X-661-71-443

PREPRINT

NASA TM X-65759

THE EQUILIBRIUM AND STABILITY OF THE GASEOUS COMPONENT OF THE GALAXY. II.

SANFORD A. KELLMAN

NOVEMBER 1971

GSFC

GODDARD SPACE FLIGHT CENTER
GREENBELT, MARYLAND

THE EQUILIBRIUM AND STABILITY OF THE GASEOUS COMPONENT OF THE GALAXY. II.

Sanford A. Kellman
Department of Physics and Astronomy
University of Maryland, College Park, Maryland, 20742
and
NASA-Goddard Space Flight Center, Greenbelt, Maryland, 20771

ABSTRACT

A time-independent, linear, plane and axially-symmetric stability analysis is performed on a self-gravitating,plane-parallel, isothermal layer of non-magnetic, non-rotating gas. The gas layer is immersed in a plane-stratified isothermal layer of stars which supply a self-consistent gravitational field. Only the gaseous component is perturbed. Expressions are derived for the perturbed gas potential and perturbed gas density that satisfy both the Poisson and hydrostatic equilibrium equations. The equation governing the size of the perturbations in the mid-plane is found to be analogous to the one-dimensional time-independent Schrodinger equation for a particle bound by a potential well, and with similar boundary conditions. The radius of the neutral state is computed numerically and compared with the Jeans' and Ledoux radius. The inclusion of a rigid stellar component increases the Ledoux radius, though only slightly. Isodensity contours of the neutral or marginally unstable state are constructed. Large flattened objects with masses of 5×10^6 $M_\odot$ and radii of 1-2 kpc result. These numbers are not inconsistent with the large-scale structure observed in the gaseous component of spiral arms in the Galaxy. The possibility is discussed that the gravitational instability of the gaseous component excites density waves of the type described by Lin.

I. INTRODUCTION

The best-known perturbation analysis with applications in galactic astronomy was carried out by Jeans (1928) and appropriately is called the Jeans' instability problem. The initial equilibrium state was taken to be a self-gravitating infinite uniform gas with a uniform gravitational potential field throughout. The last two assumptions are, however, inconsistent with the Poisson equation. It is this inconsistency which has prompted others to reinvestigate Jeans' analysis with self-consistent equilibrium density and potential distributions. Jeans applied perturbations to the momentum, continuity, Poisson, and heat equation, and after linearization and Fourier analysis found the condition for marginal stability to be $\lambda_J = c_s(\pi/G\rho_o)^{1/2}$, where c_s is the isothermal sound speed of the medium and ρ_o is its density. Any disturbance with length greater than λ_J is gravitationally unstable and will collapse in a finite time; any disturbance with length smaller than λ_J is stable.

The first modification to Jeans' analysis of importance to galactic astronomy was undertaken by Ledoux (1951), who considered the more realistic initial equilibrium state of a plane-parallel, non-rotating, isothermal, self-gravitating gas layer (no stars). The gas density at the plane of symmetry assumes its maximum value ρ_o and decreases with distance from the mid-plane according to the well-known formula $\rho(z)/\rho_o = \mathrm{sech}^2(z/H)$, where H is the scale height of the gas layer in the z direction. Using these self-consistent initial conditions, Ledoux finds the condition for marginal stability to be $\lambda_L = c_s(2\pi/G\rho_o)^{1/2}$, precisely the same relation found by Jeans except that ρ_o is replaced by $\rho_o/2$.

Goldreich and Lynden-Bell (1965) considered the gravitational stability of a plane-stratified, self-gravitating, uniformly-rotating disk of gas (no stars). They found that (i) pressure effects stabilize short wavelength disturbances while rotation stabilizes long wavelength disturbances and (ii) when the quantity $\pi G\bar{\rho}/4\Omega^2$ is greater than about 1.0, where $\bar{\rho}$ is the mean gas density and Ω is the angular velicity, disturbances of dimension several times the layer thickness become unstable.

In what follows, we consider a stability analysis similar to that of Ledoux (1951), the principal difference being that the gas layer is immersed in a rigid non-perturbable star layer which supplies a self-consistent gravitational field. The effect of a magnetic field on the stability will be considered in Paper III of this series (Kellman 1972a); the effect of a magnetic field plus cosmic-ray gas will be investigated in Paper IV (Kellman 1972b). We will not address ourselves directly to the grand overall design of spiral structure observed in galaxies, and discussed by Lin and Shu (1964, 1966) and Lin, Yuan, and Shu (1969). This is probably a collective stellar phenomenon and therefore lies outside the range of our analysis. Rather, we limit our attention to the existence of the large-scale structure within the gaseous component of spiral arms.

A 21-cm southern survey by McGee and Milton (1964) revealed that the principal elements of the gaseous component of spiral arms in the

Galaxy are enormous flattened objects, 1-2 kpc in size and 10^7 $M_\odot$ in mass. In addition, these structures were observed to be strung out along the length of a spiral arm like beads on a string, their size increasing with increasing distance from the galactic center beyond the Sun's distance R_o. Because the projected central density of these structures is observed to be only slightly larger than the projected ensity of regions between the structures, a linearized stability analysis of the type carried out below may shed light on their existence.

II. STABILITY ANALYSIS

With these ideas in mind, we proceed to examine the stability of a self-gravitating plane-stratified distribution of gas to linear, time-independent, plane and axially-symmetric perturbations. The plane of symmetry is the mid-plane z=0 where the gas and star densities attain their maximum values. The axis of symmetry is perpendicular to the symmetry plane and arbitrarily positioned. The gas is immersed in the gravitational field of a plane-parallel distribution of stars. Both components of the two-fluid mixture are assumed to be isothermal, but each has a different 'temperature'. We should stress that only the gaseous component is perturbed; the stars merely supply a self-consistent gravitational field which adds to the gravitational field of the gas. This initial equilibrium state is essentially that calculated by Kellman (1972c) in Paper I of this series. By restricting the analysis to be time-independent, only the marginally unstable or neutral state is determined. The inclusion of a stellar component should lend the analysis an increased sense of reality as concerns the stability of the gaseous component of the Galaxy.

