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THE
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CORPORATION

GUIDANCE
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DIVISION

TETERBORO
NEW JERSEY 07608

MODULAR DESIGN ATTITUDE CONTROL SYSTEM

FINAL REPORT
5 OCTOBER 1964

PREPARED FOR:

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HUNTSVILLE, ALABAMA

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FOREWORD

This final report is submitted in accordance with "Scope of Work, Exhibit D" for Contract NAS8-33979. The study was directed from the Guidance Systems Division (GSD) of the Allied Bendix Corporation. The engineering manager at this location was Mr. Joel Levinthal. Most of the analytical effort in support of this project was provided by Dr. Frederick Chichester, who wrote all sections of this report. Most of the algebraic development for this report was verified using the Symbolic Manipulation Program (SMP) software package on a VAX-11 computer by Mr. Alan Reynolds who also prepared this report using the TEX typesetting software package on the same computer. The guidance of Dr. Henry B. Waites and Mr. Stan Carroll of MSFC during the course of this study is gratefully acknowledged.

ABSTRACT

The problem of applying modular attitude control to a rigid body - flexible suspension model of a flexible spacecraft with some state variables inaccessible was addressed by developing a sequence of single-axis models and generating a series of reduced state linear observers of minimum order to reconstruct those scalar state variables that were inaccessible. The specific single axis models treated consisted of two, three, four and five rigid bodies, respectively, interconnected by a flexible shaft passing through the mass centers of the bodies. Reduced state linear observers of all orders up to one less than the total number of scalar state variables were generated for each of the four single-axis models cited. Each of the single-axis models was then transformed to a corresponding modal model to which modal damping was added. Each of the damped modal models was written in state variable form. With the assumption that at least one of the scalar modal state variables was accessible, reduced state linear observers were developed for sythesizing the inaccessible modal state variables for each modal model.

SECTION 1
INTRODUCTION

This report is submitted in compliance with the Scope of Work under contract NAS8-33979. The period of performance covered by the contract is from October 1, 1983 to August 31, 1984. The submission and approval of this report constitute the successful completion of the "Exhibit D" portion of the contract.

This report is a sequel to five others, two of them previously submitted under a different contract number. The two prior reports, under a different contract number, references (1-1) and (1-2), were submitted in October 1978 and September, 1979 and covered the periods from July 27, 1977 to July 27, 1978 and from August 26, 1978 to August 26, 1979, respectively, in compliance with "Exhibit A" of contract NAS8-32660. Three prior final reports were prepared under contract NAS-33979. Reference (1-3) was submitted on March 8, 1982 and covered the period from August 15, 1980 to October 15, 1981 in compliance with "Exhibit A" of the contract. Reference (1-4) was submitted on March 18, 1983 and covered the period from October 16, 1981 to October 31, 1982 in compliance with "Exhibit B". Reference (1-5) was submitted on January 24, 1984 and covered the period from November 1, 1982 to September 30, 1983 in compliance with "Exhibit C".

1.1 OBJECTIVE

The sections that follow summarize the effort expended on the Modular Design Attitude Control System Study contract from October 1, 1983 to August 31, 1984. In prior applications of modular attitude control to rigid body-flexible suspension approximations of the rotational dynamics of prototype flexible spacecraft, it was assumed that all of the scalar state variables of the linearized models were accessible for measurement and/or control. Actual spacecraft to be controlled almost never satisfy such a broad condition. Therefore, the principal objective of the development of modular attitude control, completed August 31, 1984, was the generation of a series of linear observers to support the application of control to state variable models of flexible spacecraft with damping for which one or more state variables are inaccessible.

1.2 SCOPE

Study effort was concentrated in four main areas:

A. Development of a series of single axis state variable models of flexible spacecraft with damping to be utilized in the comparison of different approaches to the development of modular attitude control systems. These models consisted of two, three, four or five rigid bodies serially connected by a flexible suspension in such a way that motion was restricted to rotation about a common axis through the mass centers of the bodies.

B. Generation of reduced state linear observers for each damped single axis model developed in Task A corresponding to various numbers and distributions of inaccessible state variables following the approaches presented in Luenberger (1-6), (1-7), (1-8), and Sage (1-9).

C. Transformation of the undamped versions of the single axis models developed in Task A to their corresponding modal models with modal damping following the approach presented in Thomson (1-10).

D. Generation of reduced state linear observers for each modal model developed in Task C with various numbers of inaccessible modal state variables utilising direct matrix products as described in Lancaster (1-11).

1.3 GENERAL

This report is comprised of seven sections. Sections 2 through 5 describe the development of the two-, three-, four- and five-body single-axis state variable models, respectively, of a prototype flexible spacecraft with damping and the generation of the minimum order reduced state linear observers for the reconstruction of inaccessible scalar state variables of these models. Section 6 begins with the transformation of the single-axis models of Sections 2 through 5 to modal forms to which modal damping is added and concludes with the development of reduced state linear observers for these models when one or more modal state variables are inaccessible. Section 7 lists a number of conclusions and recommendations drawn from generation of linear observers for the series of single-axis state variable models described above. References are listed at the end of each section.

The original RFQ requested that the International System of units (designated as SI) be used in the program and in any reporting. Torques, moments, angular momentum, moments of inertia and distances, however, are stated in English units since this was the system of units used in presenting all of the vehicle data in the RFQ.

1.4 REFERENCES

1-1 Guidance Systems Division, The Bendix Corporation, "Space Construction Base Control System", Final Report, Contract NAS8-32660 for George C. Marshall Space Flight Center, October 27, 1978

1-2 Guidance Systems Division, The Bendix Corporation, "Space Construction Base Control System", Final Report, Contract NAS8-32660 for George C. Marshall Space Flight Center, September 1, 1979.

1-3 Guidance Systems Division, The Bendix Corporation, "Modular Design Attitude Control System", Final Report, Contract NAS8-33979 for George C. Marshall Space Flight Center, March 8, 1982.

1-4 Guidance Systems Division, The Bendix Corporation, "Modular Design Attitude Control System", Final Report, Contract NAS8-33979 for George C. Marshall Space Flight Center, March 18, 1983.

1-5 Guidance Systems Division, The Bendix Corporation, "Modular Design Attitude Control System", Final Report, Contract NAS8-33979 for George C. Marshall Space Flight Center, January 24, 1984.

1-6 Luenberger, D.G., "Determining the State of a Linear System with Observers of Low Dynamic Order", Ph.D. dissertation, Stanford University, 1963.

1-7 Luenberger, D.G., "Observers for Multivariable Systems", *IEEE Transactions on Automatic Control*, Vol. AC-11, No. 2, April 1966, pp. 190–197.

1-8 Luenberger, D.G., "An Introduction to Observers", *IEEE Transactions on Automatic Control*, Vol. AC-16, No. 6, December 1971, pp. 596–602.

1-9 Sage, A.P., *Optimum Systems Control.* Englewood Cliffs, N.J.: Prentice–Hall, Inc. 1968, pp. 306–312.

1-9 Thomson, W.T., *Theory of Vibrations with Applications.* Englewood Cliffs, N.J.: Prentice–Hall, Inc., 1972, pp. 226–230.

1-10 Lancaster, Peter, *Theory of Matrices.* New York: Academic Press, 1959, p. 256.

SECTION 2

DEVELOPMENT OF TWO-BODY SINGLE-AXIS MODEL AND ITS REDUCED STATE LINEAR OBSERVERS

2.1 ORIGINAL DAMPED MODEL

The rotational dynamics of the two-body single-axis model of a flexible spacecraft with damping shown in Fig. 2–1 may be represented by the following set of equations:

$$I_1\ddot{\theta}_1 = -c_1(\dot{\theta}_1 - \dot{\theta}_2) - k_1(\theta_1 - \theta_2) + q_1 \tag{2-1}$$

$$I_2\ddot{\theta}_2 = c_1(\dot{\theta}_1 - \dot{\theta}_2) + k_1(\theta_1 - \theta_2) + q_2 \tag{2-2}$$

where:

I_i = rotational inertia of body i; $i = 1, 2$
θ_i = angular displacement of body i
$\dot{\theta}_i$ = angular rate of body i
q_i = torque applied to body i
k_1 = rotational spring coefficient at the interface between the bodies
c_1 = rotational damping coefficient at the interface between the bodies

2.2 STATE VARIABLE MODEL

The state variable form of the two-body single-axis model of a flexible spacecraft with damping shown in Fig. 2–1 may be expressed as follows:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{2-3}$$

$$\mathbf{x}_A = \mathbf{C}\mathbf{x} \tag{2-4}$$

where:

$\mathbf{x}$ $= [\theta_1 \quad \dot{\theta}_1 \quad \theta_2 \quad \dot{\theta}_2]^T = [x_1 \quad x_2 \quad x_3 \quad x_4]^T = [\mathbf{x}_A^T \quad \mathbf{x}_I^T]^T$ = state vector
$\mathbf{x}_A$ = m-vector of accessible scalar states
$\mathbf{x}_I$ = p-vector of inaccessible scalar states
$\mathbf{u}$ $= [u_1 \quad u_2]^T = \left[\frac{q_1}{I_1} \quad \frac{q_2}{I_2}\right]^T$ = control vector
$\mathbf{A}$ = 4×4 state vector coefficient matrix
$\mathbf{B}$ = $4 \times r$ control vector coefficient matrix ($r = 1$ or 2)
$\mathbf{C}$ = $m \times 4$ measurement or observation matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -a_{23} & -a_{23}r_1 & a_{23} & a_{23}r_1 \\ 0 & 0 & 0 & 1 \\ a_{41} & a_{41}r_1 & -a_{41} & -a_{41}r_1 \end{bmatrix} \tag{2-5}$$

$$r_1 = \frac{c_1}{k_1} \tag{2-6}$$

$$a_{23} = \frac{k_1}{I_1} \tag{2-7}$$

$$a_{41} = \frac{k_1}{I_2} \tag{2-8}$$

$$\mathbf{B} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \text{ for } r = 2 \tag{2-9}$$

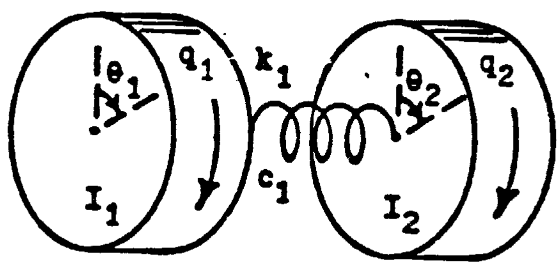


FIGURE 2-1

TWO-BODY SINGLE-AXIS MODEL WITH DAMPING

The block diagram corresponding to this model is depicted in Fig. 2-2.

2.3 REDUCED STATE LINEAR OBSERVERS

2.3.1 Introduction

The minimum order (number of scalar state variables) of a reduced state linear observer required to reconstruct the 4-m inaccessible scalar states of the two-body single-axis model represented by equations (2-3) through (2-9) is $p = 4 - m$. This reconstruction was accomplished for a given state variable model in three main stages.

1) Synthesizing a linear observer of minimum required order (p).
2) Defining a synthesized variable corresponding to each of the inaccessible state variables of the given state variable model.
3) Expressing each synthesized variable as a function of the state variables of the reduced state observer and the accessible state variables of the given state variable model.

The relationship between the single axis model and its corresponding reduced state observer is depicted in Fig. 2-3.

The equations for the reduced state observers corresponding to the state variable model of equations (2-3) through (2-9) are the following:

$$\dot{\mathbf{z}} = \mathbf{D}\mathbf{z} + \mathbf{E}\mathbf{u} + \mathbf{G}\mathbf{y} \tag{2-10}$$

$$\mathbf{z} = \mathbf{T}\mathbf{x} \tag{2-11}$$

$$\mathbf{E} = \mathbf{T}\mathbf{B} \tag{2-12}$$

where:

$\mathbf{D}$ = $p \times p$ observer coefficient matrix (assumed diagonal)
$\mathbf{E}$ = $p \times r$ observer control vector coefficient matrix
$\mathbf{G}$ = $p \times m$ observer vector of observed states coefficient matrix
$\mathbf{T}$ = $p \times 4$ observer weighting matrix

The corresponding block diagram appears in Fig. 2-4.

