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## FOPEWORD

This final report is submittod in accordasce with "Seope of Wisk, Exhibit D" for Contract NAS8-33979. The ctedy wee firreted from the Guidance Systems Dirision (GSD) of the Allied Bendix Corporation. The enginearing manager at this location wae Mr. Joel Levinthal. Mont of the analytical effort in support of this project was providod by Dr. Fredarict Chicheater, who wrote all sections of this reyort. Most of the algebraic dovelopment for this report wac verifed uing the Symbolic Manipulation Program (SMP) software package on a VAX-11 compater by Mr. Alan Reynolds who aleo prepared this report using the TEX typesetting softrare package on the same computer. The gridance of Dr. Henry B. Waites and Mr. Stan Carroll of MSFC during the course of thin study is gratefully acknowledged.

## ABETRACT

The problem of applying modelar attitude control to a rigid body. flexible mapension model of a flaxible epacecrast with some state variables inscecwible was addreseod by doveloping a sequence of singo-axis modele and grearnting a saries of roduced atato linosr obearvers of minimom order to reconstruct those scalar state variabler that wese inacceaible. The spocific singin axie modele treated consirted of two, throe, four and ive rigid bodies, respectively, interconnocted by a fexible shaft pascing through the maee centers of the bodias. Rectuced state linear obeervers of all orders of to soe lese than the total number of scalar state variables were generated for each of the four singlo-avis modole cited. Each of the single-axis models was then transformed to a corropoading modal model to which modal dwaping was addod. Each of the damped modal models we written in state vaciable form. With the accumption that at least one of the sealar modal s ite varibles was accescible, rudnced atate linear obearvare were developed for sythesising the insecessible modal otate variables for each modal model.

This roport is sabmitted in complizeee with the Scope of Wort under contract NAS8-35979. The pariod of parformance covered by the contract is from October 1, 1983 to Anguat 31, 1984. The submiesion and approval of this report constitute the recenoful completion of the "Erihibit D" portion of the contract.

This roport is a soquel to five others, two of them previously sebmitted undar a different contract mani er. The two prior reports, nador a diffesunt contract namber, references ( $1-1$ ) and (1-2), were submitted in October 1978 and September, 1979 and coverod the periode from July 27, 1977 to July 27, 1978 and from Augut 26, 1978 to Angust 28, 1979, respectively, in compliance with "Exhibit A" of contract NAS8-32660. Throe prior final reporte were propared under comaract NAS-33979. Refervace ( $1-3$ ) was submitted on March 8, 1982 and covered the pariod from Augurt 15, 1980 to October 15, 1081 in complinnce with "Exhibit A" of the contract. Reforence (1-4) was submitted on March 18, 1983 and covered the period from October 16, 1981 to October 31, 1982 in compliance with "Exhibit B". Reference (1-5) was submitted on January 24, 1984 and covered the period from November 1, 1982 to September 30, 1983 in compliance with "Exhibit $\mathrm{C}^{\prime \prime}$ ".

### 1.1 OBJECTIVE

The sections that follow summarise the effort expended on the Modular Design Attitude Control System Study contrac : from October 1, 1983 to Augcet 31, 1984. In prior applications of modular attitude control to rigid body-flerible sumpension approximations of the rotational dynamics of prototype flexible spacecraft, it was asswred that all of the scalar state variables of the linearized models were accessible for measurement and/or control. Actual spacecraft to be controlled almost never satisfy such a broad condition. Therefore, the priscipal objective of the development of modular attitude control, completed August 31, 1984, was the generation of a series of linear observers to support the application of control to state variable models of flexible spacecraft with damping for which one or more state variable: are inaccessible.

### 1.2 SCOPE

Study effort was concentrated in four main areas:
A. Development of a series of single axis state variable models of flexib.e spacecraft with damping to be utilized in the comparison of different approaches to the development of modular attitade control systems. These models consisted of two, three, four or five rigid bodies serially connected by a lexible sumpeasion in such a way that motion was restricted to rotation about a common axis through the mase centers of the bodies.
B. Generaiisn of reduced state linear obaervers for each damped single axis model developed in Task A corresponding to various numbers und distributions of inaccessible state variables following the approaches presented in Luenberger (1-6), (1-7), (1-8), and Sage (1-9).
C. Iransformation of the undamped versions of the single axis models developed in Task $A$ to their corresponding modal models with modal damping following the approach presented in Thomson (110).
D. Generation of reduced state linear obeervers for each modal model developed in Task C with various numbers of inacessoible modal state variables ntilising direct matrix products as described in Lancaoter (1-11).

### 1.3 GENERAL

This report is comprised of seven sections. Secrions 2 through 5 describe the development of the two-, three-, four- and five-body single-axia state variable models, respectively, of a prototype flexible spaceeraft with damping and the generation of the minimam order reduced state linear observers for the reconstruction of inacceacible scalar state variables of these modelo. Section 6 begins with the transiormation of the singleaxis models of Sections 2 through 5 to modal forms to which modal damping is added and coacludes with the development of reduced state linear observers for these models when one or more modal state variableg are inaccessible. Section 7 lists a number of conclusions and recommendations drawn from generation of linear observers for the series of single-axis state variable models described above. References are listed at the end of ench section.
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## JECMION 2

## DEVILOPMENT OF TWO-BODY SLNGLE-AXIS MODEL AND IT8 RHDUCED ETATE LTNEAR OBEEYNES

### 2.1 ORIGINAL DAMPED MODEL

The rotational dynamices of the two-body singlo-acis model of a lexible spacocratt with damping shown in Fis. 2-1 mas be ruproenatod by the following ant of equations:

$$
\begin{align*}
& I_{1} \dot{\theta}_{1}=-c_{1}\left(\dot{\theta}_{1}-\dot{\theta}_{2}\right)-h_{1}\left(\theta_{1}-\theta_{2}\right)+q_{1}  \tag{2-1}\\
& I_{1} \dot{\delta}_{2}=c_{1}\left(\dot{\theta}_{1}-\dot{\theta}_{2}\right)+h_{1}\left(\theta_{1}-\theta_{2}\right)+\varphi_{2} \tag{2-2}
\end{align*}
$$

- inemer
$I_{i}=$ rotational inertia of body $i ; i=1,2$
$\theta_{i}=$ angular dieplacemone of body $i$
$\dot{o}_{i}=$ angular rate of body $i$
$q_{i}=$ torqae applied to body $i$
$h_{1}$ = rotational spring coefficient at the interface between the bodies
$c_{1}=$ rotational damping coefficicat at the interiace between the bodies


### 2.2 STATE VARIABLE MODEL

The state variable form of the two-body singlo-axis model of a fexible spacecrait with damping shown in Fig. 2-1 may be expresed an follows:

$$
\begin{align*}
& x=\Delta x+B u  \tag{2-3}\\
& x_{A}=C x \tag{2-4}
\end{align*}
$$

where:
$x=\left[\begin{array}{llll}\theta_{1} & \dot{\theta}_{1} & \theta_{2} & \dot{\theta}_{2}\end{array}\right]^{T}=\left[\begin{array}{llll}x_{1} & x_{2} & x_{3} & x_{4}\end{array}\right]^{T}=\left[\begin{array}{ll}x_{A}^{T} & x_{1}^{T}\end{array}\right]^{T}=$ state vector
$x_{A}=m$-vector of accesrible acalar states
$x_{1}=p$-vector of inacremibie scalar states
$u=\left[\begin{array}{ll}u_{1} & u_{2}\end{array}\right]^{T}=\left[\begin{array}{ll}\frac{q_{1}}{I_{1}} & \frac{q_{2}}{I_{2}}\end{array}\right]^{T}=$ control vactor
A $=4 \times 4$ state vector coefficient matrix
B $=4 \times r$ control vector coefficient matrix ( $r=1$ or 2 )
C $=m \times 4$ mearurewent or observation matrix
$A=\left[\begin{array}{cccc}0 & 1 & 0 & 0 \\ -a_{22} & -a_{23 r_{1}} & a_{23} & a_{23 r_{1}} \\ 0 & 0 & 0 & 1 \\ a_{41} & a_{41} r_{1} & -a_{41} & -a_{41} r_{1}\end{array}\right]$
$r_{1}=\frac{c_{1}}{k_{1}}$
$a_{22}=\frac{k_{1}}{I_{1}}$
$a_{41}=\frac{h_{1}}{I_{2}}$
B $=\left[\begin{array}{ll}0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1\end{array}\right]$ for $r=2$

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FIGURE 2-1

## 2 2 EDUCED STATE INSAR OB8BEVERS

## 2.8 .2 Preserdurition

The minimon order (mumber of scalar state variables) of a roduced state linear observer required to recomatruct the 4-m inscescoible sealar states of the two-body singlo-axis model represented by equationa (2-3) through (2-9) is $p=4-\mathrm{m}$. Thin recompruction was accomplinhed for a given state variable model in thrve main stegrea.

1) Symbuviring a linear obourver of minimam required order $(p)$.
2) Defining a synthmined vaiable corresponding to each of the insccessible state variables of the given atate variable model.
3) Expresing each synthesised variable as a function of the state variables of the reduced state observer aed the acceaible state variables of the given state variable model.
The relationship between the single axis model and its corresponding reduced state observer is depicted in Fig. 2-3.

The equations for the reduced stat: observers corresponding to the state variable model of equations (2-3) through (2-9) are the following:

$$
\begin{align*}
& s=D_{z}+\mathrm{Eu}+\mathrm{Gy}  \tag{2-10}\\
& \mathrm{E}=\mathrm{Tx}  \tag{2-11}\\
& \mathbf{E}=\mathbf{T B} \tag{2-12}
\end{align*}
$$

where:
$\mathbf{D}=p \times p$ observer coc ${ }^{\text {flcient matix }}$ (assumed diagonal)
$E=p \times r$ observer control vector coefficiemt matrix
$\mathbf{G}=p \times m$ observer vector of observed states coefficient matrix
$\mathbf{T}=p \times 4$ observer weighting matrix
The corresponding block diagram appears in Fig. 2-4.

### 2.3.2 Observer Synthesis Equations

The equations for synthesixing the reduced state linear observers, based on those appearing in Luenberger (1-1), (1-2), (1-3) and Sage (1-4), were written in the following form.

$$
\begin{gather*}
\mathbf{P A}-\mathbf{D T}=\mathbf{F}  \tag{2-13}\\
\mathbf{F}=\mathbf{G C} \tag{2-14}
\end{gather*}
$$

For

$$
\begin{align*}
& \mathbf{T}=\left[\begin{array}{cccc}
t_{11} & t_{12} & t_{18} & t_{14} \\
\vdots & \vdots & \vdots & \vdots \\
t_{p, 1} & t_{p, 2} & t_{p, 2} & t_{p, 4}
\end{array}\right]  \tag{2-15}\\
& T=\left[\begin{array}{cccc}
f_{11} & f_{12} & f_{18} & f_{14} \\
\vdots & \vdots & \vdots & \vdots \\
f_{p, 1} & f_{p, 2} & j_{p, 2} & f_{p, 4}
\end{array}\right]  \tag{2-16}\\
& D=\left[\begin{array}{lll}
d_{11} & & 0 \\
& \ddots & \\
0 & & d_{p, p}
\end{array}\right] \tag{2-17}
\end{align*}
$$

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State and Observation Equations:

$$
\begin{aligned}
& \underline{\underline{x}}=A \underline{x}+B \underline{u} \\
& \underline{x}_{A}=C \underline{x}
\end{aligned}
$$



$\underline{x}_{A}=$ vector of accessible scalar states of model
$z=v e c t o r$ of scalar states of observer
T = observer weighting matrix
$\hat{X}_{I}=v e c t o r$ of reconstructed scalar states of model
$\hat{\underline{x}}=\left[\begin{array}{c}\underline{x}_{A} \\ -\hat{\underline{x}}_{I}-\end{array}\right]=\begin{gathered}\text { resonstructed vector of all scalar state variables of } \\ \text { vehicle model }\end{gathered}$

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Observer Equations:

$$
\underline{z}=D_{\underline{z}}+G_{\underline{x}_{A}}+E \underline{u}
$$

$$
\text { Since } G \underline{X}_{A}=G C \underline{x}=F \underline{x}
$$

$$
\underline{\underline{z}}=D \underline{z}+F \underline{x}+E \underline{\underline{u}}
$$

and the form of the $\mathbf{A}$ matrix givea in equation (2-5) the obeurver synthesin equatione retuce to the following seatel forma.

$$
\begin{align*}
& t_{i 1}=d_{i 12} t_{i 2}-a_{41} r_{1} t_{i 4}+f_{i 2} \\
& t_{i 2}=-a_{23 r_{1} t_{i 2}}+d_{i i 2} t_{i 4}+f_{i 4}  \tag{2-19}\\
& {\left[\begin{array}{cc}
-\left(a_{23} p_{1}+d_{i i}^{2}\right) & a_{41} p_{1} \\
a_{2 i p_{1}} & -\left(a_{41 p_{1}}+d_{i i}^{p}\right)
\end{array}\right]\left[\begin{array}{l}
t_{i 2} \\
t_{i 4}
\end{array}\right]=\left[\begin{array}{l}
f_{i 1}+d_{i i} f_{i 2} \\
f_{i 2}+d_{i i} f_{i 4}
\end{array}\right]} \tag{2-20}
\end{align*}
$$

where:

$$
\begin{align*}
d_{i i 1} & =a_{28} r_{1}+i i  \tag{2-21}\\
d_{i i 2} & =a_{41} r_{1}+d_{i i}  \tag{2-22}\\
p_{1} & =1+r_{1} d_{i i}  \tag{2-23}\\
\Delta_{i 2}^{\prime} & =\left(a_{23} p_{1}+d_{i i}^{2}\right)\left(a_{41} p_{1}+d_{i i}^{2}\right)-a_{23} a_{41} p_{1}^{2} \\
& =d_{i i}^{2}\left(a_{23} p_{1}+a_{41} p_{1}+d_{i i}^{2}\right)  \tag{2-24}\\
t_{i 2} & =-\frac{\left(a_{21} p_{1}+d_{i i}^{2}\right)\left(f_{i 1}+d_{i i} f_{i 2}\right)+a_{41} p_{1}\left(f_{i 3}+d_{i i} f_{i 4}\right)}{\Delta_{i 2}^{\prime}} ;  \tag{2-25}\\
t_{i 4} & =-\frac{a_{23} p_{1}\left(f_{i 1}+d_{i i} f_{i 2}\right)+\left(a_{23} p_{1}+d_{i j}^{2}\right)\left(f_{i 3}+d_{i i} f_{i 4}\right)}{\Delta_{i 2}^{\prime}} \\
t_{i 1} & =-\frac{d_{i i}\left[a_{41}+\left(a_{23}+a_{41}\right) r_{1} d_{i i}+d_{i i}^{2}\right]}{\Delta_{i 2}^{\prime}}\left(f_{i 1}+d_{i i} f_{i 2}\right)+f_{i 2}+\frac{a_{41} d_{i i}}{\Delta_{i 2}^{\prime}}\left(f_{i 3}+d_{i i} f_{i 4}\right)  \tag{2-26}\\
t_{i 2} & =-\frac{a_{38} d_{i i}}{\Delta_{i 2}^{\prime}}\left(f_{i 1}+d_{i i} f_{i 2}\right)+\frac{d_{i i}\left[a_{23}+\left(a_{23}+a_{41}\right) r_{1} d_{i i}+d_{i i}^{2}\right]}{\Delta_{i 2}^{\prime}}\left(f_{i 3}+d_{i i} f_{i 4}\right)+f_{i 4} \tag{2-27}
\end{align*}
$$

### 2.3.3 Comparison of T Matrices for Damped and Undamped Modele

If damping is removed from the model, $r_{1} \rightarrow 0, d_{i i 1} \rightarrow d_{i i}, d_{i i 2} \rightarrow d_{i i}, p_{1} \rightarrow 1$ and $\Delta_{i 2}^{\prime} \rightarrow d_{i i}^{2}\left(a_{28}+a_{41}+d_{i i}^{2}\right)=\Delta_{i 2}$.

$$
\begin{align*}
& t_{i 2}=-\frac{\left(a_{41}+d_{i 2}^{2}\right)\left(f_{i 1}+d_{i i} f_{i 2}\right)+a_{41}\left(f_{i 3}+d_{i i} f_{i 4}\right)}{\Delta_{i 2}} ; \quad i=1,2, \ldots, p  \tag{2-29}\\
& t_{i 4}=-\frac{a_{22}\left(f_{i 1}+d_{i i} f_{i 2}\right)+\left(a_{22}+d_{i j}^{2}\right)\left(f_{i a}+d_{i i} f_{i 4}\right)}{\Delta_{i 2}}  \tag{2-30}\\
& t_{i 1}=f_{i 2}-\frac{d_{i i}\left[\left(a_{41}+d_{i 2}\right)\left(f_{i 1}+d_{i i} f_{i 2}\right)-a_{41}\left(f_{i 2}+d_{i i} f_{i 4}\right)\right]}{\Lambda_{i 2}}  \tag{2-31}\\
& t_{i 3}=f_{i 4}-\frac{d_{i 3}\left[a_{23}\left(f_{i 1}+d_{i i} f_{i 2}\right)-\left(a_{22}+d_{i i}^{R}\right)\left(f_{i 2}+d_{i i} f_{i 4}\right)\right]}{\Delta_{i 2}} . \tag{2-32}
\end{align*}
$$

Comparison of the corresponding equations for generating the elements of the $\mathbf{T}$ matrix, $t_{i j}(i=1,2, \ldots, p$; $j=1,2,3,4$ ) revealed that the addition of damping at the interface between the two bodies had the following effects:

1. Fi the cquations expocoing $t_{i 2}$ and $t_{i 4}$, the elomonte of the even columan of the $T$ matrix as a function of the fij, the clemente of the $F$ matrix, the form of each equation remains the aume under addition of drmping with casp1 and $a_{41} p_{1}$ being subotituted for each scalar $a_{28}$ and $a_{41}$ appearing in the corrmponding equations for the undermped two body model.
2. In the equation expreving $t_{i 1}$, the elemonts of the first column of the $T$ matrix as a function of the $f_{i j}$, the demouste of the $\overline{\text { I matrix, the form of the equation remsins the same under addition of }}$ damping except that the expreacion, $a_{41} p_{1}+a_{32} r_{1} d_{i f}$, appears in the place of $a_{41}$ in the coethcient of ( $f_{i 1}+d_{i f} f_{i 2}$ ) in the mamerator and $a_{98} p_{1}$ and $a_{41} p_{1}$ appear in the place of $a_{28}$ and $a_{41}$ respectively in the denominator.
3. In the equation exprouing $t_{i s}$, the clements of the third column of the $\mathbf{T}$ matrix as a functinon of the fij, the elamante of the $F$ matrix, the form of the equation remsins the same under addition of damping except that the expression, $a_{23} p_{1}+a_{41} r_{1} d_{i f}$, appears in the place of $a_{23}$ in the coefficient of ( $f_{i 8}+d_{i i} f_{i 4}$ ) in the numerator and $a_{23} p_{1}$ and $a_{41} p_{1}$ appear in the place of $a_{23}$ and $a_{41}$, recpectively, in the denominator.

### 2.4 SOLUTION FOR SYNTELESLED STATE VARIABLES

### 2.4.1 Imtroduction

Inaccessibility of a state variable in the model equations (2-3), (2-4) is reflected by a corresponding null column in the obeervation matrix, $\mathbf{C}$, and a corresponding null colvmn in the $\mathbf{F}$ matrix as implied by equation (2-14). For the generation of reduced order observers for the two body model the number of inacressible state variables can be 1,2 or 3.

### 2.42 Firat Order Observery ( $p=1$ )

A first order linear observer corresponds to inaccessibility of one of the four scalar state variables of the two body model. The observer equation then reduces to:

$$
\begin{equation*}
\dot{z}=d z+\mathbf{E u}+\mathbf{G y} \tag{2-33}
\end{equation*}
$$

the $\mathbf{F}$ and $\mathbf{T}$ matrices reduce to:

$$
\begin{align*}
& F=\left[\begin{array}{llll}
f_{1} & f_{2} & f_{2} & f_{4}
\end{array}\right]  \tag{2-34}\\
& \mathbf{T}=\left[\begin{array}{llll}
t_{1} & t_{2} & t_{3} & t_{4}
\end{array}\right] \tag{2-35}
\end{align*}
$$

and the observer synthesis equations reduce to the following forms:

$$
\begin{align*}
& t_{2}=-\frac{\left(a_{41} p_{1}+d^{2}\right)\left(f_{1}+d f_{2} j+a_{41} p_{1}\left(f_{2}+d f_{4}\right)\right.}{\Delta_{1}^{\prime}}  \tag{2-36}\\
& t_{4}=-\frac{a_{23} p_{1}\left(f_{1}+d f_{2}\right)+\left(a_{23} p_{1}+d^{2}\right)\left(f_{3}+d f_{4}\right)}{\Delta_{2}^{\prime}}  \tag{2-37}\\
& t_{1}=-\frac{d\left(a_{41} p_{1}+a_{23} r_{1} d+d^{2}\right)\left(f_{1}+d f_{2}\right)}{\Delta_{2}^{\prime}}+f_{2}+\frac{a_{41} d}{\Delta_{2}^{\prime}}\left(f_{3}+d f_{4}\right)  \tag{2-38}\\
& t_{3}=-\frac{a_{23} d}{\Delta_{2}^{\prime}}\left(f_{1}+d f_{2}\right)+\frac{d\left(a_{23} p_{1}+a_{41} r_{1} d+d^{2}\right)\left(f_{3}+d f_{4}\right)}{\Delta_{2}^{\prime}}+f_{4}  \tag{2-39}\\
& p_{1}=1+r_{1} d  \tag{2-40}\\
& \Delta_{2}^{\prime}=d^{2}\left(a_{23} p_{1}+a_{41} p_{1}+d^{2}\right) \tag{2-41}
\end{align*}
$$

Since this case corresponds to inaccessibility of one state variable, one of the $f_{i}(i=1,2,3,4)=0$.

## Bymale

Suppoee $a_{4}$, the sealar state roprowesting the angular rate of body 2 , is imacemaible. Then it is acoumed thet:

$$
C=\left[\begin{array}{llll}
1 & 0 & 0 & 0  \tag{2-42}\\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{array}\right]
$$

for which:

$$
F=\left[\begin{array}{llll}
f_{1} & f_{2} & f_{2} & 0 \tag{2-48}
\end{array}\right]
$$

and T is of the form shown in equation (2-35).
From equations (2-14), (2-42) and (2-43),

$$
G=\left[\begin{array}{lll}
f_{1} & f_{2} & f_{3} i \tag{2-44}
\end{array}\right.
$$

and from equations (2-9), (2-12) and (2-35)

$$
E=\left[\begin{array}{ll}
t_{2} & t_{4} \tag{2-45}
\end{array}\right]
$$

This equation corresponds to $r=2$, control torques applied to both bodies. For control troque applied only to body 1,

$$
E=\left[\begin{array}{ll}
t_{2} & 0 \tag{2-46}
\end{array}\right]
$$

and for control torque applied only to body 2 ,

$$
E=\left[\begin{array}{ll}
0 & t_{4} \tag{2-47}
\end{array}\right]
$$

The equations for determining the elements of the $\mathbf{T}$ matrix reduce to the following forms:

$$
\begin{align*}
& t_{2}=-\frac{\left(a_{41} p_{1}+d^{2}\right)\left(f_{1}+d f_{2}\right)+a_{41} p_{1} f_{3}}{\Delta_{2}^{\prime}}  \tag{2-48}\\
& t_{4}=-\frac{a_{23} p_{1}\left(f_{1}+f_{2}\right)+\left(a_{23} p_{1}+d^{2}\right) f_{3}}{\Delta_{2}^{\prime}}  \tag{2-49}\\
& t_{1}=-\frac{d\left(a_{41} p_{1}+a_{23} r_{1} d+d^{2}\right)\left(f_{1}+d f_{2}\right)}{\Delta_{2}^{\prime}}+f_{2}+\frac{a_{41} d}{\Delta_{2}^{\prime}} f_{3}  \tag{2-50}\\
& t_{2}=-\frac{a_{9} d\left(f_{1}+d f_{3}\right)+d\left(a_{23} p_{1}+a_{41} r_{1} d+d^{2}\right) f_{2}}{\Delta_{2}^{\prime}} \tag{2-51}
\end{align*}
$$

From equations (2-11) and (2-35),

$$
\begin{equation*}
z=t_{1} x_{1}+t_{2} x_{2}+t_{3} x_{8}+t_{4} \hat{x}_{4} \tag{2-52}
\end{equation*}
$$

$w$ nere $\hat{x}_{4}=$ cine synthesised $x_{4}$.
Solving for $\hat{x}_{4}$ yields:

$$
\begin{equation*}
\hat{x}_{4}=\frac{1}{t_{4}}\left(x-\sum_{i=1}^{3} t_{i} x_{i}\right) \tag{2-53}
\end{equation*}
$$

For insecmelbility of $z_{1}, s_{2}$ or $z_{i}$, the equatione for detarmining $t_{i},(2-30)$ through (2-30) are appropriately mediliod.

## 2048 Eacond Order Observers ( $p=2$ )

The equation for a linear observer of order two corresponde to two of the four scalar state variables being inscrumible. It in ruprosented here as equation (2-10). If the oboerver coefficient matrix is assumed to be diagonal in this cace it appoase as followe:

$$
D=\left[\begin{array}{cc}
d_{11} & 0  \tag{2-54}\\
0 & d_{32}
\end{array}\right]
$$

Since the observer is of order two,
and,

$$
\begin{align*}
& ==\left[z_{1}, s_{5}\right]^{T}  \tag{2-55}\\
& T=\left[\begin{array}{llll}
f_{11} & f_{12} & f_{12} & f_{14} \\
f_{21} & f_{22} & f_{23} & f_{24}
\end{array}\right] \tag{2-56}
\end{align*}
$$

$$
\mathbf{T}=\left[\begin{array}{llll}
t_{11} & t_{12} & t_{12} & t_{14}  \tag{2-57}\\
t_{21} & t_{22} & t_{22} & t_{24}
\end{array}\right]
$$

The specific form of the equations for generating the elements of $I$ depend apon which two of the scalar states are inaccesnile. For each inaccessible state the corresponding columns in the $\mathbf{C}$ and $\mathbf{F}$ matrices are noll.