The time-independent perturbations may be written as follows:

$$\rho_g(r,z) = \rho_{eg}(z) + \Delta\rho_g(r,z) \tag{1}$$

$$p_g(r,z) = p_{eg}(z) + \Delta p_g(r,z) = \langle v_{tz}^2 \rangle \rho_g(r,z) \tag{2}$$

$$\varphi_g(r,z) + \varphi_*(r,z) = \varphi_{eg}(z) + \varphi_{e*}(z) + \Delta\varphi_g(r,z). \tag{3}$$

ρ, p, and φ are the density, pressure, and gravitational potential. $\langle v_{tz}^2 \rangle$ is the mean square z turbulent gas velocity dispersion. The subscripts g and * refer to the gas and star components, respectively. The subscript e denotes the equilibrium state described above. Δ denotes the perturbed quantities, functions of both r (the distance from the symmetry axis) and z (the distance from the symmetry plane). Since only the gaseous component is perturbed, we can write that

$$\varphi_*(r,z) = \varphi_{e*}(z), \tag{4}$$

so that equation (3) becomes

$$\varphi_g(r,z) = \varphi_{eg}(z) + \Delta\varphi_g(r,z). \tag{5}$$

Equations (1), (4), and (5) are substituted into the gas hydrostatic equilibrium equation

$$\frac{\langle v_{tz}^2 \rangle}{\rho_g} \nabla\rho_g = \nabla(\varphi_g + \varphi_*), \tag{6}$$

and retaining terms only to first order in $\Delta\rho_g$ and $\Delta\varphi_g$ we find that

$$\frac{1}{\rho_{eg}} \nabla(\Delta\rho_g) - \frac{\nabla\rho_{eg}}{\rho_{eg}^2} \Delta\rho_g = \frac{1}{\langle v_{tz}^2 \rangle} \nabla(\Delta\varphi_g). \tag{7}$$

Goon (1966) has suggested the usefulness of defining a new variable $\epsilon(r,z)$:

$$\epsilon(r,z) = \frac{\Delta\rho_g(r,z)}{\rho_{eg}(z)} . \tag{8}$$

Equation (7) can now be written in terms of ϵ:

$$\nabla\epsilon = \frac{1}{\langle v_{tz}^2\rangle}\nabla(\Delta\varphi_g). \tag{9}$$

Equation (9) is integrated to give

$$\epsilon = \frac{1}{\langle v_{tz}^2\rangle}\Delta\varphi_g + C, \tag{10}$$

where C is a constant. We apply the Laplacian operator ∇^2 to equation (10) with the result that

$$\nabla^2\epsilon = \frac{1}{\langle v_{tz}^2\rangle}\nabla^2(\Delta\varphi_g) = -\frac{4\pi G}{\langle v_{tz}^2\rangle}\Delta\rho_g = -\frac{\epsilon}{2H_g^2}\rho_{eg}/\rho_{ego}, \tag{11}$$

where the Poisson equation

$$\nabla^2(\Delta\varphi_g) = -4\pi G\Delta\rho_g \tag{12}$$

has been used and where

$$H_g = \left(\frac{\langle v_{tz}^2\rangle}{8\pi G\rho_{ego}}\right)^{1/2} \tag{13}$$

is a scale height for the gas in the absence of stars. Equation (11) may be solved by the method of separation of variables, in which case ϵ is written as

$$\epsilon(r,z) = X(z)\,Y(r). \tag{14}$$

After some simplification, equation (11) becomes

$$-\frac{(Y''(r) + \frac{1}{r}Y'(r))}{Y(r)} = k^2 = \frac{(X''(z) + X(z)\;1/2H_g^2\;\rho_{eg}(z)/\rho_{ego})}{X(z)} \tag{15}$$

The separation constant k^2 arises because the left-hand side of equation (15) is a function only of r while the right-hand side is a function only of z.

The differential equation for Y(r)

$$Y''(r) + \frac{1}{r}Y'(r) + k^2Y(r) = 0 \tag{16}$$

has the solution

$$Y(r) = J_o(kr), \tag{17}$$

where J_o is the zero order Bessel function. The other linearly independent solution to equation (16) $\rightarrow \infty$ as $r \rightarrow 0$ and is therefore not considered further. The differential equation for $X(z) = X_k(z)$ is

$$X_k''(z) + \left(\frac{1}{2Hg^2} \rho_{eg}(z)/\rho_{ego} - k^2\right) X_k(z) = 0, \tag{18}$$

with the boundary conditions

$$X_k'(0) = 0 \tag{19}$$

$$\lim_{|z| \rightarrow \infty} X_k(z) = 0. \tag{20}$$

In addition, $X_k(0)$ is normalized to unity. Equation (19) arises because $\Delta\rho_g(r,z)$ is an even function of z. Equation (20) expresses the constraint that $\Delta\rho_g(r,z)/\rho_{eg}(z) \rightarrow 0$ as $|z| \rightarrow \infty$.

The general solution to equation (11) can be expressed as an integral over k of the product $X_k(z)J_o(kr)$ with the appropriate expansion amplitudes A(k)

$$\epsilon(r,z) = \int_0^\infty A(k)X_k(z)J_o(kr)dk, \tag{21}$$

where $J_o(kr)$ is the solution to equation (16) and $X_k(z)$ is the solution to equation (18) subject to the boundary conditions imposed by equations (19) and (20). Equations (10) and (21) can be combined to give an expression for the perturbed gas potential:

$$\Delta\varphi_g(r,z) = \langle v_{tz}^2 \rangle \int_0^\infty A(k)X_k(z)J_o(kr)dk - \langle v_{tz}^2 \rangle C. \tag{22}$$

The solution of the Poisson equation for the perturbed gas potential is just the sum of the particular solution of the inhomogeneous equation and the general solution of the homogeneous equation. Equation (22) with C=0 corresponds to the former. The general solution of the homogeneous equation is equivalent to the general solution of the Laplace equation

$$\nabla^2 \Delta\varphi_{gh} = 0, \tag{23}$$

where the subscript h denotes homogeneous.

To solve equation (23) we expand $\Delta\varphi_{gh}$ in terms of $J_o(kr)$ with expansion amplitudes $s_k(z)$:

$$\Delta\varphi_{gh}(r,z) = \langle v_{tz}^2 \rangle \int_o^\infty s_k(z)\ J_o(kr) dk. \tag{24}$$

Expressed in terms of cylindrical coordinates, equation (23) assumes the form

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial}{\partial r}\,\Delta\varphi_{gh}\right) + \frac{\partial^2}{\partial z^2}\,\Delta\varphi_{gh} = 0. \tag{25}$$

Equation (24) is substituted into equation (25), and after some differentiation and algebraic manipulation we find that

$$\int_o^\infty \left[s_k''(z)\ J_o(kr) + ks_k(z)\left(\frac{J_o'(kr)}{r} + kJ_o''(kr)\right)\right] dk = 0. \tag{26}$$

We can make use of the well-known relations between the zeroth and first order Bessel functions J_o and J_1 and their derivatives

$$J_o'(kr) = -J_1(kr) \tag{27}$$

$$J_o''(kr) = -J_o(kr) + \frac{1}{kr}\,J_1(kr) \tag{28}$$

to simplify equation (26):

$$0 = \int_0^\infty (s_k''(z)\, J_o(kr) - k^2 s_k(z)\, J_o(kr))dk. \qquad (29)$$

Since the J_o functions are orthogonal, it follows that

$$s_k''(z) - k^2 s_k(z) = 0, \qquad (30)$$

which has the solution

$$s_k(z) = w_{1k} e^{-k|z|} + w_{2k} e^{+k|z|}. \qquad (31)$$

The coefficients w_{2k} must vanish if the boundary condition $\lim_{|z|\to\infty} \Delta\varphi_{gh} = 0$ is to be satisfied.