2.3.2 Observer Synthesis Equations

The equations for synthesizing the reduced state linear observers, based on those appearing in Luenberger (1-1), (1-2), (1-3) and Sage (1-4), were written in the following form.

$$\mathbf{TA} - \mathbf{DT} = \mathbf{F} \tag{2-13}$$

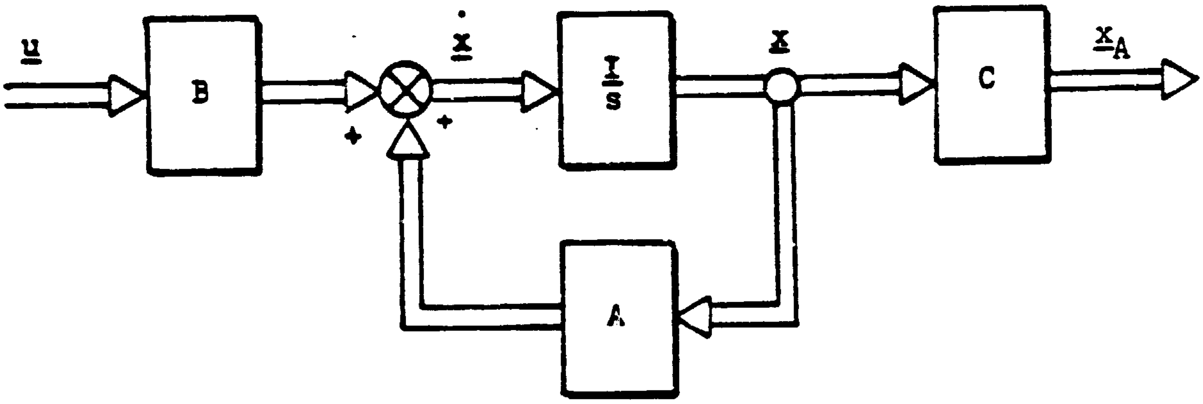
$$\mathbf{F} = \mathbf{GC} \tag{2-14}$$

For

$$\mathbf{T} = \begin{bmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ \vdots & \vdots & \vdots & \vdots \\ t_{p,1} & t_{p,2} & t_{p,3} & t_{p,4} \end{bmatrix} \tag{2-15}$$

$$\mathbf{F} = \begin{bmatrix} f_{11} & f_{12} & f_{13} & f_{14} \\ \vdots & \vdots & \vdots & \vdots \\ f_{p,1} & f_{p,2} & f_{p,3} & f_{p,4} \end{bmatrix} \tag{2-16}$$

$$\mathbf{D} = \begin{bmatrix} d_{11} & & 0 \\ & \ddots & \\ 0 & & d_{p,p} \end{bmatrix} \tag{2-17}$$



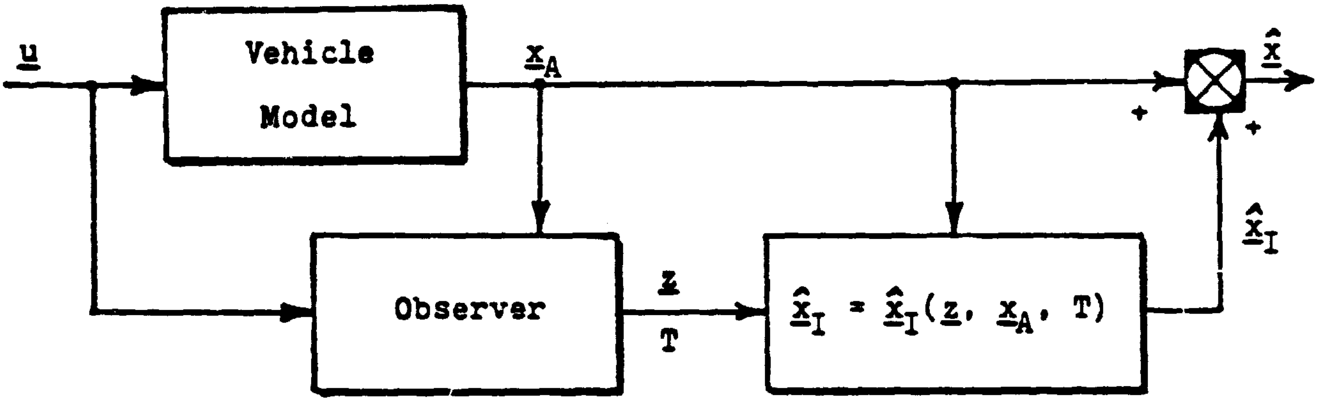
State and Observation Equations:

$$\dot{\underline{x}} = A\underline{x} + B\underline{u}$$

$$\underline{x}_A = C\underline{x}$$

FIGURE 2-2

BLOCK DIAGRAM OF THE STATE VARIABLE MODEL
OF A FLEXIBLE SPACECRAFT



$\underline{u}$ = vector of scalar inputs to vehicle model

$\underline{x}_A$ = vector of accessible scalar states of model

$\underline{z}$ = vector of scalar states of observer

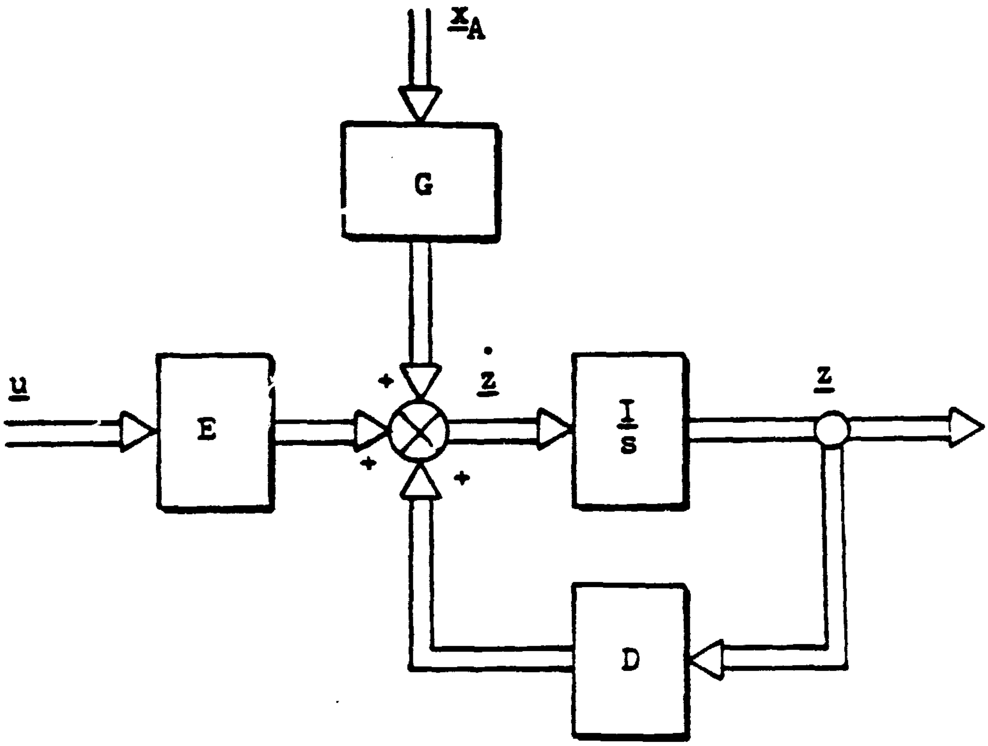
T = observer weighting matrix

$\hat{\underline{x}}_I$ = vector of reconstructed scalar states of model

$\hat{\underline{x}} = \begin{bmatrix} \underline{x}_A \\ \hline \hat{\underline{x}}_I \end{bmatrix}$ = reconstructed vector of all scalar state variables of vehicle model

FIGURE 2-3

BLOCK DIAGRAM OF SPACECRAFT MODEL
AND ITS REDUCED STATE LINEAR OBSERVER



Observer Equations:

$$\dot{\underline{z}} = D\underline{z} + G\underline{x}_A + E\underline{u}$$

Since $G\underline{x}_A = GC\underline{x} = F\underline{x}$,

$$\dot{\underline{z}} = D\underline{z} + F\underline{x} + E\underline{u}$$

FIGURE 2-4

BLOCK DIAGRAM OF LINEAR OBSERVER
FOR STATE VARIABLE MODEL OF FIGURE 2-2

and the form of the A matrix given in equation (2–5) the observer synthesis equations reduce to the following general forms.

$$t_{i1} = d_{ii1} t_{i2} - a_{41} r_1 t_{i4} + f_{i2} \qquad i = 1, 2, \ldots, p \qquad (2\text{–}18)$$

$$t_{i3} = -a_{23} r_1 t_{i2} + d_{ii2} t_{i4} + f_{i4} \qquad (2\text{–}19)$$

$$\begin{bmatrix} -(a_{23}p_1 + d_{ii}^2) & a_{41}p_1 \\ a_{23}p_1 & -(a_{41}p_1 + d_{ii}^2) \end{bmatrix} \begin{bmatrix} t_{i2} \\ t_{i4} \end{bmatrix} = \begin{bmatrix} f_{i1} + d_{ii} f_{i2} \\ f_{i3} + d_{ii} f_{i4} \end{bmatrix} \qquad (2\text{–}20)$$

where:

$$d_{ii1} = a_{23} r_1 + d_{ii} \qquad (2\text{–}21)$$

$$d_{ii2} = a_{41} r_1 + d_{ii} \qquad (2\text{–}22)$$

$$p_1 = 1 + r_1 d_{ii} \qquad (2\text{–}23)$$

$$\begin{aligned} \Delta'_{i2} &= (a_{23}p_1 + d_{ii}^2)(a_{41}p_1 + d_{ii}^2) - a_{23}a_{41}p_1^2 \\ &= d_{ii}^2(a_{23}p_1 + a_{41}p_1 + d_{ii}^2) \end{aligned} \qquad (2\text{–}24)$$

$$t_{i2} = -\frac{(a_{41}p_1 + d_{ii}^2)(f_{i1} + d_{ii} f_{i2}) + a_{41}p_1(f_{i3} + d_{ii} f_{i4})}{\Delta'_{i2}}; \qquad i = 1, 2, \ldots, p \qquad (2\text{–}25)$$

$$t_{i4} = -\frac{a_{23}p_1(f_{i1} + d_{ii} f_{i2}) + (a_{23}p_1 + d_{ii}^2)(f_{i3} + d_{ii} f_{i4})}{\Delta'_{i2}} \qquad (2\text{–}26)$$

$$t_{i1} = -\frac{d_{ii}\left[a_{41} + (a_{23} + a_{41}) r_1 d_{ii} + d_{ii}^2\right]}{\Delta'_{i2}}(f_{i1} + d_{ii} f_{i2}) + f_{i2} + \frac{a_{41} d_{ii}}{\Delta'_{i2}}(f_{i3} + d_{ii} f_{i4}) \qquad (2\text{–}27)$$

$$t_{i3} = -\frac{a_{23} d_{ii}}{\Delta'_{i2}}(f_{i1} + d_{ii} f_{i2}) + \frac{d_{ii}\left[a_{23} + (a_{23} + a_{41}) r_1 d_{ii} + d_{ii}^2\right]}{\Delta'_{i2}}(f_{i3} + d_{ii} f_{i4}) + f_{i4} \qquad (2\text{–}28)$$

2.3.3 Comparison of T Matrices for Damped and Undamped Models

If damping is removed from the model, $r_1 \to 0$, $d_{ii1} \to d_{ii}$, $d_{ii2} \to d_{ii}$, $p_1 \to 1$ and $\Delta'_{i2} \to d_{ii}^2(a_{23} + a_{41} + d_{ii}^2) = \Delta_{i2}$.

$$t_{i2} = -\frac{(a_{41} + d_{ii}^2)(f_{i1} + d_{ii} f_{i2}) + a_{41}(f_{i3} + d_{ii} f_{i4})}{\Delta_{i2}}; \qquad i = 1, 2, \ldots, p \qquad (2\text{–}29)$$

$$t_{i4} = -\frac{a_{23}(f_{i1} + d_{ii} f_{i2}) + (a_{23} + d_{ii}^2)(f_{i3} + d_{ii} f_{i4})}{\Delta_{i2}} \qquad (2\text{–}30)$$

$$t_{i1} = f_{i2} - \frac{d_{ii}\left[(a_{41} + d_{ii}^2)(f_{i1} + d_{ii} f_{i2}) - a_{41}(f_{i3} + d_{ii} f_{i4})\right]}{\Delta_{i2}} \qquad (2\text{–}31)$$

$$t_{i3} = f_{i4} - \frac{d_{ii}\left[a_{23}(f_{i1} + d_{ii} f_{i2}) - (a_{23} + d_{ii}^2)(f_{i3} + d_{ii} f_{i4})\right]}{\Delta_{i2}}. \qquad (2\text{–}32)$$

Comparison of the corresponding equations for generating the elements of the T matrix, t_{ij} ($i = 1, 2, \ldots, p$; $j = 1, 2, 3, 4$) revealed that the addition of damping at the interface between the two bodies had the following effects:

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1. In the equations expressing t_{i2} and t_{i4}, the elements of the even columns of the **T** matrix as a function of the f_{ij}, the elements of the **F** matrix, the form of each equation remains the same under addition of damping with $a_{23}p_1$ and $a_{41}p_1$ being substituted for each scalar a_{23} and a_{41} appearing in the corresponding equations for the undamped two body model.

2. In the equation expressing t_{i1}, the elements of the first column of the **T** matrix as a function of the f_{ij}, the elements of the **F** matrix, the form of the equation remains the same under addition of damping except that the expression, $a_{41}p_1 + a_{23}r_1 d_{ii}$, appears in the place of a_{41} in the coefficient of $(f_{i1} + d_{ii} f_{i2})$ in the numerator and $a_{23}p_1$ and $a_{41}p_1$ appear in the place of a_{23} and a_{41} respectively in the denominator.

3. In the equation expressing t_{i3}, the elements of the third column of the **T** matrix as a functiuon of the f_{ij}, the elements of the **F** matrix, the form of the equation remains the same under addition of damping except that the expression, $a_{23}p_1 + a_{41}r_1 d_{ii}$, appears in the place of a_{23} in the coefficient of $(f_{i3} + d_{ii} f_{i4})$ in the numerator and $a_{23}p_1$ and $a_{41}p_1$ appear in the place of a_{23} and a_{41}, recpectively, in the denominator.

2.4 SOLUTION FOR SYNTHESIZED STATE VARIABLES

2.4.1 Introduction

Inaccessibility of a state variable in the model equations (2–3), (2–4) is reflected by a corresponding null column in the observation matrix, **C**, and a corresponding null column in the **F** matrix as implied by equation (2–14). For the generation of reduced order observers for the two body model the number of inaccessible state variables can be 1, 2 or 3.