## Example

Corresponding to the angular position and rate, respectively, of body 2, suppose that the scalar states $x_{3}$ and $x_{4}$ are inaccessible. Then the equations for generating the elements of the $\mathbf{T}$ matrix assume the following forms.

$$
\begin{align*}
& t_{i 2}=-\frac{\left(a_{41} p_{1}+d_{i 2}^{2}\right)}{\Delta_{i 3}^{\prime}}\left(f_{i 1}+d_{i i} f_{i 2}\right) \quad i=1,2  \tag{2-58}\\
& t_{i 4}=-\frac{a_{2 g} p_{1}}{\Delta_{i 2}^{\prime}}\left(f_{i 1}+d_{i i} f_{i 2}\right)  \tag{y}\\
& t_{i 1}=-\frac{\left.d_{i 1}\left(a_{41} p_{1}+a_{23} r_{1} d_{i 2}\right)+d_{i i 2}^{2}\right)}{\Delta_{i 2}^{\prime}}\left(f_{i 1}+d_{i 1} f_{i 2}\right)+f_{i 2}  \tag{2-60}\\
& t_{i 3}=-\frac{a_{38} d_{i j}}{\Delta_{i 2}^{\prime}}\left(f_{i 1}+d_{i i} f_{i 2}\right) \tag{2-81}
\end{align*}
$$

where $p_{1}$ and $\Delta_{i 2}^{\prime}$ are defined in equations (2-23) and (2-24).
From equation (2-11),

$$
\left[\begin{array}{ll}
t_{12} & t_{14}  \tag{2-62}\\
t_{23} & t_{24}
\end{array}\right]\left[\begin{array}{l}
\hat{x}_{3} \\
\hat{z}_{4}
\end{array}\right]=\left[\begin{array}{l}
z_{1}-t_{12} x_{1}-t_{12} x_{2} \\
z_{2}-t_{21} x_{1}-t_{32} x_{2}
\end{array}\right]
$$

where $\hat{x}_{8}$ and $\hat{x}_{4}$ are synthesised state varinbles.

$$
\text { Let } \Delta_{2}=\left|\begin{array}{ll}
t_{13} & t_{14} \\
t_{23} & t_{34}
\end{array}\right|=t_{18} t_{34}-t_{14} t_{28} \neq 0
$$

where:

$$
\left(\Delta_{3}\right)_{i j}=\Delta_{2} \text { without elements of ith row and jth column }
$$

$$
\begin{align*}
& \omega_{0}=\frac{\left(\Delta_{1}\right)_{h_{1} 1}\left(x_{1}-t_{11} \varepsilon_{1}-t_{12} \varepsilon_{2}\right)-\left(\Delta_{2}\right)_{2_{2} 1}\left(x_{2}-t_{21} \varepsilon_{1}-t_{29} \varepsilon_{2}\right)}{\Delta_{2}}  \tag{2-8}\\
& a_{4}=\frac{-\left(\Delta_{2}\right)_{12}\left(x_{1}-t_{11} x_{1}-t_{12} z_{2}\right)+\left(\Delta_{2}\right)_{2,}\left(z_{2}-t_{21} x_{1}-t_{22} a_{1}\right)}{\Delta_{2}} \tag{2-84}
\end{align*}
$$

For $a_{3}$ and as inscersible, it is acoumod that:

$$
\begin{align*}
& C=\left[\begin{array}{llll}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{array}\right]  \tag{2-65}\\
& F=\left[\begin{array}{llll}
f_{11} & f_{12} & 0 & 0 \\
f_{21} & f_{22} & 0 & 0
\end{array}\right]
\end{align*}
$$

From $\mathbf{F}=\mathbf{G C}$,

$$
\mathbf{G}=\left[\begin{array}{ll}
f_{11} & f_{12}  \tag{2-67}\\
f_{21} & f_{22}
\end{array}\right]
$$

Frmm $\mathbf{F}=\mathbf{T B}$,

$$
\begin{align*}
& \mathbf{E}=\left[\begin{array}{ll}
t_{12} & t_{14} \\
t_{22} & t_{24}
\end{array}\right] \quad \text { for } r=2 \text { (control torques applied to both bodies) }  \tag{2-68}\\
& \mathbf{E}=\left[\begin{array}{ll}
t_{12} & 0 \\
t_{22} & 0
\end{array}\right] \quad \text { for control restricted to body } 1  \tag{2-69}\\
& E=\left[\begin{array}{ll}
0 & t_{14} \\
0 & t_{24}
\end{array}\right] \quad \text { for control restricted to body } 2 \tag{2-70}
\end{align*}
$$

### 2.4.4 Third Order Observers ( $p=3$ )

The equation for the linear observer of order one less than the system's dimension corresponds to three of the four scalar state varisbles being inaccessible. It is represented here as equation (2-10). If the observer coefficient matrix is asomed to be diagonal in this case it appears an follows,

$$
D=\left[\begin{array}{ccc}
d_{11} & 0 & 0  \tag{2}\\
c & d_{i 2} & 0 \\
0 & 0 & d_{3 B}
\end{array}\right]
$$

Since the observer is of order 3 ,

$$
\begin{align*}
I & =\left[x_{1}, x_{2}, x_{8}\right]^{T}  \tag{2-72}\\
\mathbf{T} & =\left[\begin{array}{llll}
f_{11} & f_{12} & f_{13} & f_{14} \\
f_{21} & f_{22} & f_{28} & f_{24} \\
f_{21} & f_{22} & f_{28} & f_{24}
\end{array}\right] \tag{2-73}
\end{align*}
$$

and,

$$
T=\left[\begin{array}{llll}
t_{11} & t_{12} & t_{12} & t_{14}  \tag{2-74}\\
t_{21} & t_{22} & t_{22} & t_{24} \\
t_{31} & t_{22} & t_{22} & t_{24}
\end{array}\right]
$$

The specific forms of the equations for generating the elements of $\mathbf{T}$ depend upon which three of the scalar states are inaccessible. For each inaccessible state the corresponding columns in th $\mathbf{C}$ and $\mathbf{F}$ matrices are null.

## Erample

Sappoes the acalar ataten, a3, as and a4, roproseating the angular rate of body 1 and the angular position and rate of body 2, are inaceocible. Then the equations for generating the elemente of the $\mathbf{T}$ matrix moum the following form ince $f_{i 1}=f_{i s}=f_{i 4}=0$ for $i=1,2,3$.

$$
\begin{align*}
& t_{i 2}=-\frac{\left(a_{41} p_{1}+d_{i i}\right)}{\Delta_{i 2}^{\prime}} f_{i 1} \quad i=1,2,3  \tag{2-78}\\
& t_{i 4}=-\frac{a_{23} p_{1}}{\Delta_{i 2}^{\prime}} f_{i 1}  \tag{2-76}\\
& t_{i 1}=-\frac{d_{i 1}\left(a_{i 1} p_{1}+a_{23} r_{1} d_{i 1}+d_{i i}^{\prime}\right)}{\Delta_{i 2}^{\prime}} f_{i 1}  \tag{2-77}\\
& t_{i 2}=-\frac{a_{23} d_{i i}}{\Delta_{i 2}^{\prime}} f_{i 1} \tag{2-78}
\end{align*}
$$

where $p_{1}$ and $\Delta_{i 2}^{\prime}$ are defmed in equations (2-23) and (2-24).
From equation (2-11),

$$
\left[\begin{array}{lll}
t_{12} & t_{12} & t_{14}  \tag{2-79}\\
t_{22} & t_{23} & t_{24} \\
t_{22} & t_{32} & t_{34}
\end{array}\right]\left[\begin{array}{l}
\hat{x}_{2} \\
\hat{z}_{2} \\
\hat{x}_{4}
\end{array}\right]=\left[\begin{array}{l}
x_{1}-t_{11} x_{1} \\
z_{2}-t_{21} x_{1} \\
z_{3}-t_{31} x_{1}
\end{array}\right]
$$

where $\hat{x}_{2}, \hat{x}_{2}$ and $\hat{x}_{4}$ are synthecised state variables.

$$
\text { Let } \Delta_{8}=\left|\begin{array}{lll}
t_{12} & t_{12} & t_{14} \\
t_{22} & t_{23} & t_{34} \\
t_{22} & t_{32} & t_{34}
\end{array}\right| \psi 0
$$

where,

$$
\left(\Delta_{3}\right)_{i, j}=\Delta_{8} \text { without elemente of ith row and jth column }
$$

$$
\begin{equation*}
=\frac{\sum_{i=1}^{3}(-1)^{i+j}\left(\Delta_{8}\right)_{i, j}\left(z_{j}-a_{1}\right)}{\Delta_{3}} \quad j=1,2,3 \tag{2-80}
\end{equation*}
$$

For $x_{2}, x_{2}$ and $x_{4}$ inaccesnible, it is aspumed that:

$$
\begin{align*}
& C=\left[\begin{array}{llll}
1 & 0 & 0 & 0
\end{array}\right]  \tag{2-81}\\
& F=\left[\begin{array}{llll}
f_{11} & 0 & 0 & 0 \\
f_{12} & 0 & 0 & 0 \\
f_{a 1} & 0 & 0 & 0
\end{array}\right] \tag{2-82}
\end{align*}
$$

From $\boldsymbol{F}=\mathbf{G C}$,

$$
\boldsymbol{G}=\left[\begin{array}{l}
f_{11}  \tag{2-83}\\
f_{21} \\
f_{21}
\end{array}\right]
$$

From $E=T B$,

$$
\mathbf{E}=\left[\begin{array}{ll}
t_{12} & t_{14}  \tag{2-84}\\
t_{22} & t_{24} \\
t_{32} & t_{34}
\end{array}\right] \quad \text { for } r=2 \text { (control torques applied to both bodies) }
$$

## 28 2etranarces

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## CETION 8

## DEVELOPMENT OF TERE TEREFBODY BINGLD-ATIS MODEL AND IIS REDUCED STAME LINEAR OBAEEVERS

### 8.1 OEIGINAL DAMRED MODEL

The rotational dymanien of the throebody singlo-axis model of a flexible spacecraft with dampins shown in Fig. 3-1 may be reprevented by the following eet of equationa:

$$
\begin{align*}
& I_{1} \dot{\theta}_{1}=-c_{1}\left(\dot{\theta}_{1}-\dot{\theta}_{2}\right)-h_{1}\left(\theta_{1}-\theta_{2}\right)+q_{1}  \tag{8-1}\\
& I_{2} \dot{\theta}_{2}=c_{1}\left(\dot{\theta}_{1}-\dot{\theta}_{2}\right)+h_{1}\left(\theta_{1}-\theta_{2}\right)+c_{2}\left(\dot{\theta}_{2}-\dot{\theta}_{2}\right)+h_{2}\left(\theta_{2}-\theta_{2}\right)+q_{2}  \tag{3-2}\\
& I_{8} \dot{\theta}_{2}=-c_{2}\left(\dot{\theta}_{2}-\dot{\theta}_{2}\right)-h_{2}\left(\theta_{2}-\theta_{1}\right)+q_{3} \tag{3-3}
\end{align*}
$$

where
$I_{i}=$ rotational inertia of body $i ; i=1,2,3$
$\theta_{i}=$ angular dislpacement of body $i$
$\dot{\theta}_{i}=$ angular rate of body $;$
$g_{i}=$ torque applied to body $i$
$h_{j}=$ rotational spring coefficient at interface $j ; j=1,2$
$c_{j}=$ rotational dampoing coefficient at interface $j$

## s. $\operatorname{sTATE}$ VARLABLE MODEL

Th. state variable form of the three-body singlo-axis model of a flexible : pacecraft shown in Fig. 3-1 may be expressed as follows:

$$
\begin{align*}
& \dot{k}=A x+B u  \tag{3-4}\\
& x_{A}=C x \tag{3-5}
\end{align*}
$$

where:

$$
\begin{aligned}
& x=\left[\begin{array}{llll}
x_{1} & x_{2} & \cdots & x_{6}
\end{array}\right]^{T}=\left[\begin{array}{llllll}
\theta_{1} & \dot{\theta}_{1} & \theta_{2} & \dot{\theta}_{2} & \theta_{3} & \dot{\theta}_{3}
\end{array}\right]^{T}=\left[\begin{array}{ll}
x_{A}^{T} & x_{1}^{T}
\end{array}\right]^{T}=\text { state vector } \\
& x_{A}=m \text {-vector of accessible scalar states } \\
& x_{1}=p \text {-vector of inaccessible scalar states } \\
& u=\left[\begin{array}{llll}
u_{1} & \cdots & u_{r}
\end{array}\right]^{T}=\left[\begin{array}{lll}
\frac{q_{1}}{I_{1}} & \cdots & \frac{q_{r}}{I_{r}}
\end{array}\right]^{T} \quad(r=1,2 \text { or } 3)
\end{aligned}
$$

$C=$ obecration matrix of dirmanions $m \times 6, m=1,2, \ldots, 5$ (Minimom dimen + ion of reduced order obeerver required $=6-\mathrm{m})$.
Partitioning of this model by rigid body revalte in the following forms for its coeflicient matrices.

$$
\mathbf{A}=\left[\begin{array}{cccccc}
0 & 1 & 0 & 0 & 0 & 0  \tag{3-6}\\
-a_{28} & -a_{23} r_{1} & a_{38} & a_{28} r_{1} & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
a_{41} & a_{41} r_{1} & a_{49} & a_{44} & a_{46} & a_{45} r_{2} \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & a_{63} & a_{62} r_{2} & -a_{63} & -a_{63} r_{3}
\end{array}\right]
$$

ORIGNAL PRGE : OF POOR QUALITY


FIGURE 3-1

$$
\begin{align*}
& a_{28}=\frac{h_{1}}{I_{1}} \\
& a_{41}=\frac{h_{1}}{I_{2}} \\
& a_{45}=\frac{L_{2}}{I_{2}}  \tag{3-7}\\
& a_{48}=-\left(a_{41}+a_{45}\right) \\
& a_{41}=-\left(a_{41} r_{1}+a_{45} r_{2}\right) \\
& a_{62}=\frac{h_{2}}{I_{2}} \\
& r_{j}=\frac{c_{j}}{h_{j}} ; \quad j=1,2  \tag{3-8}\\
& B=\left[\begin{array}{rrr}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & .0 & 1
\end{array}\right] \text { for } r=3 \text { (control torques applied to all three bodies) } \tag{3-9}
\end{align*}
$$

The block diagram corresponding to this model is shown in Fig. 2-2.

## 3.s REDUCED STATE LINEAR OBSERYERS

## s.s.1 Introduction

For the three-body single-axis model represenced by equations (3-4) through (3-9), the minimmon order of a reduced state linear observer required to generate the inaccessible states is $p=0-m(m=1,2, \ldots, j)$. All of the roduced state linear observers for the three body model may be written in the form represented by equations ( $2-10$ ) and ( $2-11$ ) where, in this case, the observer coefficient matrix, $\mathbf{D}$, is assumed to be diagonal and of dimensions $p \times p$. The corresponding observer weighting matrix is of the following form:

$$
\mathbf{T}=\left[\begin{array}{ccc}
t_{11} & \ldots & t_{16}  \tag{3-10}\\
\vdots & & \vdots \\
t_{p, 1} & \ldots & t_{p, 6}
\end{array}\right]
$$

From equations (2-12), (3-9) and (3-10).

$$
\begin{align*}
& \mathbf{E}=\left[\begin{array}{ccc}
t_{12} & t_{14} & t_{16} \\
\vdots & \vdots & \vdots \\
t_{p, 2} & t_{p, 4} & t_{p, 6}
\end{array}\right] \text { for } r=3 \text { (control torques applied to all } 3 \text { bodies) }  \tag{3-11}\\
& \mathbf{F}=\left[\begin{array}{ccc}
f_{11} & \ldots & f_{16} \\
\vdots & & \vdots \\
f_{p, 1} & \ldots & f_{p, 6}
\end{array}\right] \tag{3-12}
\end{align*}
$$

The corresponding observer block diagram appears in Fig 2-4.

### 28.3 Oberver Eynithomic Equations

Trom Lamburger (3-1), (3-2), (3-3) and Sage (3-4) the equations for aynthenicing the reduced state Finear obvecvers for the throe-body singloware modal repreceated by equation (3-4) through (3-9) are given by equations (2-13) and (2-14). With coefficiest matrivies of the forms listed in 3.3.1 this set of observer ayathoie equations rodeces to the following forme:

$$
\begin{align*}
& t_{i 2}=\frac{\left(\Delta_{i a}^{\prime}\right)_{1,1}\left(f_{i 1}+d_{i i} f_{i 2}\right)-\left(\Delta_{i a}^{\prime}\right)_{2,1}\left(f_{i a}+d_{i i} f_{i 4}\right)+\left(\Delta_{i a}^{\prime}\right)_{2,1}\left(f_{i 6}+d_{i i} f_{i 6}\right)}{\Delta_{i s}^{\prime}} \\
& =\frac{\left[\left(a_{41} p_{1}+d_{i i}^{2}\right)\left(a_{83} p_{2}+d_{i i}\right)+a_{45} p_{3} d_{i i}^{2}\right]}{\Delta_{i 8}^{\prime}}\left(f_{i 1}+d_{i i} f_{i 2}\right) \\
& +\frac{a_{41} p_{1}\left[\left(a_{83} p_{2}+d_{i i}^{2}\right)\left(f_{i 3}+d_{i i} f_{i 4}\right)+a_{63} p_{2}\left(f_{i 6}+d_{i i} f_{i 6}\right)\right]}{\Delta_{i 3}^{\prime}} ; \quad i=1,2, \ldots, p  \tag{3-13}\\
& t_{i 4}=-\frac{\left(\Delta_{i 8}^{\prime}\right)_{1,2}\left(f_{i 1}+d_{i i} f_{i 2}\right)-\left(\Delta_{i 3}^{\prime}\right)_{2,2}\left(f_{i a}+d_{i i} f_{i 4}\right)+\left(\Delta_{i 3}^{\prime}\right)_{3,2}\left(f_{i 6}+d_{i i} f_{i 6}\right)}{\Delta_{i 3}^{\prime}} \\
& =\frac{a_{23} p_{1}\left(a_{83} p_{2}+d_{i i}^{2}\right)\left(f_{i 1}+d_{i i} f_{i 3}\right)+\left(a_{23} p_{1}+d_{i i}^{2}\right)\left(a_{63} p_{2}+d_{i i}^{2}\right)\left(f_{i a}+d_{i i} f_{i 4}\right)}{\Delta_{i 3}^{\prime}} \\
& +\frac{a_{69} p_{2}\left(a_{2 a p_{1}}+d_{i j}^{2}\right)\left(f_{i 5}+d_{i i} f_{i 6}\right)}{\Delta_{i 2}^{\prime}}  \tag{3-14}\\
& t_{i 6}=\frac{\left(\Delta_{i 3}^{\prime}\right)_{1,3}\left(f_{i 1}+d_{i i} f_{i a}\right)-\left(\Delta_{i 3}^{\prime}\right)_{2,3}\left(f_{i a}+d_{i i} f_{i 4}\right)+\left(\Delta_{i 3}^{\prime}\right)_{3,3}\left(f_{i 6}+d_{i i} f_{i 6}\right)}{\Delta_{i 3}^{\prime}} \\
& =\frac{a_{23} a_{+6} p_{1} p_{2}\left(f_{i 1}+d_{i i} f_{i 2}\right)+a_{45} p_{2}\left(a_{23} p_{1}+d_{i j}^{2}\right)\left(f_{i 3}+d_{i i} f_{i 4}\right)}{\Delta_{i 8}^{\prime}} \\
& +\frac{\left[\left(a_{23} p_{1}+d_{i i}^{2}\right)\left(a_{45} p_{2}+d_{i i}^{2}\right)+a_{41} p_{1} d_{i i}^{2}\right]\left(f_{i 6}+d_{i i} f_{i 6}\right)}{\Delta_{i 3}^{\prime}}  \tag{3-15}\\
& t_{i 1}=\frac{d_{i i}\left\{\left[\left(a_{41} p_{1}+d_{i i}^{2}\right)\left(a_{8 n} p_{2}+d_{i i}^{2}\right)+a_{46} p_{y} d_{i i}^{2}\right]\right.}{\Delta_{i 3}^{\prime}} \\
& +\frac{\left.a_{23} r_{1} d_{i i}\left[\left(a_{46}+a_{69}\right) p_{2}+d_{i i}\right]\right\}}{\Delta_{i 2}^{\prime}}\left(f_{i 1}+d_{i i} f_{i 2}\right)+f_{i 2} \\
& +\frac{a_{41} d_{i i}\left[\left(a_{83} p_{2}+d_{i i}^{2}\right)\left(f_{i 8}+d_{i i} f_{i 4}\right)+a_{68} p_{2}\left(f_{i 8}+d_{i i} f_{i 8}\right)\right]}{\Delta_{i 2}^{\prime}} \tag{3-16}
\end{align*}
$$

$$
\begin{align*}
& t_{i 3}=\frac{\operatorname{cosiz}_{i i}\left[-\alpha_{40} d_{i i}\left(r_{1}-r_{7}\right)+\left(\alpha_{8 a} p_{2}+d_{i i}\right)\right]}{\Delta_{i 3}^{\prime}}\left(f_{i 1}+d_{i i} f_{i 3}\right) \\
& +\frac{d_{i i}\left[\left(a_{33} p_{1}+d_{i i}\right)\left(a_{s 3} p_{2}+d_{i n}\right)+a_{41} r_{1} d_{i i}\left(a_{83} p_{7}+d_{i i}^{2}\right)\right.}{\Lambda_{i 8}^{\prime}} \\
& +\frac{\left.a_{45} r_{2} d_{i i}\left(a_{2}\right\lrcorner f_{1}+d_{i i}\right)}{\Delta_{i 2}^{\prime}}\left(f_{i 8}+d_{i i} f_{i 4}\right)+f_{i 4} \\
& +\frac{\cos _{i i}\left[a_{41} \dot{d}_{i i}\left(r_{1}-r_{3}\right)+\left(a_{33} p_{1}+d_{i i}^{\prime}\right)\right]}{\Delta_{i 8}^{\prime}}\left(f_{i 6}+d_{i i} f_{i 6}\right)  \tag{3-17}\\
& t_{i 6}=\frac{a_{45} d_{i i}\left[a_{23} p_{1}\left(f_{i 1}+d_{i i} f_{i 3}\right)+\left(a_{23} p_{1}+d_{i i}\right)\left(f_{i 3}+d_{i i} f_{i 4}\right)\right]}{\Delta_{i 3}^{\prime}} \\
& +\frac{d_{i i}\left\{a_{63} r_{2} d_{i i}\left[\left(a_{23}+a_{41}\right) p_{1}+d_{i i}^{2}\right]\right.}{\Delta_{i 3}^{\prime}} \\
& +\frac{\left.\left[\left(a_{2 a} p_{1}+d_{i i}^{P}\right)\left(a_{45} p_{2}+d_{i i}^{p}\right)+a_{41} p_{1} d_{i i}^{2}\right]\right\}}{\Delta_{i 3}^{\prime}}\left(f_{i 6}+d_{i i} f_{i 6}\right)+f_{i 6} \tag{3-18}
\end{align*}
$$

where:

$$
\begin{align*}
& r_{j}=\frac{c_{j}}{k_{j}} ; \quad j=1,2  \tag{3-8}\\
& p_{j}=1+r_{j} d_{i i} ; \quad j=1,2  \tag{3-19}\\
& d_{i i 1}=a_{33 r_{1}}+d_{i i}  \tag{3-20}\\
& d_{i i 2}=a_{41} r_{1}+a_{46} r_{2}+d_{i i}  \tag{3-21}\\
& d_{i i 3}=a_{63} r_{3}+d_{i i}  \tag{3-22}\\
& \Delta_{i 3}^{\prime}=\left|\begin{array}{ccc}
-\left(a_{28} p_{1}+d_{i i}^{2}\right) & a_{41} p_{1} & 0 \\
a_{23} p_{1} & -\left(a_{41} p_{1}+a_{46} p_{2}+d_{i i}^{2}\right) & a_{63} p_{2} \\
0 & a_{46} p_{2} & -\left(a_{63} p_{2}+d_{i i}^{2}\right)
\end{array}\right| \\
& =-\left(a_{22} p_{1}+d_{i i}^{2}\right)\left(\Delta_{i a}^{\prime}\right)_{1,1}-a_{28} p_{1}\left(\Delta_{i a}^{\prime}\right)_{2,1} \\
& =-\alpha_{i i}\left[\left(a_{38} a_{46}+a_{28} a_{83}+a_{41} a_{88}\right) p_{1} p_{2}\right. \\
& \left.+\left(a_{33} p_{1}+a_{4 i} p_{1}+a_{46} p_{2}+a_{63} p_{3}\right) d_{i i}^{2}+d_{i i}\right] \tag{3-23}
\end{align*}
$$

### 3.8.8 Comparison of T Matrices for Damping at Various Interfaces

The observer synthesis equations for the three-body single-axis model with damping were compared with those for the same model without damping. A general form was developed for these equations that encompassed the synthesis of the elements of the observer I matrix for the following conditions with respect to damping in the model.

1. No damping;
2. Damping only at the interface between bodies 1 and 2;
3. Darroping only at the intarice botwone bodies 2 and 3;

## 4. Dampiag at both intenfaces.

Elimination of damping at interface $j$ of the model corresponde to setting $r_{j}=0$ and $p_{j}=1$ in the equations for guarating the elementes of the $T$ matrix with damping present at both interfaces, equations (3-13) through (3-23). If all denping is removed from the three body model, $r_{j} \rightarrow 0, p_{j} \rightarrow 1, d_{i i 1} \rightarrow d_{i i 2} \rightarrow d_{i i a} \rightarrow d_{i i}$ and $\Delta_{i 8}^{\prime} \rightarrow-d_{i i}^{1}\left[a_{28} a_{45}+a_{28} a_{83}+a_{41} a_{88}+\left(a_{28}+a_{41}+a_{45}+a_{83}\right) d_{i i}^{2}+d_{i i}^{4}\right]=\Delta_{i 8}$.

$$
\begin{align*}
t_{i 2}= & \frac{\left[\left(\alpha_{4}+d_{i i}^{2}\right)\left(a_{63}+d_{i i}^{2}\right)+\alpha_{45} d_{i i}^{2}\right]}{\Delta_{i 2}}\left(f_{i 1}+d_{i i} f_{i 2}\right) \\
& +\frac{a_{41}\left[\left(a_{63}+d_{i i}^{2}\right)\left(f_{i 3}+d_{i i} f_{i 4}\right)+a_{63}\left(f_{i 8}+d_{i i} f_{i 6}\right)\right]}{\Delta_{i 2}} \quad i=1,2, \ldots, p \tag{3-24}
\end{align*}
$$

$$
\begin{gathered}
t_{i 4}=\frac{a_{23}\left(a_{63}+a_{i i}\right)\left(f_{i 1}+d_{i i} f_{i 2}\right)+\left(a_{23}+d_{i i}^{2}\right)\left(a_{63}+d_{i i}^{2}\right)\left(f_{i 3}+d_{i i} f_{i 4}\right)}{\Delta_{i 3}} \\
+\frac{a_{63}\left(a_{23}+d_{i i}^{2}\right)}{\Delta_{i 3}}\left(f_{i 6}+d_{i i} f_{i 6}\right)
\end{gathered}
$$

$$
\begin{aligned}
t_{i 6}= & \frac{a_{23} a_{45}\left(f_{i 1}+d_{i i} f_{i 3}\right)+a_{45}\left(a_{23}+d_{i i}^{2}\right)\left(f_{i 3}+d_{i i} f_{i 4}\right)}{\Delta_{i 3}} \\
& +\frac{\left[a_{23} a_{46}+\left(a_{28}+a_{41}+a_{46}\right) d_{i i}^{2}+d_{i 1}^{4}\right]}{\Delta_{i 8}}\left(f_{i 5}+d_{i i} f_{i 6}\right)
\end{aligned}
$$

$$
t_{i 1}=\frac{d_{i i}\left[\left(a_{4 i}+d_{i i}^{2}\right)\left(a_{63}+d_{i i}^{2}\right)+a_{46} d_{i i}^{2}\right]}{\Delta_{i 2}}\left(f_{i 1}+d_{i i} f_{i 2}\right)+f_{i 2}
$$

$$
+\frac{a_{41} d_{i i}\left[\left(a_{63}+d_{i i}^{2}\right)\left(f_{i a}+d_{i i} f_{i 4}\right)+a_{63}\left(f_{i B}+d_{i i} f_{i 6}\right)\right]}{\Delta_{i 3}}
$$

$$
t_{i 8}=\frac{a_{23} d_{i i}\left(a_{63}+d_{i i}^{2}\right)}{\Delta_{i 8}}\left(f_{i 1}+d_{i i} f_{i 2}\right)+\frac{d_{i i}\left(a_{23}+d_{i j}^{2}\right)\left(a_{83}+d_{i i}^{2}\right)}{\Delta_{i 3}}\left(f_{i 3}+d_{i i} f_{i 4}\right)+f_{i 4}
$$

$$
\begin{equation*}
+\frac{a_{63} d_{i i}\left(a_{23}+d_{i i}^{P}\right)}{\Delta_{i 3}}\left(f_{i 3}+d_{i i} f_{i 6}\right) \tag{3-28}
\end{equation*}
$$

$$
t_{i 6}=\frac{a_{48} d_{i i}\left[a_{33}\left(f_{i 1}+d_{i i} f_{i 2}\right)+\left(a_{33}+d_{i i}^{?}\right)\left(f_{i 8}+d_{i i} f_{i 4}\right)\right]}{\Delta_{i 8}}
$$

$$
\begin{equation*}
+\frac{d_{i i}\left[\left(a_{23}+d_{i i}\right)\left(a_{45}+d_{i i}^{2}\right)+a_{41} d_{i i}^{2}\right]}{\Delta_{i 3}}\left(f_{i 6}+d_{i i} f_{i 6}\right)^{\prime}+f_{i B} \tag{3-29}
\end{equation*}
$$

The elements, $t_{i 1}$ through $t_{i 6}(i=1, \ldots, p)$, of the $T$ matrix of the linear observer of order $p$ corresponding to this model with one or more inaccessible states were found to be affected by the addition of damping at the interface between bodies 1 and 2 as follows.

1. The scalars, $a_{23}$ and $a_{41}$, were modified to $a_{23} p_{1}$ and $a_{41} p_{1}$, respectively, in the equations for generating $t_{i 2}, t_{i 4}, t_{i 6}$ and $t_{i 6}$ and in the denominators of the equations for generating $t_{i 1}$ and $t_{i 3}$ where $p_{j}$ was defined in equation (3-19).
2. In the yamerator of the equation for gromatiog $t_{i 1}$, (3-16), the followias changen occurred.
 equation (3-8).
b. The scalar, $a_{41}$, in the coeflicient of $f_{i 1}$ was changed to $a_{41} p_{1}$.
3. In the mamerator of the equation for generating $t_{i s},(3-17)$, the following chnnges occorred.
4. The term, $a_{38} a_{45}{ }_{1} d_{i i}$, wae sabtracted from the coefficient of $\left(f_{i 1}+d_{i i} f_{i 3}\right)$.
b. The term, $a_{41} r_{1} d_{i i}^{f}\left(a_{68}+d_{i i}^{f}\right)$, was added to the coefficient of $f_{i 3}$.
c. Esch $a_{28}$ in the coefficients of $f_{i 8}, f_{i 6}$ and $f_{i 6}$ was mdified to $a_{23} p_{1}$.
d. The term, $a_{11} a_{c g r} d_{i i}$, wan added to the coeficienst of ( $f_{i s}+d_{i i} f_{i 6}$ ).

