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COMPUTATIONAL PROCEDURE FOR VINTI'S ACCURATE REFERENCE ORBIT WITH INCLUSION OF THE THIRD ZONAL HARMONIC

BY

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SUMMARY

Vinti has recently modified his spheroidal potential so as to permit exact inclusion of the effects of the third zonal harmonic of the planet's gravitational field. This corresponds to a potential fitted exactly through the third zonal harmonic and about two-thirds of the fourth. The present paper treats the method for obtaining the position and velocity coordinates of a satellite moving in the field corresponding to this accurate modified potential.

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INTRODUCTION

Vinti (Reference 1) has found a gravitational potential for an axially symmetric planet in oblate spheroidal coordinates. This solution accounts for all of the second zonal harmonic and more than half of the fourth zonal harmonic. This potential which simultaneously satisfies Laplace's equation and separates the Hamilton Jacobi equation, succeeds in reducing the problem of satellite motion to quadratures.

More, recently however, Vinti (Reference 3) has generalized his potential by means of a metric preserving transformation of the associated Cartesian system. This preserves separability of the problem of orbital motion when the potential coefficients $J_{2,1}$ and J_3 are taken into account (Reference 4). The inclusion of J_3 is of considerable practical importance, permitting a more accurate treatment than that given by perturbation theory. This leads to the computing procedure for obtaining the position and velocity coordinates of a drag free satellite from a knowledge of its initial conditions.

STATEMENT OF THE PROBLEM

If we take r_e as the earth's equatorial radius, and if,

$$c^2 = r_e^2 J_2 \left(1 - \frac{1}{4} J_3^2 J_2^{-3}\right),$$

$$\delta = -\frac{1}{2} r_e J_2^{-1} J_3 ,$$

then the gravitational potential,

$$V = -\mu\left(\rho^2 + c^2 \eta^2\right)^{-1} (\rho + \eta\delta)$$

leads to a separability of the problem of satellite motion. Here, $\delta \approx +7$ kilometers for the earth and the above potential leads to a fit of

$$V = -\frac{\mu}{r}\left[1 - \sum_{2}^{\infty}\left(\frac{r_e}{r}\right)^n J_n P_n(\sin\theta)\right]$$

exactly through the third zonal harmonic, about two-thirds of the fourth zonal harmonic, and negligible values of order J_2^3 for the higher harmonics.

If we take ρ, η, and ϕ as oblate spheroidal coordinates satisfying the equations

$$\begin{aligned} x + iy &= r\cos\theta\, e^{i\phi} \\ &= \left[(\rho^2 + c^2)(1 - \eta^2)\right]^{1/2} e^{i\phi}, \end{aligned}$$

and

$$z = r\sin\theta = \rho\eta - \delta,$$

then x, y, and z are the rectangular coordinates of a satellite in a Cartesian frame, with the origin at the center of mass of the planet. Also, r, θ, and ϕ are the planetocentric distance, latitude, and right ascencion.

From Vinti (Reference 3), if α_1 is the energy of the system, α_3, the z component of angular momentum, and α_2, the separation constant, the generalized momenta are given by, $p_\phi = \alpha_3$,

$$p_\rho = \pm\left(\rho^2 + c^2\right)^{-1} F^{1/2}(\rho)$$

and

$$p_\eta = \pm\left(1 - \eta^2\right)^{-1} G^{1/2}(\eta).$$

Here $F(\rho)$ and $G(\eta)$ are the quartics

$$F(\rho) = c^2\alpha_3^2 + \left(\rho^2 + c^2\right)\left(-\alpha_2^2 + 2\mu\rho + 2\alpha_1\rho^2\right),$$

and

$$G(\eta) = -\alpha_3^2 + \left(1 - \eta^2\right)\left(\alpha_2^2 + 2\mu\eta\delta + 2\alpha_1 c^2\eta^2\right).$$

The Hamilton-Jacobi function $W(\rho, \eta, \phi)$ is then,

$$W = \int p_\phi \, d\phi + \int p_\rho \, d\rho + \int p_\eta \, d\eta ,$$

or

$$W = \alpha_3 \phi + \int_{\rho_1}^{\rho} \pm \left(\rho^2 + c^2\right)^{-1} F^{1/2}(\rho) \, d\rho + \int_{\eta_1}^{\eta} \pm \left(1 - \eta^2\right)^{-1} G^{1/2}(\eta) \, d\eta$$

If β_1, β_2' and β_3' are constants of the motion, the orbit is then given by

$$t + \beta_1 = \frac{\partial W}{\partial \alpha_1} , \qquad \beta_2' = \frac{\partial W}{\partial \alpha_2}$$

and

$$\beta_3 = \frac{\partial W}{\partial \alpha_3} .$$

From Vinti's solution of these equations (Reference 3), together with the expressions for the generalized momenta, we can describe a computational procedure similar to that described in Reference 6.