2.4.2 First Order Observers ($p = 1$)

A first order linear observer corresponds to inaccessibility of one of the four scalar state variables of the two body model. The observer equation then reduces to:

$$\dot{z} = dz + \mathbf{E}\mathbf{u} + \mathbf{G}\mathbf{y}, \tag{2-33}$$

the **F** and **T** matrices reduce to:

$$\mathbf{F} = [f_1 \quad f_2 \quad f_3 \quad f_4] \tag{2-34}$$

$$\mathbf{T} = [t_1 \quad t_2 \quad t_3 \quad t_4] \tag{2-35}$$

and the observer synthesis equations reduce to the following forms:

$$t_2 = -\frac{(a_{41}p_1 + d^2)(f_1 + df_2) + a_{41}p_1(f_3 + df_4)}{\Delta_2'} \tag{2-36}$$

$$t_4 = -\frac{a_{23}p_1(f_1 + df_2) + (a_{23}p_1 + d^2)(f_3 + df_4)}{\Delta_2'} \tag{2-37}$$

$$t_1 = -\frac{d(a_{41}p_1 + a_{23}r_1 d + d^2)(f_1 + df_2)}{\Delta_2'} + f_2 + \frac{a_{41}d}{\Delta_2'}(f_3 + df_4) \tag{2-38}$$

$$t_3 = -\frac{a_{23}d}{\Delta_2'}(f_1 + df_2) + \frac{d(a_{23}p_1 + a_{41}r_1 d + d^2)(f_3 + df_4)}{\Delta_2'} + f_4 \tag{2-39}$$

$$p_1 = 1 + r_1 d \tag{2-40}$$

$$\Delta_2' = d^2(a_{23}p_1 + a_{41}p_1 + d^2) \tag{2-41}$$

Since this case corresponds to inaccessibility of one state variable, one of the f_i $(i = 1, 2, 3, 4) = 0$.

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Example

Suppose x_4, the scalar state representing the angular rate of body 2, is inaccessible. Then it is assumed that:

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{2-42}$$

for which:

$$\mathbf{F} = [f_1 \quad f_2 \quad f_3 \quad 0] \tag{2-43}$$

and $\mathbf{T}$ is of the form shown in equation (2-35).

From equations (2-14), (2-42) and (2-43),

$$\mathbf{G} = [f_1 \quad f_2 \quad f_3] \tag{2-44}$$

and from equations (2-9), (2-12) and (2-35)

$$\mathbf{E} = [t_2 \quad t_4] \tag{2-45}$$

This equation corresponds to $r = 2$, control torques applied to both bodies. For control troque applied only to body 1,

$$\mathbf{E} = [t_2 \quad 0] \tag{2-46}$$

and for control torque applied only to body 2,

$$\mathbf{E} = [0 \quad t_4] \tag{2-47}$$

The equations for determining the elements of the $\mathbf{T}$ matrix reduce to the following forms:

$$t_2 = -\frac{(a_{41}p_1 + d^2)(f_1 + df_2) + a_{41}p_1 f_3}{\Delta_2'} \tag{2-48}$$

$$t_4 = -\frac{a_{23}p_1(f_1 + f_2) + (a_{23}p_1 + d^2) f_3}{\Delta_2'} \tag{2-49}$$

$$t_1 = -\frac{d(a_{41}p_1 + a_{23}r_1 d + d^2)(f_1 + df_2)}{\Delta_2'} + f_2 + \frac{a_{41}d}{\Delta_2'} f_3 \tag{2-50}$$

$$t_3 = -\frac{a_{23}d(f_1 + df_2) + d(a_{23}p_1 + a_{41}r_1 d + d^2) f_3}{\Delta_2'} \tag{2-51}$$

From equations (2-11) and (2-35),

$$z = t_1 x_1 + t_2 x_2 + t_3 x_3 + t_4 \hat{x}_4 \tag{2-52}$$

where $\hat{x}_4 =$ the synthesized x_4.

Solving for $\hat{x}_4$ yields:

$$\hat{x}_4 = \frac{1}{t_4}\left(z - \sum_{i=1}^{3} t_i x_i\right) \tag{2-53}$$

For inaccessibility of x_1, x_2 or x_4, the equations for determining t_i, (2–36) through (2–39) are appropriately modified.

2.4.3 Second Order Observers ($p = 2$)

The equation for a linear observer of order two corresponds to two of the four scalar state variables being inaccessible. It is represented here as equation (2–10). If the observer coefficient matrix is assumed to be diagonal in this case it appears as follows:

$$\mathbf{D} = \begin{bmatrix} d_{11} & 0 \\ 0 & d_{22} \end{bmatrix} \tag{2–54}$$

Since the observer is of order two,

$$\mathbf{z} = [z_1, z_2]^T \tag{2–55}$$

$$\mathbf{F} = \begin{bmatrix} f_{11} & f_{12} & f_{13} & f_{14} \\ f_{21} & f_{22} & f_{23} & f_{24} \end{bmatrix} \tag{2–56}$$

and,

$$\mathbf{T} = \begin{bmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ t_{21} & t_{22} & t_{23} & t_{24} \end{bmatrix} \tag{2–57}$$

The specific forms of the equations for generating the elements of T depend upon which two of the scalar states are inaccessible. For each inaccessible state the corresponding columns in the C and F matrices are null.

<u>Example</u>

Corresponding to the angular position and rate, respectively, of body 2, suppose that the scalar states x_3 and x_4 are inaccessible. Then the equations for generating the elements of the **T** matrix assume the following forms.

$$t_{i2} = -\frac{(a_{41}p_1 + d_{ii}^2)}{\Delta'_{i2}}(f_{i1} + d_{ii}f_{i2}) \qquad i = 1,2 \tag{2–58}$$

$$t_{i4} = -\frac{a_{23}p_1}{\Delta'_{i2}}(f_{i1} + d_{ii}f_{i2}) \tag{2–59}$$

$$t_{i1} = -\frac{d_{ii}(a_{41}p_1 + a_{23}r_1d_{ii}) + d_{ii}^2)}{\Delta'_{i2}}(f_{i1} + d_{ii}f_{i2}) + f_{i2} \tag{2–60}$$

$$t_{i3} = -\frac{a_{23}d_{ii}}{\Delta'_{i2}}(f_{i1} + d_{ii}f_{i2}) \tag{2–61}$$

where p_1 and Δ'_{i2} are defined in equations (2–23) and (2–24).
From equation (2-11),

$$\begin{bmatrix} t_{13} & t_{14} \\ t_{23} & t_{24} \end{bmatrix} \begin{bmatrix} \hat{x}_3 \\ \hat{x}_4 \end{bmatrix} = \begin{bmatrix} z_1 - t_{11}x_1 - t_{12}x_2 \\ z_2 - t_{21}x_1 - t_{22}x_2 \end{bmatrix} \tag{2–62}$$

where $\hat{x}_3$ and $\hat{x}_4$ are synthesized state variables.

Let $\Delta_2 = \begin{vmatrix} t_{13} & t_{14} \\ t_{23} & t_{24} \end{vmatrix} = t_{13}t_{24} - t_{14}t_{23} \neq 0$

where:

$(\Delta_2)_{i,j} = \Delta_2$ without elements of ith row and jth column

$$\hat{z}_3 = \frac{(\Delta_2)_{1,1}(z_1 - t_{11}x_1 - t_{12}x_2) - (\Delta_2)_{2,1}(z_2 - t_{21}x_1 - t_{22}x_2)}{\Delta_2} \qquad (2\text{–}63)$$

$$\hat{z}_4 = \frac{-(\Delta_2)_{1,2}(z_1 - t_{11}x_1 - t_{12}x_2) + (\Delta_2)_{2,2}(z_2 - t_{21}x_1 - t_{22}x_2)}{\Delta_2} \qquad (2\text{–}64)$$

For x_3 and x_4 inaccessible, it is assumed that:

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad (2\text{–}65)$$

$$\mathbf{F} = \begin{bmatrix} f_{11} & f_{12} & 0 & 0 \\ f_{21} & f_{22} & 0 & 0 \end{bmatrix} \qquad (2\text{–}66)$$

From $\mathbf{F} = \mathbf{GC}$,

$$\mathbf{G} = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \qquad (2\text{–}67)$$

From $\mathbf{E} = \mathbf{TB}$,

$$\mathbf{E} = \begin{bmatrix} t_{12} & t_{14} \\ t_{22} & t_{24} \end{bmatrix} \quad \text{for } r = 2 \text{ (control torques applied to both bodies)} \qquad (2\text{–}68)$$

$$\mathbf{E} = \begin{bmatrix} t_{12} & 0 \\ t_{22} & 0 \end{bmatrix} \quad \text{for control restricted to body 1} \qquad (2\text{–}69)$$

$$\mathbf{E} = \begin{bmatrix} 0 & t_{14} \\ 0 & t_{24} \end{bmatrix} \quad \text{for control restricted to body 2} \qquad (2\text{–}70)$$

2.4.4 Third Order Observers ($p = 3$)

The equation for the linear observer of order one less than the system's dimension corresponds to three of the four scalar state variables being inaccessible. It is represented here as equation (2–10).
If the observer coefficient matrix is assumed to be diagonal in this case it appears as follows,

$$\mathbf{D} = \begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix} \qquad (2\text{–}71)$$

Since the observer is of order 3,

$$\mathbf{z} = [z_1, z_2, z_3]^T \qquad (2\text{–}72)$$

$$\mathbf{F} = \begin{bmatrix} f_{11} & f_{12} & f_{13} & f_{14} \\ f_{21} & f_{22} & f_{23} & f_{24} \\ f_{31} & f_{32} & f_{33} & f_{34} \end{bmatrix} \qquad (2\text{–}73)$$

and,

$$\mathbf{T} = \begin{bmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ t_{21} & t_{22} & t_{23} & t_{24} \\ t_{31} & t_{32} & t_{33} & t_{34} \end{bmatrix} \qquad (2\text{–}74)$$

The specific forms of the equations for generating the elements of $\mathbf{T}$ depend upon which three of the scalar states are inaccessible. For each inaccessible state the corresponding columns in th $\mathbf{C}$ and $\mathbf{F}$ matrices are null.

Example

Suppose the scalar states, x_2, x_3 and x_4, representing the angular rate of body 1 and the angular position and rate of body 2, are inaccessible. Then the equations for generating the elements of the T matrix assume the following form since $f_{i1} = f_{i3} = f_{i4} = 0$ for $i = 1,2,3$.

$$t_{i2} = -\frac{(a_{41}p_1 + d_{ii}^2)}{\Delta'_{i2}} f_{i1} \qquad i = 1,2,3 \tag{2-75}$$

$$t_{i4} = -\frac{a_{23}p_1}{\Delta'_{i2}} f_{i1} \tag{2-76}$$

$$t_{i1} = -\frac{d_{ii}(a_{41}p_1 + a_{23}r_1 d_{ii} + d_{ii}^2)}{\Delta'_{i2}} f_{i1} \tag{2-77}$$

$$t_{i3} = -\frac{a_{23}d_{ii}}{\Delta'_{i2}} f_{i1} \tag{2-78}$$

where p_1 and Δ'_{i2} are defined in equations (2-23) and (2-24).
From equation (2-11),

$$\begin{bmatrix} t_{12} & t_{13} & t_{14} \\ t_{22} & t_{23} & t_{24} \\ t_{32} & t_{33} & t_{34} \end{bmatrix} \begin{bmatrix} \hat{x}_2 \\ \hat{x}_3 \\ \hat{x}_4 \end{bmatrix} = \begin{bmatrix} z_1 - t_{11}x_1 \\ z_2 - t_{21}x_1 \\ z_3 - t_{31}x_1 \end{bmatrix} \tag{2-79}$$

where $\hat{x}_2$, $\hat{x}_3$ and $\hat{x}_4$ are synthesized state variables.

$$\text{Let } \Delta_3 = \begin{vmatrix} t_{12} & t_{13} & t_{14} \\ t_{22} & t_{23} & t_{24} \\ t_{32} & t_{33} & t_{34} \end{vmatrix} \neq 0$$

where,

$(\Delta_3)_{i,j}$ = Δ_3 without elements of ith row and jth column

$$\hat{x}_{j+1} = \frac{\sum_{i=1}^{3} (-1)^{i+j} (\Delta_3)_{i,j} (z_j - x_1)}{\Delta_3} \qquad j = 1,2,3 \tag{2-80}$$

For x_2, x_3 and x_4 inaccessible, it is assumed that:

$$\mathbf{C} = [1 \quad 0 \quad 0 \quad 0] \tag{2-81}$$

$$\mathbf{F} = \begin{bmatrix} f_{11} & 0 & 0 & 0 \\ f_{21} & 0 & 0 & 0 \\ f_{31} & 0 & 0 & 0 \end{bmatrix} \tag{2-82}$$

From $\mathbf{F} = \mathbf{GC}$,

$$\mathbf{G} = \begin{bmatrix} f_{11} \\ f_{21} \\ f_{31} \end{bmatrix} \tag{2-83}$$

From $\mathbf{E} = \mathbf{TB}$,

$$\mathbf{E} = \begin{bmatrix} t_{12} & t_{14} \\ t_{22} & t_{24} \\ t_{32} & t_{34} \end{bmatrix} \qquad \text{for } r = 2 \text{ (control torques applied to both bodies)} \tag{2-84}$$

2.5 REFERENCES

2-1 Luenberger, D.G., "Determining the State of a Linear System with Observers of Low Dynamic Order", Ph.D. dissertation. Stanford University, 1963.

2-2 Luenberger, D.G., "Observers for Multivariable Systems", *IEEE Transactions on Automatic Control*, Vol. AC-11, No. 2, April 1966, pp. 190–197.

2-3 Luenberger, D.G., "An Introduction to Observers", *IEEE Transactions on Automatic Control*, Vol. AC-16, No. 6, December 1971, pp. 596–602.

2-4 Sage, A.P., *Optimum Systems Control*. Englewood Cliffs, N.J.: Prentice-Hall, Inc. 1968, pp. 306–312.