Addition of daxnping at the interface between bodies 2 and 3 had the following effects.

1. The scalars, $a_{46}$ and $a_{62}$, were modified to $a_{45} p_{2}$ and $a_{63} p_{2}$, respectively, in the equations for generating $t_{i 1}, t_{i 3}, t_{i 4}$ and $t_{i 6}$ and the denominators of the equations for generating $t_{i 3}$ and $t_{i 5}$ where $p_{j}$ was defined in equation (3-19).
2. In the nomerator of the equation for generating $t_{i 3},(3-17)$, the following changes occrired.
a. The term, $a_{33} a_{45} r_{2} d_{i i}$, was added to the coefficient of $\left(f_{i 1}+d_{i i} f_{i 2}\right)$ where $r_{j}$ is defined in equation (3-8).
b. Each $\sigma_{63}$ in the coefficients of $f_{i 1}, f_{i 2}$ and $f_{i a}$ was modified to $a_{63} p_{2}$.
c. The term, $a_{45} f_{2} d_{i i}^{P}\left(a_{28}+d_{i i}^{R}\right)$, was added to the coefficient of $f_{i 3}$.
d. The term, $a_{41} a_{63} r_{3} d_{i i}^{2}$, was subtracted from the coefficient of ( $f_{i 5}+d_{i i} f_{i 6}$ ).
3. In the numerator of the equation for $t_{i B},(3-18)$, the following changes occurred
4. The term, $a_{68} r_{2} d_{i i}^{2}\left(a_{2 a}+a_{41}+d_{i i}^{2}\right)$, was added to the coefficient of $f_{i 6}$.
b. The scalar, $a_{45}$, in the coefficient of fis was changed to $a_{46} p_{2}$.

Addition of damping at both the interface between bodies 1 and 2 and the interface between bodies 2 and 3 had the following effects.

1. The scalars, $a_{23}, a_{41}, a_{45}$ and $a_{63}$, were modified to $a_{23} p_{1}, a_{41} p_{1}, a_{45} p_{2}$ and $a_{63} F_{2}$ respectively, in the equations for generating $t_{i 2}, t_{i 4}$ and $t_{i 8}$ and in the denominators of the equations for generating $t_{i 1}$, $t_{i 3}$ and $t_{i b}$ where $p_{j}$ are defined in equation (3-19).
2. In the nomerator of the equation for generating $t_{i 1},(3-16)$, the following changes occurred.
3. The term, $a_{22} r_{1} d_{i i}^{P}\left[\left(a_{45}+a_{63}\right) p_{2}+d_{i i}^{2}\right]$, was added to the coefficient of $f_{i 1}$, where $r_{j}$ is defined in equation (3-8) and $p_{j}$ is defined in equation (3-19).
b. The remaining sealars, $a_{41}, a_{45}$ and $a_{68}$, were modified to $a_{41} p_{1}, a_{45} p_{2}$ and $a_{63} p_{7}$, respectively, with the exception of the $a_{41}$ common to the coefficients of $f_{i 4}$ through fis.
4. In the numerator of the equation for gene-ting $t_{i s}$, ( $3-17$ ), the following changes occurred.
a. The term, $a_{23} a_{48} d_{i j}\left(r_{2}-r_{1}\right)$, was added to the coefficient of $\left(f_{i 1}+d_{i i} f_{i 2}\right)$.
b. The terms, $a_{41} r_{1} d_{i i}^{2}\left(a_{63} p_{2}+d_{i j}^{2}\right)$ and $a_{46} r_{2} d_{i i}^{2}\left(c_{23} p_{1}+d_{i i}^{p}\right)$, were added to the coefficient of $f_{i a}$.
c. The term, $a_{41} a_{68} d_{i i}^{2}\left(r_{1}-r_{2}\right)$, was added to the coefficient of $\left(f_{i 6}+d_{i i} f_{i 6}\right)$.
d. The remaining sealars, $a_{32}$ and $a_{83}$, were modified to $a_{23} p_{1}$ and $a_{83} p_{3}$, respectively.
5. In the namerator of the equation for generating $t_{i 6},(3-18)$, the following changes occurred.
a. The term, $a_{63} r_{2} d_{i j}\left[\left(a_{23}+a_{41}\right) p_{1}+d_{i i}^{2}\right]$, was added to the coefficient of $f_{i 5}$.
b. The remaining scalars, $a_{23}, a_{41}, a_{45}$ and $a_{83}$ were modified tc $a_{23} p_{1}, a_{41} p_{1}, a_{46} p_{2}$ and $a_{63} p_{2}$ respectively with the exception of the $a_{46}$ common to the coefficients of $f_{i 1}, f_{i 2}, f_{i 3}$ and $f_{i 4}$.
8.41 mensoduction

Insecomibility of a sealar state vriable in equstion sot (3-1), (3-2) is reflected by a corresponding mall columan in the oboervation matrix, $C$ and, as implied by equation (2-11), in the $\mathbf{F}$ matrix for the generation of reducod order obearvers for the thro-body model. The number of inaccessible sealar states can be 1,2 , 3, 4 or 5.
3.12 Firot Order Observese ( $p=1$ )

A fint order obeorver is required when any one of the six scalar state variables of the three body model is inaceuable. The frost ordar form of the linear oboarver equation is:

$$
\begin{equation*}
\dot{s}=d s+I_{u}+G y \tag{3-30}
\end{equation*}
$$

The $\boldsymbol{F}$ and $\mathbf{T}$ matrices associated with a first order observer for the three body model then reduce to the following row forma.

$$
\begin{align*}
& T=\left[\begin{array}{llllll}
f_{1} & f_{2} & f_{3} & f_{4} & f_{6} & f_{6}
\end{array}\right]^{T}  \tag{3-31}\\
& T=\left[\begin{array}{llllll}
t_{1} & t_{2} & t_{3} & t_{4} & t_{5} & t_{6}
\end{array}\right]^{T} \tag{3-32}
\end{align*}
$$

The observer sypthesis equations are then given by equation (3-13) through equation (3-23) with $i=1$.
Since a first order observer corresponds to one of the scalar state variables being inaccessible, one of the $f_{i}(i=1,2,3,4,5,6)=0$.
Example
Sappose that the scalar state representing the angular racte of body $3, x_{6}$, is inaccessible. Then $f_{6}=0$ and the observer synthesis equations reduce to the following forms.

$$
\begin{align*}
& t_{2}=\frac{\left[\left(a_{41} p_{1}+d^{2}\right)\left(a_{63} p_{2}+d^{2}\right)+a_{45} p_{2} d^{2}\right]\left(f_{1}+d f_{2}\right)}{\Delta_{8}^{\prime}} \\
& +\frac{a_{41} p_{1}\left[\left(a_{63} p_{2}+d^{2}\right)\left(f_{3}+d f_{4}\right)+a_{63} p_{2} f_{5}\right.}{\Delta_{3}^{\prime}}  \tag{3-33}\\
& t_{4}=\frac{a_{23} p_{1}\left(a_{63} p_{3}+d^{2}\right)\left(f_{1}+d f_{3}\right)+\left(a_{33} p_{1}+d^{2}\right)\left(a_{63} p_{2}+d^{2}\right)\left(f_{3}+d f_{4}\right)}{\Delta_{8}^{\prime}} \\
& +\frac{a_{83} p_{2}\left(a_{23} p_{1}+d^{2}\right) f_{5}}{\Delta_{8}^{\prime}}  \tag{3-34}\\
& t_{8}=\frac{a_{38} a_{48} p_{1} p_{2}\left(f_{1}+d f_{2}\right)+\left(a_{2} p_{1}+d^{2}\right) a_{48} p_{2}\left(f_{8}+d f_{4}\right)}{\Delta_{8}^{\prime}} \\
& \left.+\frac{\left[\left(a_{28} p_{1}+d^{2}\right)\left(a_{45} p_{2}+d^{2}\right)+a_{41} p_{1} d^{2}\right]}{\Delta_{8}^{\prime}}\right)  \tag{3-35}\\
& t_{1}=\frac{d\left\{\left[\left(a_{41} p_{1}+d^{2}\right)\left(a_{63} p_{3}+d^{2}\right)+a_{48} p_{2} d^{2}\right]+a_{23} r_{1} d\left[\left(a_{48}+a_{38}\right) p_{2}+d^{2}\right]\right\}}{\Delta_{i 8}^{\prime}}\left(f_{1}+d f_{2}\right)+f_{2} \\
& +\frac{a_{41} d\left[\left(a_{83} p_{2}+d^{2}\right)\left(f_{3}+d f_{4}\right)+a_{83} p_{2} f_{6}\right]}{\Delta_{8}^{\prime}} \tag{3-36}
\end{align*}
$$

$$
\begin{align*}
& +\frac{d\left[\left(a_{9 a p_{1}}+d^{2}\right)\left(a_{89} p_{2}+d^{2}\right)+a_{41} r_{1} d\left(a_{68} p_{2}+d^{2}\right)+a_{46} r_{2} d\left(a_{28} p_{1}+d^{2}\right)\right]}{\Delta_{3}^{\prime}}\left(f_{3}+d f_{4}\right)+f_{4} \\
& +\frac{\operatorname{ang} d\left[a_{41} d\left(r_{1}-r_{7}\right)+\left(a_{38} p_{1}+d^{2}\right)\right]}{\Delta_{8}^{\prime}} f_{5}  \tag{3-37}\\
& t_{5}=\frac{a_{4 s} d\left[a_{33} p_{1}\left(f_{1}+d f_{2}\right)+\left(a_{3 a} p_{1}+d^{2}\right)\left(f_{2}+d f_{4}\right)\right.}{\Delta_{1}^{\prime}} \\
& +\frac{d\left\{a_{69} r_{2} d\left[\left(a_{28}+a_{41}\right) p_{1}+d^{2}\right]+\left[\left(a_{23} p_{1}+d^{2}\right)\left(a_{45} p_{2}+d^{2}\right)+a_{41} p_{1} d^{2}\right]\right\}}{\Delta_{3}^{\prime}} \tag{3-38}
\end{align*}
$$

where:

$$
\begin{align*}
r_{j}= & \frac{c_{j}}{h_{j}}  \tag{3-8}\\
p_{j} & =1+r_{j} d  \tag{3-39}\\
\Delta_{8}^{\prime} & =-d^{2}\left[\left(a_{28} a_{48}+a_{28} a_{68}+a_{41} a_{83}\right) p_{1} p_{2}\right. \\
& \left.\quad+\left(a_{23} p_{1}+a_{41} p_{1}+a_{45} p_{2}+a_{63} p_{2}\right) d^{2}+d^{4}\right] \tag{3-40}
\end{align*}
$$

From equation (2-11) the synthesized scalar state, $\hat{x}_{6}$, is expressed in terms of the observer state variable, $\boldsymbol{z}$, and the accessible scalar state variables as follows,

$$
\begin{equation*}
\hat{z}_{6}=\frac{1}{t_{8}}\left[x-\sum_{i=1}^{5} t_{i} x_{i}\right] \tag{3-41}
\end{equation*}
$$

In this case, it is assumed that:

$$
\mathbf{C}=\left[\begin{array}{ll|c} 
& & 0  \tag{3-42}\\
& \mathbf{I}_{5} & \\
& & \vdots \\
& &
\end{array}\right]
$$

where $\mathbf{I}_{5}=\mathbf{5} \times 5$ identity matrix.
From $\mathbf{F}=\mathbf{G C}$,

$$
G=\left[\begin{array}{lllll}
f_{1} & f_{2} & f_{3} & f_{4} & f_{6} \tag{3-43}
\end{array}\right]
$$

From $\mathbf{E}=\mathbf{T B}$,

$$
\begin{align*}
& \mathbf{E}=\left[\begin{array}{lll}
t_{2} & t_{4} & t_{5}
\end{array}\right] \text { for } r=3 \text { (control torques applied to all three bodies) }  \tag{3-44}\\
& \mathbf{E}=\left[\begin{array}{lll}
t_{2} & t_{4} & 0
\end{array}\right] \text { for control applied to bodies } 1 \text { and } 2  \tag{3-45}\\
& \mathbf{E}=\left[\begin{array}{lll}
t_{2} & 0 & 0
\end{array}\right] \text { for control applied to body } 1  \tag{3-46}\\
& \mathbf{E}=\left[\begin{array}{lll}
t_{2} & 0 & t_{6}
\end{array}\right] \text { for control appliod to bodies } 1 \text { and } 3 \tag{3-47}
\end{align*}
$$

## 

In the cacos in which an inesumediate mamber of the nix scalar states of the throcbody singlo-axis model is inacceseible the minimmm order of the rodecod otate linear obowrver required to reconstruct these inacecomible states is given by $p$. In each cace the momber of nuil columon in the meanorement or obsorvation matrix, $C$, and the $\Gamma$ matrix also is eqaal to $p$. The genoral forms of the $E, T$ and $T$ mastrices are given in equation $(3-10),(3-11)$ and $(3-12)$ for $p=2,3$ or 4 where $p$ represente the number of inaccossible state variablee of the model.

## Example

Sappose the scalar staten, 25 and 36 , correoponding to the aggular position and rate, respectively, of body 3, are inscesaible. Thes $f_{i s}=f_{i s}=0$ for $i=1,2$ and the oboerver syntherin equatione reduce to the form of equations (3-13) through (3-23) with $f_{i 6}=f_{i 6}=0$. From equation (2-11) the synthesised scalar states, $\hat{z}_{6}$ and $\hat{z}_{5}$ are expresed in termes of the observer variables, $x_{1}$ and $z_{2}$, and the accessible state variables as tollow.

$$
\begin{align*}
& \hat{A}_{6}=\frac{\left(\Delta_{2}\right)_{1,1}\left(z_{1}-\sum_{j=1}^{4} t_{1 j} x_{j}\right)-\left(\Delta_{2}\right)_{2,1}\left(z_{2}-\sum_{j=1}^{4} t_{2 j} x_{j}\right)}{\Delta_{2}}  \tag{3-48}\\
& \hat{z}_{6}=\frac{-\left(\Delta_{2}\right)_{1,2}\left(x_{1}-\sum_{j=1}^{4} t_{1} z_{j}\right)+\left(\Delta_{2}\right)_{2,2}\left(x_{2}-\sum_{j=1}^{4} t_{2 j} x_{j}\right)}{\Delta_{3}} \tag{3-49}
\end{align*}
$$

where,

$$
\Delta_{2}=\left|\begin{array}{ll}
t_{15} & t_{16}  \tag{3-50}\\
t_{38} & t_{28}
\end{array}\right|=t_{15} t_{26}-t_{16} t_{25} \neq 0
$$

and $\left(\Delta_{2}\right)_{i, j}=\Delta_{\mathbf{2}}$ without the elements of the $\mathrm{i}^{\text {th }}$ row and $\mathrm{j}^{\text {th }}$ column.
For as and $n_{s}$ insecessibis, it is assumed that:

$$
\mathbf{C}=\left[\begin{array}{llll} 
& 1 & 0 & 0  \tag{3-51}\\
\mathbf{L} & & \vdots & \vdots \\
& & 0 & 0
\end{array}\right]
$$

where $I_{4}=4 \times 4$ identity matrix.
From $\mathbf{F}=\mathbf{G C}$,

$$
G=\left[\begin{array}{llll}
f_{11} & f_{12} & f_{13} & f_{14}  \tag{3-52}\\
f_{21} & f_{22} & f_{23} & f_{24}
\end{array}\right]
$$

From $\mathbf{E}=\mathbf{T B}$,
$E=\left[\begin{array}{lll}t_{12} & t_{14} & t_{16} \\ t_{22} & t_{24} & t_{26}\end{array}\right] \quad$ for $r=3$ (control torques applied to all three bodies)
$\boldsymbol{I}=\left[\begin{array}{lll}t_{13} & t_{14} & 0 \\ t_{22} & t_{24} & 0\end{array}\right] \quad$ for control applied to bodies 1 and 2.
$\mathbf{E}=\left[\begin{array}{lll}t_{12} & 0 & t_{16} \\ t_{22} & 0 & t_{26}\end{array}\right] \quad$ for control applied to bodies 1 and 3.
$E=\left[\begin{array}{lll}t_{12} & 0 & 0 \\ t_{12} & 0 & 0\end{array}\right] \quad$ for comerel rutricted to body 1.

## S.cet Fith Oxiter Oberevers ( $p=6$ )

An oberver of at least order five is requirod when any five of the six scalar state variables of the three body modele are incecemible. The observer synthesis equations are given in equations (3-13) through (3-23) with $i=1,2, \ldots, \delta$. Since a ethth order obearver corresponde to five of the six scalar states being inaccessible, $f_{i j}=0$ for five of the sir viluee of the subecript, $j$.

## Example

Suppoee that the acalar states, $x_{2}, x_{3}, x_{4}, x_{5}$ and $x_{6}$, roprosenting the aggular rate of body 1 and the asgular dirplacemente and rates of bodies 2 and 3 are inscessrible. Then $f_{i 2}=f_{i 8}=f_{i 4}=f_{i 6}=f_{i 6}=0$ for $i=1,2, \ldots, 5$ and the obocrver synthesis equations reduce to the form of equations (3-13) through (3-23) with only $f_{i 1} \neq 0$. The synthesized scalar state variables, $\hat{x}_{2}, \hat{x}_{8}, \hat{y}_{4}, \hat{x}_{6}$ and $\hat{\boldsymbol{p}}_{6}$ are expressed in terms of the observer scalar variables, $z_{1}, z_{2}, \ldots, z_{5}$, and the accessible state variables, using equation (2-11) as follows:

$$
\begin{align*}
\hat{t}_{k+1} & =\frac{\sum_{i=1}^{5}(-1)^{i+1}\left(\Delta_{6}\right)_{i, k}\left(x_{i}-t_{i 1} x_{1}\right)}{\Delta_{5}} \quad k=1,2, \ldots, 5  \tag{3-57}\\
\Delta_{5} & =\left|\begin{array}{ccccc}
t_{12} & t_{13} & t_{14} & t_{16} & t_{16} \\
t_{23} & \vdots & \vdots & \vdots & t_{28} \\
t_{32} & \vdots & \vdots & \vdots & t_{38} \\
t_{49} & \vdots & \vdots & \vdots & t_{46} \\
t_{62} & t_{58} & t_{54} & t_{65} & t_{56}
\end{array}\right| \\
& =t_{12}\left(\Delta_{6}\right)_{1,1}-t_{22}\left(\Delta_{5}\right)_{2,1}+t_{32}\left(\Delta_{5}\right)_{3,1}-t_{42}\left(\Delta_{6}\right)_{4,1}+t_{63}\left(\Delta_{6}\right)_{5,1} \tag{3-58}
\end{align*}
$$

where $\left(\Delta_{6}\right)_{i j}=\Delta_{6}$ withont the elements of the $i^{\text {th }}$ row and $j^{\text {th }}$ colvmn.
For only $x_{1}$ accessible, it is assumed that:

$$
C=\left[\begin{array}{llllll}
1 & 0 & 0 & 0 & 0 & 0 \tag{3-59}
\end{array}\right]
$$

From $F=\mathbf{G C}$,

$$
\mathbf{G}=\left[\begin{array}{l}
f_{11}  \tag{3-80}\\
f_{21} \\
f_{31} \\
f_{41} \\
f_{61}
\end{array}\right]
$$

From $E=\mathbf{T B}$,

$$
E=\left[\begin{array}{ccc}
t_{12} & t_{14} & t_{15}  \tag{3-81}\\
\vdots & \vdots & \vdots \\
t_{53} & t_{54} & t_{56}
\end{array}\right] \text { for } r=3 \text { (conkrol torques applied to all three bodies). }
$$

### 3.5 RRFPRENCES

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3-2 Luenberger, D.G., "Observers for Multivariable Systems", IEEE Transactions on Automatic Control, Vol. AC-11, No. 2, April 1966, pp. 190-197.

3-3 Luenberger, D.G., "An Introduction to Observers", IEEE Transactions on Automatic Control, Vol. AC16, No. 6, Deceminer 1971, pp. 596-602.
3-4 Sage, A.P., Optimum Systems Control. Englewood Cliff, N.J.: Prentice-Hall, Inc. 1968, pp. 306-312.

## DEVELOPMENT OF TEE TOUR-BODY SINGLT-AXIS MODEL AND IFS RBDUCED ETAME LINEAR OBSEEVER

## 41 ORIGNJAL DAMPED MODEL

The rotational dyanmics of the foer-body singlo-axis model of a fexible spacecraft with damping thown in Fis. 4-1 may be represented by the following set of equations.

$$
\begin{align*}
& I_{1} \dot{\theta}_{1}=-c_{1}\left(\dot{\theta}_{1}-\dot{\theta}_{2}\right)-h_{1}\left(\theta_{1}-\theta_{3}\right)+q_{1}  \tag{4-1}\\
& I_{2} \dot{\theta}_{2}=c_{1}\left(\dot{\theta}_{1}-\dot{\theta}_{2}\right)+h_{1}\left(\theta_{1}-\theta_{2}\right)+c_{2}\left(\dot{\theta}_{3}-\dot{\theta}_{2}\right)+h_{7}\left(\theta_{3}-\theta_{2}\right)+q_{2}  \tag{4-2}\\
& I_{8} \dot{\theta}_{2}=c_{2}\left(\dot{\theta}_{2}-\dot{\theta}_{3}\right)+h_{2}\left(\theta_{2}-\theta_{3}\right)+c_{8}\left(\dot{\theta}_{4}-\dot{\theta}_{3}\right)+h_{6}\left(\theta_{4}-\theta_{3}\right)+q_{3}  \tag{4-3}\\
& I_{4} \dot{\theta}_{4}=-c_{3}\left(\dot{\theta}_{4}-\dot{\theta}_{8}\right)-h_{8}\left(\theta_{4}-\theta_{3}\right)+q_{4} \tag{4-4}
\end{align*}
$$

where:

$$
I_{i}=\text { rotational inertia of body } i ; \quad i=1,2,3,4
$$

$\theta_{i}=$ angular ärplace nent of body $i$
$\dot{\theta}_{i}=$ angular rate of body $i$
$\Phi_{i}=$ torque applied to body $i$
$k_{j}=$ rotational spring coefficient at interface $j ; \quad j=1,2,3$
$c_{j}=$ rotational damping coefficiert at interface $j$

### 4.2 STATE VARLABLE MODEL

The state variable form of the four-body single-axis model of a flexible spacecraft depicted in Fig. 4-1 was written in the following form.

$$
\begin{align*}
& \dot{x}=A x+B u  \tag{4-5}\\
& x_{A}=C x
\end{align*}
$$

where:
$x=\left[\begin{array}{lll}x_{1} & \cdots & x_{8}\end{array}\right]^{T}=\left[\begin{array}{llllllll}\theta_{1} & \dot{\theta}_{1} & \theta_{2} & \dot{\theta}_{2} & \theta_{3} & \dot{\theta}_{3} & \theta_{4} & \dot{\theta}_{4}\end{array}\right]^{T}=\left[\begin{array}{lll}x_{A}^{T} & x_{1}^{T}\end{array}\right]^{T}=$ state vector
$x_{A}=p$ vector of accessible scalar states
$x_{1}=m$ vector of inaccessible scalar states
$u=\left[\begin{array}{lll}u_{1} & \cdots & u_{r}\end{array}\right]^{T}=\left[\begin{array}{lll}\frac{q_{1}}{I_{1}} & \cdots & \frac{q_{r}}{I_{r}}\end{array}\right]^{T} \quad(r=1,2,3$ or 4$)$
$\mathbf{C}=m \times 8$ mescurement or observation matrix
Partitioning of this model by rigid body yields the following forms for its coefficient matrices:

$$
\mathbf{A}=\left[\begin{array}{cccccccc}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0  \tag{4-7}\\
-a_{23} & -a_{23} r_{1} & a_{28} & a_{28} r_{1} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
a_{41} & a_{41} r_{1} & a_{42} & a_{44} & a_{48} & a_{46} r_{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & a_{88} & a_{88} r_{2} & a_{88} & a_{86} & a_{87} & a_{373} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & a_{85} & a_{86} r_{3} & -a_{85} & -a_{86} r_{3}
\end{array}\right]
$$

ORIGEM
OF POOK QURE:


FIGURE 4-1

$$
\begin{align*}
& B=\left[\begin{array}{llll}
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & i
\end{array}\right] f o r r=4  \tag{4-8}\\
& a_{22}=\frac{h_{1}}{I_{1}} \\
& a_{41}=\frac{h_{1}}{l_{2}}, \quad a_{48}=\frac{h_{2}}{l_{2}}, \quad a_{42}=-\left(a_{41}+a_{48}\right), \quad a_{44}=-\left(a_{41} r_{1}+a_{48} r_{2}\right) \\
& a_{69}=\frac{k_{2}}{I_{8}}, \quad a_{67}=-\left(a_{69}+a_{67}\right), \quad a_{66}=-\left(a_{68}+a_{67}\right), \quad a_{68}=-\left(a_{689}+a_{67} r_{3}\right)  \tag{4-9}\\
& a_{68}=\frac{h_{7}}{L_{4}} \\
& r_{j}=\frac{c_{i}}{h_{j}} \quad j=1,2,3 \tag{4-10}
\end{align*}
$$

The corresponding block diagram appears in Fig. 2-2.