COMPUTATIONAL PROCEDURE

Enter the initial conditions $x_i, y_i, z_i, \dot{x}_i, \dot{y}_i$ and $\dot{z}_i$ for a time t_i, with the constants μ, r_e, J_2, $J_3 < 0$, and,

$$c^2 = r_e^2 J_2 \left(1 - \frac{1}{4} J_3^2 J_2^{-3}\right) \text{ and } \delta = -\frac{1}{2} r_e J_2^{-1} J_3 > 0 \tag{1.1}$$

Compute,

$$r_i = \sqrt{x_i^2 + y_i^2 + z_i^2} , \qquad r_i \dot{r}_i = x_i \dot{x}_i + y_i \dot{y}_i + z_i \dot{z}_i \tag{1.2}$$

$$\rho_i^2 = \frac{r_i^2 + 2 z_i \delta + \delta^2 - c^2}{2} \left\{ 1 + \sqrt{1 + \frac{4c^2 (z_i + \delta)^2}{(r_i^2 + 2 z_i \delta + \delta^2 - c^2)^2}} \right\} \tag{1.3}$$

$$\eta_i^2 = \frac{(z_i + \delta)^2}{\rho_i^2}, \text{ (the sign of } \eta_i = \text{sign of } (z_i + \delta) \tag{1.4}$$

(the sign of η_i = sign of z_i)

$$\dot{\rho}_i = \frac{1}{2\rho_i}\left\{(r_i \dot{r}_i + \delta \dot{z}_i) + \frac{(r_i \dot{r}_i + \delta \dot{z}_i)(r_i^2 + 2z_i \delta + \delta^2 - c^2) + 2c^2(z_i + \delta)\dot{z}}{\sqrt{(r_i^2 + 2z_i \delta + \delta^2 - c^2)^2 + 4c^2(z_i + \delta)^2}}\right\} \tag{1.5}$$

$$\dot{\eta}_i = \frac{1}{2c^2 \eta}\left\{-(r_i \dot{r}_i + \delta \dot{z}_i) + \frac{(r_i \dot{r}_i + \delta \dot{z}_i)(r_i^2 + 2z_i \delta + \delta^2 - c^2) + 2c^2(z_i + \delta)\dot{z}}{\sqrt{r_i^2 + 2z_i \delta + \delta^2 - c^2)^2 + 4c^2(z_i + \delta)^2}}\right\} \tag{1.6}$$

Compute:

$$\alpha_1 = \frac{1}{2} U_i^2 - \mu(\rho_i + \eta_i \delta)(\rho_i^2 + c^2 \eta_i^2)^{-1} \tag{1.7}$$

where

$$U_i^2 = \dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2 ,$$

$$\alpha_3 = x_i \dot{y}_i - y_i \dot{x}_i ,$$

and

$$\alpha_2^2 = (1 - \eta_i^2)^{-1}\left[(\rho_i^2 + c^2 \eta_i^2)^2 \; \dot{\eta}_i^2 + \alpha_3^2 - (1 - \eta_i^2)(2\alpha_1 c^2 \eta_i^2 + 2\mu\eta_i \delta)\right]$$

Then,

$$a_0 = -\frac{1}{2}\frac{\mu}{\alpha_1}, \qquad e_0 = \left(1 + \frac{2\alpha_1 \alpha_2^2}{\mu^2}\right)^{1/2},$$

$$p_0 = a_0(1 - e_0^2), \quad \text{and} \quad i_0 = \cos^{-1}\frac{\alpha_3}{\alpha_2} \tag{1.8}$$

Prime Constants

Compute:

$$X_D^2 = -2\alpha_1 \alpha_2^2 \mu^{-2} \quad \text{and} \quad X_D^4 \tag{2.1}$$

$$p_0^2 \quad \text{and} \quad Y_D^2 = \left(\frac{\alpha_3}{\alpha_2}\right)^2 \tag{2.2}$$

$$K_0 = \frac{c^2}{p_0^2} \quad \text{and} \quad K_0^2 \tag{2.3}$$

$$\left(\rho_1 + \rho_2\right) = 2p_0 X_D^{-2}\left[1 - K_0 X_D^2 Y_D^2 - K_0^2 X_D^2 Y_D^2\left(2X_D^2 - 3X_D^2 Y_D^2 - 4 + 8Y_D^2\right)\right] \tag{2.4}$$

$$\rho_1 \rho_2 = p_0^2 X_D^{-2}\left[1 + K_0 Y_D^2\left(X_D^2 - 4\right) - K_0^2 Y_D^2\left(12X_D^2 - X_D^4 - 20X_D^2 Y_D^2 - 16 + 32Y_D^2 + X_D^4 Y_D^2\right)\right] \tag{2.5}$$

$$a = \left(\frac{\rho_1 + \rho_2}{2}\right) \tag{2.6}$$

$$g = \frac{4\rho_1 \rho_2}{\left(\rho_1 + \rho_2\right)^2} \tag{2.7}$$

$$e = \sqrt{1 - g} \tag{2.8}$$

$$\eta_0^{-2} = \frac{\alpha_2^2 - 2\alpha_1 c^2}{2\left(\alpha_2^2 - \alpha_3^2\right)}\left\{1 + \left[1 + \frac{8\alpha_1 c^2\left(\alpha_2^2 - \alpha_3^2\right)}{\left(\alpha_2^2 - 2\alpha_1 c^2\right)^2}\right]^{1/2}\right\} \tag{2.9}$$