SECTION 3

DEVELOPMENT OF THE THREE-BODY SINGLE-AXIS MODEL AND ITS REDUCED STATE LINEAR OBSERVERS

3.1 ORIGINAL DAMPED MODEL

The rotational dynamics of the three-body single-axis model of a flexible spacecraft with damping shown in Fig. 3–1 may be represented by the following set of equations:

$$I_1\ddot{\theta}_1 = -c_1(\dot{\theta}_1-\dot{\theta}_2) - k_1(\theta_1-\theta_2) + q_1 \tag{3–1}$$

$$I_2\ddot{\theta}_2 = c_1(\dot{\theta}_1-\dot{\theta}_2) + k_1(\theta_1-\theta_2) + c_2(\dot{\theta}_3-\dot{\theta}_2) + k_2(\theta_3-\theta_2) + q_2 \tag{3–2}$$

$$I_3\ddot{\theta}_3 = -c_2(\dot{\theta}_3-\dot{\theta}_2) - k_2(\theta_3-\theta_1) + q_3 \tag{3–3}$$

where

I_i = rotational inertia of body i; $i = 1,2,3$
θ_i = angular dislpacement of body i
$\dot{\theta}_i$ = angular rate of body i
q_i = torque applied to body i
k_j = rotational spring coefficient at interface j; $j = 1,2$
c_j = rotational dampoing coefficient at interface j

3.2 STATE VARIABLE MODEL

The state variable form of the three-body single-axis model of a flexible spacecraft shown in Fig. 3–1 may be expressed as follows:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{3–4}$$

$$\mathbf{x}_A = \mathbf{C}\mathbf{x} \tag{3–5}$$

where:

$\mathbf{x} = [x_1 \;\; x_2 \;\; \cdots \;\; x_6]^T = [\theta_1 \;\; \dot{\theta}_1 \;\; \theta_2 \;\; \dot{\theta}_2 \;\; \theta_3 \;\; \dot{\theta}_3]^T = [\mathbf{x}_A^T \;\; \mathbf{x}_I^T]^T$ = state vector

$\mathbf{x}_A$ = m-vector of accessible scalar states

$\mathbf{x}_I$ = p-vector of inaccessible scalar states

$\mathbf{u} = [u_1 \;\; \cdots \;\; u_r]^T = \left[\frac{q_1}{I_1} \;\; \cdots \;\; \frac{q_r}{I_r}\right]^T$ $\quad (r = 1,2 \text{ or } 3)$

$\mathbf{C}$ = observation matrix of dimensions $m \times 6$, $m = 1,2,\ldots,5$ (Minimum dimension of reduced order observer required $= 6 - m$).

Partitioning of this model by rigid body results in the following forms for its coefficient matrices.

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -a_{23} & -a_{23}r_1 & a_{23} & a_{23}r_1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ a_{41} & a_{41}r_1 & a_{43} & a_{44} & a_{45} & a_{45}r_2 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & a_{63} & a_{63}r_2 & -a_{63} & -a_{63}r_2 \end{bmatrix} \tag{3–6}$$

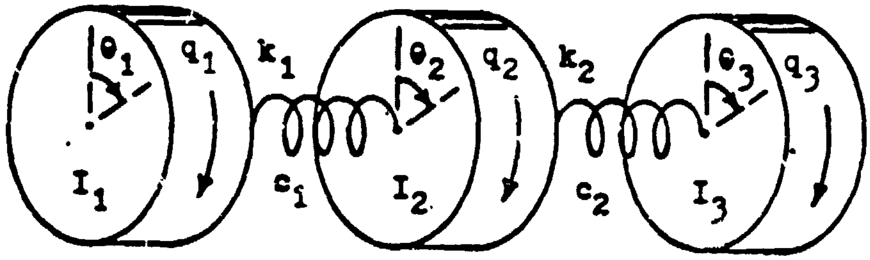

FIGURE 3-1

THREE-BODY SINGLE-AXIS MODEL WITH DAMPING
AT BOTH INTERFACES

$$
\begin{aligned}
a_{23} &= \frac{k_1}{I_1} \\
a_{41} &= \frac{k_1}{I_2} \\
a_{45} &= \frac{k_2}{I_2} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (3\text{-}7)\\
a_{43} &= -(a_{41} + a_{45}) \\
a_{44} &= -(a_{41}r_1 + a_{45}r_2) \\
a_{63} &= \frac{k_2}{I_3}
\end{aligned}
$$

$$
r_j = \frac{c_j}{k_j}; \qquad j = 1,2 \tag{3-8}
$$

$$
\mathbf{B} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{for } r = 3 \text{ (control torques applied to all three bodies)} \tag{3-9}
$$

The block diagram corresponding to this model is shown in Fig. 2-2.

3.3 REDUCED STATE LINEAR OBSERVERS

3.3.1 Introduction

For the three-body single-axis model represented by equations (3-4) through (3-9), the minimum order of a reduced state linear observer required to generate the inaccessible states is $p = 6 - m$ ($m = 1, 2, \ldots, 5$). All of the reduced state linear observers for the three body model may be written in the form represented by equations (2-10) and (2-11) where, in this case, the observer coefficient matrix, $\mathbf{D}$, is assumed to be diagonal and of dimensions $p \times p$. The corresponding observer weighting matrix is of the following form:

$$
\mathbf{T} = \begin{bmatrix} t_{11} & \cdots & t_{16} \\ \vdots & & \vdots \\ t_{p,1} & \cdots & t_{p,6} \end{bmatrix} \tag{3-10}
$$

From equations (2-12), (3-9) and (3-10).

$$
\mathbf{E} = \begin{bmatrix} t_{12} & t_{14} & t_{16} \\ \vdots & \vdots & \vdots \\ t_{p,2} & t_{p,4} & t_{p,6} \end{bmatrix} \quad \text{for } r = 3 \text{ (control torques applied to all 3 bodies)} \tag{3-11}
$$

$$
\mathbf{F} = \begin{bmatrix} f_{11} & \cdots & f_{16} \\ \vdots & & \vdots \\ f_{p,1} & \cdots & f_{p,6} \end{bmatrix} \tag{3-12}
$$

The corresponding observer block diagram appears in Fig 2-4.

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3.3.2 Observer Synthesis Equations

From Luenburger (3–1), (3–2), (3–3) and Sage (3–4) the equations for synthesizing the reduced state linear observers for the three-body single-axis model represented by equations (3–4) through (3–9) are given by equations (2–13) and (2–14). With coefficient matrices of the forms listed in 3.3.1 this set of observer synthesis equations reduces to the following forms:

$$
\begin{aligned}
t_{i2} &= \frac{(\Delta'_{i3})_{1,1}(f_{i1}+d_{ii}f_{i2}) - (\Delta'_{i3})_{2,1}(f_{i3}+d_{ii}f_{i4}) + (\Delta'_{i3})_{3,1}(f_{i5}+d_{ii}f_{i6})}{\Delta'_{i3}} \\
&= \frac{\left[(a_{41}p_1+d_{ii}^2)(a_{63}p_2+d_{ii}^2)+a_{45}p_2d_{ii}^2\right]}{\Delta'_{i3}}(f_{i1}+d_{ii}f_{i2}) \\
&\quad + \frac{a_{41}p_1\left[(a_{63}p_2+d_{ii}^2)(f_{i3}+d_{ii}f_{i4})+a_{63}p_2(f_{i5}+d_{ii}f_{i6})\right]}{\Delta'_{i3}}; \qquad i=1,2,\ldots,p
\end{aligned}
\tag{3–13}
$$

$$
\begin{aligned}
t_{i4} &= -\frac{(\Delta'_{i3})_{1,2}(f_{i1}+d_{ii}f_{i2}) - (\Delta'_{i3})_{2,2}(f_{i3}+d_{ii}f_{i4}) + (\Delta'_{i3})_{3,2}(f_{i5}+d_{ii}f_{i6})}{\Delta'_{i3}} \\
&= \frac{a_{23}p_1(a_{63}p_2+d_{ii}^2)(f_{i1}+d_{ii}f_{i2})+(a_{23}p_1+d_{ii}^2)(a_{63}p_2+d_{ii}^2)(f_{i3}+d_{ii}f_{i4})}{\Delta'_{i3}} \\
&\quad + \frac{a_{63}p_2(a_{23}p_1+d_{ii}^2)(f_{i5}+d_{ii}f_{i6})}{\Delta'_{i3}}
\end{aligned}
\tag{3–14}
$$

$$
\begin{aligned}
t_{i6} &= \frac{(\Delta'_{i3})_{1,3}(f_{i1}+d_{ii}f_{i2}) - (\Delta'_{i3})_{2,3}(f_{i3}+d_{ii}f_{i4}) + (\Delta'_{i3})_{3,3}(f_{i5}+d_{ii}f_{i6})}{\Delta'_{i3}} \\
&= \frac{a_{23}a_{45}p_1p_2(f_{i1}+d_{ii}f_{i2})+a_{45}p_2(a_{23}p_1+d_{ii}^2)(f_{i3}+d_{ii}f_{i4})}{\Delta'_{i3}} \\
&\quad + \frac{\left[(a_{23}p_1+d_{ii}^2)(a_{45}p_2+d_{ii}^2)+a_{41}p_1d_{ii}^2\right](f_{i5}+d_{ii}f_{i6})}{\Delta'_{i3}}
\end{aligned}
\tag{3–15}
$$

$$
\begin{aligned}
t_{i1} &= \frac{d_{ii}\left\{\left[(a_{41}p_1+d_{ii}^2)(a_{63}p_2+d_{ii}^2)+a_{45}p_2d_{ii}^2\right]\right.}{\Delta'_{i3}} \\
&\quad + \frac{\left.a_{23}r_1d_{ii}\left[(a_{45}+a_{63})p_2+d_{ii}^2\right]\right\}}{\Delta'_{i3}}(f_{i1}+d_{ii}f_{i2})+f_{i2} \\
&\quad + \frac{a_{41}d_{ii}\left[(a_{63}p_2+d_{ii}^2)(f_{i3}+d_{ii}f_{i4})+a_{63}p_2(f_{i5}+d_{ii}f_{i6})\right]}{\Delta'_{i3}}
\end{aligned}
\tag{3–16}
$$

$$
\begin{aligned}
t_{i3} &= \frac{a_{23}d_{ii}\Big[-a_{45}d_{ii}(r_1 - r_2) + (a_{63}p_2 + d_{ii}^2)\Big]}{\Delta'_{i3}}(f_{i1} + d_{ii}f_{i2}) \\
&\quad + \frac{d_{ii}\Big[(a_{23}p_1 + d_{ii}^2)(a_{63}p_2 + d_{ii}^2) + a_{41}r_1d_{ii}(a_{63}p_2 + d_{ii}^2)}{\Delta'_{i3}} \\
&\quad + \frac{a_{45}r_2d_{ii}(a_{23}p_1 + d_{ii}^2)\Big]}{\Delta'_{i3}}(f_{i3} + d_{ii}f_{i4}) + f_{i4} \\
&\quad + \frac{a_{63}d_{ii}\Big[a_{41}d_{ii}(r_1 - r_2) + (a_{23}p_1 + d_{ii}^2)\Big]}{\Delta'_{i3}}(f_{i5} + d_{ii}f_{i6})
\end{aligned}
\tag{3-17}
$$

$$
\begin{aligned}
t_{i5} &= \frac{a_{45}d_{ii}\Big[a_{23}p_1(f_{i1} + d_{ii}f_{i2}) + (a_{23}p_1 + d_{ii}^2)(f_{i3} + d_{ii}f_{i4})\Big]}{\Delta'_{i3}} \\
&\quad + \frac{d_{ii}\Big\{a_{63}r_2d_{ii}\Big[(a_{23} + a_{41})p_1 + d_{ii}^2\Big]}{\Delta'_{i3}} \\
&\qquad + \frac{\Big[(a_{23}p_1 + d_{ii}^2)(a_{45}p_2 + d_{ii}^2) + a_{41}p_1d_{ii}^2\Big]\Big\}}{\Delta'_{i3}}(f_{i5} + d_{ii}f_{i6}) + f_{i6}
\end{aligned}
\tag{3-18}
$$

where:

$$r_j = \frac{c_j}{k_j}; \qquad j = 1, 2 \tag{3-8}$$

$$p_j = 1 + r_jd_{ii}; \qquad j = 1, 2 \tag{3-19}$$

$$d_{ii1} = a_{23}r_1 + d_{ii} \tag{3-20}$$

$$d_{ii2} = a_{41}r_1 + a_{45}r_2 + d_{ii} \tag{3-21}$$

$$d_{ii3} = a_{63}r_2 + d_{ii} \tag{3-22}$$

$$
\begin{aligned}
\Delta'_{i3} &= \begin{vmatrix} -(a_{23}p_1 + d_{ii}^2) & a_{41}p_1 & 0 \\ a_{23}p_1 & -(a_{41}p_1 + a_{45}p_2 + d_{ii}^2) & a_{63}p_2 \\ 0 & a_{45}p_2 & -(a_{63}p_2 + d_{ii}^2) \end{vmatrix} \\
&= -(a_{23}p_1 + d_{ii}^2)(\Delta'_{i3})_{1,1} - a_{23}p_1(\Delta'_{i3})_{2,1} \\
&= -d_{ii}^2\Big[(a_{23}a_{45} + a_{23}a_{63} + a_{41}a_{63})p_1p_2 \\
&\qquad + (a_{23}p_1 + a_{41}p_1 + a_{45}p_2 + a_{63}p_2)d_{ii}^2 + d_{ii}^4\Big]
\end{aligned}
\tag{3-23}
$$

3.3.3 Comparison of T Matrices for Damping at Various Interfaces

The observer synthesis equations for the three-body single-axis model with damping were compared with those for the same model without damping. A general form was developed for these equations that encompassed the synthesis of the elements of the observer T matrix for the following conditions with respect to damping in the model.