We have now obtained the functional form of $s_k(z)$ in terms of which $\Delta\varphi_{gh}$ has been expanded. Substitution of equation (31) into equation (24) easily gives

$$\Delta\varphi_{gh}(r,z) = \langle v_{tz}^2 \rangle \int_0^\infty w_{1k} e^{-k|z|}\, J_o(kr)dk. \qquad (32)$$

Note that $\Delta\varphi_{gh}(r,z)$ expressed here satisfies the boundary condition that the perturbed gas potential be symmetric with respect to the symmetry plane $z = 0$. Adding equations (22) (with $C = 0$) and (32) then gives the general solution to the Poisson equation satisfying, of course, the gas hydrostatic equilibrium equation:

$$\Delta\varphi_g(r,z) = \langle v_{tz}^2 \rangle \int_0^\infty (A(k)X_k(z) + w_{1k} e^{-k|z|})\, J_o(kr)dk. \qquad (33)$$

For the sake of completeness we may easily derive from equations (8) and (21) an expression for the perturbed gas density satisfying the Poisson and gas hydrostatic equilibrium conditions:

$$\Delta\rho_g(r,z) = \rho_{eg}(z) \int_0^\infty A(k)\, X_k(z)\, J_o(kr)dk. \qquad (34)$$

It is appropriate here to consider more closely the nature of equations (18), (19), and (20) as regards the nature of the marginally unstable state. Equations (18) - (20) define what is known as the Sturm-Liouville problem from the theory of differential equations. This is the problem of determining (a) the relation between the separation constant k and the function $X_k(z)$ and (b) the influence on k of the boundary conditions imposed on $X_k(z)$. Stated more simply, we are searching for those values of k that lead to $X_k(z)$ satisfying both equation (18) and the boundary conditions imposed by equations (19) and (20). It is worthwhile to note the mathematical similarity of our problem and the quantum mechanical problem of determining the allowed energies E of a particle bound by a one-dimensional potential well V(z). The particle wave function $\Psi(z)$, analogous to $X_k(z)$, satisfies the one-dimensional time-independent Schrodinger equation similar to equation (18), and is subject to the square integrability condition that $\int_{-\infty}^{\infty}|\Psi^2(z)|dz$ be finite, similar to equation (20). To carry the comparison one step further, k^2 corresponds to the particle energy E and $1/2H_g^2\ \rho_{eg}(z)/\rho_{ego}$ corresponds to the form of the potential well V(z). Only certain discrete values of k^2 will yield $X_k(z)$ that obey the boundary conditions. Since $\Delta\rho_g(r,z)/\rho_{eg}(z) = \int_0^{\infty} A(k)\ X_k(z)\ J_o(kr)dk$, the statement $X_k(z) \rightarrow 0$ as $|z| \rightarrow \infty$ is equivalent to the statement $\Delta\rho_g(r,z)/\rho_{eg}(z) \rightarrow 0$ as $|z| \rightarrow \infty$. It is the latter condition then that makes this an eigenvalue problem.

The stellar component will make its presence felt only as it effects the depth and or width of the potential well $1/2H_g^2\rho_{eg}(z)/\rho_{ego}$, or equivalently, as it effects the distribution of gas $\rho_{eg}(z)$ above the galactic plane. It is at this point then that the present stability analysis differs from the earlier work of Ledoux (1951). Ledoux considered an isothermal layer of gas (no stars), so that the density distribution obeys the well-known relation $\rho_{eg}(z)/\rho_{ego} = \mathrm{sech}^2(z/H)$. When a stellar component is included, $\rho_{eg}(z)/\rho_{ego}$ can be found by solving simultaneously the gas and stellar hydrostatic equilibrium equations and the Poisson equation and by choosing appropriate values for ρ_{ego}, ρ_{*o}, $\langle v_{tz}^2\rangle^{1/2}$, and $\langle v_{*z}^2\rangle^{1/2}$. This has been done in Paper I, from which we recall equation (11):

$$\frac{d^2}{dz^2}\,\ell n\, \rho_{eg}(z)/\rho_{ego} = -\frac{4\pi G}{\langle v_{tz}^2\rangle}\left[\rho_{eg}(z) + \rho_{*o}(\rho_{eg}(z)/\rho_{ego})^{\langle v_{tz}^2\rangle/\langle v_{*z}^2\rangle}\right]. \quad (35)$$

The observed values $\rho_{ego} = 1$ H atom/cm^3 = 0.025 $M_\odot/pc^3$ (Weaver 1970), $\rho_{*o} = 0.064\ M_\odot/pc^3$ (Luyten 1968), and $\langle v_{*z}^2\rangle^{1/2} = 18$ km/sec (Woolley 1958) are chosen. $\langle v_{tz}^2\rangle^{1/2}$ is allowed to vary between 0 and 20 km/sec.

At this point we digress slightly to discuss the numerical approach employed in computing the discrete value or values of the separation constant k. It can be shown (see Appendix, equation (A16)) that the solution to equation (18) in the limit $z \to +\infty$ is just

$$X_k(z) = \alpha_k^{(+)} e^{-kz} + \beta_k^{(+)} e^{+kz}. \quad (36)$$

We wish to find the restriction on k and $X_k(z)$ if $\beta_k^{(+)}$ is to vanish. If $\beta_k^{(+)} = 0$, it is clear that

$$X_k'(z) + kX_k(z) = 0. \quad (37)$$

To derive the eigenvalue k from the restriction dictated by equation (37), a sufficiently large z is considered, denoted by z_c, for which equation

(36) holds with good accuracy. Employing numerical methods, equations (18) and (35) are solved simultaneously for $X_k(z)$, allowing us to tabulate $X_k(z_c)$ and $X_k'(z_c)$. The quantity $P = X_k'(z_c) + kX_k(z_c)$ is then calculated at each of four equally spaced points covering the range of k. The eigenvalue k lies in the interval over which P changes sign. This interval is divided into three subintervals and the process repeated until the desired accuracy is attained.