$$\hat{S} = \left(\sin^2 I\right) = \eta_0^2 \tag{2.10}$$

Mutual Constants

$$p = a\left(1 - e^2\right) \tag{3.1}$$

$$\hat{A} = \frac{-2ac^2\left(ap - c^2\hat{S}\right)(1 - \hat{S}) + \dfrac{8a^2c^2}{p}\,\delta^2\left\{1 + \dfrac{c^2}{ap}\left(3\hat{S} - 2\right)\right\}\hat{S}(1 - \hat{S})}{\left(ap - c^2\right)\left(ap - c^2\hat{S}\right) + 4a^2c^2\hat{S} + \dfrac{4c^2}{p^2}\,\delta^2\left(3ap - 4a^2 - c^2\right)\hat{S}(1 - \hat{S})} \tag{3.2}$$

$$\hat{B} = (2a)^{-1}\left(ap - c^2\right)\hat{A} + c^2 \tag{3.3}$$

$$\hat{a}_0' = a - \frac{1}{2}\hat{A}, \quad \hat{p}_0' = \frac{\left(\hat{B} + ap - 2\hat{A}a - c^2\right)}{\hat{a}_0'}, \quad \hat{\alpha}_2' = \left(\mu\hat{p}_0'\right)^{1/2} \tag{3.4}$$

$$\hat{\epsilon} = \frac{\left(\dfrac{2\delta}{\hat{p}_0'}\right)^2 (1 - \hat{S})\left(1 - \dfrac{c^2}{\hat{a}_0'\hat{p}_0'}\hat{S}\right)}{\left[1 + \left(c^2/\hat{a}_0'\hat{p}_0'\right)(1 - 2\hat{S})\right]^2}, \quad \hat{U}^{-1} = 1 + \left(\frac{c^2}{\hat{a}_0'\hat{p}_0'}\right)(1 - \hat{S}) + \hat{\epsilon} \tag{3.5}$$

$$\hat{C}_2 = \frac{c^2}{\hat{a}_0{}' \hat{p}_0{}'} \hat{U}, \qquad \hat{C}_1 = \left(1 - \frac{c^2 \hat{S}}{\hat{a}_0{}' \hat{p}_0{}'} \hat{U}\right)^{-1} \frac{2\delta}{\hat{p}_0{}'} \hat{U} \left(1 - \frac{c^2 \hat{U}}{\hat{a}_0{}' \hat{p}_0{}'}\right) \tag{3.6}$$

$$\hat{p} = \left(1 - \frac{c^2 \hat{S} \hat{U}}{\hat{a}_0{}' \hat{p}_0{}'}\right) \frac{\delta}{\hat{p}_0{}'} \hat{U}(1 - \hat{S}) \tag{3.7}$$

$$\hat{\eta}_0{}' = \hat{P} + (\hat{P}^2 + \hat{S})^{1/2}, \qquad \hat{\eta}_1{}' = \hat{P} - (\hat{P}^2 + \hat{S})^{1/2} \tag{3.8}$$

$$\hat{S}' = - \hat{\eta}_0' \hat{\eta}_1' \tag{3.9}$$

Using $\hat{S}'$, we now repeat steps (3.2) through (3.9) to obtain the quantities,

$$A,\ B,\ a_0{}',\ p_0{}'\ \alpha_2{}',\ \epsilon,\ U,\ C_2,\ C_1,\ P,\ \eta_0{}',\ \eta_1{}' \ \text{and}\ S\ . \tag{3.10}$$

Note: Equation (3.9) for $\hat{S}'$ together with the one step iteration (3.10) has now provided for us an accurate value of the element S that was originally approximated by η_0^2. (Equation 2.10)

$$b_1 = -\frac{1}{2} A, \qquad b_2 = B^{1/2}, \qquad \alpha_1{}' = -\frac{\mu}{2a_0{}'} \tag{3.11}$$

Using formulas of Reference 6, pages twelve to fifteen, we compute,

$$A_1 = (1 - e^2)^{1/2} p \sum_{n=2}^{\infty} \left(\frac{b_2}{p}\right)^n P_n\left(\frac{b_1}{b_2}\right) R_{n-2} \left[(1 - e^2)^{1/2}\right]$$

$$A_2 = (1 - e^2)^{1/2} p^{-1} \sum_{n=0}^{\infty} \left(\frac{b_2}{p}\right)^n P_n\left(\frac{b_1}{b_2}\right) R_n \left[(1 - e^2)^{1/2}\right] \tag{3.12}$$

where $P_n(b_1/b_2)$ is the Legendre Polynomial of degree n, $R_n(X_s) = X_s^n P_n(X_s^{-1})$ is a ploynomial of degree $(n/2)$ in X_s^2, and $X_s = (1 - e^2)^{1/2}$.