1. No damping;
2. Damping only at the interface between bodies 1 and 2;

3. Damping only at the interface between bodies 2 and 3;
4. Damping at both interfaces.

Elimination of damping at interface j of the model corresponds to setting $r_j = 0$ and $p_j = 1$ in the equations for generating the elements of the **T** matrix with damping present at both interfaces, equations (3–13) through (3–23). If all damping is removed from the three body model, $r_j \rightarrow 0$, $p_j \rightarrow 1$, $d_{ii1} \rightarrow d_{ii2} \rightarrow d_{ii3} \rightarrow d_{ii}$ and $\Delta'_{i3} \rightarrow -d_{ii}^2\left[a_{23}a_{45} + a_{23}a_{63} + a_{41}a_{63} + (a_{23} + a_{41} + a_{45} + a_{63})d_{ii}^2 + d_{ii}^4\right] = \Delta_{i3}$.

$$t_{i2} = \frac{\left[(a_{41} + d_{ii}^2)(a_{63} + d_{ii}^2) + a_{45}d_{ii}^2\right]}{\Delta_{i3}}(f_{i1} + d_{ii}f_{i2}) + \frac{a_{41}\left[(a_{63} + d_{ii}^2)(f_{i3} + d_{ii}f_{i4}) + a_{63}(f_{i5} + d_{ii}f_{i6})\right]}{\Delta_{i3}} \qquad i = 1, 2, \ldots, p \tag{3–24}$$

$$t_{i4} = \frac{a_{23}(a_{63} + d_{ii}^4)(f_{i1} + d_{ii}f_{i2}) + (a_{23} + d_{ii}^2)(a_{63} + d_{ii}^2)(f_{i3} + d_{ii}f_{i4})}{\Delta_{i3}} + \frac{a_{63}(a_{23} + d_{ii}^2)}{\Delta_{i3}}(f_{i5} + d_{ii}f_{i6}) \tag{3–25}$$

$$t_{i6} = \frac{a_{23}a_{45}(f_{i1} + d_{ii}f_{i2}) + a_{45}(a_{23} + d_{ii}^2)(f_{i3} + d_{ii}f_{i4})}{\Delta_{i3}} + \frac{\left[a_{23}a_{45} + (a_{23} + a_{41} + a_{45})d_{ii}^2 + d_{ii}^4\right]}{\Delta_{i3}}(f_{i5} + d_{ii}f_{i6}) \tag{3–26}$$

$$t_{i1} = \frac{d_{ii}\left[(a_{41} + d_{ii}^2)(a_{63} + d_{ii}^2) + a_{45}d_{ii}^2\right]}{\Delta_{i3}}(f_{i1} + d_{ii}f_{i2}) + f_{i2} + \frac{a_{41}d_{ii}\left[(a_{63} + d_{ii}^2)(f_{i3} + d_{ii}f_{i4}) + a_{63}(f_{i5} + d_{ii}f_{i6})\right]}{\Delta_{i3}} \tag{3–27}$$

$$t_{i3} = \frac{a_{23}d_{ii}(a_{63} + d_{ii}^2)}{\Delta_{i3}}(f_{i1} + d_{ii}f_{i2}) + \frac{d_{ii}(a_{23} + d_{ii}^2)(a_{63} + d_{ii}^2)}{\Delta_{i3}}(f_{i3} + d_{ii}f_{i4}) + f_{i4} + \frac{a_{63}d_{ii}(a_{23} + d_{ii}^2)}{\Delta_{i3}}(f_{i5} + d_{ii}f_{i6}) \tag{3–28}$$

$$t_{i5} = \frac{a_{45}d_{ii}\left[a_{23}(f_{i1} + d_{ii}f_{i2}) + (a_{23} + d_{ii}^2)(f_{i3} + d_{ii}f_{i4})\right]}{\Delta_{i3}} + \frac{d_{ii}\left[(a_{23} + d_{ii}^2)(a_{45} + d_{ii}^2) + a_{41}d_{ii}^2\right]}{\Delta_{i3}}(f_{i5} + d_{ii}f_{i6}) + f_{i6} \tag{3–29}$$

The elements, t_{i1} through t_{i6} $(i = 1, \ldots, p)$, of the **T** matrix of the linear observer of order p corresponding to this model with one or more inaccessible states were found to be affected by the addition of damping at the interface between bodies 1 and 2 as follows.

1. The scalars, a_{23} and a_{41}, were modified to $a_{23}p_1$ and $a_{41}p_1$, respectively, in the equations for generating t_{i2}, t_{i4}, t_{i5} and t_{i6} and in the <u>denominators</u> of the equations for generating t_{i1} and t_{i3} where p_j was defined in equation (3–19).

2. In the numerator of the equation for generating t_{i1}, (3–16), the following changes occurred.
   a. The term, $a_{23}r_1d_{ii}^2(a_{45}+a_{63}+d_{ii}^2)$ was added to the coefficient of f_{i1} where r_j is defined in equation (3–8).
   b. The scalar, a_{41}, in the coefficient of f_{i1} was changed to $a_{41}p_1$.
3. In the numerator of the equation for generating t_{i3}, (3–17), the following changes occurred.
   a. The term, $a_{23}a_{45}r_1d_{ii}^2$, was subtracted from the coefficient of $(f_{i1}+d_{ii}f_{i2})$.
   b. The term, $a_{41}r_1d_{ii}^2(a_{63}+d_{ii}^2)$, was added to the coefficient of f_{i3}.
   c. Each a_{23} in the coefficients of f_{i3}, f_{i5} and f_{i6} was mdified to $a_{23}p_1$.
   d. The term, $a_{41}a_{63}r_1d_{ii}^2$, was added to the coefficient of $(f_{i5}+d_{ii}f_{i6})$.

Addition of damping at the interface between bodies 2 and 3 had the following effects.

1. The scalars, a_{45} and a_{63}, were modified to $a_{45}p_2$ and $a_{63}p_2$, respectively, in the equations for generating t_{i1}, t_{i2}, t_{i4} and t_{i6} and the denominators of the equations for generating t_{i3} and t_{i5} where p_j was defined in equation (3–19).
2. In the numerator of the equation for generating t_{i3}, (3–17), the following changes occurred.
   a. The term, $a_{23}a_{45}r_2d_{ii}^2$, was added to the coefficient of $(f_{i1}+d_{ii}f_{i2})$ where r_j is defined in equation (3–8).
   b. Each a_{63} in the coefficients of f_{i1}, f_{i2} and f_{i3} was modified to $a_{63}p_2$.
   c. The term, $a_{45}r_2d_{ii}^2(a_{23}+d_{ii}^2)$, was added to the coefficient of f_{i3}.
   d. The term, $a_{41}a_{63}r_2d_{ii}^2$, was subtracted from the coefficient of $(f_{i5}+d_{ii}f_{i6})$.
3. In the numerator of the equation for t_{i5}, (3–18), the following changes occurred
   a. The term, $a_{63}r_2d_{ii}^2(a_{23}+a_{41}+d_{ii}^2)$, was added to the coefficient of f_{i5}.
   b. The scalar, a_{45}, in the coefficient of f_{i5} was changed to $a_{45}p_2$.

Addition of damping at both the interface between bodies 1 and 2 and the interface between bodies 2 and 3 had the following effects.

1. The scalars, a_{23}, a_{41}, a_{45} and a_{63}, were modified to $a_{23}p_1$, $a_{41}p_1$, $a_{45}p_2$ and $a_{63}r_2$ respectively, in the equations for generating t_{i2}, t_{i4} and t_{i6} and in the denominators of the equations for generating t_{i1}, t_{i3} and t_{i5} where p_j are defined in equation (3–19).
2. In the numerator of the equation for generating t_{i1}, (3–16), the following changes occurred.
   a. The term, $a_{23}r_1d_{ii}^2\left[(a_{45}+a_{63})p_2+d_{ii}^2\right]$, was added to the coefficient of f_{i1}, where r_j is defined in equation (3–8) and p_j is defined in equation (3–19).
   b. The remaining scalars, a_{41}, a_{45} and a_{63}, were modified to $a_{41}p_1$, $a_{45}p_2$ and $a_{63}p_2$, respectively, with the exception of the a_{41} common to the coefficients of f_{i4} through f_{i6}.
3. In the numerator of the equation for generating t_{i3}, (3–17), the following changes occurred.
   a. The term, $a_{23}a_{45}d_{ii}^2(r_2-r_1)$, was added to the coefficient of $(f_{i1}+d_{ii}f_{i2})$.
   b. The terms, $a_{41}r_1d_{ii}^2(a_{63}p_2+d_{ii}^2)$ and $a_{45}r_2d_{ii}^2(a_{23}p_1+d_{ii}^2)$, were added to the coefficient of f_{i3}.
   c. The term, $a_{41}a_{63}d_{ii}^2(r_1-r_2)$, was added to the coefficient of $(f_{i5}+d_{ii}f_{i6})$.
   d. The remaining scalars, a_{23} and a_{63}, were modified to $a_{23}p_1$ and $a_{63}p_2$, respectively.
4. In the numerator of the equation for generating t_{i5}, (3–18), the following changes occurred.
   a. The term, $a_{63}r_2d_{ii}^2\left[(a_{23}+a_{41})p_1+d_{ii}^2\right]$, was added to the coefficient of f_{i5}.
   b. The remaining scalars, a_{23}, a_{41}, a_{45} and a_{63} were modified to $a_{23}p_1$, $a_{41}p_1$, $a_{45}p_2$ and $a_{63}p_2$ respectively with the exception of the a_{45} common to the coefficients of f_{i1}, f_{i2}, f_{i3} and f_{i4}.

3.4 SOLUTION FOR SYNTHESIZED STATE VARIABLES

3.4.1 Introduction

Inaccessibility of a scalar state variable in equation set (3–1), (3–2) is reflected by a corresponding null column in the observation matrix, **C** and, as implied by equation (2–11), in the **F** matrix for the generation of reduced order observers for the three-body model. The number of inaccessible scalar states can be 1, 2, 3, 4 or 5.

3.4.2 First Order Observers ($p = 1$)

A first order observer is required when any one of the six scalar state variables of the three body model is inaccessible. The first order form of the linear observer equation is:

$$\dot{z} = dz + \mathbf{E}u + \mathbf{G}y \qquad (3\text{–}30)$$

The **F** and **T** matrices associated with a first order observer for the three body model then reduce to the following row forms.

$$\mathbf{F} = [f_1 \quad f_2 \quad f_3 \quad f_4 \quad f_5 \quad f_6]^T \qquad (3\text{–}31)$$

$$\mathbf{T} = [t_1 \quad t_2 \quad t_3 \quad t_4 \quad t_5 \quad t_6]^T \qquad (3\text{–}32)$$

The observer synthesis equations are then given by equation (3–13) through equation (3–23) with $i = 1$. Since a first order observer corresponds to one of the scalar state variables being inaccessible, <u>one</u> of the f_i $(i = 1,2,3,4,5,6) = 0$.

<u>Example</u>

Suppose that the scalar state representing the angular rate of body 3, x_6, is inaccessible. Then $f_6 = 0$ and the observer synthesis equations reduce to the following forms.

$$t_2 = \frac{\left[(a_{41}p_1 + d^2)(a_{63}p_2 + d^2) + a_{45}p_2d^2\right](f_1 + d\,f_2)}{\Delta_3'} + \frac{a_{41}p_1\left[(a_{63}p_2 + d^2)(f_3 + d\,f_4) + a_{63}p_2f_5\right.}{\Delta_3'} \qquad (3\text{–}33)$$

$$t_4 = \frac{a_{23}p_1(a_{63}p_2 + d^2)(f_1 + d\,f_2) + (a_{23}p_1 + d^2)(a_{63}p_2 + d^2)(f_3 + d\,f_4)}{\Delta_3'} + \frac{a_{63}p_2(a_{23}p_1 + d^2)f_5}{\Delta_3'} \qquad (3\text{–}34)$$

$$t_6 = \frac{a_{23}a_{45}p_1p_2(f_1 + d\,f_2) + (a_{23}p_1 + d^2)a_{45}p_2(f_3 + d\,f_4)}{\Delta_3'} + \frac{\left[(a_{23}p_1 + d^2)(a_{45}p_2 + d^2) + a_{41}p_1d^2\right]}{\Delta_3'}f_5 \qquad (3\text{–}35)$$

$$t_1 = \frac{d\left\{\left[(a_{41}p_1 + d^2)(a_{63}p_2 + d^2) + a_{45}p_2d^2\right] + a_{23}r_1d\left[(a_{45} + a_{63})p_2 + d^2\right]\right\}}{\Delta_{i3}'}(f_1 + d\,f_2) + f_2 + \frac{a_{41}d\left[(a_{63}p_2 + d^2)(f_3 + d\,f_4) + a_{63}p_2f_5\right]}{\Delta_3'} \qquad (3\text{–}36)$$

$$t_4 = \frac{a_{23}d\left[a_{45}d(r_2-r_1)+(a_{63}p_2+d^2)\right]}{\Delta_3'}(f_1+d\,f_2)$$

$$+\frac{d\left[(a_{23}p_1+d^2)(a_{63}p_2+d^2)+a_{41}r_1d(a_{63}p_2+d^2)+a_{45}r_2d(a_{23}p_1+d^2)\right]}{\Delta_3'}(f_3+d\,f_4)+f_4$$

$$+\frac{a_{63}d\left[a_{41}d(r_1-r_2)+(a_{23}p_1+d^2)\right]}{\Delta_3'}f_5 \qquad (3\text{–}37)$$

$$t_6 = \frac{a_{45}d\left[a_{23}p_1(f_1+d\,f_2)+(a_{23}p_1+d^2)(f_3+d\,f_4)\right.}{\Delta_3'}$$

$$+\frac{d\left\{a_{63}r_2d\left[(a_{23}+a_{41})p_1+d^2\right]+\left[(a_{23}p_1+d^2)(a_{45}p_2+d^2)+a_{41}p_1d^2\right]\right\}}{\Delta_3'}f_5 \qquad (3\text{–}38)$$

where:

$$r_j = \frac{c_j}{k_j} \qquad (3\text{-}8)$$

$$p_j = 1+r_j\,d \qquad (3\text{–}39)$$

$$\Delta_3' = -d^2\Big[(a_{23}a_{45}+a_{23}a_{63}+a_{41}a_{63})p_1p_2$$

$$+(a_{23}p_1+a_{41}p_1+a_{45}p_2+a_{63}p_2)d^2+d^4\Big] \qquad (3\text{–}40)$$

From equation (2–11) the synthesized scalar state, $\hat{x}_6$, is expressed in terms of the observer state variable, z, and the accessible scalar state variables as follows,

$$\hat{x}_6 = \frac{1}{t_6}[z-\sum_{i=1}^{5}t_ix_i] \qquad (3\text{–}41)$$

In this case, it is assumed that:

$$\mathbf{C} = \left[\begin{array}{c|c} \mathbf{I}_5 & \begin{matrix}0\\ \vdots\\ 0\end{matrix}\end{array}\right] \qquad (3\text{–}42)$$

where $\mathbf{I}_5 = 5\times 5$ identity matrix.
From $\mathbf{F}=\mathbf{GC}$,

$$\mathbf{G} = [f_1 \quad f_2 \quad f_3 \quad f_4 \quad f_5] \qquad (3\text{–}43)$$

From $\mathbf{E}=\mathbf{TB}$,

$\mathbf{E} = [t_2 \quad t_4 \quad t_6]$ for $r=3$ (control torques applied to all three bodies) (3–44)

$\mathbf{E} = [t_2 \quad t_4 \quad 0]$ for control applied to bodies 1 and 2 (3–45)

$\mathbf{E} = [t_2 \quad 0 \quad 0]$ for control applied to body 1 (3–46)

$\mathbf{E} = [t_2 \quad 0 \quad t_6]$ for control applied to bodies 1 and 3 (3–47)

3.4.2 Observers of Intermediate Order ($p = 2$, 3 or 4)

In the cases in which an intermediate number of the six scalar states of the three-body single-axis model is inaccessible the minimum order of the reduced state linear observer required to reconstruct these inaccessible states is given by p. In each case the number of null columns in the measurement or observation matrix, **C**, and the **F** matrix also is equal to p. The general forms of the **E**, **F** and **T** matrices are given in equations (3–10), (3–11) and (3–12) for $p = 2$, 3 or 4 where p represents the number of inaccessible state variables of the model.