To determine how the eigenvalue k relates to the linear size of the neutral state, we recall equation (17) which expresses the solution to the second order differential equation for Y(r):

$$Y(r) = J_o(kr). \tag{17}$$

We note that k must have units of reciprocal length, and that for a given value of k the first zero in Y(r) occurs when

$$r_1 = \alpha_1/k_1, \tag{38}$$

where α_1 is the first zero of the zero order Bessel function J_o and k_1 is the smallest eigenvalue found by the numerical methods enumerated above. More generally, if more than one eigenvalue k is found, we have that

$$r_n = \alpha_n/k_n, \tag{39}$$

where α_n is the nth zero of J_o. Focusing on equation (38) we observe that k_1 determines the radius of the 'perturbation', or more accurately the radius r_1 of the marginally unstable state in the plane of symmetry z=0, since

$$\epsilon(r_1 = \alpha_1/k_1,\ z=0) = X_k(0)\ J_o(\alpha_1) = \Delta\rho_g(\alpha_1/k_1, 0)/\rho_{eg}(0) = 0. \quad (40)$$

When equations (18) and (35) are solved simultaneously subject to the boundary conditions imposed by equations (19) and (20) and subject to the values chosen for ρ_{ego}, ρ_{*o}, $<v^2_{*z}>^{1/2}$, and $<v^2_{tz}>^{1/2}$, only one eigenvalue k is found. Since $<v^2_{tz}>^{1/2}$ is allowed to vary between 0 and 20 km/sec, k_1 is determined as a function of $<v^2_{tz}>^{1/2}$. r_1 may be found as a function of $<v^2_{tz}>^{1/2}$ from the relation $r_1 = \alpha_1/k_1$. The results are plotted in Figure 1. We see that r_1 is directly proportional to $<v^2_{tz}>^{1/2}$ and increases from 1 to 2 kpc as $<v^2_{tz}>^{1/2}$ varies between about 8 and 16 km/sec.

It is useful to inquire how ρ_{ego}, ρ_{*o}, $<v^2_{*z}>^{1/2}$, and $<v^2_{tz}>^{1/2}$ effect k_1 and thus r_1. Recall that the equivalent potential well has the form $V(z) = 1/2H_g^2\ \rho_{eg}(z)/\rho_{ego}$. The central depth is just $V(0) = 1/2H_g^2$ since $\rho_{eg}(0)/\rho_{ego} = 1$. Since H_g is a function only of ρ_{ego} and $<v^2_{tz}>$, ρ_{*o} and $<v^2_{*z}>$ have no effect on the central depth. Increasing the depth of the potential well by decreasing H_g^2 (and thus increasing ρ_{ego} or decreasing $<v^2_{tz}>$) will increase k_1 and decrease r_1. ρ_{*o} and $<v^2_{*z}>$ will, however, effect the width of the potential well, in the sense that increasing ρ_{*o} or decreasing $<v^2_{*z}>$ decreases the width which causes k_1 to decrease and r_1 to increase. In summary, the depth is determined only by ρ_{ego} and $<v^2_{tz}>$, while the width is determined primarily by ρ_{*o}, and to a lesser extent by ρ_{ego}, $<v^2_{*z}>$, and $<v^2_{tz}>$. The inclusion of a stellar component will increase the radius r_1 of the marginally unstable state above its value when $\rho_{*o} = 0$.

It is instructive to compare r_1 calculated with $\rho_{ego} = 0.025\ M_\odot/pc^3$, $\rho_{*o} = 0.064\ M_\odot/pc^3$, and $\langle v_{*z}^2\rangle^{1/2} = 18$ km/sec with the Jeans' radius $(\lambda_J/2)$ and the Ledoux radius $(\lambda_L/2)$, each calculated with $\rho_{ego} = 0.025\ M_\odot/pc^3$. $\langle v_{tz}^2\rangle^{1/2}$ is a free parameter, allowed to vary between 1 and 20 km/sec. The results are presented in Table 1. It is particularly useful to compare r_1 (Kellman) to r_1 (Ledoux), since both assume plane-parallel gas distributions in the equilibrium state. The presence of a rigid stellar component with $\rho_{*o} = 0.064\ M_\odot/pc^3$ and $\langle v_{*z}^2\rangle^{1/2} = 18$ km/sec increases the radius of the marginally unstable state, though only slightly.

Isodensity contours of the marginally unstable or neutral state may be derived by combining equations (1) and (34), with the result that

$$\frac{\rho_g(r,z)}{\rho_{ego}} = \frac{\rho_{eg}(z)}{\rho_{ego}} \left(1 + A(k)\ X_k(z)\ J_o(kr)\right). \tag{41}$$

The integral over k in equation (34) reduces to just one term since only the neutral state is being considered and since only one eigenvalue was found. At the center of the perturbation $r = z = 0$, $X_k(z) = 1$, $J_o(kr) = 1$, and $\rho_{eg}(z)/\rho_{ego} = 1$, and equation (41) becomes

$$\frac{\rho_g(0,0)}{\rho_{ego}} = 1 + A(k). \tag{42}$$

21-cm observations of large gas structures in spiral arms (McGee and Milton 1964) indicate that $1.0 \leq \rho_g(0,0)/\rho_{ego} \leq 2.0$, and therefore

$0 \leq A(k) \leq 1.0$. Making use of equation (35) for $\rho_{eg}(z)/\rho_{ego}$, equation (18) for $X_k(z)$, and numerical tables for $J_o(kr)$, equation (41) is used to construct an (r,z) grid of values for $\rho_g(r,z)/\rho_{ego}$ from which contours of equal $\rho_g(r,z)/\rho_{ego}$ may be derived. Figure 2 displays three isodensity contours $\rho_g(r,z)/\rho_{ego} = 1.3$, 0.7, and 0.3 calculated with $A(k) = 0.5$. The neutral state is clearly quite flattened, and is reminiscent of contours presented by McGee and Milton (1964) and Westerhout (1956), the latter referring to cross sections of spiral arms.