If m is an even integer compute,

$$D_m = D_{2i} = \sum_{n=0}^{i} (-1)^{i-n} \left(\frac{c}{p}\right)^{2i-2n} \left(\frac{b_2}{p}\right)^{2n} P_{2n}\left(\frac{b_1}{b_2}\right)$$

- If m is an odd integer compute,

$$D_m = D_{2i+1} = \sum_{n=0}^{i} (-1)^{i-n} \left(\frac{c}{p}\right)^{2i-2n} \left(\frac{b_2}{p}\right)^{2n+1} P_{2n+1}\left(\frac{b_1}{b_2}\right)$$

Then,

$$A_3 = \left(1-e^2\right)^{1/2} p^{-3} \sum_{m=0}^{\infty} D_m R_{m+2}\left[\left(1-e^2\right)^{1/2}\right]$$

$$A_{11} = \frac{3}{4}\left(1-e^2\right)^{1/2} p^{-3} e\left(-2b_1 b_2^2 p + b_2^4\right)$$

$$A_{12} = \frac{3}{32}\left(1-e^2\right)^{1/2} b_2^4 e^2 p^{-3}$$

$$A_{21} = \left(1-e^2\right)^{1/2} p^{-1} e\left[b_1 p^{-1} + \left(3b_1^2 - b_2^2\right) p^{-2} - \frac{9}{2} b_1 b_2^2 \left(1+\frac{e^2}{4}\right) p^{-3} + \frac{3}{8} b_2^4 \left(4+3e^2\right) p^{-4}\right]$$

$$A_{22} = \left(1-e^2\right)^{1/2} p^{-1}\left[\frac{e^2}{8}\left(3b_1^2 - b_2^2\right) p^{-2} - \frac{9}{8} e^2 b_1 b_2^2 p^{-3} + \frac{3}{32} b_2^4 \left(6e^2+e^4\right) p^{-4}\right]$$

$$A_{23} = \left(1-e^2\right)^{1/2} p^{-1} \frac{e^3}{8}\left(-b_1 b_2^2 p^{-3} + b_2^4 p^{-4}\right)$$

$$A_{24} = \frac{3}{256}\left(1-e^2\right)^{1/2} p^{-5} b_2^4 e^4$$

$$A_{31} = \left(1-e^2\right)^{1/2} p^{-3} e\left[2 + b_1 p^{-1}\left(3+\frac{3}{4}e^2\right) - p^{-2}\left(\frac{1}{2}b_2^2 + c^2\right)\left(4+3e^2\right)\right]$$

$$A_{32} = \left(1-e^2\right)^{1/2} p^{-3}\left[\frac{e^2}{4} + \frac{3}{4} b_1 p^{-1} e^2 - p^{-2}\left(\frac{b_2^2}{2} + c^2\right)\left(\frac{3}{2}e^2 + \frac{e^4}{4}\right)\right]$$

$$A_{33} = \left(1-e^2\right)^{1/2} e^3 \left[\frac{b_1}{12p} - \frac{p^{-2}}{3}\left(\frac{b_2^2}{2} + c^2\right)\right] p^{-3}$$

$$A_{34} = -\frac{1}{32}\left(1-e^2\right)^{1/2} p^{-5} e^4 \left(\frac{1}{2} b_2^2 + c^2\right)$$

$$Q = \left(P^2 + S\right)^{1/2} \tag{3.13}$$

Then through third order in J_2:

$$B_2 = 1 - \frac{1}{2} C_1 P + \left(\frac{3}{8} C_1^2 + \frac{1}{2} C_2\right)\left(P^2 + \frac{1}{2} Q^2\right) + \frac{9}{64} C_2^2 Q^4$$

$$- \frac{3}{8} C_1 C_2 P Q^2 + \frac{45}{128} C_1^2 C_2 Q^4 + \frac{25}{256} C_2^3 Q^6 \qquad (3.14)$$

$$B_1' = \frac{1}{2} Q^2 + P^2 - \frac{3}{4} C_1 P Q^2 + \frac{3}{2} P^2 Q^2 + \frac{3}{64} \left(4C_2 + 3C_1^2\right) Q^4$$

$$- \frac{15}{16} C_1 C_2 P Q^4 + \frac{5}{256} \left(6C_2^2 + 15C_1^2 C_2\right) Q^6 + \frac{175}{2048} C_2^3 Q^8 \qquad (3.15)$$

$$B_3 = -\frac{1}{2} C_2 - \frac{3}{8} C_1^2 - \left(\frac{15}{16} C_1^2 C_2 + \frac{3}{8} C_2^2\right)\left(1 + \frac{1}{2} Q^2 + \frac{3}{8} Q^4\right) - \frac{1}{4} C_1 C_2 P \qquad (3.16)$$

$$\zeta = \frac{P}{1 - S}, \qquad h_1 = \frac{1}{2} \left(1 + C_1 - C_2\right)^{-1/2}, \qquad h_2 = \frac{1}{2} \left(1 - C_1 - C_2\right)^{-1/2} \qquad (3.17)$$

$$e_2 = Q(1 - P)^{-1}, \qquad e_3 = Q(1 + P)^{-1} \qquad (3.18)$$

$$2\pi\nu_1 = \left(-2\alpha_1'\right)^{1/2} \left(a + b_1 + A_1 + c^2 A_2 B_1' B_2^{-1}\right)^{-1},$$

