponential term **must decay** with time, having a time constant, τ , as illustrated in the figure below.



For the observation of transient waveforms on an oscilloscope,

it is typical to use the fact that, after a time equal to five time constants, the response is 0.7% of its initial value, i.e. essentially zero.

10

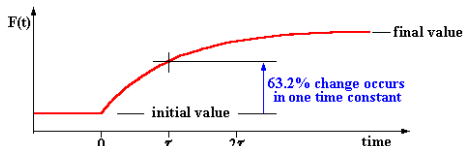
the initial point from which the decay time constant is evaluated may be taken anywhere on the exponential curve. It is often desirable to avoid the region close to the step change where second order effects may, in practice, cause a rounding of the curve. For example, at the transition the interturn capacitance of an inductor may interfere with the R-L transient decay curve for a short time.

it may be seen that the derivative of the curve with respect to time is proportional to $1/\tau$ and that this will pass through the final value one time constant later. This fact is useful in sketching good representations of an exponential curve.

Caution: Do not use the fact that the function is approximately zero after five time constants to estimate, or worse calculate, the value of the time constant!

An exponential change to a higher level

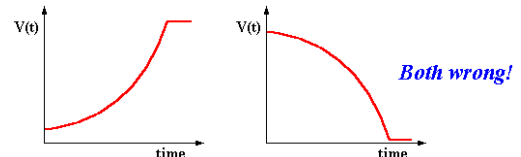
In a first order transient response, exponentially decaying waveforms change rapidly from the initial state and approach the final value as an asymptotic limit. The figure below illustrates a waveform where the initial value is less than the final value. Here, as always for a first order transient response, the waveform completes 63.2%, i.e. $(1 - 0.368) \times 100\%$, of the change towards the final value in one time constant.



Incorrect curves for first-order transient circuits:

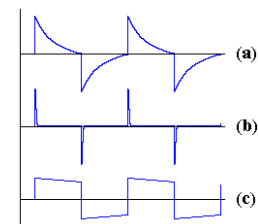
Curves which are expressed mathematically by an exponentially increasing function, in either the positive or negative direction, are wrong and do not occur in a typical first order transient response. Curves of this nature may occur in unstable systems.

11



If the time scale is all wrong:

For the observation of a first order transient response on an oscilloscope, it is desirable to set the time base such that there are about 5 time constants between successive transitions. Consider the waveform which is illustrated below:



A typical waveform for the current flow in a capacitor is shown in (a), with about five time constants between each transition. Now, if the time between transitions is increased to 50 time constants and the time base on the oscilloscope is slowed down by a factor of ten, the response is as shown in (b). The third curve, (c), shows what happens if the time between transitions is decreased to 0.5 time constants and the time base on the oscilloscope is increased by a factor of ten. **Thus it is important that the correct time between transitions is chosen for the observation of transient waveforms on an oscilloscope.**

12