The mass of the marginally unstable state may be expressed as an integral of the gas density over a cylinder perpendicular to the galactic plane, of infinite height and of radius $r_1 = \alpha_1/k_1$:

$$M_g = 2\pi \int_{-\infty}^{\infty} \int_{-r_1}^{+r_1} \rho_g(r,z) r \, dr \, dz. \tag{43}$$

Equation (43) can be evaluated numerically with the aid of equation (41). Choosing $\langle v_{tz}^2 \rangle^{1/2} = 10.0$ km/sec (see Paper I) and values quoted previously for ρ_{ego}, ρ_{*o}, and $\langle v_{*z}^2 \rangle^{1/2}$, we find that $M_g \approx 5 \times 10^6 M_\odot$.

Equations (18) and (35) may be written in dimensionless form

$$\frac{d^2}{d\beta^2} X_k = - \left(\frac{\alpha}{2} - C_k^2 \right) X_k \tag{44}$$

$$\frac{d^2}{d\beta^2} \ln\alpha = - \frac{1}{2} (\alpha + \gamma\alpha^\delta), \tag{45}$$

where the dimensionless quantities C_k, α, β, γ, and δ are defined as follows:

$$C_k = kH_g \tag{46}$$

$$\alpha = \rho_{eg}(z)/\rho_{ego} \tag{47}$$

$$\beta = z/H_g \tag{48}$$

$$\gamma = \rho_{*o}/\rho_{ego} \tag{49}$$

$$\delta = \langle v_{tz}^2\rangle/\langle v_{*z}^2\rangle. \tag{50}$$

The radius r_1 of the neutral state in the plane of symmetry can be written as $r_1 = \frac{2.405}{C_k}H_g$, but since $H_g = (\langle v_{tz}^2\rangle/8\pi G\rho_{ego})^{1/2}$, r_1 depends on the ratio $(\langle v_{tz}^2\rangle/\rho_{ego})^{1/2}$, precisely the functional dependence found by Jeans and Ledoux. If $\langle v_{tz}^2\rangle^{1/2}$ is independent of distance from the galactic center beyond the solar distance R_o (see Paper I), r_1 would be expected to increase with increasing distance from the galactic center since ρ_{ego} decreases with increasing R. Westerhout (1956) finds that (corrected to the new galactic distance scale) $\rho_{ego}(R = 11 \text{ kpc})/\rho_{ego}(R = 15 \text{ kpc}) \approx 4.7$ from which it follows that $r_1(R = 15 \text{ kpc})/r_1(R = 11 \text{ kpc}) \approx 2.2$ due to a change in ρ_{ego} alone. In conclusion, since $r_1 = \frac{2.405}{C_k} H_g$, r_1 depends principally on $\langle v_{tz}^2\rangle^{1/2}$ and ρ_{ego} through H_g, to a smaller extent on ρ_{*o} through C_k, and to an even smaller extent on $\langle v_{*z}^2\rangle^{1/2}$ through C_k.

III. COMPARISON WITH THE OBSERVATIONS

It is interesting that a 21-cm southern survey by McGee and Milton (1964) revealed that the principal elements of the gaseous component of spiral arms in our galaxy are large flattened structures with dimensions 10^7 $M_\odot$ and 1-2 kpc, strung out along the length of the arms much like beads on a string. More recently, Kerr (1968) and Westerhout (1971) have pointed to the existence of large gas condensations forming the major components of spiral arms. The size and mass of the individual gas structures observed by McGee and Milton are close to the values we have calculated for the marginally unstable state. Further, their linear size is observed to increase markedly with increasing distance from the galactic center beyond R_o, consistent with our findings above, while their mass is observed to remain essentially constant at about 10^7 $M_\odot$. In addition, they occur predominantly in the outer arms beyond R_o (McGee 1964). It is also of interest that McGee and Milton (1966) have observed similar gas structures in the Large Magellanic Cloud with a mean linear size and mass of about 600 pc and 4×10^6 $M_\odot$, respectively. The smaller size may indicate that ρ_{ego} is larger, $\langle v_{tz}^2 \rangle^{1/2}$ is smaller, and or ρ_{*o} is smaller than corresponding values in our galaxy, if the theory is to be believed.

IV. DISCUSSION

Recent work by Lin (1970) has refocused attention on the importance of classic Jeans' type stability analyses of the gaseous component of galaxies. It is Lin's opinion that density waves (primarily in the stellar component) are initiated or excited by the gravitational instability of the gas in the outer regions of galaxies, where the star density no longer overwhelmingly dominates the gas density and where the gas turbulence, which tends to inhibit instability, may be small because of reduced energy inputs. Excited in the outer regions, the density waves travel inward, subsequently giving rise to two-armed spiral patterns. Once formed, the large gas condensations are stretched by differential galactic rotation into trailing structures, and because they lie near or outside the corotation distance, form material arms. The pattern speed Ω_p over the whole galactic disk is determined then, if Lin is correct, by the angular velocity of these outer gas condensations around the galactic center. In support of Lin's ideas, Shu, Stachnick, and Yost (1971) obtained a good fit to the spiral structure observed in M33, M51, and M81 by assuming that Ω_p is equal to the angular rotation speed of the outermost HII regions.

We mentioned above that the large gas condensations observed by McGee and Milton (1964) are preferentially located at distances beyond the solar distance R_o. Not only does this fact agree nicely with Lin's ideas, but it follows directly from results obtained in Paper I of this series. There we found evidence that the rms z turbulent gas velocity dispersion $\langle v_{tz}^2 \rangle^{1/2}$ (or more correctly $Q = (\langle v_{tz}^2 \rangle + B_o^2/8\pi\rho_{ego} + P_{c-ro}/\rho_{ego})^{1/2}$) increases with decreasing distance from the galactic center in the range 4 kpc $\leq R \leq$ 10 kpc. It follows then that gravitational instability of the gas disk is inhibited when $R < R_o$.

ACKNOWLEDGMENTS

The author is grateful to Professor J. P. Ostriker and Professor G. B. Field for their advice and encouragement, and to Dr. L. A. Fisk for a critical reading of the manuscript. This work was supported in part by a National Science Foundation Graduate Fellowship while the author was a student at the University of California at Berkeley, and by NASA Grant No. NGL 21-002-033.

REFERENCES

Goldreich, P., and Lynden-Bell, D. 1965, M.N.R.A.S., 130, 97.

Goon, G. 1966, private communication.

Jeans, J. H. 1928, Astronomy and Cosmology, p. 337.

Kellman, S. A. 1972a, in preparation.

Kellman, S. A. 1972b, in preparation.

Kellman, S. A. 1972c, in preparation.

Kerr, F. J. 1968, Nebulae and Interstellar Matter, ed. B. M. Middlehurst and L. H. Aller (Chicago: The University of Chicago Press), p. 575.