$$2\pi\nu_2 = \alpha_2' U^{-1/2} A_2 B_2^{-1} \left(a + b_1 + A_1 + c^2 A_2 B_1' B_2^{-1}\right)^{-1} \qquad (3.19)$$

$$e' = \frac{ae}{a + b_1} \quad \text{where} \quad e' < e < 1 \qquad (3.20)$$

$$\alpha_3' = \operatorname{sgn} \alpha_3 \left[\mu p_0' (1 - S)\right]^{1/2} \left[1 - \frac{c^2 S}{a_0' p_0'} \frac{-\left(\frac{2\delta}{p_0'}\right)^2 \left(1 - \frac{c^2 S}{a_0' p_0'}\right) S}{\left[1 + \frac{c^2}{a_0' p_0'} (1 - 2S)\right]^2}\right]^{1/2} \qquad (3.21)$$

Here $\operatorname{sgn} \alpha_3 \gtrless 0$ for a direct or retrograde orbit.

Jacobi Constants

$$B_{11} = 2PQ - \frac{3}{8} C_1 Q^3 \qquad (4.1)$$

$$B_{12} = -\left(\frac{Q^2}{4} + \frac{1}{8} C_2 Q^4\right) \tag{4.2}$$

$$B_{13} = C_1 Q^3/24 \tag{4.3}$$

$$B_{14} = C_2 Q^4/64 \tag{4.4}$$

$$B_{21} = C_2 P Q - \frac{3}{16} C_1 C_2 Q^3 \tag{4.5}$$

$$B_{22} = -\frac{1}{32}\left[\left(4C_2 + 3C_1^2\right) Q^2 + 3C_2^2 Q^4\right] \tag{4.6}$$

$$B_{23} = -\frac{1}{48} C_1 C_2 Q^3 \tag{4.7}$$

$$B_{24} = \frac{3}{256} C_2^2 Q^4 \tag{4.8}$$

$$\cos\phi_i = \frac{x_i}{\sqrt{\rho_i^2 + c^2}\,\sqrt{1-\eta_i^2}}$$

$$\sin\phi_i = \frac{y_i}{\sqrt{\rho_i^2 + c^2}\,\sqrt{1-\eta_i^2}} \tag{4.9}$$

$$h_{\rho_i}^2 = \frac{\rho_i^2 + \eta_i^2 c^2}{\rho_i^2 + c^2}, \qquad h_{\eta_i}^2 = \frac{\rho_i^2 + \eta_i^2 c^2}{1-\eta_i^2} \tag{4.10}$$

$$\cos E_i = \frac{1}{e}\left(1 - \frac{\rho_i}{a}\right),$$

$$\sin E_i = \frac{\dot{\rho}_i h_{\rho_i}^2 \left(\rho_i^2 + c^2\right)}{ae\sqrt{-2\alpha_1'\left(\rho_i^2 + A\rho_i + B\right)}} \tag{4.11}$$

$$\sin\psi_i = \frac{\eta_i - P}{Q}$$

$$\cos\psi_i = \frac{\dot{\eta}_i h_{\eta_i}^2 \left(1-\eta_i^2\right)}{Q\sqrt{\frac{\left(\alpha_2'^2 - \alpha_3'^2\right)}{S}\left(1 + C_1\eta_i - C_2\eta_i^2\right)}} \tag{4.12}$$

$$\cos v_i = \frac{\cos E_i - e}{1 - e\cos E_i} ,$$

$$\sin v_i = \frac{\left(1 - e^2\right)^{1/2} \sin E_i}{1 - e \cos E_i} \tag{4.13}$$

$$\sin n v_i \quad \text{for} \quad n = 2, 3, 4$$

$$\sin n\psi_i \quad \text{for} \quad n = 2, 4 \tag{4.14}$$

and

$$\cos 3\psi_i$$

$$\cos E'_{2i} = \frac{e_2 + \cos\left(\psi_i + \frac{\pi}{2}\right)}{1 + e_2 \cos\left(\psi_i + \frac{\pi}{2}\right)}$$

$$\sin E'_{2i} = \frac{\left(1 - e_2^2\right)^{1/2} \sin\left(\psi_i + \frac{\pi}{2}\right)}{1 + e_2 \cos\left(\psi_i + \frac{\pi}{2}\right)}$$

$$\cos E'_{3i} = \frac{e_3 + \cos\left(\psi_i - \frac{\pi}{2}\right)}{1 + e_3 \cos\left(\psi_i - \frac{\pi}{2}\right)}$$

$$\sin E'_{3i} = \frac{\left(1 - e_3^2\right)^{1/2} \sin\left(\psi_i - \frac{\pi}{2}\right)}{1 + e_3 \cos\left(\psi_i - \frac{\pi}{2}\right)} \tag{4.15}$$

$$\chi_{0i} = \frac{E'_{2i}\left(\psi_i + \frac{\pi}{2}\right)}{2\sqrt{1 - 2\zeta}} + \frac{E'_{3i}\left(\psi_i - \frac{\pi}{2}\right)}{2\sqrt{1 + 2\zeta}} ,$$