## 4s REDUCCD STATE LINEAR OBSERVERS

## 4s.1 Introduetion

The minimam order of a reduced state linear observer required to reconstruct the $8-m$ inaccessible scalar state varisbles of the four body single axis model of a fexible spacecraft represented by equationo ( $4-5$ ) through ( $4-10$ ) is $p=8-m$ where $m=1,2,3,4,5,6$ or 7 . All of the reduced state linear observers for this four body model may be written in the form of equations (2-10) and (2-11) ander the assumption that the observer coefficient matrix, $D$, is diagonal and of dimensions $p \times p$. The corresponding observer weighting matrix is of the following form.

$$
\mathbf{T}=\left[\begin{array}{ccc}
t_{11} & \cdots & t_{18}  \tag{4-11}\\
\vdots & & \vdots \\
t_{p, 1} & \cdots & t_{p, 8}
\end{array}\right]
$$

From equations (2-12), (4-8) and (4-11).

$$
\begin{align*}
& \mathbf{L}=\left[\begin{array}{cccc}
t_{12} & t_{14} & t_{16} & t_{18} \\
\vdots & \vdots & \vdots & \vdots \\
t_{p, 2} & t_{p, 4} & p_{p, 6} & p_{p, 8}
\end{array}\right]  \tag{4-12}\\
& \mathbf{F}=\left[\begin{array}{ccc}
f_{11} & \ldots & f_{18} \\
\vdots & & \vdots \\
f_{p, 1} & \ldots & f_{p, 8}
\end{array}\right] \tag{4-13}
\end{align*}
$$

## 4.as Obearver Bymbede Equatione

Prom Laeabexy $(4-1),(4-2),(4-8)$ sed' Suge (4-4) the equatione for gyutherising the reduced state Bienar obenvers for the fourbody singlo-acis modul repromented by equations (4-5) throngh (4-10) are given by cquations $(2-18)$ and (2-14). With cooflicant matrices of the formo listed in 4.3 .1 this set of obeorver aymthais equatione roteces to the followies.

$$
\begin{aligned}
t_{i 4}= & -\frac{\left(\Delta_{i 4}^{\prime}\right)_{1,2}\left(f_{i 1}+d_{i i} f_{i 2}\right)-\left(\Delta_{i 4}^{\prime}\right)_{2,2}\left(f_{i 3}+d_{i i} f_{i 4}\right)+\left(\Delta_{i 4}^{\prime}\right)_{2,2}\left(f_{i 8}+d_{i i} f_{i 6}\right)}{\Delta_{i 4}^{\prime}} \\
& +\frac{\left(\Delta_{i 4}^{\prime}\right)_{1,2}\left(f_{i 7}+d_{i i} f_{i 8}\right)}{\Delta_{i 4}^{\prime}} \\
= & -\frac{\left[\left(a_{82 p_{2}}+d_{i i}^{p}\right)\left(a_{85 p_{3}}+d_{i i}^{2}\right)+a_{\left.67 p_{3} f_{i i}^{2}\right]\left[a_{2 a p_{1}}\left(f_{i 1}+d_{i i} f_{i 2}\right)+\left(a_{23}+d_{i i}^{2}\right)\left(f_{i 3}+d_{i 1} f_{i 4}\right)\right]}^{\Delta_{i 4}^{\prime}}\right.}{}
\end{aligned}
$$

$$
t_{i 6}=\frac{\left(\Delta_{i 4}^{\prime}\right)_{1,3}\left(f_{i 1}+d_{i i} f_{i 2}\right)-\left(\Delta_{i 4}^{\prime}\right)_{2,3}\left(f_{i 3}+d_{i i} f_{i 4}\right)+\left(\Delta_{i 4}^{\prime}\right)_{2,3}\left(f_{i 6}+d_{i i} f_{i 6}\right)}{\Delta_{i 4}^{\prime}}
$$

$$
-\frac{\left(\Delta_{i 4}^{\prime}\right)_{4, a}\left(f_{i 7}+d_{i i} f_{i B}\right)}{\Delta_{i 4}^{\prime}}
$$

$$
=-\frac{a_{46 p_{3}}\left(a_{65} p_{2}+d_{i i}^{2}\right)\left[a_{23 p_{1}}\left(f_{i 1}+d_{i i} f_{i 3}\right)+\left(a_{22 p_{1}}+d_{i i}^{2}\right)\left(f_{i 3}+d_{i i} f_{i 4}\right)\right]}{\Delta_{i 4}^{\prime}}
$$

$$
-\frac{\left[\left(a_{39} p_{1}+d_{i i}\right)\left(a_{46} p_{2}+d_{i i}^{2}\right)+a_{41} p_{1} d_{i i}^{2}\right]\left[\left(a_{86} p_{7}+d_{i i}^{2}\right)\left(f_{i 8}+d_{i i} f_{i 6}\right)\right.}{\Delta_{i 4}^{\prime}}
$$

$$
\begin{equation*}
+\frac{\left.d_{65 p_{3}}\left(f_{i 7}+d_{i i} f_{i 8}\right)\right]}{\Delta_{i 4}^{\prime}} \tag{4-16}
\end{equation*}
$$

$$
\begin{align*}
& t_{i 2}=\frac{\left(\Delta_{i 4}^{\prime}\right)_{1,1}\left(f_{i 2}+d_{i i} f_{i 2}\right)-\left(\Delta_{i 4}^{\prime}\right)_{2,1}\left(f_{i a}+d_{i i} f_{i 4}\right)+\left(\Delta_{i 4}^{\prime}\right)_{8,1}\left(f_{i 8}+d_{i i} f_{i 4}\right)}{\Delta_{i 4}^{\prime}} \\
& -\frac{\left(\Delta_{i 4}^{\prime}\right)_{i, 2}\left(f_{17}+d_{i 1} f_{i B}\right)}{\Delta_{i 4}^{\prime}} \\
& =\frac{\left\{a_{45} \sigma_{88} p_{2}^{2}\left(a_{86} p_{8}+\alpha_{i i}\right)\right.}{\Delta_{i 4}^{\prime}} \\
& -\frac{\left.\left(a_{41} p_{1}+a_{48} p_{2}+d_{i i}^{P}\right)\left[\left(a_{63} p_{2}+d_{i i}^{P}\right)\left(a_{88} p_{2}+d_{i i}^{P}\right)+a_{67} p_{3} d_{i i}^{2}\right]\right\}\left(f_{i 1}+d_{i i} f_{i 2}\right)}{\Delta_{i 4}^{\prime}} \\
& -\frac{a_{41} p_{1}\left[\left(a_{43} p_{2}+a_{i i}^{a}\right)\left(a_{65 p_{3}}+d_{i i}\right)+a_{67} p_{7} d_{i i}^{2}\right]\left(f_{i 8}+\alpha_{i i} f_{i 4}\right)}{\Delta_{i 4}^{\prime}} \\
& -\frac{a_{41} a_{83} p_{1} p_{2}\left[\left(a_{81} p_{2}+d_{i i}\right)\left(f_{i 8}+d_{i i} f_{i 8}\right)+a_{8 s} p_{2}\left(f_{i 7}+d_{i i} f_{i 8}\right)\right]}{\Delta_{i 4}^{\prime}} \quad i=1,2, \ldots, i F \tag{4-14}
\end{align*}
$$

$$
\begin{aligned}
& \sigma_{i 8}=-\frac{\left(\Delta_{i 4}^{\prime}\right)_{1,1}\left(f_{i 1}+d_{i 1} f_{i 9}\right)-\left(\Delta_{i 4}^{\prime}\right)_{2_{4}}\left(f_{i 6}+d_{i 1} f_{i 4}\right)+\left(\Delta_{i 4}^{\prime}\right)_{2,1}\left(f_{i 6}+d_{i i 3} f_{i 6}\right)}{\Delta_{i 4}^{\prime}}
\end{aligned}
$$

$$
\begin{align*}
& -\frac{a_{17} p_{1}\left[\left(a_{33} p_{1}+d_{i j}\right)\left(a_{4 s} p_{2}+d_{i 1}\right)+a_{41} p_{1} d_{i j}\right]\left(f_{i 3}+d_{i i} f_{i}\right)}{\Delta_{i 4}^{\prime}} \\
& -\frac{\left\{\alpha_{i 1}\left[a_{1} p_{1}\left(a_{41} p_{2}+a_{01} p_{2}+d_{i i}\right)+i_{8} p_{2} d_{i i}\right]\right.}{\Delta_{i 4}^{\prime}} \\
& +\frac{\left.\left(a_{22} p_{1}+d_{i i}^{2}\right)\left(a_{66} p_{2}+d_{i i}^{2}\right)\left(a_{6} 7 p_{9}+d_{i i}^{2}\right)+a_{83} a_{83} p_{1} p_{2} d_{i i}^{2}\right\}\left(f_{i 7}+\Lambda_{i i} f_{i 8}\right)}{\Delta_{i 4}^{i}}  \tag{4-17}\\
& t_{i 1}=-\frac{\left\{[ ( a _ { 2 3 } r _ { 1 } + d _ { i i } ) ( a _ { 4 8 } p _ { 7 } + d _ { i i } ^ { R } ) + a _ { 4 1 } d _ { i i } ] \left[\left(a_{83} p_{2}+d_{i i}^{2}\right)\left(a_{68} p_{3}+d_{i i}^{R}\right)\right.\right.}{\Delta_{i 4}^{\prime}} \\
& +\frac{\left.\left.a_{07} p_{9} a_{i t}^{i}\right]-a_{46} a_{83} p_{3}^{2}\left(a_{65} p_{9}+d_{i i}\right)\right\}\left(f_{i 1}+d_{i i} f_{i 2}\right)}{\Delta_{i 4}^{\prime}}+f_{i 2} \\
& -\frac{a_{41} d_{i i}\left\{\left[\left(a_{\Delta 4} p_{2}+d_{i i}^{2}\right)\left(a_{18} p_{7}+d_{i i}\right)+a_{67} p_{9} d_{i i}^{2}\right]\left(f_{i 3}+d_{i i} f_{i 4}\right)\right.}{\Delta_{i 4}^{\prime}} \\
& +\frac{\left.a_{62} p_{2}\left(a_{85} p_{3}+d_{i 1}^{2}\right)\left(f_{i 8}+d_{i i} f_{i 6}\right)+a_{83} a_{88} p_{2} p_{3}\left(f_{i 7}+d_{i i} f_{i 8}\right)\right\}}{\Delta_{i 4}^{\prime}} \tag{4-18}
\end{align*}
$$

$$
\begin{align*}
& t_{i 8}=\frac{a_{28}\left\{\left[a_{48}\left(r_{1}-r_{2}\right)-d_{i i}\right]\left[\left(a_{39} p_{2}+d_{i i}^{R}\right)\left(a_{68} p_{8}+d_{i i}^{R}\right)+a_{67} p_{9} A_{i i}\right]\right.}{\Delta_{i 4}^{\prime}} \\
& -\frac{\left.a_{46} a_{43} p_{7}\left(a_{66} p_{2}+d_{i i}\right)\left(r_{1}-r_{2}\right)\right\}\left(f_{i 1}+d_{i i} f_{i 2}\right)}{\Delta_{i 4}^{\prime}} \\
& -\frac{\left\{[ ( a _ { 3 } p _ { 1 } + d _ { i i } ^ { R } ) ( a _ { s } r _ { 2 } + d _ { i i } ) + a _ { 1 1 } r _ { 1 } d _ { i i } ^ { R } ] \left[\left(a_{43} p_{2}-d_{i i}^{R}\right)\left(a_{n s} p_{2}+d_{i i}^{R}\right)\right.\right.}{\Delta_{14}^{\prime}} \\
& +\frac{\left.\left.3_{07 p} p_{0} d_{i i}\right]-a_{46} a_{83} p_{3} r_{2}\left(a_{32} p_{1}+d_{i i}\right)\left(a_{66} p_{3}+d_{i i}\right)\right\}\left(f_{i a}+d_{i i} f_{i 4}\right)}{\Delta_{i 4}^{\prime}}+f_{i 4} \\
& +\frac{a_{83} d_{i i}\left[a_{23} p_{1}+a_{41} d_{i i}\left(r_{1}-r_{2}\right)+d_{i i}^{R}\right]\left[\left(a_{66} p_{2}+d_{i i}^{R}\right)\left(f_{i 6}+d_{i i} f_{i 6}\right)+a_{86} p_{8}\left(f_{i 7}+d_{i i} f_{i 8}\right)\right]}{\Delta_{i 4}^{\prime}} \tag{4-19}
\end{align*}
$$

$$
\begin{align*}
& t_{i 8}=-\frac{\alpha_{43} d_{i i}\left[a_{8 s} p_{3}+a_{67} d_{i i}\left(r_{3}-r_{3}\right)+\alpha_{i i}^{2}\right]\left[a_{23} p_{1}\left(f_{i 1}+d_{i i} f_{i 3}\right)+\left(a_{39} p_{1}+d_{i i}^{2}\right)\left(f_{i a}+d_{i i} f_{i 4}\right)\right]}{\Delta_{i 4}^{\prime}} \\
& +\frac{d_{i i}\left\{\left(a_{381}+d_{i i}\right)\left(a_{45} p_{2}+d_{i f}\right)\left(a_{66} p_{8}+a_{67 r_{3} d_{i i}}+d_{i i}\right)\right.}{\Delta_{i 4}^{\prime}} \\
& -\frac{\left.d_{i i}\left[\left(a_{23} p_{1}+a_{41} p_{1}+d_{i i}^{2}\right) a_{83} r_{2}+a_{41} p_{1} d_{i i}\right]\left(a_{65} p_{8}+d_{i i}^{p}\right)+a_{67 r_{3}} d_{i i}\right\}\left(f_{i 6}+d_{i i} f_{i 6}\right)}{\Delta_{i 4}^{\prime}}+f_{i 6} \\
& +\frac{a_{6 s} d_{i i}\left\{d_{i i}\left[a_{38} a_{832}\left(r_{2}-r_{1}\right)+a_{63}\left(\rho_{8}-p_{2}\right)\left(a_{41} p_{1}+d_{i i}^{2}\right)-a_{41} p_{1} d_{i i}\right]\right.}{\Delta_{i 4}^{\prime}} .  \tag{4-20}\\
& t_{i 7}=-\frac{a_{37} d_{i i}\left\{a_{46} p_{2}\left[a_{23} p_{1}\left(f_{i 1}+d_{i i} f_{i 2}\right)+\left(a_{23} p_{1}+d_{i i}^{2}\right)\left(f_{i 3}+d_{i i} f_{i 4}\right)\right]\right.}{\Delta_{i 4}^{\prime}} \\
& +\frac{d_{i i}\left\{\left[\left(a_{87} p_{3}+a_{85} r_{3} d_{i i}+d_{i i}^{2}\right)\left(a_{33} p_{1}+d_{i i}^{2}\right)\left(a_{45} p_{2}+d_{i i}^{2}\right)\right.\right.}{\Delta_{i 4}^{\prime}} \\
& +\frac{\left.a_{41} a_{67} a_{85} p_{1} p_{3} r_{3} d_{i i}\right]+d_{i i}\left[a_{33} a_{63} p_{1} p_{2}+a_{41} a_{63} p_{1} p_{2}\right.}{\Delta_{i 4}^{\prime}} \\
& +\frac{\left.\left.a_{41} a_{6} / p_{1} p_{3}+\left(a_{41} r_{1}+a_{63} p_{2}\right) d_{i i}^{2}\right]\right\}\left(f_{i 7}+d_{i i} f_{i 8}\right)}{\Delta_{i 4}^{\prime}}+f_{i 8} \tag{4-21}
\end{align*}
$$

where:

$$
\begin{align*}
& r_{j}=\frac{c_{j}}{k_{i}}, \quad j=1,2,3  \tag{4-22}\\
& p_{j}=1+r_{j} d_{i i}, \quad i=1,2, \ldots, p \quad j=1,2,3  \tag{4-23}\\
& d_{i i 1}=a_{23} r_{1}+d_{i i}  \tag{4-24}\\
& \begin{array}{l}
d_{i i 2}=a_{41} r_{1}+a_{45} r_{3}+d_{i i} \\
d_{i i 3}=a_{63} r_{2}+a_{67 r_{3}}+d_{i i}
\end{array}  \tag{4-25}\\
& \begin{array}{l}
d_{i i 2}=a_{41} r_{1}+a_{45} r_{3}+d_{i i} \\
d_{i i 3}=a_{63} r_{2}+a_{67 r_{3}}+d_{i i}
\end{array}  \tag{4-26}\\
& d_{i i 4}=a_{85} r_{a}+d_{i i}  \tag{4-27}\\
& \Delta_{i 4}^{\prime}=\left|\begin{array}{cccc}
-\left(a_{23} p_{1}+d_{i i}^{4}\right) & a_{41} p_{1} & 0 & 0 \\
a_{38} p_{1} & -\left(a_{41} p_{1}+a_{45} p_{2}+d_{i i}^{R}\right) & a_{68} p_{2} & 0 \\
0 & a_{45} p_{2} & -\left(a_{60} p_{2}+a_{87} p_{8}+d_{i i}^{2}\right) & a_{85} p_{3} \\
0 & 0 & a_{67} & -\left(a_{85} p_{3}+d_{i i}^{2}\right)
\end{array}\right| \\
& =d_{i i}^{2}\left[\left(a_{28} a_{45} a_{67}+a_{33} a_{45} a_{85}+a_{28} a_{83} a_{85}+a_{41} a_{83} a_{85}\right) p_{1} p_{2} p_{3}\right. \\
& +\left(a_{38} a_{45} p_{1} p_{2}+2 a_{28} a_{63} p_{1} p_{2}+2 a_{33} a_{8} \cdot p_{1} p_{3}+2 a_{38} a_{85} p_{1} p_{3}+2 a_{41} a_{68} p_{1} p_{2}\right. \\
& +2 a_{41} a_{67} p_{1} p_{3}+2 a_{41} a_{85} p_{1} p_{8}+a_{45} a_{63} p_{2}^{2}+2 a_{46} a_{67} p_{2} p_{3}+2 a_{46} a_{85} p_{2} p_{3} \\
& \left.\left.+a_{63} a_{85} p_{2} p_{9}\right) d_{i i}^{2}+2\left(a_{32} p_{1}+a_{41} p_{1}+a_{45} p_{2}+a_{63} p_{2}+a_{67} p_{3}+a_{80} p_{3}\right) d_{i i}^{4}+2 d_{i i}^{f}\right] \tag{4-28}
\end{align*}
$$

### 4.3.3 Comparison of T Matrices Fir Elimination of Damping at Various Interfaces

Elimination of damping at interface $j$ of the model corresponds to setting $r_{j}=0$ and $p_{j}=1$ in the
equations for gemerating the elemeats of the I matrix, equation (4-14) through (4-28). The following draping conditions have bexa troated for this set of equation.

1. Damping diminated at the interface betwoen bodies 1 and 2;
2. Damping climinated at the interface between bodies 2 and 3 ;
3. Damping eliminated at the interface between bodies 3 and 4 ;
4. Damping ciminated at the interfaces between bodies 2,3 and 4;
5. Damping eliminated at the interfaces between bodies 1 and 2 , and 3 and 4.
6. Damping aliminated at the interfaces between bodies 1,2 and 3 ;
7. Damping eliminated from all interfaces.

Example: All interface damping eliminated.
If damping is removed from all three interfaces of the four body model, $r_{j} \rightarrow 0, p_{j} \rightarrow 1, d_{i i 1} \rightarrow d_{i i}, d_{i i 3} \rightarrow d_{i i}$, $d_{i i 2} \rightarrow d_{i i}, d_{i i 4} \rightarrow d_{i i}$ and

$$
\begin{aligned}
\Delta_{i 4}^{\prime}-d_{i i}^{2} & {\left[a_{23} a_{45} a_{87}+a_{23} a_{46} a_{85}+a_{23} a_{63} a_{85}+a_{41} a_{63} a_{85}\right.} \\
& +\left(a_{23} a_{45}+2 a_{23} a_{88}+2 a_{23} a_{67}+2 a_{23} a_{85}+2 a_{41} a_{68}+2 a_{41} a_{67}\right. \\
& \left.+2 a_{41} a_{85}+a_{46} a_{63}+2 a_{45} a_{6:}+2 a_{45} a_{85}+a_{63} a_{85}\right) d_{i i}^{R} \\
& \left.+2\left(a_{23} \cdot a_{41}+a_{45}+a_{63}+a_{67}+a_{86}\right) d_{i i}^{4}+d_{i i}^{6}\right]=\Delta_{i 4}
\end{aligned}
$$

$$
t_{i 2}=\frac{\left\{a_{45} a_{68}\left(a_{85}+d_{i i}^{2}\right)-\left(a_{41}+a_{46}+d_{i i}^{2}\right)\left[\left(a_{63}+d_{i i}^{2}\right)\left(a_{86}+d_{i i}^{2}\right)+a_{67} d_{i i}^{2}\right]\right\}\left(f_{i 1}+d_{i i} f_{i 2}\right)}{\Delta_{i 1}}
$$

$$
-\frac{a_{41}\left[\left(a_{82}+d_{i i}^{2}\right)\left(a_{85}+d_{i i}^{2}\right)+a_{67} d_{i i}^{2}\right]\left(f_{i 3}+d_{i i} f_{i 4}\right)}{\Delta_{i 4}}
$$

$$
\begin{equation*}
-\frac{a_{41} a_{88}\left[\left(a_{85}+d_{i i}^{P}\right)\left(f_{i 6}+d_{i i} f_{i 8}\right)+a_{86}\left(f_{i 7}+d_{i i} f_{i 8}\right)\right]}{\Delta_{i 4}} \quad i=1,2, \ldots, p \tag{4-29}
\end{equation*}
$$

$$
t_{i 4}=-\frac{\left[\left(a_{63}+d_{i i}^{2}\right)\left(a_{g z}+d_{i i}^{2}\right)+a_{67} d_{i i}^{2}\right]\left[a_{23}\left(f_{i 1}+d_{i i} f_{i 2}\right)+\left(a_{23}+d_{i i}^{2}\right)\left(f_{i 3}+d_{i i} f_{i 4}\right)\right]}{\Delta_{i 4}}
$$

$$
\begin{equation*}
-\frac{\left(a_{38}+d_{i i}^{2}\right) a_{83}\left[\left(a_{85}+d_{i i}^{2}\right)\left(f_{i 6}+d_{i i} f_{i 6}\right)+a_{85}\left(f_{i 7}+d_{i i} f_{i 8}\right)\right]}{\Delta_{i 4}} \tag{4-30}
\end{equation*}
$$

$$
t_{i 6}=-\frac{a_{45}\left(a_{85}+d_{i i}^{2}\right)\left[a_{23}\left(f_{i 1}+d_{i i} f_{i 2}\right)+\left(a_{23}+d_{i i}^{2}\right)\left(f_{i 3}+d_{i i} f_{i 4}\right)\right]}{\Delta_{i 4}}
$$

$$
-\frac{\left.\left[\left(a_{23}+d_{i i}^{2}\right)\left(a_{48}+d_{i i}^{2}\right)+a_{41} d_{i i}^{2}\right)+a_{41} d_{i i}^{2}\right]\left[\left(a_{85}+d_{i i}^{2}\right)\left(f_{i 5}+d_{i i} f_{i 6}\right)+a_{85}\left(f_{i 7}+d_{i i} f_{i 8}\right)\right]}{\Delta_{i 4}}(4-31)
$$

$$
\begin{align*}
& t_{i s}=-\frac{a_{3 c} a_{07}\left[a_{n 3}\left(f_{i 1}+d_{i i} f_{i 2}\right)+\left(a_{33}+d_{i i}\right)\left(f_{i 2}+d_{i i} f_{i 4}\right)\right]}{\Delta_{i 4}} \\
& -\frac{a_{07}\left[\left(a_{28}+d_{i i}^{2}\right)\left(a_{48}+d_{i i}^{2}\right)+a_{41} d_{i t}^{2}\right]\left(f_{i 8}+d_{i i} f_{i 8}\right)}{\Delta_{i 4}} \\
& -\frac{\left\{d_{i i}\left[a_{41}\left(a_{83}+a_{67}+d_{i i}^{2}\right)+a_{68} d_{i i}\right]\right.}{\Delta_{i 1}} \\
& +\frac{\left.\left(a_{33}+d_{i}^{R}\right)\left(a_{46}+d_{i i}\right)\left(a_{67}+d_{i i}\right)+a_{38} a_{68} d_{i i}^{R}\right\}\left(f_{i 7}+d_{i i} f_{i 8}\right)}{\Delta_{i 4}}  \tag{4-32}\\
& t_{i 1}=-\frac{\left\{d_{i i}\left(a_{41}+a_{46}+d_{i i}^{2}\right)\left[\left(a_{63}+d_{i i}^{2}\right)\left(a_{86}+d_{i i}^{2}\right)+a_{87} d_{i i}^{2}\right]\right.}{\Delta_{i 4}} \\
& -\frac{\left.a_{45} a_{68}\left(a_{85}+d_{i i}^{P}\right)\right\}\left(f_{i 1}+d_{i i} f_{i 2}\right)}{\Delta_{i 4}}+f_{i 2} \\
& -\frac{a_{41} d_{i i}\left\{\left[\left(a_{63}+d_{i i}^{2}\right)\left(a_{85}+d_{i i}^{2}\right)+a_{37} d_{i i}^{P}\right]\left(f_{i 3}+d_{i i} f_{i 4}\right)\right.}{\Delta_{14}} \\
& +\frac{\left.a_{63}\left(a_{85}+d_{i i}^{2}\right)\left(f_{i 6}+d_{i i} f_{i 6}\right)+a_{68} a_{85}\left(f_{i 7}+d_{i i} f_{i 8}\right)\right\}}{\Delta_{i 4}}  \tag{4-33}\\
& t_{i 3}=-\frac{a_{23} d_{i i}\left[\left(a_{83}+d_{i i}^{2}\right)\left(a_{85}+d_{i j}^{2}\right)+a_{87} d_{i i}^{2}\right]\left(f_{i 1}+d_{i i} f_{i 2}\right)}{\Delta_{i 4}} \\
& -\frac{d_{i i}\left(a_{23}+d_{i i}^{2}\right)\left[\left(a_{63}+d_{i i}^{2}\right)\left(a_{85}+d_{i i}^{2}\right)+a_{87} d_{i i}^{2}\right]\left(f_{i 8}+d_{i i} f_{i 4}\right)}{\Delta_{i 4}}+f_{i 4} \\
& +\frac{a_{63} d_{i i}\left(a_{23}+d_{i i}^{2}\right)\left[\left(a_{85}+d_{i i}^{2}\right)\left(f_{i 6}+d_{i i} f_{i 6}\right)+a_{85}\left(f_{i 7}+d_{i i} f_{i 8}\right)\right]}{\Delta_{i 4}} \tag{4-34}
\end{align*}
$$

$$
\begin{align*}
& t_{i s}=-\frac{\operatorname{casc}_{i j}\left(c_{05}+f_{i t}\right)\left[a_{2 s}\left(f_{i 1}+d_{i i} f_{i 2}\right)+\left(a_{93}+d_{i i}\right)\left(f_{i 3}+d_{i i} f_{i 4}\right)\right]}{\Delta_{i 4}} . \\
& +\frac{d_{i i}\left[\left(a_{98}+d_{i i}^{R}\right)\left(a_{46}+d_{i i}^{p}\right)\left(a_{85}+d_{i i}^{R}\right)-a_{41} d_{i i}\left(a_{85}+d_{i i}\right]\left[\left(f_{i 0}+d_{i i} f_{i 6}\right)\right.\right.}{\Delta_{i 4}}+f_{i 6} \\
& -\frac{a_{66} d_{i i}\left[a_{41} d_{i i}+\left(a_{93}+d_{i i}\right)\left(a_{48}+d_{i i}\right)\right]\left(f_{i 7}+d_{i i} f_{i 8}\right)}{\Delta_{i 4}}  \tag{4-35}\\
& t_{i 7}=-\frac{a_{05 d} d_{i j}\left\{a_{45}\left[a_{33}\left(f_{i 1}+d_{i i} f_{i 3}\right)+\left(a_{33}+d_{i i}\right)\left(f_{i 3}+d_{i i} f_{i 4}\right)\right]\right.}{\Delta_{i 4}} \\
& +\frac{\left.\left[\left(a_{38}+d_{i i}^{2}\right)\left(a_{46}+d_{i i}^{2}\right)+a_{41} d_{i i}^{2}\right]\left(f_{i 8}+d_{i i} f_{i 6}\right)\right\}}{\Delta_{i 4}} \\
& +\frac{d_{i i}\left\{\left(a_{33}+d_{i i}^{2}\right)\left(a_{46}+d_{i i}^{2}\right)\left(a_{87}+d_{i i}^{2}\right)\right.}{\Delta_{i 4}} \\
& +\frac{\left.d_{i i}\left[a_{38} a_{68}+a_{41} a_{68}+a_{41} a_{67}+\left(a_{41}+a_{68}\right) a_{i 7}\right]\right\}\left(f_{i 7}+d_{i i} f_{i 8}\right)}{\Delta_{i 4}}+f_{18} \tag{4-36}
\end{align*}
$$

## 44 SOLUTION FOR SYNTELSEAED STATE VARIABLES

### 4.4.1 Introduction

Inaccessibility of a scalar state variable in the model equations ( $4-5$ ), ( 4.0 ) is reflected by a corresponding nall column in the $\mathbf{C}$ and $\mathbf{F}$ matrices as implied in equation (2-14). For the generation of reduced state observers for the four body model the number of inaccessible state variables, $p$, can be $1,2,3,4,5,6$ or 7 .

### 4.4.2 Firet Order Observers ( $p=1$ )

An observer of order at least ine is required when only one of the eight scalar state variables of the four body model is inaccessible. The first order form of the linear observer equation is as follows:

$$
\begin{equation*}
\dot{z}=d z+\mathbf{F u}+\mathbf{G y} \tag{4-37}
\end{equation*}
$$

The $\mathbf{F}$ and $\mathbf{T}$ matrices associated with a frast ordur observer for the four body model then reduce to the following row forms.

$$
\begin{align*}
& \mathbf{F}=\left[\begin{array}{llll}
f_{1} & f_{2} & \ldots & f_{8}
\end{array}\right]  \tag{4-38}\\
& \mathbf{T}=\left[\begin{array}{llll}
t_{1} & t_{2} & \ldots & t_{8}
\end{array}\right] \tag{4-39}
\end{align*}
$$

The observer synthesis equations are then of the form of equations (4-14) through (4-28) with $i=1$. Since a first order observer corresponds to one of the scalar state variables being inaccessible, one of the $f_{i}(i=1,2, \ldots, 8)=0$.

## Example

Suppoee the scalar state reprowasting the angular rate of body $4, x_{8}$ is inaccessible. Then $f_{8}=0$ and the oborver syathesis equations reduce to the form of equations (4-14) through (4-28) with fis $=0$ and $i=1$. From equation (2-11), the synthesised sealar state, $\hat{x}_{8}$, is expresoed in terms of the sealar observer variablo, $x$, and the accesible scalar state variables as follows.

$$
\begin{equation*}
\hat{x}_{8}=\frac{1}{t_{4}}\left[x-\sum_{i=1}^{7} t_{i} x_{i}\right] \tag{4-40}
\end{equation*}
$$

For $x_{8}$ insecessible, it is ascumed that:

$$
\mathbf{C}=\left[\begin{array}{lll} 
& & 1  \tag{4-41}\\
I_{7} & & \vdots \\
& & 0
\end{array}\right]
$$

where $I_{7}=7 \times 7$ identity matrix.
From $\mathbf{F}=\mathbf{G C}$,

$$
\mathbf{G}=\left[\begin{array}{lllllll}
f_{1} & f_{2} & f_{3} & f_{4} & f_{5} & f_{6} & f_{7} \tag{4-42}
\end{array}\right]
$$

From $\mathbf{E}=\mathbf{T B}$,

$$
E=\left[\begin{array}{llll}
t_{2} & t_{4} & t_{8} & t_{8} \tag{4-43}
\end{array}\right] \quad \text { for } r=4 \text { (control torques on all } 4 \text { bodies) }
$$

## 4.4.s Observers of Intarmediate Order ( $p=2,3,4,5$ or 6 )

For those cases in which an intermediate number of the eight scalar states of the four-body single-axis model is inaccessible, the minimum order of the reduced state linear observer required to reconstract these inaccessibie states is given by $p$. In each case the number of null columns in the measurement or observation matrix, $\mathbf{C}$, and the $\mathbf{F}$ matrix also is equal to $p$. The general forms of the $\mathbf{E}, \mathbf{F}$ and $\mathbf{T}$ matrices are given in equations (4-11), (4-12) and (4-13) for $p=2,3,4,5$ or 6 where $p$ represents the number of inaccessible scalar state variables of the model.