Ledoux, P. 1951, Ann. d'Ap., 14, 438.

Lin, C. C. 1970, The Spiral Structure of Our Galaxy, ed. W. Becker and G. Contopoulos (Dordrecht: D. Reidel Publishing Co.), p. 373.

Lin, C. C., and Shu, F. H. 1964, Ap. J., 140, 646.

Lin, C. C. 1966, Proc. Nat. Acad. Sci., 55, 229.

Lin, C. C., Yuan, C., and Shu, F. H. 1969, Ap. J., 155, 721.

Luyten, W. J. 1968, M.N.R.A.S., 139, 221.

McGee, R. X. 1964, The Galaxy and the Magellanic Clouds, ed. F. J. Kerr and A. W. Rodgers (Canberra: Australian Academy of Science), p. 126.

McGee, R. X., and Milton, J. A. 1964, Aust. J. Phys., 17, 128.

McGee, R. X. 1966, Aust. J. Phys., 19, 343.

Shu, F. H., Stachnik, R. V., and Yost, J. C. 1971, Ap. J., 166, 465.

Weaver, H. F. 1970, private communication.

Westerhout, G. 1956, B. A. N., 13, 201.

Westerhout, G. 1971, private communication.

Woolley, R. v.d. R. 1958, M.N.R.A.S., 118, 45.

APPENDIX

ALTERNATE CALCULATION OF $\Delta\varphi_g(r,z)$

As a check on the correctness of equation (22), $\Delta\varphi_g(r,z)$ may be derived by a different approach, beginning directly with the solution to the Poisson equation for the perturbed gas potential. Expressed in cylindrical coordinates, we have that

$$\Delta\varphi_g(r_o, \theta_o, z_o) = G\int_{-\infty}^{\infty}\int_0^{2\pi}\int_0^{\infty}\frac{1}{R}\;\Delta\rho_g(r, \theta, z)\,r\,dr\,d\theta\,dz. \tag{A1}$$

(r_o, θ_o, z_o) and (r, θ, z) are the coordinates of any two points and R is the distance between them. It can be shown that

$$\frac{1}{R} = \sum_{m=0}^{\infty}\epsilon_m\cos[m(\theta-\theta_o)]\int_0^{\infty}J_m(kr)\;J_m(kr_o)\;e^{-k|z-z_o|}dk, \tag{A2}$$

and noting that

$$\int_0^{2\pi}\cos[m(\theta-\theta_o)]d\theta = 2\pi\delta_{mo}, \tag{A3}$$

where δ is the Kronecker delta symbol, it follows that

$$\Delta\varphi_g(r_o, z_o) = 2\pi G\int_0^{\infty} r\,dr\int_{-\infty}^{\infty}dz\;\Delta\rho_g(r,z)\int_0^{\infty}J_o(kr)J_o(kr_o)e^{-k|z-z_o|}dk. \tag{A4}$$

The perturbed gas density $\Delta\rho_g(r,z)$ may be expanded in terms of the zero order Bessel function $J_o(k'r)$ and the appropriate expansion amplitudes $T_{k'}(z)$

$$\Delta\rho_g(r,z) = \int_0^{\infty}T_{k'}(z)\;J_o(k'r)\;dk', \tag{A5}$$

and if we note that

$$\int_0^{\infty}r\;J_o(kr)\;J_o(k'r)dr = \frac{1}{k}\,\delta(k'-k), \tag{A6}$$

it follows that

$$\Delta\varphi_g(r_o,z_o) = 2\pi G\int_{-\infty}^{\infty}dz\int_0^{\infty}\frac{1}{k}\,T_k(z)e^{-k|z-z_o|}J_o(kr)dk. \tag{A7}$$

Note that two expressions have been derived for $\Delta\rho_g(r,z)$, represented by equations (34) and (A5). If they are equated we find that

$$T_k(z) = A(k)\rho_{eg}(z)X_k(z), \tag{A8}$$

and equation (A7) becomes

$$\Delta\varphi_g(r_o,\ z_o) = 2\pi G\int_o^\infty dk\ \frac{A(k)J_o(kr_o)}{k}\int_{-\infty}^\infty \rho_{eg}(z)X_k(z)e^{-k|z-z_o|}dz. \tag{A9}$$

Equation (18) can be recalled to supply an expression for the quantity $\rho_{eg}(z)X_k(z)$ appearing in equation (A9):

$$\rho_{eg}(z)X_k(z) = 2\rho_{ego}H_g^2\ (k^2X_k(z)-X_k''(z)). \tag{A10}$$

It is this condition that, when substituted into equation (A9), constrains $\Delta\varphi_g(r_o,\ z_o)$ to satisfy the gas hydrostatic equilibrium equation:

$$\Delta\varphi_g(r_o,\ z_o) = 4\pi GH_g^2\ \rho_{ego}\int_o^\infty dk\ \frac{A(k)J_o(kr_o)}{k}\int_{-\infty}^\infty (k^2X_k(z)-X_k''(z))e^{-k|z-z_o|}dz. \tag{A11}$$

To proceed further it becomes necessary to evaluate the integral over z in equation (A11). This can be accomplished by splitting the range into two parts:

$$\int_{-\infty}^\infty (k^2X_k(z)-X_k''(z))e^{-k|z-z_o|}dz = -\int_{-\infty}^{z_o} X_k''(z)e^{+k(z-z_o)}dz$$

$$+\ k^2\int_{-\infty}^{z_o} X_k(z)e^{+k(z-z_o)}dz - \int_{z_o}^\infty X_k''(z)e^{-k(z-z_o)}dz + k^2\int_{z_o}^\infty X_k(z)e^{-k(z-z_o)}dz. \tag{A12}$$

The first and third terms on the right hand side of equation (A12) are integrated twice by parts and added to the second and fourth terms, and after some simplification we find that

$$\begin{aligned}\int_{-\infty}^\infty (k^2X_k(z)-X_k''(z))e^{-k|z-z_o|}dz &= -X_k'(z)e^{+k(z-z_o)}\Big|_{-\infty}^{z_o} \\ &+ kX_k(z)e^{+k(z-z_o)}\Big|_{-\infty}^{z_o} - X_k'(z)e^{-k(z-z_o)}\Big|_{z_o}^{\infty} - kX_k(z)e^{-k(z-z_o)}\Big|_{z_o}^{\infty}.\end{aligned} \tag{A13}$$