$$\chi_{1i} = \frac{E'_{2i}\left(\psi_i + \frac{\pi}{2}\right)}{2\sqrt{1 - 2\zeta}} - \frac{E'_{3i}\left(\psi_i - \frac{\pi}{2}\right)}{2\sqrt{1 + 2\zeta}} \tag{4.16}$$

$$\beta_1 = \left(-2\alpha_1'\right)^{-1/2}\left[b_1 E_i + a\left(E_i - e \sin E_i\right) + A_1 v_i + A_{11} \sin v_i + A_{12} \sin 2v_i\right]$$

$$+ \frac{c^2 U^{1/2}}{\alpha_2'}\left[B_1' \psi_i + B_{11} \cos\psi_i + B_{12} \sin 2\psi_i + B_{13} \cos 3\psi_i + B_{14} \sin 4\psi_i\right] - t_i , \tag{4.17}$$

$$\beta_2' = -\alpha_2'\left(-2\alpha_1'\right)^{-1/2}\left[A_2 v_i + \sum_{n=1}^{4} A_{2n} \sin nv_i\right] + U^{1/2}\left[\left(\psi_i + \frac{\pi}{2}\right)B_2 + B_{21}\cos\psi_i\right.$$

$$\left. + B_{22}\sin 2\psi_i + B_{23}\cos 3\psi_i + B_{24}\sin 4\psi_i\right] \tag{4.18}$$

$$\beta_3 = \phi_i + c^2\alpha_3'\left(-2\alpha_1'\right)^{-1/2}\left[A_3 v_i + \sum_{j=1}^{4} A_{3j}\sin jv_i\right] - \frac{\alpha_3'}{\alpha_2'}U^{1/2}\left\{(1-S)^{-1/2}\left[\left(h_1+h_2\right)\chi_{0i}\right.\right.$$

$$\left.\left. + \left(h_1 - h_2\right)\chi_{1i}\right] + B_3\psi_i + \frac{1}{4}C_1C_2Q\cos\psi_i + \frac{3}{32}C_2^2Q^2\sin 2\psi_i\right\} \tag{4.19}$$

$$\beta_2 = \beta_2' - U^{1/2}B_2\frac{\pi}{2} \tag{4.20}$$

$$\ell_0 = 2\pi\nu_1\left(\beta_1 - \frac{c^2\beta_2 B_1'}{\alpha_2' B_2}\right) \tag{4.21}$$

$$\ell_0 + g_0 = 2\pi\nu_2\left[\beta_1 + \left(\frac{\beta_2}{\alpha_2'}\right)\left(a + b_1 + A_1\right)A_2^{-1}\right] \tag{4.22}$$

<u>Orbit Generator</u>

$$\begin{aligned} M_s &= \ell_0 + 2\pi\nu_1 t \\ \psi_s &= \ell_0 + g_0 + 2\pi\nu_2 t \end{aligned} \tag{5.1}$$

By Newton-Raphson iteration we solve $M_s + E_0 - e'\sin\left(M_s + E_0\right) = M_s$ where $M_s + E_0 = \mathcal{E}$; therefore,

$$\mathcal{E} = \mathcal{E}_{n+1} = \mathcal{E}_n - \frac{\left[\mathcal{E}_n - e'\sin\mathcal{E}_n - M_s\right]}{\left(1 - e'\cos\mathcal{E}_n\right)} - \frac{1}{2}\left[\frac{\mathcal{E}_n - e'\sin\mathcal{E}_n - M_s}{1 - e'\cos\mathcal{E}_n}\right]^2\left[\frac{e'\sin\mathcal{E}_n}{1 - e'\cos\mathcal{E}_n}\right] \tag{5.2}$$

and $\mathcal{E}_n = M_s$ initially.

$$\begin{aligned} \cos v' &= (\cos\mathcal{E} - e)(1 - e\cos\mathcal{E})^{-1} \\ \sin v' &= \left(1 - e^2\right)^{1/2}(1 - e\cos\mathcal{E})^{-1}\sin\mathcal{E} \end{aligned} \tag{5.3}$$

$$v_0 = v' - M_s \tag{5.4}$$

$$\psi_0 = \alpha_2' \left(-2\alpha_1'\right)^{-1/2} U^{-1/2} A_2 B_2^{-1} v_0 \tag{5.5}$$

$$M_1 = -\left(a + b_1\right)^{-1} \left[\left(A_1 + c^2 A_2 B_1' B_2^{-1}\right) v_0 + \frac{c^2}{\alpha_2'} \left(-2\alpha_1'\right)^{1/2} U^{1/2} B_{12} \sin\left(2\psi_s + 2\psi_0\right)\right] \tag{5.6}$$

$$E_1 = \frac{M_1}{1 - e' \cos\left(M_s + E_0\right)} - \frac{e'}{2} \frac{M_1^2 \sin\left(M_s + E_0\right)}{\left[1 - e' \cos\left(M_s + E_0\right)\right]^3} \tag{5.7}$$

$$\cos v'' = \left[\cos\left(\mathcal{E} + E_1\right) - e\right]\left[1 - e\cos\left(e + E_1\right)\right]^{-1}$$