## Example

Suppose the scalar states, $x_{7}$ and $x_{8}$, which represent the angular position and rate of body ${ }^{4}$, are inaccessible. Then $f_{i 7}=f_{i 8}=0$ for $i=1,2$ and the observer synthesis equations reduce to the form of equations (4-14) through ( $4-28$ ) with the preceding conditions. From equation (2-11) the synthesised sealar states, $\hat{x}_{7}$ and $\hat{x}_{8}$, are expressed in terms of the scalar observer variahles, $z$ and $z_{2}$ and the accessiable scalar state variables as follows.

$$
\begin{align*}
& \hat{x}_{7}=\frac{\sum_{i=1}^{2}(-1)^{i+1}\left(\Delta_{2}\right)_{i, 1}\left(z_{i}-\sum_{j=1}^{6} t_{i j} x_{j}\right)}{\Delta_{2}}  \tag{4-44}\\
& \hat{z}_{8}=\frac{\sum_{i=1}^{2}(-1)^{i+1}\left(\Delta_{2}\right)_{i, 2}\left(z_{i}-\sum_{j=1}^{6} t_{i j} x_{j}\right)}{\Delta_{2}} \tag{4-45}
\end{align*}
$$

for

$$
\Delta_{2}=\left|\begin{array}{ll}
t_{17} & t_{18}  \tag{4-48}\\
t_{27} & t_{28}
\end{array}\right|=t_{17} t_{28}-t_{18} t_{27} \neq 0
$$

where $\left(\Delta_{8}\right)_{i, j}=\Delta_{8}$ without the demante of the $f^{\text {th }}$ row and $j^{\text {th }}$ column.
For 9 and $x_{8}$ inscceswible, it is asoumed that:

$$
\mathbf{C}=\left[\begin{array}{llll} 
& & 1 & 0  \tag{4-47}\\
0 \\
\mathbf{I}_{\mathbf{5}} & & \vdots & \vdots \\
& & 0 & 0
\end{array}\right]
$$

where $I_{6}=6 \times 6$ identity matrix.
Since $\mathbf{F}=\mathbf{G C}$,
$\mathbf{G}=\left[\begin{array}{llllll}f_{11} & f_{12} & f_{12} & f_{14} & f_{15} & f_{16} \\ f_{21} & f_{22} & f_{23} & f_{24} & f_{25} & f_{26}\end{array}\right]$
From $\mathbf{E}=\mathbf{T B}$,
$\mathbf{E}=\left[\begin{array}{llll}t_{12} & t_{14} & t_{16} & t_{18} \\ t_{22} & t_{24} & t_{26} & t_{28}\end{array}\right]$ for $r=4$ (control torques applied to all four bodies)
$\mathbf{I}=\left[\begin{array}{llll}t_{12} & t_{14} & t_{16} & 0 \\ t_{22} & t_{24} & t_{26} & 0\end{array}\right]$ for $r=3$ (control torques applied to bodies 1,2 and 3 )
$\mathbf{E}=\left[\begin{array}{llll}t_{12} & t_{14} & 0 & 0 \\ t_{22} & t_{34} & 0 & 0\end{array}\right]$ fur $r=2$ (control torques applied to bodies 1 and 2 )
$\mathbf{E}=\left[\begin{array}{llll}t_{12} & 0 & 0 & 0 \\ t_{22} & 0 & 0 & 0\end{array}\right]$ for $r=1$ (control torque applied to body t )

### 4.4.4 Seventh Order Observers ( $p=7$ )

When any seven of the eight scalar state variables of the four body model are inaccessible, a linear observer of at least order seven is required. The observer synthesis equations are as presented in equations (4-14) through (4-28) with $i=1,2, \ldots, 7$. Since a seventh order observer corresponds to seven of the scalar states being inaccessible, $f_{1 j}=f_{2 j}=\ldots=f_{7 j}=0$ for seven of the eight values of the subscript, $j$.

## Example

Suppose only the scalar state variable representing the angular position of body $1, x_{1}$, is accessible. Then the remaining scalar states, $x_{2}, x_{3}, \ldots, x_{8}$ are inaceessible, $f_{i 2}=f_{i 3}=\ldots=f_{i 8}=0$ for $i=1,2,3,4,5$, 6 and 7 and the observer synthesis equations reduce to the form of equations (4-14) through (4-28) with $f_{i 3}=f_{i 8}=\ldots=f_{i s}=0$ and $i=1,2, \ldots, 7$. The synthesized scalar state variables, $\hat{x}_{2}$ through $\hat{x}_{8}$, are expressed in terms of the observer variables, $z_{1}$ through $2_{7}$, and the accessible state variable, $x_{1}$, by atilixing equation (2-11) in the following form.

$$
\begin{align*}
\dot{x}_{k+1} & =\frac{\sum_{i=1}^{7}(-1)^{i+1}\left(\Delta_{7}\right)_{i, k}\left(x_{i}-t_{i 1} x_{1}\right)}{\Delta_{7}} \quad k=1,2, \ldots, 7  \tag{4-53}\\
\Delta_{7} & =\left|\begin{array}{ccc}
t_{12} & \cdots & t_{18} \\
\vdots & & \vdots \\
t_{72} & \cdots & t_{78}
\end{array}\right| \\
& =t_{12}\left(\Delta_{7}\right)_{1,1}-t_{22}\left(\Delta_{7}\right)_{2,1}+t_{22}\left(\Delta_{7}\right)_{3,1}-t_{42}\left(\Delta_{7}\right)_{4,1}
\end{align*}
$$

$$
\begin{equation*}
+i_{52}\left(\Delta_{7}\right)_{6,1}-t_{83}\left(\Delta_{7}\right)_{6,1}+t_{72}\left(\Delta_{7}\right)_{7,1} \tag{4-54}
\end{equation*}
$$

where $\left(\Delta_{Y}\right)_{i, j}=\Delta_{Y}$, without the elcunente of the $s^{\text {th }}$ row and the $j^{\text {th }}$ colvme.
For only $z_{1}$ accesaible, it is asomed that:

$$
C=\left[\begin{array}{llllllll}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \tag{4-55}
\end{array}\right]
$$

Prom $\boldsymbol{F}=\mathbf{G C}$,

$$
G=\left[\begin{array}{c}
f_{11}  \tag{4-56}\\
\vdots \\
f_{71}
\end{array}\right]
$$

From $\mathrm{E}=\mathbf{T B}$,

$$
\mathbf{E}=\left[\begin{array}{cccc}
t_{12} & t_{14} & t_{16} & t_{18}  \tag{4-57}\\
\vdots & \vdots & \vdots & \vdots \\
t_{72} & t_{74} & t_{78} & t_{78}
\end{array}\right] \text { for } r=4 \text { (control torques applied to all four bodies) }
$$

### 4.5 REFERENCES

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4-4 Sage, A.P., Optimum Systems Control. Englewood Cliffs, N.J.: Prentice-Hall, Inc. 1938, pp. 306-312.

## DEVELOPMRNT OF THE TIVEBODY SINGLE-AXIS MODEL AND ITS EDDUCED STATE LINEAR OBSEEVERS

## S.1 ORIGINAL DANIPED MODEL

In eartier work, Guidance Symems Division (5-1), it was shown that one axis of the three-axis flive-body approximation of a prototype flerible spacecrat can be decoupled from the uther two axes. The four-body ainglo-axis models of a flexible spacecraft developed in the previons section were therefore extended to corresponding five body models to represent the decoupled axis of the three-axis five-body model.

The rotational dynamics of the fivo-body singlo-axis modei of a flezible spacecratt with damping shown in Fig. 5-1 may be rupresented by the following set of equations.

$$
\begin{align*}
& I_{1} \dot{\theta}_{1}=-c_{1}\left(\dot{\theta}_{1}-\dot{\theta}_{2}\right)-k_{1}\left(\theta_{1}-\theta_{2}\right)+q_{1}  \tag{5-1}\\
& I_{2} \dot{\theta}_{2}=c_{1}\left(\dot{\theta}_{1}-\dot{\theta}_{2}\right)+k_{1}\left(\theta_{1}-\theta_{2}\right)+c_{2}\left(\dot{\theta}_{8}-\dot{\theta}_{2}\right)+k_{2}\left(\theta_{2}-\theta_{2}\right)+q_{2}  \tag{5-2}\\
& I_{3} \dot{\theta}_{3}=c_{2}\left(\dot{\theta}_{2}-\dot{\theta}_{3}\right)+k_{2}\left(\theta_{2}-\theta_{3}\right)+c_{3}\left(\dot{\theta}_{4}-\dot{\theta}_{3}\right)+k_{8}\left(\theta_{4}-\theta_{3}\right)+q_{3}  \tag{5-3}\\
& I_{4} \dot{\theta}_{4}=c_{3}\left(\dot{\theta}_{3}-\dot{\theta}_{4}\right)+k_{3}\left(\theta_{3}-\theta_{4}\right)+c_{4}\left(\dot{\theta}_{5}-\dot{\theta}_{4}\right)+k_{4}\left(\theta_{5}-\theta_{4}\right)+q_{4}  \tag{5-4}\\
& I_{6} \dot{\theta}_{5}=c_{4}\left(\dot{\theta}_{4}-\dot{\theta}_{5}\right)+k_{4}\left(\theta_{4}-\theta_{6}\right)+q_{5} \tag{5-8}
\end{align*}
$$

where:

$$
\begin{aligned}
& I_{i}=\text { rotational inertia of body } i, \quad i=1,2, \ldots, 5 \\
& \theta_{i}=\text { angular displacement of body } ; \\
& \dot{\theta}_{i}=\text { angular rate of body } i \\
& q_{i}=\text { torque applied to body } i \\
& k_{j}=\text { rotational spring coefficient at interface } j \quad j=1,2,3,4 \\
& c_{j}=\text { rotational damping coefficient at interface } j
\end{aligned}
$$

### 6.2 STATE VARLABLE MODEL

The state variable form of the five-body single-axis model of a flexible spacecraft depicted in Fig. 5-1 was written in the following form.

$$
\begin{align*}
& \dot{x}=A x+B u  \tag{5-8}\\
& x_{A}=C x \tag{5-7}
\end{align*}
$$

where:

$$
x=\left[\begin{array}{lll}
x_{1} & \cdots & x_{10}
\end{array}\right]^{T}=\left[\begin{array}{ll}
x_{\Lambda}^{T} & x_{1}^{T}
\end{array}\right]^{T}=\left[\begin{array}{llllllllll}
\theta_{1} & \dot{\theta}_{1} & \theta_{2} & \dot{\theta}_{2} & \theta_{3} & \dot{\theta}_{3} & \theta_{4} & \dot{\theta}_{4} & \theta_{5} & \dot{\theta}_{5}
\end{array}\right]^{T}=\text { state vector }
$$

$x_{n}=m$ vector of accesible scalar states
$x_{1}=p$ vector of inaccessible scalar states
$\mathbf{u}=\left[\begin{array}{lll}u_{1} & \cdots & u_{r}\end{array}\right]^{T}=\left[\begin{array}{lll}\frac{q_{1}}{I_{1}} & \cdots & \frac{q_{r}}{I_{r}}\end{array}\right]^{T} \quad r=1,2, \ldots, 5$
C $=m \times 10$ measurement or observation matrix.


FIGURE 5-1
FIVE-BODY SINGLE-AXIS MODEL WITH DAMPING AT ALL FOUR INTERFACES

$a_{22}=\frac{h_{1}}{I_{1}}$
$a_{41}=\frac{h_{1}}{I_{2}}, \quad a_{45}=\frac{h_{2}}{I_{2}}, \quad a_{48}=-\left(a_{41}+a_{48}\right), \quad a_{44}=-\left(a_{41} r_{1}+a_{45} r_{2}\right)$
$a_{88}=\frac{k_{2}}{I_{2}}, \quad a_{67}=-\left(a_{62}+a_{67}\right), \quad a_{68}=-\left(a_{63}+a_{67}\right), \quad a_{68}=-\left(a_{63} r_{2}+a_{6773}\right)$
$a_{86}=\frac{h_{6}}{L_{4}} \quad a_{69}=\frac{k_{4}}{L_{4}}, \quad a_{87}=-\left(a_{88}+a_{89}\right), \quad a_{88}=-\left(a_{86} r_{3}+a_{89} r_{4}\right)$
$a_{10,7}=\frac{k_{4}}{I_{5}}$
$r_{j}=\frac{c_{j}}{k_{j}} ; \quad j=1,2,3,4$
$B=\left[\begin{array}{lllll}0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1\end{array}\right]$ for $r=5$

## 5.S REDUCED STATE LINEAR OBSERVERS

## 5.s.1 Introduetion

The mininmon order of a reduced state linear observer required to reconstruct the $10-m$ inaccessible scalar state variables of the five-body singlo-axis model of a flexible spacecraft represented by equations (5-6)
through ( $8-10$ ) is $p=10-m$ whaee $m=1,2,3,4,8,6,7,8$ or 9 . All of the roduced state linear observers for this five body model may be writtoan in the form of equations (2-10) and (2-11) under the accumption that the obearver coofificiont matrix, $\mathbf{D}$, is diagonal and of dimensions $p \times p$. The corrosponding observer wighting matrix is of the following form

$$
\mathbf{T}=\left[\begin{array}{ccc}
t_{11} & \cdots & t_{1,10}  \tag{8-12}\\
\vdots & & \vdots \\
t_{p, 1} & \cdots & t_{p, 10}
\end{array}\right]
$$

From equations (2-12), (6-10) and (5-11),

$$
\begin{align*}
& E=\left[\begin{array}{ccccc}
t_{12} & t_{14} & t_{18} & t_{18} & t_{1,: 0} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
t_{p, 2} & t_{p, 4} & t_{p, 8} & t_{p, 2} & t_{p, 10}
\end{array}\right]  \tag{5-13}\\
& \mathbf{F}=\left[\begin{array}{ccc}
f_{11} & \cdots & f_{1,10} \\
\vdots & & \vdots \\
f_{p, 1} & \cdots & f_{p, 10}
\end{array}\right] \tag{5-14}
\end{align*}
$$

The corresponding block diagram appears in Fig. 2-4.

### 5.3.2 Obeerver Synthesie Equations

From Luenberger (5-2), (5-3) and (5-4) and Sage (5-5) the equations for synthesizing the reduced state linear observers for the five-body single-axis model represented by equations (5-8) through ( $5-10$ ) are given by equations (2-13) and (2-14). With coefficient matrices of the form listed in 5.3 .1 this set of equations for generating the elements of the $\mathbf{T}$ matrix reduces to the following.

$$
\begin{align*}
& t_{i 2}=\frac{\left\{\left[a_{41} a_{63} p_{1} p_{2}+{ }_{-11} a_{67} p_{1} p_{3}+a_{48} a_{67} p_{2} p_{3}+\left(a_{41} p_{1}+a_{48} p_{2}+a_{63} p_{2}+a_{67} p_{3}\right) d_{i i}^{2}+d_{i i}^{4}\right]\right.}{a_{23} a_{48} p_{1} p_{2} \Delta_{i 6}^{\prime}}\left(\Delta_{i 8}^{\prime}\right)_{1,3} \\
& -\frac{\left.a_{67} a_{88} p_{3}^{2}\left(a_{41} p_{1}+a_{46} p_{2}+d_{i i}^{2}\right)\left(a_{10,7} p_{4}+d_{i i}^{2}\right)\right\}}{\Delta_{i 6}^{\prime}}\left(f_{i 1}+d_{i i} f_{i 2}\right) \\
& +a_{41}\left[\frac{\left(a_{63} p_{2}+a_{67} p_{3}+d_{i i}^{p}\right)\left(\Delta_{i 5}^{\prime}\right)_{1,3}}{a_{23} a_{46} p_{7} \Delta_{i 5}^{\prime}}-\frac{a_{67} a_{35} p_{1} p_{3}^{2}\left(a_{10,7 p_{4}}+d_{i i}^{2}\right.}{\Delta_{i 6}^{\prime}}\right]\left(f_{3}+d_{i i} f_{i 4}\right) \\
& +\frac{a_{41} a_{83}\left(\Lambda_{i 6}^{\prime}\right)_{1,3}}{a_{23} a_{48} \Delta_{i 5}^{\prime}}\left(f_{i 6}+d_{i i} f_{i 6}\right) \\
& +\frac{a_{41} a_{68} a_{85} p_{1} p_{2} p_{8}}{\Delta_{i 6}^{\prime}}\left[\left(a_{10,7 p_{4}}+d_{i i}\right)\left(f_{i 7}+d_{i i} f_{i 8}\right)+a_{10,7 p_{4}}\left(f_{i 9}+d_{i i} f_{i 18} i\right] ; i=1,2, \ldots, p\right.  \tag{5-15}\\
& t_{i 4}=\left[\frac{\left(a_{83} p_{3}+a_{87} p_{3}+d_{i i}^{R}\right)\left(\Delta_{i 6}^{\prime}\right)_{1,2}}{a_{33} a_{45} p_{1} p_{5} \Delta_{i 6}^{\prime}}-\frac{a_{87} a_{85} p_{3}^{2}\left(a_{10,7} p_{4}+d_{i i)}^{2}\right)}{\Delta_{i 6}^{\prime}}\right] \\
& \times\left[a_{33} p_{1}\left(f_{i 1}+d_{i i} f_{i 2}\right)+\left(a_{33} p_{1}+d_{i i}^{2}\right)\left(f_{i a}+d_{i i} f_{i 4}\right)\right]+\frac{\left(a_{33} p_{1}+d_{i i}^{2}\right) a_{83}\left(\Delta_{i 8}^{\prime}\right)_{1,3}}{a_{23} a_{48} p_{1} \Delta_{i 5}^{\prime}}\left(f_{i 6}+d_{i i} f_{i 6}\right) \\
& +\frac{a_{63} a_{66} p_{9} p_{3}\left(a_{33} p_{1}+d_{i i}^{2}\right)}{\Delta_{i 6}^{\prime}}\left[\left(a_{10,7 p_{4}}+d_{i i}^{2}\right)\left(f_{i 7}+d_{i i} f_{i 8}\right)+\alpha_{10,7 p_{4}}\left(f_{i 0}+d_{i i} f_{i 10}\right)\right] \tag{5-16}
\end{align*}
$$

$$
\begin{align*}
& +\frac{\left(\Delta_{i 3}^{\prime}\right)_{1,2}\left(\Delta_{i s}^{\prime}\right)_{6,8}}{a_{25} a_{46} a_{88} a_{10,}, 7 p_{1} p_{2} p_{9} p_{4} \Delta_{i 6}^{\prime}}\left(f_{i 6}+d_{16} f_{i 6}\right) \\
& +\frac{\left(\Delta_{i 3}^{\prime}\right)_{6,2}}{a_{10,7 p_{4}} \Delta_{i 6}^{\prime}}\left[\left(a_{10,7 P_{4}}+d_{i i}\right)\left(f_{i 7}+d_{i i} f_{i 8}\right)+a_{10,7 p 4}\left(f_{i 0}+d_{i i} f_{i 10}\right)\right]
\end{align*}
$$

$$
\begin{align*}
& t_{i 1}=\frac{\alpha_{48} a_{93} p_{21} p_{1}\left(a_{10,7 p_{4}}+d_{i i}\right)}{\Delta_{i 5}^{i}}\left[a_{32 p_{1}}\left(f_{i 1}+d_{i i} f_{i i}\right)+\left(a_{33} p_{1}+\alpha_{i i}\right)\left(f_{i 8}+d_{i i} f_{i 4}\right)\right] \\
& +\frac{a_{87}\left(\Delta_{i 6}^{\prime}\right)_{5,3}\left(a_{10,7 p_{4}}+d_{i i}^{R}\right)}{a_{86} a_{10,7}, 74 \Delta_{i 6}^{\prime}}\left(f_{i 6}+d_{i i} f_{i 6}\right) \\
& +\left[\frac{\left(a_{83} p_{7}+a_{8} 7 p_{7}+d_{i i}^{2}\right)\left(\Delta_{i 6}^{\prime}\right)_{5,8}}{a_{86} a_{10,7 p p} p_{4} \Delta_{i 6}^{\prime}}-\frac{\left(a_{28} p_{1}+d_{i j}^{2}\right) a_{48} a_{63} p_{2}^{2}}{\Delta_{i 5}^{\prime}}\right] \\
& \times\left[\left(s_{15,7 p_{4}}+d_{i i 1}\right)\left(f_{i 7}+d_{i i} f_{i 3}\right)+a_{10,7 p_{4}}\left(f_{i 0}+d_{i i} f_{i 10}\right)\right] \tag{5-18}
\end{align*}
$$

$$
\begin{aligned}
& t_{i 20}=\frac{a_{46} a_{87} a_{80} p_{2} p_{9} p_{4}\left[a_{23} p_{1}\left(f_{i 1}+d_{i i} f_{i 2}\right)+\left(a_{23} p_{1}+d_{i i}^{2}\right)\left(f_{i 8}+d_{i i} f_{i 4}\right)\right]}{\Delta_{i 6}^{\prime}} \\
& +\frac{a_{67} a_{00}\left(\Delta_{i 6}^{\prime}\right)_{5,2}}{a_{85} a_{10,7} \Delta_{i 6}^{\prime}}\left(f_{i 5}+d_{i i} f_{i 6}\right) \\
& +a_{60}\left[\frac{\left(a_{63} p_{7}+a_{67} p_{2}+d_{i i}\right)\left(\Delta_{i 6}^{\prime}\right)_{6,3}}{a_{88} a_{10,7} p_{3} \Delta_{i 6}^{\prime}}-\frac{\left(a_{93} p_{1}+d_{i 8}^{2}\right) a_{46} a_{83} p_{2}^{2} p_{4}}{\Delta_{i 8}^{\prime}}\right]\left(f_{i 7}+d_{i i} f_{i 8}\right)
\end{aligned}
$$

$$
\begin{align*}
& \left.-\frac{\left(a_{23} p_{1}+d_{i i}\right) a_{48} a_{63} p_{3}\left(a_{86} p_{2}+q_{80} p_{4}+d_{i i}^{2}\right)}{\Delta_{i 5}^{\prime}}\right\}\left(f_{i 9}+d_{i i} f_{i 10}\right) \tag{5-19}
\end{align*}
$$

$$
\begin{aligned}
& t_{i 1}=\left[\left\{\frac{d_{i j}\left[\left(a_{11} p_{1}+d_{i i}^{2}\right)\left(a_{43} p_{2}+a_{67} p_{8}+d_{i i}\right)+a_{46} p_{2}\left(a_{67} p_{2}+d_{i i}\right)\right]}{a_{33} a_{46} p_{1} p_{2} \Delta_{i 6}^{\prime}}\right.\right. \\
& \left.+\frac{a_{32} r_{1}\left[a_{83} p_{3} d_{i 1}+\left(a_{46} p_{2}+d_{i i}^{R}\right)\left(a_{87} p_{9}+d_{i i}^{2}\right)\right]}{a_{22} a_{46} p_{1} p_{2} \Delta_{i 5}^{\prime}}\right\}\left(\Delta_{i 5}^{\prime}\right)_{1,3} \\
& \left.-\frac{a_{87} a_{66} p_{3}^{2}\left(a_{10,7} p_{4}+d_{i i}^{p}\right)\left[d_{i i}\left(a_{41} p_{1}+a_{46} p_{2}+d_{i i}^{2}\right)+a_{23} r_{1}\left(a_{46} p_{2}+d_{i i}^{p}\right)\right]}{\Delta_{i 5}^{\prime}}\right]\left(f_{i 1}+d_{i i} f_{i 2}\right) \\
& +f_{i 2}+\frac{a_{41} d_{i i}}{a_{33} a_{65} p_{1} p_{2} \Delta_{i b}^{\prime}}\left\{\left[\left(c_{A 3} p_{3}+a_{87} p_{3}+d_{i i}^{2}\right)\left(\Delta_{i 6}^{\prime}\right)_{1,3}\right.\right.
\end{aligned}
$$


$+\operatorname{asap}_{3}\left(\Delta_{i s}^{\prime}\right)_{1,8}\left(f_{i 6}+d_{i 8} f_{i 6}\right)$
$\left.+a_{28} a_{48} a_{68} a_{65} p_{1} p_{3}^{2} p_{0}\left[\left(a_{10,7 p_{4}}+d_{i i}\right)\left(f_{i 7}+d_{i i} f_{i 6}\right)+a_{10,7 p_{4}}\left(f_{i 0}+d_{i} f_{i 10}\right)\right]\right\}$

$$
+f_{i 4}+\frac{a_{43} d_{i i}\left[a_{33} p_{1}+a_{41} d_{i i}\left(r_{1}-r_{2}\right)+d_{i i}^{2}\right]\left(\Delta_{i b}^{\prime}\right)_{1,3}}{a_{23} a_{46} p_{1} p_{2} \Delta_{i s}^{\prime}}\left(f_{i B}+d_{i i} \delta_{i B}^{\prime}\right)
$$

$$
+a_{63}\left\{\frac{a_{65 p_{3} p_{2}}\left[a_{32} p_{1}\left(a_{46} r_{2}+d_{i i}\right)+d_{i i}^{2} d_{i i 2}\right]}{\Delta_{i 6}^{\prime}}-\frac{r_{2}\left(\Delta_{i B}^{\prime}\right)_{5,3}}{a_{10,7 p_{4} \Delta_{i 5}^{\prime}}}\right\}
$$

$$
\begin{equation*}
\times\left[\left(a_{10,7 p_{4}}+d_{i i}^{2}\right)\left(f_{i 7}+d_{i i} f_{i \theta}\right)+a_{10,7 p_{4}}\left(f_{i \theta}+d_{i i} f_{i 10}\right)\right] \tag{5-21}
\end{equation*}
$$

$$
\begin{aligned}
& t_{i 8}=-\left\{\frac{\left\{\left[a_{48}\left(r_{1}-r_{2}\right)+d_{i i}\right]\left(a_{67} p_{9}+d_{i i}\right)+a_{40} p_{3} d_{i i}\right\}\left(\Delta_{i 8}^{\prime}\right)_{1,2}}{a_{45 p_{1} p_{2} \Delta_{i s}^{\prime}}}\right. \\
& \left.+\frac{a_{38} a_{47} a_{66} p_{i}^{2}\left(a_{10,7} p_{4}+d_{i i}^{2}\right)\left[a_{48}\left(r_{1}-r_{2}\right)-d_{i i}\right]}{\Delta_{i 6}^{\prime}}\right]\left(f_{i 1}+d_{i i} f_{i 3}\right) \\
& +\left[\frac{\left\{\left(a_{67} p_{2}+d_{i i}^{2}\right)\left[\left(a_{23} p_{1}+d_{i i}^{2}\right)\left(33_{45} r_{3}+d_{i i}\right)+a_{41} r_{1} d_{i i}^{2}\right]\right.}{a_{23} a_{46} p_{1} p_{2} \Delta_{i 5}^{\prime}}\right. \\
& +\frac{\left.a_{83} d_{i i} F_{2}\left(a_{23} p_{1}+a_{41} r_{1} d_{i i}+d_{i 1}^{p}\right)\right\}\left(\Delta_{i 6}^{\prime}\right)_{1, p}}{a_{38} a_{46} p_{1} p_{3} \Delta_{i 6}^{\prime}} \\
& \left.+\frac{a_{67} a_{36} p_{3}^{2}\left(a_{10,7 p_{4}}+d_{i i}^{P}\right)\left[\left(a_{23} p_{1}+d_{i i}^{2}\right)\left(a_{46} r_{3}+d_{i i}\right)+a_{41} r_{1} d_{i i}^{2}\right]}{\Delta_{i 6}^{\prime}}\right]\left(f_{i 3}+d_{i i} f_{i 4}\right)
\end{aligned}
$$

$$
\begin{aligned}
& t_{i s}=\left\{\frac{\left[d_{1}-a_{07}\left(r_{2}-r_{i}\right)\right]\left(\Delta_{i B}^{\prime}\right)_{1 r_{2}}}{a_{33} p_{1} p_{2} \Delta_{i B}^{\prime}}+\frac{a_{4 B} a_{07} a_{81} p_{1}\left(a_{10,2 p_{4}}+d_{i j}\right)\left(r_{2}-r_{3}\right)}{\Delta_{i s}^{\prime}}\right\} \\
& \times\left[a_{31} p_{1}\left(f_{i 1}+d_{i i} f_{i 2}\right)+\left(a_{38} p_{1}+d_{i i}\right)\left(f_{i 1}+d_{i i} f_{i 4}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
& \left.+\frac{a_{s 7 p_{3}}\left(a_{10,7 p_{4}}+A_{i}\right)\left(\Delta_{i s}^{\prime}\right)_{6,2}}{a_{10,5 p_{4} \Delta_{i s}^{\prime}}}\right]\left(f_{i 8}+d_{i i} f_{i s}\right)+f_{i s}
\end{aligned}
$$