<u>Example</u>

Suppose the scalar states, x_5 and x_6, corresponding to the angular position and rate, respectively, of body 3, are inaccessible. Then $f_{i5} = f_{i6} = 0$ for $i = 1, 2$ and the observer synthesis equations reduce to the form of equations (3–13) through (3–23) with $f_{i5} = f_{i6} = 0$. From equation (2–11) the synthesized scalar states, $\hat{x}_5$ and $\hat{x}_6$, are expressed in terms of the observer variables, z_1 and z_2, and the accessible state variables as follows.

$$\hat{x}_5 = \frac{(\Delta_2)_{1,1}(z_1 - \sum_{j=1}^{4} t_{1j}x_j) - (\Delta_2)_{2,1}(z_2 - \sum_{j=1}^{4} t_{2j}x_j)}{\Delta_2} \tag{3-48}$$

$$\hat{x}_6 = \frac{-(\Delta_2)_{1,2}(z_1 - \sum_{j=1}^{4} t_{1j}x_j) + (\Delta_2)_{2,2}(z_2 - \sum_{j=1}^{4} t_{2j}x_j)}{\Delta_2} \tag{3-49}$$

where,

$$\Delta_2 = \begin{vmatrix} t_{15} & t_{16} \\ t_{25} & t_{26} \end{vmatrix} = t_{15}t_{26} - t_{16}t_{25} \neq 0 \tag{3-50}$$

and $(\Delta_2)_{i,j} = \Delta_2$ without the elements of the ith row and jth column.

For x_5 and x_6 inaccessible, it is assumed that:

$$\mathbf{C} = \left[\begin{array}{c|cc} & 0 & 0 \\ \mathbf{I}_4 & \vdots & \vdots \\ & 0 & 0 \end{array}\right] \tag{3-51}$$

where $\mathbf{I}_4 = 4 \times 4$ identity matrix.
From $\mathbf{F} = \mathbf{GC}$,

$$\mathbf{G} = \begin{bmatrix} f_{11} & f_{12} & f_{13} & f_{14} \\ f_{21} & f_{22} & f_{23} & f_{24} \end{bmatrix} \tag{3-52}$$

From $\mathbf{E} = \mathbf{TB}$,

$$\mathbf{E} = \begin{bmatrix} t_{12} & t_{14} & t_{16} \\ t_{22} & t_{24} & t_{26} \end{bmatrix} \quad \text{for } r = 3 \text{ (control torques applied to all three bodies)} \tag{3-53}$$

$$\mathbf{E} = \begin{bmatrix} t_{12} & t_{14} & 0 \\ t_{22} & t_{24} & 0 \end{bmatrix} \quad \text{for control applied to bodies 1 and 2.} \tag{3-54}$$

$$\mathbf{E} = \begin{bmatrix} t_{12} & 0 & t_{16} \\ t_{22} & 0 & t_{26} \end{bmatrix} \quad \text{for control applied to bodies 1 and 3.} \tag{3-55}$$

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$$\mathbf{E} = \begin{bmatrix} t_{12} & 0 & 0 \\ t_{22} & 0 & 0 \end{bmatrix} \quad \text{for control restricted to body 1.} \tag{3-56}$$

3.4.4 Fifth Order Observers ($p = 5$)

An observer of at least order five is required when any five of the six scalar state variables of the three body models are inaccessible. The observer synthesis equations are given in equations (3–13) through (3–23) with $i = 1, 2, \ldots, 5$. Since a fifth order observer corresponds to five of the six scalar states being inaccessible, $f_{ij} = 0$ for five of the six values of the subscript, j.

<u>Example</u>

Suppose that the scalar states, x_2, x_3, x_4, x_5 and x_6, representing the angular rate of body 1 and the angular displacements and rates of bodies 2 and 3 are inaccessible. Then $f_{i2} = f_{i3} = f_{i4} = f_{i5} = f_{i6} = 0$ for $i = 1, 2, \ldots, 5$ and the observer synthesis equations reduce to the form of equations (3–13) through (3–23) with only $f_{i1} \neq 0$. The synthesized scalar state variables, $\hat{x}_2$, $\hat{x}_3$, $\hat{x}_4$, $\hat{x}_5$ and $\hat{x}_6$ are expressed in terms of the observer scalar variables, $z_1, z_2, \ldots, z_5$, and the accessible state variables, using equation (2–11) as follows:

$$\hat{x}_{k+1} = \frac{\sum_{i=1}^{5} (-1)^{i+1} (\Delta_5)_{i,k} (z_i - t_{i1} x_1)}{\Delta_5} \qquad k = 1, 2, \ldots, 5 \tag{3-57}$$

$$\Delta_5 = \begin{vmatrix} t_{12} & t_{13} & t_{14} & t_{15} & t_{16} \\ t_{22} & \vdots & \vdots & \vdots & t_{26} \\ t_{32} & \vdots & \vdots & \vdots & t_{36} \\ t_{42} & \vdots & \vdots & \vdots & t_{46} \\ t_{52} & t_{53} & t_{54} & t_{55} & t_{56} \end{vmatrix}$$

$$= t_{12}(\Delta_5)_{1,1} - t_{22}(\Delta_5)_{2,1} + t_{32}(\Delta_5)_{3,1} - t_{42}(\Delta_5)_{4,1} + t_{52}(\Delta_5)_{5,1} \tag{3-58}$$

where $(\Delta_5)_{i,j} = \Delta_5$ without the elements of the i^{th} row and j^{th} column.

For only x_1 accessible, it is assumed that:

$$\mathbf{C} = [1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0] \tag{3-59}$$

From $\mathbf{F} = \mathbf{GC}$,

$$\mathbf{G} = \begin{bmatrix} f_{11} \\ f_{21} \\ f_{31} \\ f_{41} \\ f_{51} \end{bmatrix} \tag{3-60}$$

From $\mathbf{E} = \mathbf{TB}$,

$$\mathbf{E} = \begin{bmatrix} t_{12} & t_{14} & t_{16} \\ \vdots & \vdots & \vdots \\ t_{52} & t_{54} & t_{56} \end{bmatrix} \quad \text{for } r = 3 \text{ (control torques applied to all three bodies).} \tag{3-61}$$

3.5 REFERENCES

3–1 Luenberger, D.G., "Determining the State of a Linear System with Observers of Low Dynamic Order", Ph.D. dissertation, Stanford University, 1963.

3–2 Luenberger, D.G., "Observers for Multivariable Systems", *IEEE Transactions on Automatic Control*, Vol. AC-11, No. 2, April 1966, pp. 190–197.

3–3 Luenberger, D.G., "An Introduction to Observers", *IEEE Transactions on Automatic Control*, Vol. AC-16, No. 6, December 1971, pp. 596–602.

3–4 Sage, A.P., *Optimum Systems Control*. Englewood Cliffs, N.J.: Prentice-Hall, Inc. 1968, pp. 306–312.

SECTION 4

DEVELOPMENT OF THE FOUR-BODY SINGLE-AXIS MODEL AND ITS REDUCED STATE LINEAR OBSERVER

4.1 ORIGINAL DAMPED MODEL

The rotational dynamics of the four-body single-axis model of a flexible spacecraft with damping shown in Fig. 4–1 may be represented by the following set of equations.

$$I_1\ddot{\theta}_1 = -c_1(\dot{\theta}_1 - \dot{\theta}_2) - k_1(\theta_1 - \theta_2) + q_1 \tag{4–1}$$

$$I_2\ddot{\theta}_2 = c_1(\dot{\theta}_1 - \dot{\theta}_2) + k_1(\theta_1 - \theta_2) + c_2(\dot{\theta}_3 - \dot{\theta}_2) + k_2(\theta_3 - \theta_2) + q_2 \tag{4–2}$$

$$I_3\ddot{\theta}_3 = c_2(\dot{\theta}_2 - \dot{\theta}_3) + k_2(\theta_2 - \theta_3) + c_3(\dot{\theta}_4 - \dot{\theta}_3) + k_3(\theta_4 - \theta_3) + q_3 \tag{4–3}$$

$$I_4\ddot{\theta}_4 = -c_3(\dot{\theta}_4 - \dot{\theta}_3) - k_3(\theta_4 - \theta_3) + q_4 \tag{4–4}$$

where:

I_i = rotational inertia of body i; $\quad i = 1,2,3,4$

θ_i = angular displacement of body i

$\dot{\theta}_i$ = angular rate of body i

q_i = torque applied to body i

k_j = rotational spring coefficient at interface j; $\quad j = 1,2,3$

c_j = rotational damping coefficient at interface j

4.2 STATE VARIABLE MODEL

The state variable form of the four-body single-axis model of a flexible spacecraft depicted in Fig. 4–1 was written in the following form.

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{4–5}$$

$$\mathbf{x}_A = \mathbf{C}\mathbf{x} \tag{4–6}$$

where:

$\mathbf{x} = [x_1 \ \cdots \ x_8]^T = [\theta_1 \ \ \dot{\theta}_1 \ \ \theta_2 \ \ \dot{\theta}_2 \ \ \theta_3 \ \ \dot{\theta}_3 \ \ \theta_4 \ \ \dot{\theta}_4]^T = [\mathbf{x}_A^T \ \ \mathbf{x}_I^T]^T$ = state vector

$\mathbf{x}_A$ = p vector of accessible scalar states

$\mathbf{x}_I$ = m vector of inaccessible scalar states

$\mathbf{u} = [u_1 \ \cdots \ u_r]^T = \left[\frac{q_1}{I_1} \ \cdots \ \frac{q_r}{I_r}\right]^T \qquad (r = 1,2,3 \text{ or } 4)$

$\mathbf{C}$ = $m \times 8$ measurement or observation matrix

Partitioning of this model by rigid body yields the following forms for its coefficient matrices:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -a_{23} & -a_{23}r_1 & a_{23} & a_{23}r_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ a_{41} & a_{41}r_1 & a_{43} & a_{44} & a_{45} & a_{45}r_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & a_{63} & a_{63}r_2 & a_{65} & a_{66} & a_{67} & a_{67}r_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & a_{85} & a_{85}r_3 & -a_{85} & -a_{85}r_3 \end{bmatrix} \tag{4–7}$$

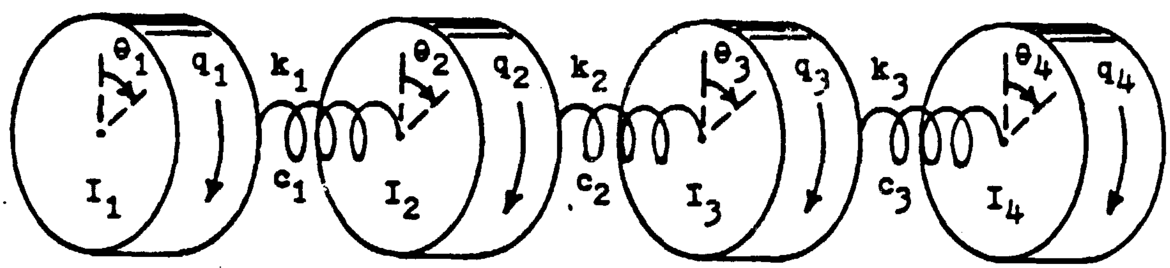

FIGURE 4-1

FOUR-BODY SINGLE-AXIS MODEL WITH DAMPING AT ALL THREE INTERFACES

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$$\mathbf{B} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ & & & \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ & & & \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ & & & \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ for } r = 4 \tag{4-8}$$

$$\begin{aligned}
a_{23} &= \frac{k_1}{I_1} \\
a_{41} &= \frac{k_1}{I_2}, \quad a_{45} = \frac{k_2}{I_2}, \quad a_{43} = -(a_{41} + a_{45}), \quad a_{44} = -(a_{41} r_1 + a_{45} r_2) \\
a_{63} &= \frac{k_2}{I_3}, \quad a_{67} = -(a_{63} + a_{67}), \quad a_{65} = -(a_{63} + a_{67}), \quad a_{66} = -(a_{63} r_2 + a_{67} r_3) \\
a_{85} &= \frac{k_3}{I_4}
\end{aligned} \tag{4-9}$$

$$r_j = \frac{c_j}{k_j} \qquad j = 1, 2, 3 \tag{4-10}$$

The corresponding block diagram appears in Fig. 2-2.