$$\sin v'' = \left(1 - e^2\right)^{1/2} \left[1 - e\cos\left(\mathcal{E} + E_1\right)\right]^{-1} \sin\left(\mathcal{E} + E_1\right) \tag{5.8}$$

$$v_1 = v'' - \left(v_0 + M_s\right) = \left(v'' - v'\right) \tag{5.9}$$

$$\psi_1 = -B_{22} B_2^{-1} \sin\left(2\psi_s + 2\psi_0\right) + \alpha_2' \left(-2\alpha_1'\right)^{-1/2} U^{-1/2} B_2^{-1} \left[A_2 v_1 + A_{21} \sin\left(M_s + v_0\right) + A_{22} \sin\left(2M_s + 2v_0\right)\right] \tag{5.10}$$

$$M_2 = -\left(a + b_1\right)^{-1} \left[A_1 v_1 + A_{11} \sin\left(M_s + v_0\right) + A_{12} \sin\left(2M_s + 2v_0\right) + \frac{c^2}{\alpha_2'} \left(-2\alpha_1'\right)^{1/2} U^{1/2} \left\{B_1' \psi_1 \right.\right.$$

$$\left.\left. + B_{11} \cos\left(\psi_s + \psi_0\right) + 2B_{12} \psi_1 \cos\left(2\psi_s + 2\psi_0\right) + B_{13} \cos\left(3\psi_s + 3\psi_0\right) + B_{14} \sin\left(4\psi_s + 4\psi_0\right)\right\}\right] \tag{5.11}$$

$$E_2 = \frac{M_2}{1 - e' \cos\left(M_s + E_0 + E_1\right)} \tag{5.12}$$

$$\cos v''' = \left[\cos\left(\mathcal{E} + E_1 + E_2\right) - e\right]\left[1 - e\cos\left(\mathcal{E} + E_1 + E_2\right)\right]^{-1}$$

$$\sin v''' = \left(1 - e^2\right)^{1/2} \left[1 - e\cos\left(\mathcal{E} + E_1 + E_2\right)\right]^{-1} \sin\left(\mathcal{E} + E_1 + E_2\right) \tag{5.13}$$

$$v_2 = v''' - \left(v_0 + M_s + v_1\right) = \left(v''' - v''\right) \tag{5.14}$$

$$\psi_2 = -B_2^{-1} \left[B_{21} \cos\left(\psi_s + \psi_0\right) + 2B_{22} \psi_1 \cos\left(2\psi_s + 2\psi_0\right) + B_{23} \cos\left(3\psi_s + 3\psi_0\right) + B_{24} \sin\left(4\psi_s + 4\psi_0\right)\right]$$

$$+ \alpha_2' U^{-1/2} \left(-2\alpha_1'\right)^{-1/2} B_2^{-1} \left[A_2 v_2 + A_{21} v_1 \cos\left(M_s + v_0\right) + 2A_{22} v_1 \cos\left(2M_s + 2v_0\right)\right.$$

$$\left. + A_{23} \sin\left(3M_3 + 3v_0\right) + A_{24} \sin\left(4M_s + 4v_0\right)\right] \tag{5.15}$$

Then,

$$E = \mathcal{E} + E_1 + E_2 ,$$

$$v = M_s + v_0 + v_1 + v_2 ,$$

and

$$\psi = \psi_s + \psi_0 + \psi_1 + \psi_2 \tag{5.16}$$

$$\cos E_2' = \frac{e_2 + \cos\left(\psi + \frac{\pi}{2}\right)}{1 + e_2 \cos\left(\psi + \frac{\pi}{2}\right)}$$

$$\sin E_2' = \frac{\left(1 - e_2^2\right)^{1/2} \sin\left(\psi + \frac{\pi}{2}\right)}{1 + e_2 \cos\left(\psi + \frac{\pi}{2}\right)}$$

and,

$$\cos E_3' = \frac{e_3 + \cos\left(\psi - \frac{\pi}{2}\right)}{1 + e_3 \cos\left(\psi - \frac{\pi}{2}\right)}$$

$$\sin E_3' = \frac{\left(1 - e_3^2\right)^{1/2} \sin\left(\psi - \frac{\pi}{2}\right)}{1 + e_3 \cos\left(\psi - \frac{\pi}{2}\right)} \tag{5.17}$$

$$\chi_0 = \frac{E_2'\left(\psi + \frac{\pi}{2}\right)}{2\sqrt{1 - 2\zeta}} + \frac{E_3'\left(\psi - \frac{\pi}{2}\right)}{2\sqrt{1 + 2\zeta}}$$

and

$$\chi_1 = \frac{E_2'\left(\psi + \frac{\pi}{2}\right)}{2\sqrt{1 - 2\zeta}} - \frac{E_3'\left(\psi - \frac{\pi}{2}\right)}{2\sqrt{1 + 2\zeta}} \tag{5.18}$$

$$\rho = a(1 - e\cos E) , \qquad \eta = P + Q\sin\psi ,$$

and,

$$\phi = \beta_3 - c^2 \alpha_3' \left(-2\alpha_1'\right)^{-1/2} \left[A_3 v + \sum_{j=1}^{4} A_{3j} \sin jv\right] + \frac{\alpha_3'}{\alpha_2'} U^{1/2} \left\{(1-S)^{-1/2}\left[\left(h_1 + h_2\right)\chi_0\right.\right.$$