$$
\begin{align*}
& \times\left[\left(a_{10,7 p_{4}}-d_{i i}^{2}\right)\left(f_{i 9}+d_{i i} f_{i 8}\right)+a_{10,7 p_{4}}\left(f_{i 0}+d_{i i} f_{i 10}\right)\right] \tag{5-22}
\end{align*}
$$

$$
\begin{align*}
& x\left[a_{23} p_{1}\left(f_{i 1}+d_{i i} f_{i 2}\right) \div\left(a_{33} p_{1}+d_{i i}^{2}\right)\left(f_{i 2}+d_{i i} f_{i 4}\right)\right] \\
& +a_{87} d_{i:} \frac{\left[a_{\left.10,7 p_{4}-a_{80} d_{i i}\left(r_{9}-r_{4}\right)+d_{i i}^{2}\right]\left(\Delta_{i 8}^{\prime}\right)_{5,2}}^{a_{85} a_{10,7 p_{3} p_{4} \Delta_{i 5}^{\prime}}}\left(f_{i 8}+d_{i i} f_{i 8}\right)()\left(d_{i 8}\right)\right.}{} \\
& +\left\{\left\{\frac{\left(a_{83} p_{2}+d_{i i}^{p}\right)\left[\left(a_{85} r_{3}+d_{i i}\right)\left(a_{10,7 p_{4}}+d_{i i}^{2}\right)+a_{80} r_{4} d_{i i}^{2}\right]}{a_{85} a_{16,7}, 7 p_{3} p_{4} \Delta_{i 5}^{\prime}}\right.\right. \\
& \left.+\frac{a_{6} T p_{3} d_{i i}\left(a_{10, T p_{4}}+a_{60} r_{4} d_{i i}+d_{i i}\right)}{a_{65} a_{10,7 p_{3} p_{4} \Delta_{i 6}^{\prime}}}\right\}\left(\Delta_{i 8}^{\prime}\right)_{s, 2} \\
& -\frac{\left(a_{23} p_{1}+d_{i i}^{2}\right) a_{46} a_{83} p_{2}^{2}\left[d_{i i}^{2} d_{i 44}+a_{\left.10,7 p_{4}\left(a_{86} r_{3}+d_{i i}\right)\right]}^{\Delta_{i 6}^{\prime}}\right]\left(J_{1:}+d_{i i} f_{i 8}\right)+f_{i 8}}{} \\
& +\left\{\frac{\left[a_{88}\left(a_{83} p_{2}+d_{i i}^{R}\right)\left(r_{4}-r_{2}\right)-\left(a_{53} p_{2}+a_{87} p_{4}+d_{i i}^{\prime}\right) d_{i i}\right]\left(\Delta_{i k}^{\prime}\right)_{6,2}}{a_{85} p_{7} p_{4} \Delta_{i 6}^{\prime}}\right. \\
& \left.-\frac{\left(a_{21} p_{1}+d_{i i}^{2}\right) a_{48} a_{88} a_{10,7 p_{2}^{2}}^{2}\left[a_{86}\left(r_{8}-r_{4}\right)+d_{i i}\right]}{\Delta_{i 8}^{\prime}}\right\}\left(f_{i 0}+d_{i i} f_{i 10}\right) \tag{5-23}
\end{align*}
$$

$$
\begin{aligned}
& \left.+\left(a_{33} p_{1}+d_{i i}^{2}\right)\left(f_{i 8}+d_{i i} f_{i 4}\right)\right]+a_{87 p_{3}}\left(\Delta_{i 8}^{\prime}\right)_{B, 2}\left(f_{i 8}+d_{i i} f_{i 6}\right)
\end{aligned}
$$

$$
\begin{aligned}
& +\left[\left\{\frac{d_{i i 6}\left[a_{88} a_{85} p_{7} p_{3}+a_{88} a_{89} p_{2} p_{4}+a_{87} a_{89} p_{3} p_{4}+\left(a_{83} p_{2}+a_{87} p_{8}+a_{85} p_{3}+a_{89 p_{4}}\right) d_{i i}^{2}+d_{i i}^{4}\right]}{a_{88} a_{10}, 7 p_{3} p_{4} \Delta_{i 6}^{\prime}}\right.\right.
\end{aligned}
$$

$$
\begin{align*}
& \left.+\frac{\left(a_{33} p_{1}+d_{i i}^{P}\right) a_{46} a_{63} p_{2}^{2}\left[a_{89} p_{4} a_{10,7 r_{4}}-d_{i i 6}\left(a_{85} p_{3}+a_{80} p_{4}+a_{i i}^{2}\right)\right]}{\Delta_{i 5}^{\prime}}\right] \\
& \times\left(f_{i 9}+d_{i i} f_{i, 10}\right)+f_{i, 10} \tag{5-24}
\end{align*}
$$

where:

$$
\begin{align*}
& r_{j}=\frac{c_{j}}{h_{j}} ; \quad j=1,2,3,4  \tag{5-10}\\
& p_{j}=1+r_{j} d_{i i} ; \quad i=1,2, \ldots, p \quad j=1,2,3,4  \tag{5-25}\\
& d_{i i 1}=a_{33} r_{1}+d_{i i}  \tag{5-28}\\
& a_{i i 2}=a_{41} r_{1}+a_{46} r_{2}+d_{i i}  \tag{5-27}\\
& d_{i i 3}=a_{83} r_{3}+a_{67 r_{3}}+d_{i i}  \tag{5-28}\\
& d_{i i 4}=a_{88} r_{3}+a_{89 r_{4}}+d_{i i}  \tag{5-29}\\
& d_{i i 6}=a_{10,744}+d_{i i} \tag{5-30}
\end{align*}
$$

$$
\begin{align*}
& \Delta_{i 6}^{\prime}=\left\lvert\, \begin{array}{ccccc}
-\left(a_{23} p_{1}+d_{i i}^{2}\right) & a_{41} p_{1} & 0 & 0 & 0 \\
a_{23} p_{1} & -\left(a_{41} p_{1}+a_{46}+p_{2}^{2}\right) & a_{63} p_{2} & 0 & 0 \\
0 & a_{46} p_{2} & -\left(a_{83} p_{2}+a_{87} p_{3}+d_{i i}^{2}\right) & a_{85} 7_{3} & 0 \\
0 & 0 & a_{675} & -\left(a_{85} p_{3}+a_{89} p_{4}+d_{i i}^{2}\right) & a_{10,7 p_{4}} \\
0 & 0 & 0 & a_{89} p_{4} & \cdots\left(a_{10,7 p_{4}+0}\right.
\end{array}\right. \\
& =-\left(a_{23 p_{1}}+d_{i i}^{2}\right)\left(\Delta_{i 6}^{\prime}\right)_{1,1}-a_{28} p_{1}\left(\Delta_{i 6}^{\prime}\right)_{2,1} \\
& =-\left(a_{23 p_{1}}+d_{i i}^{2}\right)\left(\Delta_{i 5}^{\prime}\right)_{1,1}-a_{38} p_{1}\left(\Delta_{i 6}^{\prime}\right)_{1,2} \tag{5-31}
\end{align*}
$$

## 6.3.s Elimination of Damping Prom Model Interfaces

Elimination of damping at interface $j$ ( $j=1,2,3,4$ for the five body model) corresponds to setting $r_{j}=0$ and $p_{j}=1$ in the equations for generating the elements of the $\mathbf{T}$ matrix, equations (5-16) through (5-31).

## Example All Inteaface Damping Eliminated

If damping is removed from all four interfaces of the five body model, $r_{j} \rightarrow 0, p_{j} \rightarrow 1, d_{i i 1} \rightarrow d_{i i}$, $d_{i j 2} \rightarrow d_{i i}, d_{i i 8} \rightarrow d_{i i}, d_{i i 4} \rightarrow d_{i i}, d_{i t} \rightarrow d_{i i}$ and $\Delta_{i t}^{\prime} \rightarrow \Delta_{i s}$.

$$
\begin{align*}
t_{i 2}= & \left\{\frac{\left[a_{41} a_{68}+a_{41} a_{67}+a_{48} a_{67}+\left(a_{41}+a_{46}+a_{88}+a_{67}\right) d_{i i}^{2}+d_{i i 1}^{4}\right]}{a_{28} a_{45} \Delta_{i 6}}\left(\Delta_{i 6}\right)_{1,3}\right. \\
& \left.-\frac{a_{67} a_{85}\left(a_{41}+a_{45}+d_{i i}^{2}\right)\left(a_{10,7}+d_{i i 1}^{2}\right)}{\Delta_{i 6}}\right\}\left(f_{i 1}+d_{i i} f_{i 3}\right) \\
& +a_{41}\left[\frac{\left(a_{68}+a_{67}+d_{i i}^{2}\right)\left(\Delta_{i 5}\right)_{1,3}}{a_{23} a_{45} \Delta_{i 5}}-\frac{a_{67} a_{85}\left(a_{10,7}+d_{i i}^{2}\right)}{\Delta_{i 6}}\right]\left(f_{i 3}+d_{i i} f_{i 4}\right) \\
& +\frac{a_{41} a_{63}\left(\Delta_{i 5}\right)_{1,2}}{a_{23} a_{45} \Delta_{i 6}}\left(f_{i 6}+d_{i i} f_{i 6}\right) \\
& +\frac{a_{41} a_{63} a_{85}}{\Delta_{i 5}}\left[\left(a_{10,7}+d_{i i}^{2}\right)\left(f_{i 7}+d_{i i} f_{i 8}\right)+a_{10,7}\left(f_{i 9}+d_{i i} f_{i 10}\right)\right] ; \quad i=1,2, \ldots, p \tag{5-32}
\end{align*}
$$

$$
\begin{aligned}
t_{i 4}= & {\left[\frac{\left(a_{63}+a_{67}+d_{i i}^{2}\right)\left(\Delta_{i 5}\right)_{1,3}}{a_{23} a_{45} \Delta_{i 5}}-\frac{a_{67} a_{85}\left(a_{10,7}+t_{i i}\right)}{\Delta_{i 6}}\right] } \\
& \times\left[a_{23}\left(f_{i 1}+d_{i i} f_{i 2}\right)+\left(a_{33}+d_{i i}^{2}\right)\left(f_{i 3}+d_{i i} f_{i 4}\right)\right]+\frac{\left(a_{23}+d_{i i}^{2}\right) a_{63}\left(\Delta_{i 5}\right)_{1,3}}{a_{23} a_{45} \Delta_{i 5}}\left(f_{i 5}+a_{i i} f_{i 6}\right) \\
& +\frac{a_{63} a_{86}\left(a_{23}+d_{i i}^{2}\right)}{\Delta_{i 5}}\left[\left(a_{10,7}+d_{i i}^{2}\right)\left(f_{i 7}+d_{i i} f_{i 8}\right)+a_{10,7}\left(f_{i 9}+d_{i i} f_{i 10}\right)\right]
\end{aligned}
$$

$$
t_{i 6}=\frac{\left(\Delta_{i 6}\right)_{1,3}}{a_{23} \Delta_{i 5}}\left[a_{23}\left(f_{i 1}+d_{i i} f_{i 2}\right)+\left(a_{23}+d_{i i}^{2}\right)\left(f_{i 3}+d_{i i} f_{i 4}\right)\right]
$$

$$
+\frac{\left(\Delta_{i 5}\right)_{1,3}\left(\Delta_{i 5}\right)_{5,3}}{a_{23} a_{45} a_{85} a_{10,7} \Delta_{i 5}}\left(f_{i 5}+d_{i i} f_{i 6}\right)
$$

$$
+\frac{\left(\Delta_{i 6}\right)_{5,3}}{a_{10,7} \Delta_{i 5}}\left[\left(a_{10,7}+d_{i i}^{2}\right)\left(f_{i 7}+d_{i i} f_{i 8}\right)+a_{10,7}\left(f_{i 9}+d_{i i} f_{i 10}\right)\right]
$$

$$
t_{i 8}=\frac{a_{45} a_{67}\left(a_{10,7}+d_{i i}^{2}\right)}{\Delta_{i 8}}\left[a_{23}\left(f_{i 1}+d_{i i} f_{i 2}\right)+\left(a_{2 a}+d_{i i}^{2}\right)\left(f_{i 3}+d_{i i} f_{i 4}\right)\right]
$$

$$
+\frac{a_{67}\left(\Delta_{i 5}\right)_{5,3}\left(a_{10,7}+d_{i i}^{2}\right)}{a_{85} a_{10,7} \Delta_{i 8}}\left(f_{i 6}+d_{i i} f_{i 6}\right)
$$

$$
+\left[\frac{\left(a_{62}+a_{87}+d_{i i}^{2}\right)\left(\Delta_{i 5}\right)_{5,3}}{a_{85} a_{10,7} \Delta_{i 5}}-\frac{\left(a_{23}+d_{i i 2}^{2}\right) a_{48} a_{86}}{\Delta_{i 6}}\right]
$$

$$
\begin{equation*}
\times\left[\left(a_{10,7}+d_{i i}^{2}\right)\left(f_{i 7}+d_{i i} f_{i 8}\right)+a_{10,7}\left(f_{i 9}+d_{i i} f_{i 10}\right)\right] \tag{5-35}
\end{equation*}
$$

$$
\begin{align*}
& t_{i 10}=\frac{a_{85} c_{87} a_{00}\left[a_{29}\left(f_{i 1}+d_{i i} f_{i 9}\right)+\left(a_{28}+d_{i i}\right)\left(f_{i 8}+d_{i i} f_{i 4}\right)\right]}{\Delta_{i 8}} \\
& +\frac{a_{878} a_{89}\left(\Delta_{i 6}\right)_{6,2}}{a_{85} a_{10,7} \Delta_{i 6}}\left(f_{i 6}+\alpha_{i i} f_{i 6}\right) \\
& +a_{60}\left[\frac{\left(a_{88}+a_{87}+d_{i i}^{2}\right)\left(\Delta_{i 6}\right)_{6,9}}{a_{65} a_{10,7} \Delta_{i 5}} \ldots \frac{\left(a_{28}+d_{i i}^{2}\right) a_{45} a_{88}}{\Delta_{i 6}}\right]\left(f_{i 7}+d_{i i} f_{i 8}\right) \\
& +\left\{\frac{\left[a_{83} a_{65}+a_{68} a_{80}+a_{67} a_{60}+\left(a_{89}+a_{67}+a_{85}+a_{80}\right) d_{i i}^{2}+d_{i i}^{4}\right]}{a_{85} a_{10,7} \Delta_{i 5}}\left(\Delta_{i 5}\right)_{5,3}\right. \\
& \left.-\frac{\left(a_{23}+d_{i i}^{2}\right) a_{45} a_{83}\left(a_{85}+a_{80}+d_{i i}^{2}\right)}{\Delta_{i 5}}\right\}\left(f_{i 9}+d_{i i} f_{i 10}\right)  \tag{j}\\
& \text { ORIGINA. PS: } \\
& t_{i 1}=d_{i i}\left[\left\{\frac{\left[\left(a_{41}+d_{i i}^{2}\right)\left(a_{62}+a_{67}+d_{i i}^{2}\right)+a_{45}\left(a_{67}+d_{i i}^{2}\right)\right]}{a_{38} a_{45} \Delta_{i 5}}\left(\Delta_{i 5}\right)_{1,3}\right.\right. \\
& \left.-\frac{a_{67} a_{85}\left(a_{10,7}+d_{i i}^{2}\right)\left(a_{41}+a_{45}+d_{i i}^{2}\right)}{\Delta_{i 5}}\right\}\left(f_{i 1}+d_{i i} f_{i a}\right) \\
& +\frac{a_{41}}{a_{73} a_{45} \Delta_{i 5}}\left\{\left[\left(a_{63}+a_{67}+d_{i i}^{2}\right)\left(\Delta_{i 5}\right)_{1,2}-a_{23} a_{45} a_{67} a_{85}\left(a_{10,7}+d_{i i}^{2}\right)\right]\left(f_{i 3}+d_{i i} f_{44}\right)\right. \\
& +a_{63}\left(\Delta_{i 5}\right)_{1,3}\left(f_{i s}+d_{i i} f_{i 6}\right) \\
& \left.\left.+a_{23} a_{45} a_{63} a_{85}\left[\left(a_{10,7}+d_{i i}^{R}\right)\left(f_{i 7}+d_{i i} f_{i 8}\right)+a_{10,7}\left(f_{i 9}+d_{i i} f_{i 10}\right)\right]\right\}\right]+f_{i 2} \tag{5-37}
\end{align*}
$$

$$
\begin{align*}
& t_{i 5}=-d_{i i}\left[\left[\frac{\left(a_{83}+a_{67}+d_{i i}^{2}\right)\left(\Delta_{i 6}\right)_{1,3}}{a_{45} \Delta_{i 5}}-\frac{a_{28} a_{67} a_{85}\left(a_{10,7}+d_{i i}^{2}\right)}{\Delta_{i 5}}\right]\left(f_{i 1}+d_{i i} f_{i 2}\right)\right. \\
& -\left\{\frac{\left[\left(a_{28}+d_{i i}^{P}\right)\left(a_{67}+d_{i i}^{2}\right)+a_{88}\left(a_{28}+d_{i i}^{2}\right)\right]\left(\Delta_{i 5}\right)_{1,3}}{a_{38} a_{45} \Delta_{i 5}}\right. \\
& \left.+\frac{a_{67} a_{85}\left(a_{23}+d_{i i}^{2}\right)\left(a_{10,7}+d_{i i}^{2}\right)}{\Delta_{i 6}}\right\}\left(f_{i 3}+d_{i i} f_{i 4}\right) \\
& -\frac{a_{68}\left(a_{2 a}+d_{i i}^{2}\right)\left(\Delta_{i 6}\right)_{1,3}}{a_{28} a_{45} \Delta_{i 5}}\left(f_{i 6}+d_{i i} f_{i 6}\right) \\
& \left.-\frac{a_{63} a_{85}\left(a_{23}+f_{M}^{2}\right)}{\Delta_{i 5}}\left[\left(a_{10,7}+a_{i i}^{2}\right)\left(f_{i 7}+d_{i i} f_{i 8}\right)+a_{10,7}\left(f_{i 9}+d_{i i} f_{i 10}\right)\right]\right]+f_{i 4} \tag{5-38}
\end{align*}
$$

$$
\begin{align*}
& +\frac{\left(\Delta_{i 6}\right)_{1,3}\left(\Delta_{i 6}\right)_{6,3}}{a_{38} a_{46} a_{66} a_{10,7} \Delta_{i 6}}\left(f_{i 6}+d_{i 8} f_{i 6}\right) \\
& \left.+\frac{\left(\Delta_{i 8}\right)_{5,9}}{a_{10,7} \Delta_{i 5}}\left[\left(a_{10,7}-d_{i i}\right)\left(f_{i 7}+d_{i i} f_{i B}\right)+a_{10,7}\left(f_{i 9}+d_{i i} f_{i 10}\right)\right]\right\}+f_{i 6}  \tag{5-39}\\
& t_{77}=-d_{i i}\left[\frac{a_{48} a_{67}\left(a_{10,7}+d_{i i}^{2}\right)}{a_{32} \Delta_{i 6}}\left[a_{33}\left(f_{i 1}+d_{i i} f_{i 3}\right)+\left(a_{38}+d_{i i}\right)\left(f_{i 3}+d_{i i} f_{i 4}\right)\right]\right. \\
& -\frac{a_{67}\left(a_{10,7}+d_{i i}^{2}\right)\left(\Delta_{i 6}\right)_{5,3}}{a_{85} a_{10,7} \Delta_{i 6}}\left(f_{i 6}+d_{i i} f_{i 6}\right) \\
& -\left(a_{10,7}+d_{i i}^{R}\right)\left\{\frac{\left[\left(a_{85}+d_{i i}^{2}\right)\left(a_{85}+d_{i i}^{2}\right)+a_{67}\right]}{a_{85} a_{10,7} \Delta_{i 5}}\left(\Delta_{i 5}\right)_{5,3}-\frac{\left(a_{23}+d_{i i}^{2}\right) a_{45} a_{63}}{\Delta_{i 5}}\right\}\left(f_{7}+d_{i i} f_{i 8}\right) \\
& \left.+\left[\frac{\left(a_{63}+a_{67}+d_{i i}^{2}\right)\left(\Delta_{i 5}\right)_{5,3}}{a_{65} \Delta_{i 5}}+\frac{\left(a_{23}+d_{i i}^{2}\right) a_{45} a_{63} a_{10,7}}{\Delta_{i 5}}\right]\left(f_{i 9}+d_{i i} f_{i 10}\right)\right]+f_{i 8}  \tag{5-40}\\
& t_{i 9}=d_{i i}\left[\frac { a _ { 8 0 } } { a _ { 8 5 } a _ { 1 0 , 7 } \Delta _ { i 8 } } \left\{a _ { 4 5 } a _ { 8 7 } a _ { 8 5 } a _ { 1 0 , 7 } \left[a_{23}\left(f_{i 1}+d_{i i} f_{i 2}\right)\right.\right.\right. \\
& \left.+\left(a_{22}+d_{i i}^{P}\right)\left(f_{i 8}+d_{i i} f_{i 4}\right)\right]+a_{67}\left(\Delta_{i 5}\right)_{\delta, \mathrm{a}}\left(f_{i 5}+d_{i i} f_{i 6}\right) \\
& \left.+\left[\left(a_{83}+a_{87}+d_{i i}^{2}\right)\left(\Delta_{i 5}\right)_{5,3}-\left(a_{33}+d_{i i}^{2}\right) a_{45} a_{63} a_{85} a_{10,7}\right]\left(f_{i 7}+d_{i i} f_{i 8}\right)\right\} \\
& +\left\{\frac{\left[a_{88} a_{86}+a_{68} a_{89}+a_{67} a_{89}+\left(a_{88}+a_{87}+a_{85}+a_{89}\right) d_{i i}^{2}+d_{i i}^{4}\right]\left(\Delta_{i 6}\right)_{5,3}}{a_{86} a_{10,7} \Delta_{i 8}}\right. \\
& \left.\left.-\frac{\left(a_{28}+d_{i i}^{2}\right) a_{45} a_{83}\left(a_{85}+a_{80}+d_{i i}^{2}\right)}{\Delta_{i 6}}\right\}\left(f_{i 9}+d_{i i} f_{i, 10}\right)\right]+f_{i, 10} \tag{5-41}
\end{align*}
$$

### 5.4 SOLUTION FOR SYNTELSEIED STATE VARIABLES

## 5.4 .1 Introduction

Inaccessibility of a scalar state variable in equations (5-0) and (5-7) is reflected by a corresponding noll column in the $\mathbf{C}$ and $\mathbf{F}$ matrices as implied in equation (2-14). For the generation of reduced state observers for the five body model, the number of inaccessible states can vary between one and nine.

### 6.4.2 First Order Observers ( $p=1$ )

An observer of order at least one is required when any one of the ten scalar state variables of the five body model is inaccessible. The first order form of the linear observer equation is as follows:

$$
\begin{equation*}
\dot{x}=d x+E x+G y \tag{5-42}
\end{equation*}
$$

## OF POOR QUALITX

The I and I matrice amociated with a first ordar obarver for the ive body model then reduce to the following row forme:

$$
\begin{align*}
& T=\left[\begin{array}{llll}
f_{1} & f_{2} & \cdots & f_{10}
\end{array}\right]  \tag{6-43}\\
& T=\left[\begin{array}{llll}
t_{1} & t_{2} & \cdots & t_{10}
\end{array}\right]
\end{align*}
$$

The observer synthesis equations are then of the form of equation (5-15) through (5-31) with i 31 . Since a fust order obecrver corresponds to one of the scalar state variables being inaccessible, one of the $f$; $(i=1,2, \ldots, 10)=0$.

## Example

Suppose the scalar state ropresenting the angular rate of body $5, z_{10}$, is insccessible. Then $f_{10}=0$ and the observer symthesis equations reduce to the form of equations ( $8-15$ ) through ( $5-31$ ) with the subscript, $i$, omitted and $f_{10}=0$. From equation ( $2-11$ ), the synthesized scalar state, $\hat{x}_{10}$, is expressed in terms of the scalar observer state variable $z$, and the accessible model scalar state variables as follows.

$$
\begin{equation*}
\hat{x}_{10}=\frac{1}{t_{10}}\left[z-\sum_{i=1}^{9} t_{i} x_{i}\right] \tag{5-45}
\end{equation*}
$$

For $x_{10}$ inaccessible, it is asoumed that:

$$
\mathbf{C}=\left[\begin{array}{llc} 
& & 0  \tag{5-46}\\
I_{9} & & \vdots \\
& & 0
\end{array}\right]
$$

where $I_{9}=9 \times 9$ identity matrix.
From $\mathbf{F}=\mathbf{G C}$,

$$
\mathbf{G}=\left[\begin{array}{llll}
f_{1} & f_{2} & \cdots & f_{9} \tag{5-47}
\end{array}\right]
$$

From $E=\mathbf{T B}$,

$$
I=\left[\begin{array}{lllll}
t_{2} & t_{4} & t_{6} & t_{8} & t_{10} \tag{5-48}
\end{array}\right] \text { for } r=5 \text { (control torques applied to all } 5 \text { bodies) }
$$

### 5.4.5 Observery of Order Greater Than One $(1<p<10)$

For those cases in which more than one of the ten scalar states of the five-body single-axis model are inaccessible, the minimum order of the reduced state linear observer required to reconstruct these inaccessible states is given by $p$. m each cace the number of mall colums in the measurement or observation matrix, $C$, and the $F$ matrix also is equal to $p$. The general forms of the $\mathrm{F}, \mathrm{F}$ and T matrices are given in equations $(5-12)$ through $(5-14)$ for $p=2,3,4,5,6,7,8$ or 9.

## Example

Suppose the scalar states, $x_{9}$ and $x_{10}$, which represent the angular position and rate of body 5 , are inaccessible. Then $f_{i o}=f_{i, 10}=0$ for $i=1,2$ and the observer synthesis equations reduce to the form of equations (5-14) through ( $5-3!$ ) with $i=1,2$ and $f_{i 0}=f_{i, 10}=0$. From equation (2-11) the synthesized scalar states, $\hat{z}_{9}$ and $\hat{x}_{10}$, are expressed in terms of the scaiar observer states, $z_{1}$ and $z_{2}$ and the accessible model scalar state variables as follows.

$$
\begin{equation*}
\dot{x}_{0}=\frac{\sum_{i=1}^{2}(-1)^{i+1}\left(\Delta_{2}\right)_{i, 1}\left(x_{i}-\sum_{j=1}^{8} t_{i j} x_{j}\right)}{\Delta_{2}} \tag{5-49}
\end{equation*}
$$

$$
\begin{equation*}
A_{10}=\frac{\sum_{i=1}^{2}(-1)^{i+1}\left(\Delta_{2}\right)_{i, 2}\left(s_{i}-\sum_{j=1}^{\ell} t_{i j} \varepsilon_{j}\right)}{\Delta_{2}} \tag{5-60}
\end{equation*}
$$

for

$$
\Delta_{2}=\left|\begin{array}{ll}
t_{10} & t_{1,10}  \tag{5-51}\\
t_{20} & t_{2,10}
\end{array}\right|=t_{10} t_{2,20}-t_{1,10} t_{29} \neq 0
$$

Where $\left(\Delta_{2}\right)_{i, j}=\Delta_{2}$ without the elemente of the $i^{\text {th }}$ row and $j^{\text {th }}$ column.
For $x_{0}$ and $x_{10}$ inaccessible, it in asmumed that:

$$
\mathbf{C}=\left[\begin{array}{llll} 
& 1 & 0 & 0  \tag{5-52}\\
\mathbf{I}_{4} & & \vdots & \vdots \\
& & 0 & 0
\end{array}\right]
$$

where $I_{8}$ is an $8 \times 8$ identity matrix.
Since $\mathbf{F}=\mathbf{G C}$,

$$
\mathbf{G}=\left[\begin{array}{lll}
f_{11} & \cdots & f_{18}  \tag{5-53}\\
f_{21} & \cdots & f_{28}
\end{array}\right]
$$

From $\mathrm{E}=\mathbf{T B}$,
$\mathbf{E}=\left[\begin{array}{lllll}t_{12} & t_{14} & t_{16} & t_{18} & t_{1,10} \\ t_{22} & t_{24} & t_{26} & t_{28} & t_{2,10}\end{array}\right]$ for $r=5$ (control torques applied to all five bodies)
$\mathbf{I}=\left[\begin{array}{lllll}t_{12} & t_{14} & t_{18} & t_{18} & 0 \\ t_{22} & t_{24} & t_{28} & t_{28} & 0\end{array}\right]$ for control torques applied to bodies $1,2,3$ and 4

### 6.5 REFERENCES

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## 8ECTION 6

## APPLICATION OF MODAL MODELING AND DPRCT MATREX PRODUCTS

### 6.1 INTRODUCTION

In the developmant of reduced atate obearvers for the claes of single-axis models of a fexible spacecraft presonted in sections 2 through 6 of this report, it was asomed that the atate vector coefficient matrix of the abeerver model was diagonal in order to reduce the amount of computation involved $n$ solving the observer synthesin equation,

$$
\begin{equation*}
\mathbf{T A}-\mathbf{D T}=\mathbf{F}, \tag{6-1}
\end{equation*}
$$

for the elements of the $\mathbf{T}$ matrix as a function of the elements of the $\mathbf{F}$ matrix where

$$
\begin{equation*}
\mathbf{F}=\mathbf{G C} \tag{6-2}
\end{equation*}
$$

Despite this rather arbitrary assumption, the computational effort involved in this solution grew with alarming rapidity as the number of flexibly connected rigid bodies incorporated in the single-axis model was increased. Furthermore, especially for the models incorporating both larger numbers of rigid bodies and damping, the assumption of a diagonal $\mathbf{D}$ matrix seemed a rather noor approximation in view of the considerable departure from diagonal form of the state vector coefficient matrices ( $A$ matrices) of these models. In view of these problems, Dr. Henry Waites (0-1) of Marshall Space Flight Center suggested that a more fruitful approach to synthesixing observers for this class of single-axis models of a flexible spacecraft treated in the preceding sections of this report would be based upon the following sequence of steps.