To evaluate these terms in the limit $z \to \pm\infty$ we recall the differential equation for $X_k(z)$:

$$X_k''(z) + \left(\frac{1}{2H_g^2}\,\rho_{eg}(z)/\rho_{ego} - k^2\right) X_k(z) = 0. \tag{A14}$$

In the limit $z \to \pm\infty$, equation (A14) becomes

$$X_k''(z) = k^2 X_k(z), \tag{A15}$$

since $\lim_{z\to\pm\infty} \frac{1}{2H_g^2}\,\rho_{eg}(z)/\rho_{ego} = 0$. The solution to equation (A15) can easily be written:

$$z > 0 : X_k(z) = \alpha_k^{(+)} e^{-kz} + \beta_k^{(+)} e^{+kz} \tag{A16}$$

$$z < 0 : X_k(z) = \alpha_k^{(-)} e^{-kz} + \beta_k^{(-)} e^{+kz} . \tag{A17}$$

Referring back to equation (34) for the perturbed gas density

$$\Delta\rho_g(r,z)/\rho_{eg}(z) = \int_o^\infty A(k) X_k(z) J_o(kr)\,dk, \tag{A18}$$

it is clear that unless $\lim_{z\to\pm\infty} X_k(z) = 0$, the quantity $\Delta\rho_g(r,z)/\rho_{eg}(z)$ will not $\to 0$ as $z \to \pm\infty$. This requires that $\alpha_k^{(-)} = \beta_k^{(+)} = 0$. Symmetry of $\Delta\rho_g(r,z)/\rho_{eg}(z)$ about the plane $z = 0$ results in the further restriction that $\alpha_k^{(+)} = \beta_k^{(-)}$. Equations (A16) and (A17) therefore become

$$z > 0: X_k(z) = \gamma e^{-kz} \tag{A19}$$

$$z < 0: X_k(z) = \gamma e^{+kz}, \tag{A20}$$

where $\gamma = \alpha_k^{(+)} = \beta_k^{(-)}$. It follows that

$$z > 0: \; X_k'(z) = -\gamma k e^{-kz} \tag{A21}$$

$$z < 0: \; X_k'(z) = \gamma k e^{+kz}. \tag{A22}$$

We can now evaluate each term in equation (A13) at the infinite limit. For example, consider the first term:

$$\lim_{z \to -\infty} (-X_k'(z) e^{+k(z-z_o)}) = \lim_{z \to -\infty} (-\gamma k e^{-kz_o} e^{+2kz}) = 0. \tag{A23}$$

Similarly, the other three terms evaluated at the infinite limit also vanish. We are now able to evaluate the integral over z appearing in equation (A11):

$$\int_{-\infty}^{\infty} (k^2 X_k(z) - X_k''(z)) e^{-k|z-z_o|} dz = -X_k'(z_o) + kX_k(z_o) + X_k'(z_o) + kX_k(z_o) = 2kX_k(z_o). \tag{A24}$$

Equation (A11) therefore simplifies to

$$\Delta\varphi_{g2}(r,z) = 8\pi G H_g^2 \, \rho_{ego} \int_o^{\infty} A(k) X_k(z) J_o(kr) dk, \tag{A25}$$

where the arbitrary coordinates (r_o, z_o) have been replaced by (r,z). This is, however, just the particular solution to the Poisson equation (and consistent with the gas hydrostatic equilibrium equation), for which we have derived another expression in the way of equation (22) (with $C = 0$):

$$\Delta\varphi_{g1}(r,z) = \langle v_{tz}^2 \rangle \int_o^{\infty} A(k) \; X_k(z) J_o(kr) dk. \tag{A26}$$

Recalling that $H_g^2 = \langle v_{tz}^2 \rangle / 8\pi \rho_{ego}$, it is clear that

$$\Delta\varphi_{g1}(r,z) = \Delta\varphi_{g2}(r,z). \tag{A27}$$

The purpose of the above derivation was to obtain an independent check on $\Delta\varphi_{g1}(r,z)$ derived in Section II, and the equality expressed by equation (A27) gives us confidence that the expression obtained for $\Delta\varphi_g(r,z)$ is correct.

TABLE 1

COMPARISON BETWEEN THE JEANS', LEDOUX, AND KELLMAN RADIUS AS FUNCTIONS OF $<v_{tz}^2>^{1/2}$

$<v_{tz}^2>^{1/2}$ (km/sec)	r_1(Jeans) (kpc)	r_1(Ledoux) (kpc)	r_1(Kellman) (kpc)
1.0	0.085	0.120	0.127
2.5	0.213	0.301	0.318
5.0	0.425	0.601	0.635
7.5	0.638	0.902	0.953
10.0	0.850	1.202	1.270
15.0	1.275	1.803	1.905
20.0	1.700	2.404	2.540

FIGURE CAPTIONS

1. The dimension r_1 of the marginally unstable state in the symmetry plane (z = 0) as a function of the rms z turbulent gas velocity dispersion.
2. Isodensity contours $\rho_g(r,z)/\rho_{ego}$ = 1.3, 0.7, and 0.3 of the marginally unstable state, calculated with A(k) = 0.5.