4.3 REDUCED STATE LINEAR OBSERVERS

4.3.1 Introduction

The minimum order of a reduced state linear observer required to reconstruct the $8 - m$ inaccessible scalar state variables of the four body single axis model of a flexible spacecraft represented by equations (4-5) through (4-10) is $p = 8 - m$ where $m = 1, 2, 3, 4, 5, 6$ or 7. All of the reduced state linear observers for this four body model may be written in the form of equations (2-10) and (2-11) under the assumption that the observer coefficient matrix, $\mathbf{D}$, is diagonal and of dimensions $p \times p$. The corresponding observer weighting matrix is of the following form.

$$\mathbf{T} = \begin{bmatrix} t_{11} & \cdots & t_{18} \\ \vdots & & \vdots \\ t_{p,1} & \cdots & t_{p,8} \end{bmatrix} \tag{4-11}$$

From equations (2-12), (4-8) and (4-11).

$$\mathbf{E} = \begin{bmatrix} t_{12} & t_{14} & t_{16} & t_{18} \\ \vdots & \vdots & \vdots & \vdots \\ t_{p,2} & t_{p,4} & p_{p,6} & p_{p,8} \end{bmatrix} \tag{4-12}$$

$$\mathbf{F} = \begin{bmatrix} f_{11} & \cdots & f_{18} \\ \vdots & & \vdots \\ f_{p,1} & \cdots & f_{p,8} \end{bmatrix} \tag{4-13}$$

The corresponding observer block diagram appears in Fig. 2-4.

4.3.2 Observer Synthesis Equations

From Luenberger (4-1), (4-2), (4-3) and Sage (4-4) the equations for synthesizing the reduced state linear observers for the four-body single-axis model represented by equations (4-5) through (4-10) are given by equations (2-13) and (2-14). With coefficient matrices of the form listed in 4.3.1 this set of observer synthesis equations reduces to the following.

$$
\begin{aligned}
t_{i2} = {} & \frac{(\Delta'_{i4})_{1,1}(f_{i1}+d_{ii}f_{i2}) - (\Delta'_{i4})_{2,1}(f_{i3}+d_{ii}f_{i4}) + (\Delta'_{i4})_{3,1}(f_{i5}+d_{ii}f_{i6})}{\Delta'_{i4}} \\
& - \frac{(\Delta'_{i4})_{4,1}(f_{i7}+d_{ii}f_{i8})}{\Delta'_{i4}} \\
= {} & \frac{\Big\{ a_{45}a_{63}p_2^2(a_{65}p_3+d_{ii}^2)}{\Delta'_{i4}} \\
& - \frac{(a_{41}p_1 + a_{45}p_2 + d_{ii}^2)\Big[(a_{63}p_2+d_{ii}^2)(a_{65}p_3+d_{ii}^2) + a_{67}p_3d_{ii}^2\Big]\Big\}(f_{i1}+d_{ii}f_{i2})}{\Delta'_{i4}} \\
& - \frac{a_{41}p_1\Big[(a_{63}p_2+d_{ii}^2)(a_{65}p_3+d_{ii}^2) + a_{67}p_3d_{ii}^2\Big](f_{i3}+d_{ii}f_{i4})}{\Delta'_{i4}} \\
& - \frac{a_{41}a_{63}p_1p_2\Big[(a_{65}p_3+d_{ii}^2)(f_{i5}+d_{ii}f_{i6}) + a_{65}p_3(f_{i7}+d_{ii}f_{i8})\Big]}{\Delta'_{i4}} \qquad i = 1,2,\ldots,p
\end{aligned}
\tag{4-14}
$$

$$
\begin{aligned}
t_{i4} = {} & -\frac{(\Delta'_{i4})_{1,2}(f_{i1}+d_{ii}f_{i2}) - (\Delta'_{i4})_{2,2}(f_{i3}+d_{ii}f_{i4}) + (\Delta'_{i4})_{3,2}(f_{i5}+d_{ii}f_{i6})}{\Delta'_{i4}} \\
& + \frac{(\Delta'_{i4})_{4,2}(f_{i7}+d_{ii}f_{i8})}{\Delta'_{i4}} \\
= {} & -\frac{\Big[(a_{63}p_2+d_{ii}^2)(a_{65}p_3+d_{ii}^2) + a_{67}p_3d_{ii}^2\Big]\Big[a_{23}p_1(f_{i1}+d_{ii}f_{i2}) + (a_{23}+d_{ii}^2)(f_{i3}+d_{ii}f_{i4})\Big]}{\Delta'_{i4}} \\
& - \frac{(a_{23}p_1 + d_{ii}^2)a_{63}p_2\Big[(a_{65}p_3+d_{ii}^2)(f_{i5}+d_{ii}f_{i6}) + a_{65}p_3(f_{i7}+d_{ii}f_{i8})\Big]}{\Delta'_{i4}}
\end{aligned}
\tag{4-15}
$$

$$
\begin{aligned}
t_{i6} = {} & \frac{(\Delta'_{i4})_{1,3}(f_{i1}+d_{ii}f_{i2}) - (\Delta'_{i4})_{2,3}(f_{i3}+d_{ii}f_{i4}) + (\Delta'_{i4})_{3,3}(f_{i5}+d_{ii}f_{i6})}{\Delta'_{i4}} \\
& - \frac{(\Delta'_{i4})_{4,3}(f_{i7}+d_{ii}f_{i8})}{\Delta'_{i4}} \\
= {} & -\frac{a_{45}p_2(a_{65}p_3+d_{ii}^2)\Big[a_{23}p_1(f_{i1}+d_{ii}f_{i2}) + (a_{23}p_1+d_{ii}^2)(f_{i3}+d_{ii}f_{i4})\Big]}{\Delta'_{i4}} \\
& - \frac{\Big[(a_{23}p_1+d_{ii}^2)(a_{45}p_2+d_{ii}^2) + a_{41}p_1d_{ii}^2\Big]\Big[(a_{65}p_3+d_{ii}^2)(f_{i5}+d_{ii}f_{i6})}{\Delta'_{i4}} \\
& + \frac{a_{65}p_3(f_{i7}+d_{ii}f_{i8})\Big]}{\Delta'_{i4}}
\end{aligned}
\tag{4-16}
$$

$$
\begin{aligned}
t_{i8} &= -\frac{(\Delta'_{i4})_{1,4}(f_{i1}+d_{ii}f_{i2}) - (\Delta'_{i4})_{2,4}(f_{i3}+d_{ii}f_{i4}) + (\Delta'_{i4})_{3,4}(f_{i5}+d_{ii}f_{i6})}{\Delta'_{i4}} \\
&\quad + \frac{(\Delta'_{i4})_{4,4}(f_{i7}+d_{ii}f_{i8})}{\Delta'_{i4}} \\
&= -\frac{a_{45}a_{67}p_2p_3\left[a_{23}p_1(f_{i1}+d_{ii}f_{i2}) + (a_{23}p_1+d_{ii}^2)(f_{i3}+d_{ii}f_{i4})\right]}{\Delta'_{i4}} \\
&\quad - \frac{a_{67}p_3\left[(a_{23}p_1+d_{ii}^2)(a_{45}p_2+d_{ii}^2) + a_{41}p_1d_{ii}^2\right](f_{i5}+d_{ii}f_{i6})}{\Delta'_{i4}} \\
&\quad - \frac{\left\{d_{ii}^2\left[a_{41}p_1(a_{63}p_2+a_{67}p_3+d_{ii}^2) + a_{63}p_2d_{ii}^2\right]\right.}{\Delta'_{i4}} \\
&\quad + \frac{\left.(a_{23}p_1+d_{ii}^2)(a_{45}p_2+d_{ii}^2)(a_{67}p_3+d_{ii}^2) + a_{23}a_{63}p_1p_2d_{ii}^2\right\}(f_{i7}+d_{ii}f_{i8})}{\Delta'_{i4}}
\end{aligned}
\tag{4-17}
$$

$$
\begin{aligned}
t_{i1} &= -\frac{\left\{\left[(a_{23}r_1+d_{ii})(a_{45}p_2+d_{ii}^2) + a_{41}d_{ii}\right]\left[(a_{63}p_2+d_{ii}^2)(a_{85}p_3+d_{ii}^2)\right.\right.}{\Delta'_{i4}} \\
&\quad + \frac{\left.\left. a_{67}p_3d_{ii}^2\right] - a_{45}a_{63}p_2^2(a_{85}p_3+d_{ii}^2)\right\}(f_{i1}+d_{ii}f_{i2})}{\Delta'_{i4}} + f_{i2} \\
&\quad - \frac{a_{41}d_{ii}\left\{\left[(a_{63}p_2+d_{ii}^2)(a_{85}p_3+d_{ii}^2) + a_{67}p_3d_{ii}^2\right](f_{i3}+d_{ii}f_{i4})\right.}{\Delta'_{i4}} \\
&\quad + \frac{\left. a_{63}p_2(a_{85}p_3+d_{ii}^2)(f_{i5}+d_{ii}f_{i6}) + a_{63}a_{85}p_2p_3(f_{i7}+d_{ii}f_{i8})\right\}}{\Delta'_{i4}}
\end{aligned}
\tag{4-18}
$$

$$
\begin{aligned}
t_{i3} &= \frac{a_{23}\left\{\left[a_{45}(r_1-r_2) - d_{ii}\right]\left[(a_{63}p_2+d_{ii}^2)(a_{85}p_3+d_{ii}^2) + a_{67}p_3d_{ii}^2\right]\right.}{\Delta'_{i4}} \\
&\quad - \frac{\left. a_{45}a_{63}p_2(a_{85}p_3+d_{ii}^2)(r_1-r_2)\right\}(f_{i1}+d_{ii}f_{i2})}{\Delta'_{i4}} \\
&\quad - \frac{\left\{\left[(a_{23}p_1+d_{ii}^2)(a_{45}r_2+d_{ii}) + a_{41}r_1d_{ii}^2\right]\left[(a_{63}p_2-d_{ii}^2)(a_{85}p_3+d_{ii}^2)\right.\right.}{\Delta'_{i4}} \\
&\quad + \frac{\left.\left. a_{67}p_3d_{ii}^2\right] - a_{45}a_{63}p_2r_2(a_{23}p_1+d_{ii}^2)(a_{85}p_3+d_{ii}^2)\right\}(f_{i3}+d_{ii}f_{i4})}{\Delta'_{i4}} + f_{i4} \\
&\quad + \frac{a_{63}d_{ii}\left[a_{23}p_1+a_{41}d_{ii}(r_1-r_2)+d_{ii}^2\right]\left[(a_{85}p_2+d_{ii}^2)(f_{i5}+d_{ii}f_{i6}) + a_{85}p_3(f_{i7}+d_{ii}f_{i8})\right]}{\Delta'_{i4}}
\end{aligned}
\tag{4-19}
$$

$$t_{i5} = -\frac{a_{45}d_{ii}\Bigl[a_{85}p_3 + a_{67}d_{ii}(r_3 - r_2) + d_{ii}^4\Bigr]\Bigl[a_{23}p_1(f_{i1} + d_{ii}f_{i2}) + (a_{23}p_1 + d_{ii}^2)(f_{i3} + d_{ii}f_{i4})\Bigr]}{\Delta'_{i4}}$$

$$+\frac{d_{ii}\Bigl\{(a_{23}p_1 + d_{ii}^2)(a_{45}p_2 + d_{ii}^2)(a_{85}p_3 + a_{67}r_3d_{ii} + d_{ii}^2)}{\Delta'_{i4}}$$

$$-\frac{d_{ii}\Bigl[(a_{23}p_1 + a_{41}p_1 + d_{ii}^2)a_{63}r_2 + a_{41}p_1d_{ii}\Bigr](a_{85}p_3 + d_{ii}^2) + a_{67}r_3d_{ii}^2\Bigr\}(f_{i5} + d_{ii}f_{i6})}{\Delta'_{i4}} + f_{i6}$$

$$+\frac{a_{85}d_{ii}\Bigl\{d_{ii}\Bigl[a_{23}a_{63}p_1(r_2 - r_1) + a_{63}(r_3 - r_2)(a_{41}p_1 + d_{ii}^2) - a_{41}p_1d_{ii}\Bigr]}{\Delta'_{i4}}$$

$$-\frac{(a_{23}p_1 + d_{ii}^2)(a_{45}p_2 + d_{ii}^2)\Bigr\}(f_{i7} + d_{ii}f_{i8})}{\Delta'_{i4}} \qquad (4\text{–}20)$$

$$t_{i7} = -\frac{a_{67}d_{ii}\Bigl\{a_{45}p_2\Bigl[a_{23}p_1(f_{i1} + d_{ii}f_{i2}) + (a_{23}p_1 + d_{ii}^2)(f_{i3} + d_{ii}f_{i4})\Bigr]}{\Delta'_{i4}}$$

$$+\frac{\Bigl[(a_{23}p_1 + d_{ii}^2)(a_{45}p_2 + d_{ii}^2) + a_{41}p_1d_{ii}^2\Bigr](f_{i5} + d_{ii}f_{i6})\Bigr\}}{\Delta'_{i4}}$$

$$+\frac{d_{ii}\Bigl\{\Bigl[(a_{67}p_3 + a_{85}r_3d_{ii} + d_{ii}^2)(a_{23}p_1 + d_{ii}^2)(a_{45}p_2 + d_{ii}^2)}{\Delta'_{i4}}$$

$$+\frac{a_{41}a_{67}a_{85}p_1p_3r_3d_{ii}\Bigr] + d_{ii}\Bigl[a_{23}a_{63}p_1p_2 + a_{41}a_{63}p_1p_2}{\Delta'_{i4}}$$