1. Recast the state variable forms of each undamped single-axis model into modal form.
2. Add modai damping to each modal model.
3. Recast the observer synthesis equation in terms of direct matrix products.

The advantages cited for this approach include the following.

1. The state vector coefficient matrix, A, in each modal single-axis model appears in $2 \times 2$ block diagonal form implying that the state vector coefficient matrix, $\mathbf{D}$, of the corresponding observer requived to accurately synthesire the inaccessible states would be no less sparse than $2 \times 2$ block diagonal.
2. The modal model is more amenable to truncation of less significant oscillatory modes.
3. The coefficient of damping associated with each vibrational mode can be specified at the outset of the analysis.

### 6.3 TRANSFORMATION OF TEIE TWO-BODY SINGLE-AXIS MODEL TO MODAL FORM

The approach atilized in tranoforming the undamped two-body single-axis model of Section 2 to damped modal form follows that presented in Thomson (6-2). It consists of the following steps.

1. Write original singleaxis model in ondamped form.
2. Write undamped single-acis model in terms of inertia and stiffness matrices.
3. Solve extended eigenvalue problem for eigenvalues and corresponding eigenvectors.
4. Normalize the eigenvectors.
5. Construct the modal matrix from the normalized eigenvectors.
6. Transform the model to principal coordinates (modal form) utilizing the modal matrix.
7. Add modal damping to the model in modal form.
8. Write the modal model with denoping in atate variable form.

### 6.2.1 Orighal Undamped Two-Body Singlo-Axis Model

The undamped form of equations (2-1) and (2-2) is the following.

$$
\begin{align*}
& I_{1} \theta_{1}=-h_{1} \theta_{1}+h_{1} \theta_{2}=q_{1}  \tag{6-3}\\
& I_{2} \dot{\theta}_{2}=-h_{1} \theta_{2}+h_{1} \theta_{1}=q 2 \tag{6-4}
\end{align*}
$$

whare the coefficients and variables appearing in this set of equations are defined in Fig. 2-1.

### 6.3.2 Undamped Two-Body Model in Turme of Inertia and Stifinegs Matrices

$$
\begin{equation*}
E X+K x=q \tag{6-5}
\end{equation*}
$$

where:

$$
\begin{aligned}
& \pi=\left[\begin{array}{ll}
\theta_{1} & \theta_{2}
\end{array}\right]^{T}=\left[\begin{array}{ll}
x_{1} & x_{2}
\end{array}\right]^{T} \\
& \mathbf{I}=\left[\begin{array}{cc}
I_{1} & 0 \\
0 & I_{2}
\end{array}\right]=\text { rotational inertia matrix } \\
& \mathbf{K}=k_{1}\left[\begin{array}{cc}
1 & -1 \\
-1 & 1
\end{array}\right]=\text { rotational stiffues matrix } \\
& \mathbf{q}=\left[\begin{array}{ll}
q_{1} & q_{2}
\end{array}\right]^{T}
\end{aligned}
$$

### 6.2.3 Determination of Eigenvalues and Eigenvectory

The eigenvalues for equation (6-5) are obtained by solving the extended eigenvalue problem which is eamivalent to solving the following equation for $\lambda$.

$$
\begin{equation*}
\lambda \mathbf{I} \mathbf{x}=\mathbf{K} \mathbf{x} \tag{8-8}
\end{equation*}
$$

An equivalent form of the above equation is:

$$
\begin{equation*}
|\lambda \mathbf{I}-\mathbf{K}| \boldsymbol{x}=0 \tag{6-7}
\end{equation*}
$$

A non-trivial solution of the extended eigenvalue problem exists if the following holds with the expanded forms of the rotational inertia and stifness matrices for the two-body model expressed immediately following equation (6-5)

$$
\begin{align*}
|\lambda I-\mathbf{K}| & =\left|\begin{array}{cc}
\lambda I_{1}-h_{1} & k_{1} \\
h_{1} & \lambda I_{2}-k_{1}
\end{array}\right| \\
& =I_{1} I_{2} \lambda\left(\lambda-h_{1} \frac{I_{1}+I_{2}}{I_{1} I_{2}}\right)=0 \tag{6-8}
\end{align*}
$$

The solutions for this ertended eigenvalue problem are:

$$
\begin{gather*}
\lambda_{1}=0,  \tag{6-9}\\
\lambda_{2}=\frac{k_{1}}{I_{1}}+\frac{k_{1}}{I_{2}}
\end{gather*}
$$

The ciscomectore corropending to $\lambda_{i}$ are obtained by solving equations of the following form for $\boldsymbol{v}_{i}$ whare $i=1,2$.

$$
\begin{equation*}
\left[\lambda_{i} I-E \mid V_{i}=0\right. \tag{6-10}
\end{equation*}
$$

The eigeavectors are normalised by solving the following equation for $c_{i}(i=1,2$

$$
\begin{equation*}
v_{i}^{T} I_{i}=1 \tag{0-11}
\end{equation*}
$$

The eigenvalues, eigenvectors and normalised eigenvector coefficients are displayed in Table b-1

### 6.2.4 Construction of the Modal Matrix

$$
P=\left[\begin{array}{lll}
v_{1} & \mid & v_{2}
\end{array}\right]=c_{1}\left[\begin{array}{cc}
1 & \left(\frac{I_{1}}{I_{2}}\right)^{-1 / 2}  \tag{0-12}\\
1 & -\left(\frac{I_{1}}{I_{2}}\right)^{1 / 2}
\end{array}\right]
$$

### 6.2.5 Tranaformation to Principal Coordinated

$$
\begin{align*}
& \mathbf{y}=\mathbf{p}^{x_{x}}  \tag{6-13}\\
& y_{1}=c_{1} x_{1}+c_{1} x_{2}=c_{1} \theta_{1}+c_{1} \theta_{2}  \tag{6-14}\\
& y_{2}=c_{1}\left(\frac{I_{1}}{I_{2}}\right)^{-1 / 2} x_{1}-c_{1}\left(\frac{I_{1}}{I_{2}}\right)^{1 / 2} x_{2}=c_{1}\left(\frac{I_{1}}{I_{2}}\right)^{-1 / 2} \theta_{1}-c_{1}\left(\frac{I_{1}}{I_{2}}\right)^{1 / 2} \theta_{2} \\
& \mathbf{q}^{\prime}=\mathbf{p}^{T} \mathbf{q}  \tag{6-18}\\
& q_{1}^{\prime}=c_{1} q_{1}+c_{1} q_{2}  \tag{6-17}\\
& q_{2}^{\prime}=c_{1}\left(\frac{I_{1}}{I_{2}}\right)^{-1 / 2} q_{1}-c_{1}\left(\frac{I_{1}}{I_{2}}\right)^{1 / 2} q_{2} \tag{6-18}
\end{align*}
$$

Model in Principal Coordinates (Modal Model)

$$
\begin{align*}
& \dot{y_{1}}=q_{1}  \tag{0-19}\\
& \bar{y}_{2}=-\omega_{1}^{2} y_{2}+q_{2}  \tag{6-20}\\
& \omega_{1}=\left(\frac{k_{1}}{I_{2}}+\frac{k_{1}}{I_{2}}\right)^{1 / 2} \tag{6-21}
\end{align*}
$$

### 6.2.6 Two Body Modal Model WIth Damping

Modal damping is added to the modal model described by equations (6-19) and (6-20) by adding a damping term to the equation with which the modal frequency, $w_{1}$, is associated. The two body model with damping in modal form then may be written as follows.

$$
\begin{align*}
& \dot{y}_{1}=q_{1}^{\prime}  \tag{0-22}\\
& \dot{y}_{2}=-2 \varsigma_{1} \omega_{1} \dot{y}_{2}-\omega_{1}^{2} y_{2}+q_{2}^{\prime} \tag{6-23}
\end{align*}
$$

where $s_{1}$ is the damping ratio associated with $w_{l}$.

## Ifsurvalues and Hisemvectors for Fach

## Mode of -wo Body Model

Mode No.

> Eigenvalue
> $\lambda_{i}=\omega_{i}^{2}$
Eisenvector

$$
\mathbf{v}_{i}
$$

Normalized Eigenvector Coefficient, $c_{i}$
$\pm\left(I_{1}+I_{2}\right)^{-1 / 2}$

2
0
$\frac{k_{1}}{I_{1}}+\frac{k_{1}}{I_{2}}$
$e_{1}\left[\begin{array}{l}1 \\ 1\end{array}\right]$
$c_{2}\left[\begin{array}{c}1 \\ \frac{-I_{1}}{I_{2}}\end{array}\right]$
$c_{1}\left(\frac{I_{2}}{I_{1}}\right)^{-1 / 2}$

### 6.2.7 State Variable form of the Two-Body Modal Modial with Damping

The subscript on $y_{2}$ hat been changed to " $3^{n}$ so that the following relationships can be used in constructing the state variable form of the modal model.

$$
\begin{align*}
& y_{2}=\dot{y}_{1}  \tag{6-24}\\
& y_{4}=\dot{y}_{3} \tag{8-25}
\end{align*}
$$

State Variable Modal Model

$$
\left[\begin{array}{l}
\dot{y}_{1}  \tag{6-26}\\
\dot{y}_{2} \\
\dot{y}_{8} \\
\dot{y}_{4}
\end{array}\right]=\left[\begin{array}{cccc}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -\omega_{1}^{2} & -2 s_{1} \omega_{1}
\end{array}\right]\left[\begin{array}{l}
y_{1} \\
y_{2} \\
\\
y_{8} \\
y_{4}
\end{array}\right]+\left[\begin{array}{ll}
0 & 0 \\
1 & 0 \\
0 & 0 \\
0 & 1
\end{array}\right]\left[\begin{array}{l}
q_{1} \\
q_{2}
\end{array}\right]
$$

### 6.5 TRANSFORMATION OF THRER-BODY SINGLT-AXIS MODEL TO MODAL FORM

### 6.3.1 Origmal Undamped Three-Body Singlo-Axis Modal

The undamped form of equations (3-1) through (3-3) is the following.

$$
\begin{align*}
& I_{1} \bar{\theta}_{1}=-k_{1} \theta_{1}+k_{1} \theta_{2}+q_{1}  \tag{6-27}\\
& I_{2} \bar{\theta}_{2}=k_{1} \theta_{1}+\left(k_{1}+k_{2}\right) \theta_{2}+k_{3} \theta_{3}+q_{2}  \tag{6-28}\\
& I_{3} \bar{\theta}_{3}=k_{2} \theta_{2}-k_{2} \theta_{3}+q_{3} \tag{b-29}
\end{align*}
$$

where the coefficients and varirbles appearing in this set oi equations are defined in Fig. 3-1.
6.s.2 Undamped Three-Body Model in Terms of Inortia and Stiffness Matrices

$$
\begin{equation*}
\mathbf{I x}+\mathbf{K x}=\mathbf{q} \tag{6-30}
\end{equation*}
$$

$x=\left[\begin{array}{lll}\theta_{1} & \theta_{2} & \theta_{3}\end{array}\right]^{T}=\left[\begin{array}{lll}x_{1} & x_{2} & x_{8}\end{array}\right]^{T}$
$I=\left[\begin{array}{rrr}I_{1} & 0 & 0 \\ 0 & I_{2} & 0 \\ 0 & 0 & I_{3}\end{array}\right]=$ rotational inertia matrix
$\mathbf{I}=\left[\begin{array}{ccc}h_{1} & -h_{1} & 0 \\ -h_{1} & h_{1}+h_{2} & -h_{2} \\ 0 & -i_{2} & h_{2}\end{array}\right]=$ rotational stiffucse matrix
$q=\left[\begin{array}{lll}q_{1} & q_{2} & a_{3}\end{array}\right]^{T}$

### 6.3.3 Determination of Migenvalues and Modad Irequencies

Corresponding to equation (0-30) an exteaded eigenvalue problem can be defined which consists of solving the following equation for $\lambda$.

$$
\begin{equation*}
\lambda \mathbf{I}=\mathbf{K} \mathbf{x} \tag{0-31}
\end{equation*}
$$

This is equivalent to setting the following determinant equal to zero

$$
\begin{align*}
|\lambda I-\mathbf{K}| & =\left|\begin{array}{ccc}
\lambda I_{1}-k_{1} & \dot{k}_{1} & 0 \\
k_{1} & \lambda I_{2}-\left(k_{1}+k_{2}\right) & k_{2} \\
0 & k_{2} & \lambda I_{3}-k_{2}
\end{array}\right| \\
& =I_{1} I_{2} I_{3} \lambda\left[\lambda^{2}-\left(\frac{k_{1}}{I_{1}}+\frac{k_{1}+k_{2}}{I_{2}}+\frac{k_{2}}{I_{3}}\right) \lambda+\frac{k_{1} k_{2}}{I_{1} I_{2}}+\frac{k_{1} k_{2}}{I_{1} I_{3}}+\frac{k_{1} k_{2}}{I_{2} I_{3}}\right]=0 \tag{6-32}
\end{align*}
$$

Since each solution for $\lambda$ corresponds to the square of a modal frequency, equation ( $6-32$ ) may also be written as follows.

$$
\begin{equation*}
\lambda\left(\lambda-\omega_{1}^{2}\right)\left(\lambda-\omega_{2}^{2}\right)=0 \tag{8-33}
\end{equation*}
$$

for which the solutions are: $\lambda_{1}=0, \lambda_{2}=\omega_{1}^{2}$ and $\lambda_{3}=\omega_{2}^{2}$.
The eigenvectors corresponding to $\lambda_{i}$ are obtained by solving equations of the form presented in equation ( $0-10$ ) for $\nabla_{i}$ where $i=1,2,3$. The resulting pairs of eigenvalues and eigenvectors are displayed in Table 8-2.

Application of the remaining steps in the approach utilized in Section 6.2 yields the following state variable modal form for the three-body single-axis model of a lexible spacecraft.

### 6.3.4 State Variable Form of Throo-Body Modai Model with Damping

$$
\begin{align*}
{\left[\begin{array}{l}
\dot{y}_{1} \\
\dot{y}_{2} \\
\dot{y}_{2} \\
\dot{y}_{4} \\
\dot{y_{5}} \\
\dot{y}_{6}
\end{array}\right] } & =\left[\begin{array}{lllccc}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & -\omega_{1}^{2} & -2 \varsigma_{1} \omega_{1} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & -\omega_{2}^{2} & -2 \varsigma_{2} \omega_{2}
\end{array}\right]\left[\begin{array}{l}
y_{1} \\
y_{2} \\
y_{8} \\
y_{4} \\
y_{5} \\
y_{6}
\end{array}\right] \\
& +\left[\begin{array}{lll}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{array}\right]\left[\begin{array}{l}
q_{1} \\
q_{2} \\
q_{2}^{\prime}
\end{array}\right] \tag{6-34}
\end{align*}
$$

## Thble 6-9

## Elgurviluee and Eiganwetors for Each Mode of Threo Body Model

Mode No.

$$
\begin{aligned}
& \text { Eigurviue } \\
& \lambda_{i}=\omega_{i}^{2}
\end{aligned}
$$

Normalized
Eisuavector
$V_{i}$
1
0
$c_{1}\left[\begin{array}{l}1 \\ 1 \\ 1\end{array}\right]$
$2 \quad \omega_{1}^{2} \quad c_{2}\left[\begin{array}{c}1 \\ \frac{k_{1}-\omega_{1}^{2} I_{1}}{k_{1}} \\ \frac{k_{1}-\omega_{1}^{2} I_{1}}{k_{2}-\omega_{1}^{2} I_{2}} \frac{k_{2}}{k_{1}}\end{array}\right]$
$3 \quad \omega_{3}^{2}\left[\begin{array}{c}1 \\ \frac{k_{1}-\omega_{2}^{2} I_{1}}{k_{1}} \\ \frac{k_{1}-\omega_{2}^{2} I_{1}}{h_{2}-\omega_{2}^{2} I_{3}} \frac{k_{2}}{h_{1}}\end{array}\right]$

$$
\pm\left[I_{1}+\left(\frac{k_{1}-r_{1} I_{1}}{k_{1}}\right)^{3} I_{2}+\left(\frac{k_{1}-r_{1} I_{1}}{k_{2}-r_{1} I_{3}} \frac{k_{2}}{k_{1}}\right)^{2} I_{3}\right]^{-1 / 2}
$$

$$
\pm\left[I_{1}+\left(\frac{k_{1}-r_{1} I_{1}}{k_{1}}\right)^{2} I_{2}+\left(\frac{k_{1}-r_{1} I_{1}}{k_{2}-r_{1} I_{3}} \frac{k_{1}}{k_{2}}\right)^{2} I_{3}\right]^{-1 / 2}
$$

### 6.4 EXTENSION OF RESULTS TO SINGLE-AXIS MODELS WITE FOUR OK MORE RIGID BODIES

lnepection of the state variable modal form of the three-body single-axis model with modal dampins in equation (0-34) and the corresponding modal two-body model in equation (0-20) reveals that the state vector coefficient matrix, A, in these modal models could be written in the following $2 \times 2$ block diagonal forms.

Two-Body Model:

$$
\Delta=\left[\begin{array}{cc}
\Delta_{11} & 0  \tag{6-35}\\
0 & \mathbf{A}_{22}
\end{array}\right]
$$

Three-Body Model:

$$
A=\left[\begin{array}{ccc}
\mathbf{A}_{11} & 0 & 0  \tag{6-36}\\
0 & \mathbf{A}_{22} & 0 \\
0 & 0 & \mathbf{A}_{32}
\end{array}\right]
$$

where $A_{i}$ and 0 are $2 \times 2$ anbmatrices.
Farthermore,

$$
A_{11}=\left[\begin{array}{ll}
0 & 1  \tag{6-37}\\
0 & 0
\end{array}\right]
$$

and

$$
\Delta_{u}=\left[\begin{array}{cc}
0 & 1  \tag{0-38}\\
-\omega_{-1}^{2} & -2 \theta-1 \omega_{1}
\end{array}\right] \operatorname{ter} i>1
$$

It aloo should be moted that the dimparions of each a matris are equal to twice the namber of rigid bodies in the modul. Application of the approsech atilized is traacforming the two-body and thro-body singlo-axis models to state veriable modal form with dampins yields a set of models that extend the patterns for thene modale. La particular, a singlo-mois model involving $P$ ripid bodies caa be tranformod to a modal state vaisble model with a at-to weltor coellicient matrix of the following form.

$$
\Delta=\left[\begin{array}{llll}
A_{11} & & &  \tag{-39}\\
& \Delta_{n 2} & & \cap \\
& & \ddots & \\
& & & A_{n / 2, n / 2}
\end{array}\right]
$$

which is a $2 \times 2$ block diagonal matrix of overall dimension $n \times n$ where $n=2 r$. The forms of the $2 \times 2$ mabmatriets along the principal diagonal of this coefficient matrix are given in equations (6-37) and (0-38) for $i=1,2, \ldots, n / 2$. The remaining elements in the $A$ matrix are zero.

## 6. 8 OBSERVER SYNTELSIS EQUATIONS EXPREESED IN TERMS OF DIRECT MATRIX PRODUCTS

The observer syathesis equations are exprissed in the following form in Section 2 of this report

$$
\begin{equation*}
\text { . } \mathbf{T A}-\mathbf{D T}=\mathbf{F} \equiv \mathbf{G C} \tag{0-40}
\end{equation*}
$$

for the gtate variable form of a single-axis model of a fexible spacecraft, with some scalar states inaccessible,

$$
\begin{align*}
& \dot{x}=A x+B u  \tag{0-41}\\
& x_{A}=C x \tag{0-42}
\end{align*}
$$

Where the corresponding reduced state observer is given by:

$$
\begin{align*}
& s=D s+E u+G x_{A}  \tag{6-13}\\
& s=\mathbf{I x} \tag{6-14}
\end{align*}
$$

The coefficient matrices and vectors appearing in equations (b-40) through (b-44) are defined for a linear model of dimencion $n$ with $m$ accessible sealar states and $p$ inzecessible sealar atates as follows.

A $=n \times n$ model state vector coefficient matrix
B $=n \times r$ model control vector coeficient matrix
C $=m \times n$ model measorement or observation matrix
D $=p \times p$ obeerver state vector coofficient matrix
E. $\quad \rho \times r$ observer control vector coefficient matrix

G $=p \times m$ observer oiseerved vector coefficient contrix
T $m p \times n$ tranformation matrix from model state ve.tor to oteerver gtate vector
$x=\left[\begin{array}{ll}x_{A}^{T} & x_{1}^{T}\end{array}\right]^{T}=n$-voctor of model gealar states
$x_{A}=m$-vector of accessible scalar states
$x_{1}=$ p-vector of inaccestible scalar atates

## - O procter of moranaliond conted terques

-     - precter of obearver scalar stateo

From theer definitioas, each of the matrix prodacte appearing in oquation (0-40) has the dimencions $p \times n$. If $I_{p}$ is defaed as the idontity matrix of dimenoions $p \times p$ and $I_{n}$ is defned correspondingly, then each of the matrix producta, I, TA and DXI aloo han the dimearionn $p \times n$. The obeorver synthecis equations may now be wittiat in the following form.

$$
\begin{equation*}
I_{p} T A-D I I_{n}=G C=T \tag{-45}
\end{equation*}
$$

Hownere, the defmition of a direct matrix product divea in Laseacter (0-3) may be used to write the obecrver synehesis equations is the following equivelems form

$$
\begin{equation*}
\left|I_{\rho} \otimes A^{T}-D \otimes I_{n}\right| \bar{T}=\bar{L} \tag{0-4}
\end{equation*}
$$

where:

$$
\begin{aligned}
& \overline{\mathbf{T}}=\left[\begin{array}{c}
\mathbf{T}_{1}^{T} \\
\mathbf{T}_{2}^{T} \\
\vdots \\
\mathbf{T}_{p}^{T}
\end{array}\right] \\
& \mathbf{T}_{i}^{T}=\text { vector comprised of the elemente of the } i^{\text {th }} \text { row of the } \mathbf{T} \text { matrix }
\end{aligned}
$$

and $\dot{F}$ is relaced to $F$ in the same way.
From Lancaster ( $1-3$ ), the direct matrix products appearing in equation (6-40) may be expanded ae follows.

$$
I_{p} \otimes A^{T}=\left[\begin{array}{lll}
A^{T} & & \cap  \tag{0-47}\\
& A^{T} & \bigcirc \\
& \ddots & A^{T}
\end{array}\right]_{n p \times n p}
$$

For

$$
\begin{align*}
& \text { D } \quad=\left[\begin{array}{ccc}
d_{11} & \cdots & d_{1 p} \\
\vdots & & \vdots \\
d_{p 1} & \cdots & d_{p p}
\end{array}\right]  \tag{0-48}\\
& D \otimes I_{n}=\left[\begin{array}{ccc}
d_{11} I_{n} & \cdots & d_{1}, I_{n} \\
\vdots & & \vdots \\
d_{p 1} I_{n} & \cdots & d_{p p} I_{n}
\end{array}\right]_{n p \times n p} \tag{8-49}
\end{align*}
$$

Solving for $\boldsymbol{T}$ yields

$$
\begin{equation*}
\mathbf{T}=\left[I_{p} \otimes A^{T}-D \otimes I_{n}\right]^{-1} \widetilde{G C} \tag{8-50}
\end{equation*}
$$

In general, this solution would require inversion of a matrix of dimension $n p \times n p$.

### 6.6 OBSEEVER SYNTERISTS EQUATIONS FOR SINGLE-AXIS MODELS IN MODAL FORM

If $A$ is the state vector coefficient matrix of a single-aris model in modal form, it was shown in Subsection s. 4 that it assumes the form given in equation ( $0-39$ ). The traspose of :ch $2 \times 2$ bleck diagonal matrix aso is block diagonal and of the following form.

$$
\mathbf{A}^{\boldsymbol{T}}=\left[\begin{array}{ccc}
\Delta_{11}^{T} & &  \tag{0-51}\\
& \mathbf{A}_{22}^{T} & \\
\\
0 & \ddots & \\
& & \\
\mathbf{A}_{n / 2, n / 2}^{T}
\end{array}\right]
$$

where each submatrix, $A_{i i}$, is $2 \times 2$ and the remaining elements in $A^{T}$ are zero.
For an observer of even order, $p$, of a state variable single-axis model in modal form with an $\mathbf{A}$ matrix of the $2 \times 2$ block diagonal form appearing in equation $(8-39)$ the observer state vector coefficient matrix is of the following $2 \times 2$ black diagonal form.

$$
D=\left[\begin{array}{ccc}
D_{11} & &  \tag{6-52}\\
& D_{22} & \\
& & \ddots
\end{array}\right)
$$

where $\mathbf{D}_{i i}=2 \times 2$ submatrix on the principal diagonal.

$$
D 刃 I_{n}=\left[\begin{array}{cccc}
D_{11} \otimes I_{n} & & &  \tag{6-5.3}\\
& D_{22} \otimes I_{n} & & \\
& & \ddots & \\
& & & D_{7 / 2, p / 2} \otimes I_{n}
\end{array}\right]
$$

where:

$$
\begin{align*}
\mathbf{D}_{11} & =\left[\begin{array}{ll}
d_{11} & d_{12} \\
d_{21} & d_{22}
\end{array}\right]  \tag{8-54}\\
& =\left[\begin{array}{ll}
d_{33} & d_{24} \\
\mathbf{D}_{22} & d_{44}
\end{array}\right]  \tag{8-55}\\
\vdots & =\left[\begin{array}{ll}
d_{p-2, p-2} & d_{p-2, p-1} \\
d_{p-1, p-2} & d_{p-1, p-1}
\end{array}\right]  \tag{6-56}\\
\mathbf{D}_{2-1, p-\frac{1}{2}} & =\left[\begin{array}{cc}
d_{p-1, p-1} & d_{p-1, p} \\
d_{p, p-1} & d_{p, p}
\end{array}\right] \tag{6-57}
\end{align*}
$$

The equar: an for genarating the elemente of the $T$ matrix may now be written in the followng form

where:

$$
\begin{align*}
& \mathbf{I}_{1}=\left[\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right] \\
& \mathbf{T}_{i j}=\left[\begin{array}{ll}
t_{i, 2 j-1} & t_{i, 2 j}
\end{array}\right]^{T} \quad i=1,2, \ldots, p  \tag{b-59}\\
& \mathbf{F}_{i j}=\left[\begin{array}{ll}
f_{i, 2 j-1} & f_{i, 2 j}
\end{array}\right]^{T} \tag{6-80}
\end{align*}
$$

It should be noted that all of the $2 \times 2$ submatrices appearing in the coefficient inatrices of equation set ( $6-58$ ) are commatative under maltiplication becanss this property is useful in the solution for the elements of the T matrix for $p>1$. Since all diagonal matrices commute, it is necessary only to show that the matrices, $\mathbf{A}_{j j}^{T}-d_{i k} I_{2}$ and $\mathbf{A}_{j j}^{T}-d_{r i} I_{2}$, commate for $j=1,2, \ldots, n / 2$.

$$
\begin{align*}
& {\left[A_{11}^{T}-d_{k+1} I_{2}\right]\left[A_{11}^{T}-d_{r a} I_{2}\right]=\left[\begin{array}{cc}
-d_{k t} & 0 \\
1 & -d_{k+1}
\end{array}\right]\left[\begin{array}{cc}
-d_{k H} & 0 \\
1 & -d_{k t}
\end{array}\right]} \\
& =\left[\begin{array}{cc}
d_{k H} d_{r s} & 0 \\
-\left(d_{k l}+d_{r c}\right) & d_{k l} d_{r e}
\end{array}\right]=\left[\mathbf{A}_{11}^{T}-d_{r s} I_{2} \mid\left[\mathbf{A}_{11}^{T}-1_{k l} I_{2}\right]\right. \tag{6-61}
\end{align*}
$$

$$
\begin{align*}
& =\left[\begin{array}{cc}
d_{k d} d_{r e}-\omega_{j-1}^{2} & \omega_{j-1}^{2}+2 \varsigma_{j-1} \omega_{j-1} \\
-\left(d_{k l}+d_{r e}+2 \varsigma_{j-1} \omega_{j-1}\right) & \left.d_{k l} d_{r e}+2 \varsigma_{j-1} \omega_{j-1}\left(d_{k l}+d_{r e}\right)+\left(4 \varsigma_{j-1}^{2}-1\right) \omega_{j-1}^{2}\right)
\end{array}\right] \\
& =\left[A_{j j}^{T}-d_{r o} I_{2}\right]\left[A_{j j}^{T}-C_{H H} I_{3}\right] \quad j=2,3, \ldots, n / 2 \tag{6-62}
\end{align*}
$$

Hence, the $2 \times 2$ submatrices, $A_{j j}-d_{i k} I_{2}$ and $A_{j j}-d_{r e} I_{2}$ are commutative under matrix multiplication for $j=1,2, \ldots, n / 2$.