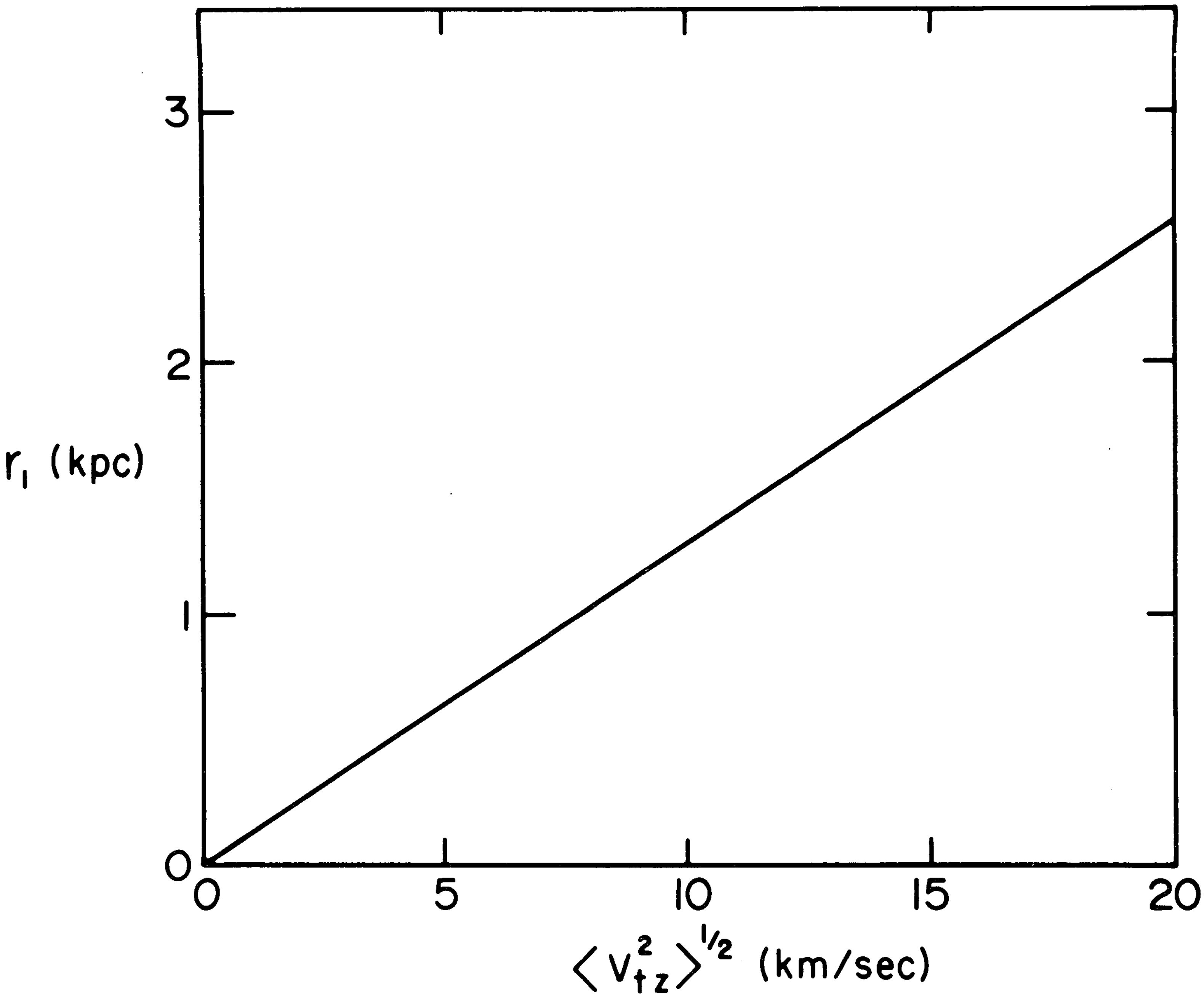

Figure 1

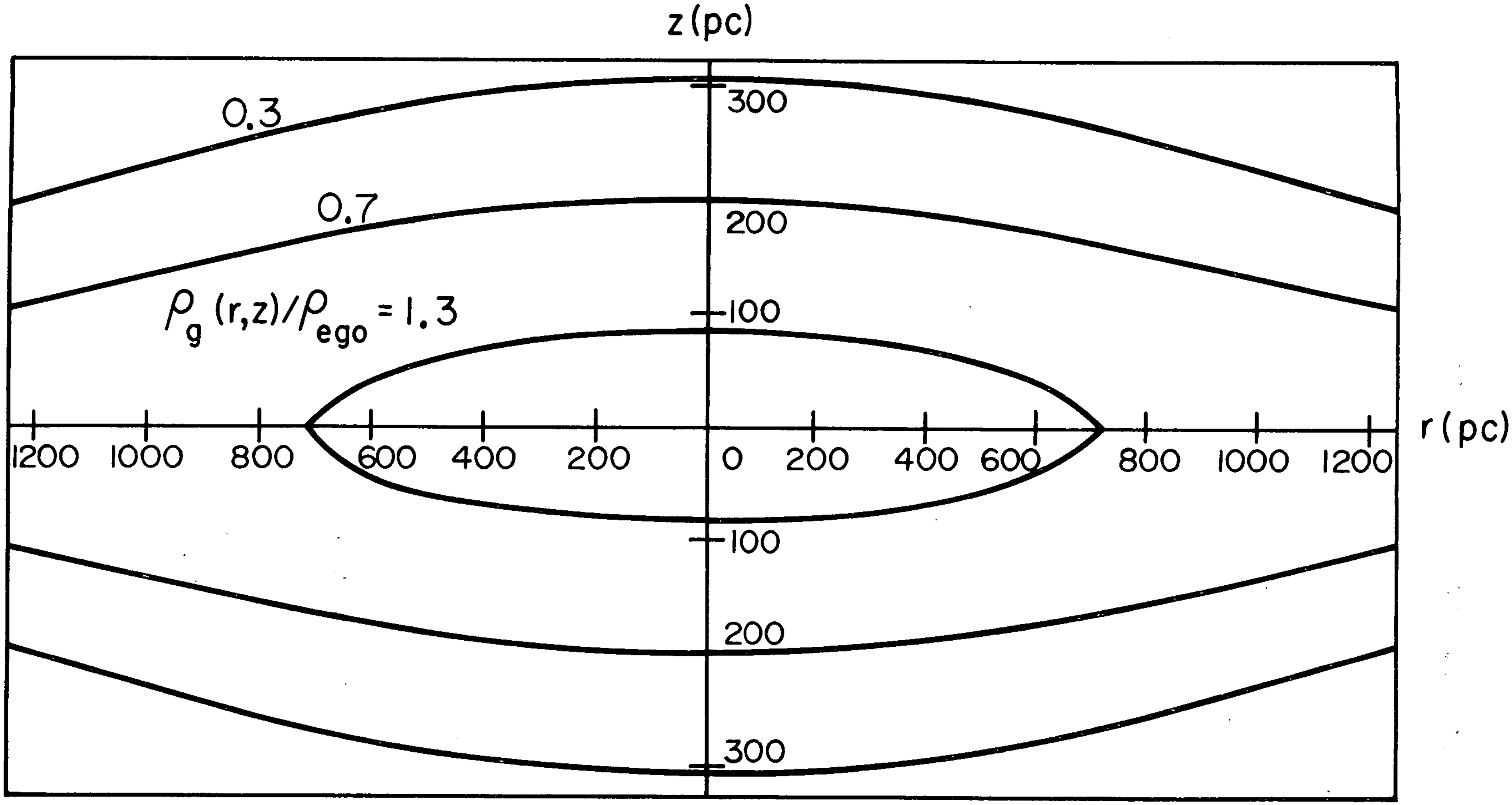
z (pc)
0.3
0.7
$\rho_g (r,z)/\rho_{ego} = 1.3$
300
200
100
100
200
300
r (pc)
1200
1000
800
600
400
200
0
200
400
600
800
1000
1200

Figure 2