$$\left.\left. + \left(h_1 - h_2\right)\chi_1\right] + B_3\psi + \frac{1}{4} C_1 C_2 Q \cos\psi + \frac{3}{32} C_2^2 Q^2 \sin 2\psi\right\} \qquad (5.19)$$

$$h_\rho^2 = \frac{\rho^2 + \eta^2 c^2}{\rho^2 + c^2}, \qquad h_\eta^2 = \frac{\rho^2 + \eta^2 c^2}{1 - \eta^2}$$

and

$$h_\phi^2 = \left(\rho^2 + c^2\right)\left(1 - \eta^2\right) \qquad (5.20)$$

$$\dot{\rho} = \frac{ae\sqrt{-2\alpha_1'\left(\rho^2 + A\rho + B\right)}}{h_\rho^2\left(\rho^2 + c^2\right)} \sin E$$

$$\dot{\eta} = \frac{Q\sqrt{\dfrac{\left(\alpha_2'^2 - \alpha_3'^2\right)}{S}\left(1 + C_1\eta - C_2\eta^2\right)}}{h_\eta^2\left(1 - \eta^2\right)} \cos\psi \qquad (5.21)$$

$$X = \sqrt{\left(\rho^2 + c^2\right)\left(1 - \eta^2\right)} \cos\phi$$

$$Y = \sqrt{\left(\rho^2 + c^2\right)\left(1 - \eta^2\right)} \sin\phi$$

$$Z = \rho\eta - \delta$$

$$\dot{X} = X\left(\frac{\rho\dot{\rho}}{\rho^2 + c^2} - \frac{\eta\dot{\eta}}{1 - \eta^2}\right) - \frac{Y \cdot \alpha_3'}{h_\phi^2}$$

$$\dot{Y} = Y\left(\frac{\rho\dot{\rho}}{\rho^2 + c^2} - \frac{\eta\dot{\eta}}{1 - \eta^2}\right) + \frac{X \cdot \alpha_3'}{h_\phi^2}$$

and

$$\dot{Z} = \rho\dot{\eta} + \eta\dot{\rho} \qquad (5.22)$$

REMARKS

The accuracy of the orbit itself as a solution for the given potential is carried out through terms of the third order in J_2, the coefficient of the second zonal harmonic. Its accuracy however, and thus that of the secular terms, may be increased at will. Periodic terms are carried through the second order, but their accuracy may also be increased. In order to carry this kind of accuracy perturbation methods presently in use would become far too cumbersome and impractical for orbit computational purposes.

An advantage of accounting for J_3 in this way is the absence of small denominators in e or $\sin I$ that occur in perturbation theories. Thus, one can easily compute polar orbits and circular equatorial orbits.

This program, similar in structure to the previous Vinti accurate intermediary orbit requires a relatively small number of storage locations throughout the entire computing procedure. Consequently, this program is expected to go as fast as the Vinti accurate intermediary orbit producing approximately 1800 minute vector points (time, x, y, z, $\dot{x}$, $\dot{y}$, and $\dot{z}$, 1800 times) each minute of IBM 7094 Mod. I computer operation with simultaneous production of BCD tape output (Reference 6).

Recent tests on the IBM 7094 have indicated that this latter Vinti method enjoys a rather sizeable advantage in computational speed over other methods which use perturbation techniques with the Vinti program producing some three to four times more minute vector points per unit time of computing machine operation.

The residual fourth harmonic term (Reference 5) has been programmed and is presently being tested as is the $J_{2,1}$ tesseral harmonic term. In addition, the orbit differential correction program written for the Vinti accurate intermediary orbit theory (Reference 7), will need only slight modification to account for the element S which replaces $\eta_0 = \sin I$ in Vinti's new theory. Consequently it is expected to go just as rapidly, producing a set of mean elements of even greater accuracy with which to predict satellite orbits over a longer interval of time. It has been shown (Reference 7) that the orbit differential correction program for the Vinti accurate intermediary orbit theory converges rapidly and with great accuracy. This latter program, tested on the Relay II Satellite with an eccentricity of 0.23597617, and Satellite ANNA with an eccentricity of 0.00671710 gave the following results:

Using radio direction cosine observation data, and for seventy equations of condition of the Relay II Satellite extending over a five hour arc following injection into orbit, the program converged on the third iteration to a standard deviation of fit criterion of 2.7×10^{-3} within thirty seconds. Using Smithsonian Astrophysical Observatory optical data from the Anna Statellite extending over an arc of seventy five hours following injection, and with twenty equations of condition, the program converged on the second iteration to a standard deviation of fit of 0.2×10^{-3} (milliradians). For seven equations of condition of Satellite ANNA covering the first forty-five hours following injection, the Vinti orbit differential correction program converged on the second iteration to a standard devia tion of fit criterion of 0.04×10^{-3} (milliradians). All tests were conducted using the IBM 7094 Mod. I electronic digital computer.

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