For an even $p$ the solution for the elements of the $\mathbf{T}$ matrix now involves $n p / 4$ inversions of the coefficient matrices of dimensions $4 \times 4$ partitioned into $2 \times 2$ submatrices. Since the $2 \times 2$ subinatrices are commutative under multiplication, each of the $n p / 4$ vector-matrix equations of the set can be solved in terms of each $\mathbf{T}_{i j}$ which hat the effect of reducing dimensions of the anatrix inversion involved by a factor of two. The definitions of $T_{i j}$ and $F_{i j}$ in equations ( $0-59$ ) and ( $0-60$ ) in terms of the indevidual elements of the $T$ and Fmatrices would then be invoked to complete the solation.

For the case in which $p$ is an odd integer, the corresponding $\mathbf{D}$ matrix is $2 \times 2$ block diagonal except at one location along its principal diagonal where there occurs a degenerate "block" in the form of a single non-zero scalar element. With the assumption that the individual sealar state variables can be reordered, this isolated principal diagonal element can be placed at the lower right hand corner so that the $\mathbf{D}$ matrix assumes the following form.
$\mathbf{D}=\left[\begin{array}{llllll}D_{11} & & & & \\ & \mathbf{D}_{21} & & & \\ & & \ddots & & \\ & & & \mathbf{D}_{\frac{1-1}{2}, \frac{c 1}{2}} & \\ & & & & & d_{p, p}\end{array}\right]$

Where $D_{i i}$ are $2 \times 2$ sebmatrices defined in the same way as for $p$ even for $i=1,2, \ldots,(p-1) / 2$ and $d, p$ is a sealar on the principal diagonal of the $D$ matrix.

$$
D \otimes I_{n}=\left[\begin{array}{ccccc}
D_{11} \otimes I_{n} & & & &  \tag{6-64}\\
& D_{22} \otimes I_{n} & & & \\
& & \ddots & & \\
& & & D_{2 \mp 1, q_{1}} \otimes I_{n} & \\
& & & & d_{p, p} \otimes I_{n}
\end{array}\right]
$$

By the aame procedure as utilized for even $p$ the equations for generating the elements of the $T$ matrix when $p$ is odd may be expressed in the following form with $F_{i j}$ and $T_{i j}$ defined in equations (6-59) and (6-60).

$$
\left[\begin{array}{ccccc}
\mathbf{A}_{j j}^{T}-d_{11} \mathbf{I}_{2} & \mathbf{A}_{j j}^{T}-d_{12} \mathbf{I}_{2} & & &  \tag{6-65}\\
\mathbf{A}_{j j}^{T}-d_{21} \mathbf{I}_{2} & \mathbf{A}_{j j}^{T}-d_{22} \mathbf{I}_{2} & & & \\
& & & \ddots & \\
& & & \mathbf{A}_{j j}^{T}-d_{p-2, p-2} \mathbf{I}_{2} & -d_{p-2, p-1} \mathbf{I}_{2} \\
& & & -d_{p-1, p-2} \mathbf{I}_{2} & \mathbf{A}_{j j}^{T}-d_{p-1, p-1} \mathbf{I}_{2} \\
& & & \\
& & & \mathbf{A}_{j j}^{T}-d_{p, p} \mathbf{I}_{2}
\end{array}\right]\left[\begin{array}{c}
\mathbf{T}_{1 j} \\
\mathbf{T}_{2 j} \\
\vdots \\
\mathbf{T}_{p-3, j} \\
\mathbf{T}_{p-1, j} \\
\mathbf{T}_{p j}
\end{array}\right]=\left[\begin{array}{c}
\mathbf{F}_{1 j} \\
\mathbf{F}_{2 j} \\
\vdots \\
\mathbf{F}_{p-2, j} \\
\mathbf{F}_{p-1, j} \\
\mathbf{F}_{p j}
\end{array}\right]
$$

From equation get ( $6-65$ ), it is evident that when $p$ is an odd integer, $n / 2$ of the equations in the set reduce to the form,

$$
\begin{equation*}
\left[\mathbf{A}_{j j}^{T}-d_{p, p} \mathbf{I}_{2}\right] \mathbf{T}_{p, j}=\mathbf{F}_{p, j} \quad j=1,2 \ldots, \frac{n}{2} \tag{6-66}
\end{equation*}
$$

where it has been assumed that the state variabie model can be rearranged, if necessary, so that this vectormatrix equation appears last in each of the $\frac{n}{2}$ sets of equations.

From equation (6-37),

$$
\left[A_{11}^{T}-d_{k l} I_{2}\right]^{-1}=\frac{\left[\begin{array}{cc}
-d_{k t} & 0  \tag{6-87}\\
-1 & -d_{k t}
\end{array}\right]}{d_{k-1}^{2}}
$$

From equation (6-38),

$$
\left[\mathbf{A}_{j j}^{T}-d_{k-1} I_{2}\right]^{-1}=\frac{\left[\begin{array}{cc}
-\left(2 \varsigma_{j-1} \omega_{j-1}+d_{k-1}\right) & \omega_{j-1}^{2}  \tag{6-88}\\
-1 & -d_{k l}
\end{array}\right]}{\omega_{j-1}^{2}+2 d_{k 1} \varsigma_{j-1} \omega_{j-1}+d_{k l}^{2}}
$$

Hence, for the case in which the order of the observer, $p$, is an odd integer the solution for the elements of the $\mathbf{T}$ matrix in terms of the elements of the $\mathbf{F}$ matrix reduces to $n(p-1) / 4$ inversions of coefficient matrices of dimension $4 \times 4$ partitioned into $2 \times 2$ submatrices and $n / 2$ inversions of coefficient matrices of dimensions $2 \times 2$. Since all of the $2 \times 2$ matrices of equaticn set ( $0-58$ ) commote under maltiplication, the dimensions of the $n(p-1) / 4$ coefficient matrices to be inverted are in effect reduced by a factor of two by first solving for the $\mathbf{T}_{i j}$ 's in terms of the $\mathbf{F}_{i j}$ 's and then applying equations (b-59) and (6-60).

### 6.6.1 First Ordar Obsarvers ( $p=1$ )

Since $p$ is an odd integer, equation (6-85) applies and reduces to the following form.

$$
\begin{equation*}
\left[A_{j j}^{T}-d_{11} I_{2}\right] \mathbf{T}_{1 j}=\mathbf{F}_{1 j}, \quad j=1,2, \ldots, n / 2 \tag{6-69}
\end{equation*}
$$

where $F_{1 j}$ and $T_{1 j}$ are defined in equations (6-59) and (6-60) and the $F$ matrix, whirh has a single row, may be written as follows:

$$
\mathbf{F}=\left[\begin{array}{llll}
\mathbf{F}_{11}^{T} & \mathbf{F}_{12}^{T} & \cdots & \mathbf{F}_{1, n / 2}^{T} \tag{6-70}
\end{array}\right]^{T}
$$

The corroponding single row $T$ matrix mov be written in the same form at the $F$ matrix. Solving oquation (b-0) for $\mathrm{T}_{i j}$ yield the following:

$$
\begin{equation*}
T_{1 j}=\left[\Lambda_{j j}^{T}-d_{11} I_{2}\right]^{-1} F_{1 j} \tag{6-71}
\end{equation*}
$$

which, in view of oquations ( $0-59$ ) and ( $0-60$ ), becomes:

$$
\left[\begin{array}{c}
i_{1}, 2_{i}-  \tag{0-72}\\
t_{1,2 j}
\end{array}\right]=\left[\Lambda_{j j}^{T}-d_{11} I_{2}\right]^{-1}\left[\begin{array}{c}
f_{2,2 j-1} \\
f_{2,2 j}
\end{array}\right]
$$

where $\left[\Lambda_{11}^{T}-d_{11} I_{2}\right]^{-1}$ is given in equation ( $0-67$ ) and $\left[\Lambda_{j j}^{T}-d_{11} I_{2}\right]^{-1}$ is given in equation set ( $0-68$ ) for
$j=2,3, \ldots, n / 2$.

### 6.6.2 8econd Order Obeervers ( $p=2$ )

Since $p$ is even, equation (0-58) applies. For $p=2$ it reduces to:

$$
\left[\begin{array}{cc}
\Lambda_{j j}^{T}-d_{11} I_{2} & -d_{12} I_{2}  \tag{0-73}\\
-d_{21} I_{2} & A_{j j}^{T}-d_{22} I_{2}
\end{array}\right]\left[\begin{array}{l}
\mathbf{T}_{1 j} \\
\mathbf{T}_{2 j}
\end{array}\right]=\left[\begin{array}{c}
\mathbf{F}_{1 j} \\
\mathbf{F}_{2 j}
\end{array}\right] ; \quad j=1,2, \ldots, \frac{n}{2}
$$

where $T_{i j}$ and $F_{i j}$ are defined in equations ( $6-59$ ) and ( $6-60$ ).

$$
\Delta_{2 j}=\left[\begin{array}{cc}
A_{j j}^{T}-d_{11} I_{2} & -d_{12} I_{2}  \tag{6-74}\\
-d_{21} I_{2} & A_{j j}^{T}-d_{22} I_{2}
\end{array}\right]
$$

Since $\boldsymbol{A}_{j j}^{T}-d_{k 1} I_{2}$ commates with $\boldsymbol{A}_{j j}^{T}-d_{r e} I_{2}$ under matrix maltiplication

$$
\begin{equation*}
\left|\Delta_{2 j}\right|=\left|\left[\Lambda_{j j}^{T}-d_{11} I_{2}\right]\left[\Lambda_{j j}^{T}-d_{22} I_{2}\right]-d_{12} d_{21} I_{2}\right| \tag{6-75}
\end{equation*}
$$

Then

$$
\begin{align*}
& \mathbf{T}_{1 j}=\frac{\Delta_{j j}^{T}-d_{22} I_{2}}{\left|\Delta_{2 j}\right|} \mathbf{F}_{1 j}+\frac{d_{12} I_{2}}{\left|\Delta_{2 j}\right|} \mathbf{F}_{2 j ;} \quad j=1,2, \ldots, \frac{n}{2}  \tag{6-76}\\
& \mathbf{T}_{2 j}=\frac{d_{21} \mathbf{I}_{2}}{\left|\Delta_{2 j}\right|} \mathbf{F}_{1 j}+\frac{A_{j i}^{T}-d_{11} I_{2}}{\left|\Delta_{2 j}\right|} \mathbf{F}_{2 j} \tag{0-77}
\end{align*}
$$

Solutions for the individual elements of the $\mathbf{T}$ matrix in terms of the elements of the $\mathbf{F}$ matrix are then obtained by application of the definitions of $T_{i j}$ and $F_{i j}$ in equations (b-59) and ( $8-60$ ).

$$
\begin{align*}
& {\left[\begin{array}{c}
t_{1,2 j-1} \\
t_{1,2 j}
\end{array}\right]=\frac{\mid \mathbf{A}_{j j}^{T}-d_{23} I_{2}}{\left|\Delta_{2 j}\right|}\left[\begin{array}{c}
f_{1,2 j-1} \\
f_{1,2 j}
\end{array}\right]+\frac{d_{12}}{\left|\Delta_{2 j}\right|}\left[\begin{array}{c}
f_{2,2 j-1} \\
f_{2,2 j}
\end{array}\right]} \\
& {\left[\begin{array}{c}
t_{2,2 j-1} \\
t_{2,2 j}
\end{array}\right]=\frac{d_{12}}{\left|\Delta_{2 j}\right|}\left[\begin{array}{c}
f_{1,2 j-1} \\
f_{1,2 j}
\end{array}\right]+\frac{\left[\mathbf{A}_{j j}^{T}-d_{11} I_{2}\right]}{\left|\Delta_{2 j}\right|}\left[\begin{array}{c}
f_{1,2 j-1} \\
f_{2,2 j}
\end{array}\right]} \tag{6-78}
\end{align*}
$$

## 6.6.s Observere of Eigher Order

For $p=3$ equation ( $0-65$ ) reduces to the following.

$$
\left[\begin{array}{cc}
\mathbf{A}_{j j}^{T}-d_{11} \mathbf{I}_{2} & -d_{12} \mathbf{I}_{2}  \tag{6-73}\\
-d_{21} \boldsymbol{I}_{2} & \mathbf{A}_{j j}^{T}-d_{22} \mathbf{I}_{2}
\end{array}\right]\left[\begin{array}{l}
\mathbf{T}_{1 j} \\
\mathbf{T}_{2 j}
\end{array}\right]=\left[\begin{array}{l}
\mathbf{T}_{1 j} \\
\mathbf{T}_{2 j}
\end{array}\right]
$$

ORiGinin sex
OF POOR QUALIT

$$
\begin{equation*}
\left[\Lambda_{j j}^{T}-d_{03} I_{H}\right] T_{a j}=P_{a j} ; \quad j=1,2, \ldots, \frac{n}{2} \tag{0-80}
\end{equation*}
$$

The obeerver synthesis equations for $p=3$ differ from thoo for $p=2$ by the addition of equation set (b80) which is of the same form as the observer synthenis equations for $p=1$ with the subscript, 3 , substituted for the subecript, 1. Therefore, the solutions for the elements of the first two rows of the $\mathbf{T}$ matrix for $p=3$ are identical with those for the two rowe of the matrix for $p=2$, equations sets ( $0-76$ ) and ( $6-77$ ). While the solutions for the elements of the third row are of the same form a thone for $T$ matrix for $p=1$, equation set (0-71) with the mumarieal subecript, 3 , substituted for the subscript, 1.

For $p=4$ equation set $(0-58)$ reduces to one equation set identical with the one for $p=2$, and another equation set of the same form with each nomerical subscript incremented by one. Hence, the solutions for the clemants for the frrst two rows of the $\mathbf{T}$ matrix for $p=4$ are identical with those for the two rows of the I matrix for $p=2$, equation set (6-76) and (6-77). The solutions for the elements of the third and fourth rows are given by the same equations with each of the numerical subscripts incremented by one.

The solutions for the elemente of the $\mathbf{T}$ matrix for larger values of $p$ follow the same pattern. Thus, they can be constructed directly by using the solutions for $p=1$ and $p=2$ as "building blocks" as was demonstrated for $p=3$ and $p=4$.

### 6.7 SOLUTION FOR SYNTHESEMTD STATE VAi工IABLES

In Subsection 8.4 it was shown that the single-axis modal models of a flexible spacecraft treated in this report can be written in the state variable form in terms of the modal state vector as follows.

$$
\begin{align*}
& \dot{\mathbf{y}}=\mathbf{A y}+B \mathbf{u}  \tag{6-81}\\
& \mathbf{y}_{\mathrm{A}}=\mathbf{C y} \tag{6-82}
\end{align*}
$$

where

A $=n \times n$ state vector coefficient matrix
B $=n \times r$ control vector coefficient matrix
C $=m \times n$ observation or measurement matrix
$y=$ modal state vector of dimension $n$
$y_{A}=$ vector of accessible modal state variables of dimensicn $m$
$\mathbf{u}=$ control vector of dimension $r$
The block diagram corresponding to this model is the same as Fig. 2-2 except that the vectors, $\mathrm{x}_{\text {and }} \mathrm{x}_{\mathrm{A}}$, are replaced by $y$ and $y_{A}$, respectively. The $2 \times 2$ block diagonal form of the $A$ matrix is shown in equation (6-39).

If the nomber of inaccessible modal model scalar states is given by $p=n-r(1<p<n)$, then the corresponding reduced modal state linear observer model is the ioilowing.

$$
\begin{align*}
& \dot{\mathrm{s}}=\mathrm{Dz}+\mathrm{Eu}+\mathrm{Gy}_{\mathrm{A}}  \tag{6-83}\\
& \mathrm{~s}=\mathrm{Ty} \tag{6-84}
\end{align*}
$$

where

D $=p \times p$ observer state vector coefficient matrix
$\mathbf{E}=p \times 9$ observer control vector coefficient matrix
G $=p \times m$ observer observed vector coefficient matrix
$\mathbf{T}=p \times n$ observer weighting matrix

## E E obearvir tante vector

The block diegram for this obeorver is the same as that shown in Fis. 2-4 except that the vector, y, is gubetitusted for the vector $\pi$.

Atter the obeorver systhecis equation given by equation (0-40) or one of its equivalent forms such as equation aet (0-58) for even $p$ and equation set ( $0-65$ ) for odd $p$ has been solved for the elements of the $T$ matrix, $t_{i j}$, equatioa (0-8t) caa be solved to exprese the symetheaired inaceesaible modal model scalar states in terme of the accesaible modal moded scalur states. This lact stop generally will require the inversion of a $p \times p$ matrix. A block diagram of the modal model of a flexible spacecraft and its reduced state linear obneaver appears in Fig. 6-1.

## Example: Solation for two synthenised modal states in the two body model.

Suppoee that for the state variable modal two-body model the modal scalar states, $y_{8}$ and $y_{4}$ are inaccessible. Then $p=2, n=4$ and the $T$ matrix is thus:

$$
T=\left[\begin{array}{llll}
t_{11} & t_{12} & t_{13} & t_{14}  \tag{6-85}\\
t_{21} & t_{22} & t_{23} & t_{24}
\end{array}\right]
$$

corresponding to:

$$
\begin{align*}
& y=\left[\begin{array}{llll}
y_{1} & y_{2} & y_{8} & y_{4}
\end{array}\right]^{T}  \tag{8-86}\\
& y=\left[\begin{array}{ll}
z_{1} & z_{2}
\end{array}\right]^{T} \tag{6-87}
\end{align*}
$$

If the remaining modal model scalar states, $y_{i}$ and $y_{2}$ and all of the elements of the $\mathbf{T}$ matrix are known then equation (0-84) can be solved to express the synthesized inaccessible modal model scalar states $\hat{y}$ a and $\hat{y}_{4}$ as follows.

$$
\begin{align*}
& \hat{y}_{3}=\frac{\left(\Delta_{2}\right)_{1,1}}{\Delta_{2}}\left(z_{1}-t_{11} y_{1}-t_{12} y_{2}\right)-\frac{\left(\Delta_{2}\right)_{2,1}}{\Delta_{2}}\left(z_{2}-t_{21} y_{1}-t_{22} y_{2}\right)  \tag{6-88}\\
& \hat{y}_{4}=\frac{\left(\Delta_{2}\right)_{1,2}}{\Delta_{2}}\left(z_{1}-t_{11} y_{1}-t_{12} y_{2}\right)-\frac{\left(\Delta_{2}\right)_{2,2}}{\Delta_{2}}\left(z_{2}-t_{21} y_{1}-t_{22} y_{2}\right) \tag{6-89}
\end{align*}
$$

where

$$
\Delta_{2}=\left|\begin{array}{ll}
t_{11} & t_{12}  \tag{6-90}\\
t_{21} & t_{22}
\end{array}\right|=t_{11} t_{23}-t_{12} t_{21} \neq 0
$$

$\left(\Delta_{2}\right)_{i, j}=\Delta_{2}$ without the elements of the $8^{\text {th }}$ row and the $j^{t h}$ column.

### 6.8 REPMRENCES

6-1 Waites, H.B., Marshall Space Flignt Center, Huntsville, AL, telephone conversation of March 8, 1984.
6-2 Thomson, W.T., Theory of Vibrations with Applications. Englewood Cliffs, N.J.: Prentice-Hall, Inc., 1972, pp. 22e-230.
6-3 Lancaster, Peter, Theory of Matrices. New York: Academic Press, 1959, p. 256.

$\underline{\underline{u}}=$ vector of scalar inputs to vehicle modal model
$y_{A}=$ vector of accessible states of modal model
$\underline{z}=$ vector of scalar states of observer
$T=$ observer weighting matrix
$\underline{\hat{y}}_{I}=$ vector of reconstructed scalar states of model


Derias the period coverod by this report, the clace of single-axis state variable models with some inaceresible states wae extended three waye.

1. The patterns involved in the prior development of the state variable forms of the two-body, three-body, and loarbody single-axie modele of a fiexible spacecrats were extended to prodnce a $\mathfrak{i v e}$-body model that could represent the single axis that wa found to be decoupled from the remaining axes of a ive-body three axis model treated in earlier work.
2. A rotational damping coefficient was added to each fexible interconnection between the ripid bodies comprising esch model.
3. Each undamped single aris model was transformed to a modal model with one or more inaccessible model state variables.

For each combination of single axis state variable model and inaccessible gealar state(s) a reduced state linear observer was generated to reconutruct those scalar states that were inaccessible. This was done because the application of linear quadratic regulator (LQR) and closely related time domain approarhes to attitude control utilize all or nearly all of the scalar gtates of the model of the spacecraft to be controlled.

## 7. 1 CONCLUSIONS

The following conclusions were drawn mainly from the development of the damped two-, thrse-, fourand 6 ive-body single-axis models with inaccessible scalar state variables of a prototype flexible spacecraft and the generation of the correspoading linear observers of minimum order required to reconstruct these inaccessible scalar states.

### 7.1.1 Observer: Generated for Singlo-Axis Models Based on Angular Displacement and Rate State Variables

1. Since, of the four coefficient matrices appearing in the observer synthesis equation, $\mathbf{A}, \mathbf{D}, \mathbf{F}$ and $\mathbf{T}$, only the state vector coefficient matrix in the single-axis model, $\mathbf{A}$, is known a priori, the following approach was used to generate the elements of the coefficient matrix, $\mathbf{T}$, for the transformation from the state vector of the model, $x$, to the state vector of the observer, $x$.
2. The elements of $\mathbf{P}$ ean be determined by utilizing the known values for the elements of $\mathbf{C}$, the abservation matrix in the single-axis model and the assumed values of the elements of $\mathbf{G}$ in conjunction with the equation, $\mathbf{F}=\mathbf{G C}$.
b. Assuming that $\mathbf{D}$ is diagonal simplifies significantly the solution of the equations for determining the elements of $T$.
3. The minimam order required for a reduced state linear observer to reconstruct $p$ inaccessible scalar otatej of a single-axis state variable model with a total of $n$ gcalar states is $p$ where $p=1,2, \ldots, n-1$. Therefore, the number of elements in the $\mathbf{T}$ matrix to be determined equals np and solving for the $p$ inaccessible synthesized scalar state variables requires the inversion of ap×p coefficient matrix.
4. The rigid-body flexible-joint single-axis models of a flexible spacecraft treated in this report are is a more general form when damping is added to each joint connecting the rigid bodies. Therefore it is far easier to develop the observer synthesis equations for the damped models than to begin with the equations for the undamped models and generalize them to account for the effects of added damping.
5. If $n-1$ of the $n$ sealar state variables of the single-axis model are accessible, a reduced state observer of order at least one ( $p=1$ ) is required to synthesize the inaccessible staie variable. The number of elements in the $\mathbf{T}$ matrix to be determined equals $n$ and solving for the one inaccessible sealar state variable does not require the inversion of a matrix.
6. As the number of accoserible sealar state vaisbles decruaces, the number of insccessible sealar states, $p$, the monber of elomente in the T matrix to be determined, $n \mathrm{p}$ and the dimensions of the coefficient matris to be invested in solviag for the insceessible syntherised scalar state variables, $p \times p$, increase, whick incroaces the namber of compatations required.
C. At leaot one of the $n$ state variables of the single axis model must be accessible in order for the imaccessible state variables to be ayntheirixed by a reduced state observer.

### 9.1.2 Obeervers Generated for Singlo-Acie Models Based on Modal State Variables

1. The state vector coefficiont matrix, A, ia each modal singlo-axis model appears in $2 \times 2$ block diagonal form implying that the state vector coefifienf matrix, $\mathbf{D}$, in the corresponding reduced state observer is $2 \times 2$ block diagonal.
2. Whan the obeerver synthesis equatios is expressed in terms of direct matrix products, solution for the clements of the $\mathbf{T}$ matrix generally requiren inversion of an np $\times n \mathrm{np}$ coefficient matrix.
3. When the number of inaccessible states of the model, $p$, is even, use of $A$ and $D$ matrices in $2 \times 2$ block diagonal form in the observer gynthesis equation redaces the solution for the elements of the $\mathbf{T}$ matrix to the inversion of $\frac{n 8}{4} 4 x+$ matrices partitioned into $2 \times 2$ submatrices all of which commute ander maultiplication.
4. When the number of inaccegsible states of the model is odd, use of $\mathbf{A}$ and $\mathbf{D}$ matrices in $2 \times 2$ block diagonal form in the observer synthesis equation reduces the solution for the elements of the $\mathbf{T}$ matrix to the inversion of $\frac{n(p-1)}{4} 4 \times 4$ matrices partitioned into $2 \times 2$ commutative submatrices and $\frac{n}{2} 2 \times 2$ matrices.
5. The modal matrix operates on only the angular displacement gtate variables and thus earh modal state variable generally is a weighted linear sam of all of the angular displacements.
a. Reduced state linear ohservers predicated upon a modal single axis model generally require that at least one of the modal state variables be accessitle which is equivalent to resuiring that all of the angular displacement state variables of the original state variable model be accessible.
b. Reduced state observers based on the modal model can be used to synthesize one or more inacceasible angular rate state variables of the original state model.
c. If no modal state variable is accessible or, equivalently, if any one of the angular displacement state variables for the original g+ ate model is inaccessible, redured state observers predicated upon the modal model cannot be used to synthesize any state variables.
6. Even if all of the necesgary conditioas required for synthesis of state variables by a reduced state observer predicated upon a modal single axis model are satisfied, two significant disadvantages of this approach are the following:
a. Modal state variables that are weighted sums of angular displacements and rate state variables are diffienlt to inte rpret physieally.
b. Transformation from the modal state variables to the angular displacement and rate state variables many be very complicated.

### 7.2 RECOMMENDATIONS

The following directions are suggested for future stady in the application of attitude control to state variable models of flexible spacecraft for which one or more sealar states are inaccessible.

1. The modular control techniques developed for the attitude control of model, of tiexible spacecraft for which all scalar state variables are accessible should be modified for arparation to seri•s of single axis models and their associated reduced state linear observers developed in the work treated in this report.
2. Selected combinations of single axis model and its associated linear observer and modular attitude control system should be simulated on a digital computer to support investigation (f effects of changes in the following single-axis model and observer characteristics.
3. Ratioe betwoen the manoen (rotational inertion) of bodiee comprising the singlo-acis model.
b. Magritudes of apring and demping coofficiente at the interfices between the ripid bodies of the singloaxis model.
4. The generation of reduced state oboervers to reconotruct inaceessible scalar states of a model of a fexible apacecrath should be extended to the thrve-axin Aivebody model of a prototype fexible spacecraft doveloped eartiar.
5. The application of modaler techniques to the attitude control of selected combinations of a single-axis model and ite corresponding roduced otate linenr observer should be extended to the combinations of the aingleaxis and two-aris Ave body modele representing the prototype flexible spececratt and the corresponding roduced state oberrvers.
6. The combination of gingloasir and two-axis five body models and their linear observers and modular attitade control systems should be simalated on a digital computer.
7. Coefficients representing the sensitivity of the scalar states to parameters of the corrbination of singie-axis and two-axis five body models and their linear observers and modular attitude control systems should be developed.
8. Since, for a model with $n$ scalar state variables, the number of elements of the $\mathbf{T}$ matrix to be deternined, $n p$, and the dimensions of the coefflient matrices to be inverted in solving for the synthesized inaccessible variables, $p \times p$, incrasee as the mumber of inaccessible variables, $p$, increases, it would be desirable to determine whether there is a value for the ratio, $\frac{p}{n}$, at which a fall state observer would be more readily implemented than a reduced state observer.
9. In view of the especially convenient forms of the modal state variable single-axis models of the flexible spacerraft and of the corresponding observer synthesis equation it appears worthwhile to investigate ways to mitigate the requirement that all rotational digplacement state variables in the original model be accessible.