$$+\frac{a_{41}a_{67}p_1p_3 + (a_{41}r_1 + a_{63}p_2)d_{ii}^2\Bigr]\Bigr\}(f_{i7} + d_{ii}f_{i8})}{\Delta'_{i4}} + f_{i8} \qquad (4\text{–}21)$$

where:

$$r_j = \frac{c_j}{k_j}, \qquad j = 1,2,3 \qquad (4\text{–}22)$$

$$p_j = 1 + r_jd_{ii}, \qquad i = 1,2,\ldots,p \qquad j = 1,2,3 \qquad (4\text{–}23)$$

$$d_{ii1} = a_{23}r_1 + d_{ii} \qquad (4\text{–}24)$$

$$d_{ii2} = a_{41}r_1 + a_{45}r_2 + d_{ii} \qquad (4\text{–}25)$$

$$d_{ii3} = a_{63}r_2 + a_{67}r_3 + d_{ii} \qquad (4\text{–}26)$$

$$d_{ii4} = a_{85}r_3 + d_{ii} \qquad (4\text{–}27)$$

$$\Delta'_{i4} = \begin{vmatrix} -(a_{23}p_1 + d_{ii}^2) & a_{41}p_1 & 0 & 0 \\ a_{23}p_1 & -(a_{41}p_1 + a_{45}p_2 + d_{ii}^2) & a_{63}p_2 & 0 \\ 0 & a_{45}p_2 & -(a_{63}p_2 + a_{67}p_3 + d_{ii}^2) & a_{85}p_3 \\ 0 & 0 & a_{67}p_3 & -(a_{85}p_3 + d_{ii}^2) \end{vmatrix}$$

$$= d_{ii}^2\Bigl[(a_{23}a_{45}a_{67} + a_{23}a_{45}a_{85} + a_{23}a_{63}a_{85} + a_{41}a_{63}a_{85})p_1p_2p_3$$

$$+ (a_{23}a_{45}p_1p_2 + 2a_{23}a_{63}p_1p_2 + 2a_{23}a_{67}p_1p_3 + 2a_{23}a_{85}p_1p_3 + 2a_{41}a_{63}p_1p_2$$

$$+ 2a_{41}a_{67}p_1p_3 + 2a_{41}a_{85}p_1p_3 + a_{45}a_{63}p_2^2 + 2a_{45}a_{67}p_2p_3 + 2a_{45}a_{85}p_2p_3$$

$$+ a_{63}a_{85}p_2p_3)d_{ii}^2 + 2(a_{23}p_1 + a_{41}p_1 + a_{45}p_2 + a_{63}p_2 + a_{67}p_3 + a_{85}p_3)d_{ii}^4 + 2d_{ii}^6\Bigr] \qquad (4\text{–}28)$$

4.3.3 Comparison of T Matrices For Elimination of Damping at Various Interfaces

Elimination of damping at interface j of the model corresponds to setting $r_j = 0$ and $p_j = 1$ in the equations for generating the elements of the T matrix, equations (4–14) through (4–28). The following damping conditions have been treated for this set of equations.

1. Damping eliminated at the interface between bodies 1 and 2;
2. Damping eliminated at the interface between bodies 2 and 3;
3. Damping eliminated at the interface between bodies 3 and 4;
4. Damping eliminated at the interfaces between bodies 2, 3 and 4;
5. Damping eliminated at the interfaces between bodies 1 and 2, and 3 and 4.
6. Damping eliminated at the interfaces between bodies 1, 2 and 3;
7. Damping eliminated from all interfaces.

Example: All interface damping eliminated.

If damping is removed from all three interfaces of the four body model, $r_j \to 0$, $p_j \to 1$, $d_{ii1} \to d_{ii}$, $d_{ii2} \to d_{ii}$, $d_{ii3} \to d_{ii}$, $d_{ii4} \to d_{ii}$ and

$$\begin{aligned}\Delta'_{i4} \to d_{ii}^2 \Big[& a_{23}a_{45}a_{67} + a_{23}a_{45}a_{85} + a_{23}a_{63}a_{85} + a_{41}a_{63}a_{85} \\ & + (a_{23}a_{45} + 2a_{23}a_{63} + 2a_{23}a_{67} + 2a_{23}a_{85} + 2a_{41}a_{63} + 2a_{41}a_{67} \\ & \quad + 2a_{41}a_{85} + a_{45}a_{63} + 2a_{45}a_{67} + 2a_{45}a_{85} + a_{63}a_{85})d_{ii}^2 \\ & + 2(a_{23} + a_{41} + a_{45} + a_{63} + a_{67} + a_{85})d_{ii}^4 + d_{ii}^6 \Big] = \Delta_{i4}\end{aligned}$$

$$\begin{aligned} t_{i2} = & \frac{\left\{a_{45}a_{63}(a_{85} + d_{ii}^2) - (a_{41} + a_{45} + d_{ii}^2)\left[(a_{63} + d_{ii}^2)(a_{85} + d_{ii}^2) + a_{67}d_{ii}^2\right]\right\}(f_{i1} + d_{ii}f_{i2})}{\Delta_{i4}} \\ & - \frac{a_{41}\left[(a_{63} + d_{ii}^2)(a_{85} + d_{ii}^2) + a_{67}d_{ii}^2\right](f_{i3} + d_{ii}f_{i4})}{\Delta_{i4}} \\ & - \frac{a_{41}a_{63}\left[(a_{85} + d_{ii}^2)(f_{i5} + d_{ii}f_{i6}) + a_{85}(f_{i7} + d_{ii}f_{i8})\right]}{\Delta_{i4}} \qquad i = 1, 2, \ldots, p \end{aligned} \tag{4–29}$$

$$\begin{aligned} t_{i4} = & -\frac{\left[(a_{63} + d_{ii}^2)(a_{85} + d_{ii}^2) + a_{67}d_{ii}^2\right]\left[a_{23}(f_{i1} + d_{ii}f_{i2}) + (a_{23} + d_{ii}^2)(f_{i3} + d_{ii}f_{i4})\right]}{\Delta_{i4}} \\ & - \frac{(a_{23} + d_{ii}^2)a_{63}\left[(a_{85} + d_{ii}^2)(f_{i5} + d_{ii}f_{i6}) + a_{85}(f_{i7} + d_{ii}f_{i8})\right]}{\Delta_{i4}} \end{aligned} \tag{4–30}$$

$$\begin{aligned} t_{i6} = & -\frac{a_{45}(a_{85} + d_{ii}^2)\left[a_{23}(f_{i1} + d_{ii}f_{i2}) + (a_{23} + d_{ii}^2)(f_{i3} + d_{ii}f_{i4})\right]}{\Delta_{i4}} \\ & - \frac{\left[(a_{23} + d_{ii}^2)(a_{45} + d_{ii}^2) + a_{41}d_{ii}^2) + a_{41}d_{ii}^2\right]\left[(a_{85} + d_{ii}^2)(f_{i5} + d_{ii}f_{i6}) + a_{85}(f_{i7} + d_{ii}f_{i8})\right]}{\Delta_{i4}} \end{aligned} \tag{4–31}$$

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$$
\begin{aligned}
t_{i8} = & -\frac{a_{45}a_{67}\left[a_{23}(f_{i1}+d_{ii}f_{i2})+(a_{23}+d_{ii}^2)(f_{i3}+d_{ii}f_{i4})\right]}{\Delta_{i4}} \\
& -\frac{a_{67}\left[(a_{23}+d_{ii}^2)(a_{45}+d_{ii}^2)+a_{41}d_{ii}^2\right](f_{i5}+d_{ii}f_{i6})}{\Delta_{i4}} \\
& -\frac{\left\{d_{ii}^2\left[a_{41}(a_{63}+a_{67}+d_{ii}^2)+a_{63}d_{ii}^2\right]\right.}{\Delta_{i4}} \\
& \qquad +\frac{\left.(a_{23}+d_{ii}^2)(a_{45}+d_{ii}^2)(a_{67}+d_{ii}^2)+a_{23}a_{63}d_{ii}^2\right\}(f_{i7}+d_{ii}f_{i8})}{\Delta_{i4}}
\end{aligned}
\tag{4-32}
$$

$$
\begin{aligned}
t_{i1} = & -\frac{\left\{d_{ii}(a_{41}+a_{45}+d_{ii}^2)\left[(a_{63}+d_{ii}^2)(a_{85}+d_{ii}^2)+a_{67}d_{ii}^2\right]\right.}{\Delta_{i4}} \\
& \qquad -\frac{\left.a_{45}a_{63}(a_{85}+d_{ii}^2)\right\}(f_{i1}+d_{ii}f_{i2})}{\Delta_{i4}}+f_{i2} \\
& -\frac{a_{41}d_{ii}\left\{\left[(a_{63}+d_{ii}^2)(a_{85}+d_{ii}^2)+a_{67}d_{ii}^2\right](f_{i3}+d_{ii}f_{i4})\right.}{\Delta_{i4}} \\
& \qquad +\frac{\left.a_{63}(a_{85}+d_{ii}^2)(f_{i5}+d_{ii}f_{i6})+a_{63}a_{85}(f_{i7}+d_{ii}f_{i8})\right\}}{\Delta_{i4}}
\end{aligned}
\tag{4-33}
$$

$$
\begin{aligned}
t_{i3} = & -\frac{a_{23}d_{ii}\left[(a_{63}+d_{ii}^2)(a_{85}+d_{ii}^2)+a_{67}d_{ii}^2\right](f_{i1}+d_{ii}f_{i2})}{\Delta_{i4}} \\
& -\frac{d_{ii}(a_{23}+d_{ii}^2)\left[(a_{63}+d_{ii}^2)(a_{85}+d_{ii}^2)+a_{67}d_{ii}^2\right](f_{i3}+d_{ii}f_{i4})}{\Delta_{i4}}+f_{i4} \\
& +\frac{a_{63}d_{ii}(a_{23}+d_{ii}^2)\left[(a_{85}+d_{ii}^2)(f_{i5}+d_{ii}f_{i6})+a_{85}(f_{i7}+d_{ii}f_{i8})\right]}{\Delta_{i4}}
\end{aligned}
\tag{4-34}
$$

$$
\begin{aligned}
t_{i5} = &-\frac{a_{45}d_{ii}(a_{65}+d_{ii}^2)\left[a_{23}(f_{i1}+d_{ii}f_{i2})+(a_{23}+d_{ii}^2)(f_{i3}+d_{ii}f_{i4})\right]}{\Delta_{i4}} \\
&+\frac{d_{ii}\left[(a_{23}+d_{ii}^2)(a_{45}+d_{ii}^2)(a_{65}+d_{ii}^2)-a_{41}d_{ii}^2(a_{65}+d_{ii}^2)\right](f_{i5}+d_{ii}f_{i6})}{\Delta_{i4}}+f_{i6} \\
&-\frac{a_{65}d_{ii}\left[a_{41}d_{ii}^2+(a_{23}+d_{ii}^2)(a_{45}+d_{ii}^2)\right](f_{i7}+d_{ii}f_{i8})}{\Delta_{i4}}
\end{aligned}
\qquad (4\text{–}35)
$$

$$
\begin{aligned}
t_{i7} = &-\frac{a_{67}d_{ii}\left\{a_{45}\left[a_{23}(f_{i1}+d_{ii}f_{i2})+(a_{23}+d_{ii}^2)(f_{i3}+d_{ii}f_{i4})\right]\right.}{\Delta_{i4}} \\
&+\frac{\left.\left[(a_{23}+d_{ii}^2)(a_{45}+d_{ii}^2)+a_{41}d_{ii}^2\right](f_{i5}+d_{ii}f_{i6})\right\}}{\Delta_{i4}} \\
&+\frac{d_{ii}\left\{(a_{23}+d_{ii}^2)(a_{45}+d_{ii}^2)(a_{67}+d_{ii}^2)\right.}{\Delta_{i4}} \\
&+\frac{\left.d_{ii}\left[a_{23}a_{63}+a_{41}a_{63}+a_{41}a_{67}+(a_{41}+a_{63})d_{ii}^2\right]\right\}(f_{i7}+d_{ii}f_{i8})}{\Delta_{i4}}+f_{i8}
\end{aligned}
\qquad (4\text{–}36)
$$

4.4 SOLUTION FOR SYNTHESIZED STATE VARIABLES

4.4.1 Introduction

Inaccessibility of a scalar state variable in the model equations (4–5), (4–6) is reflected by a corresponding null column in the **C** and **F** matrices as implied in equation (2–14). For the generation of reduced state observers for the four body model the number of inaccessible state variables, p, can be 1, 2, 3, 4, 5, 6 or 7.

4.4.2 First Order Observers ($p = 1$)

An observer of order at least one is required when only <u>one</u> of the eight scalar state variables of the four body model is inaccessible. The first order form of the linear observer equation is as follows:

$$\dot{s} = ds + \mathbf{E}\mathbf{u} + \mathbf{G}\mathbf{y} \qquad (4\text{–}37)$$

The **F** and **T** matrices associated with a first order observer for the four body model then reduce to the following row forms.

$$\mathbf{F} = [f_1 \quad f_2 \quad \ldots \quad f_8] \qquad (4\text{–}38)$$

$$\mathbf{T} = [t_1 \quad t_2 \quad \ldots \quad t_8] \qquad (4\text{–}39)$$

The observer synthesis equations are then of the form of equations (4–14) through (4–28) with $i = 1$. Since a first order observer corresponds to one of the scalar state variables being inaccessible, <u>one</u> of the $f_i \quad (i = 1, 2, \ldots, 8) = 0$.

<u>Example</u>

Suppose the scalar state representing the angular rate of body 4, x_8 is inaccessible. Then $f_8 = 0$ and the observer synthesis equations reduce to the form of equations (4–14) through (4–28) with $f_{i8} = 0$ and $i = 1$. From equation (2–11), the synthesized scalar state, $\hat{x}_8$, is expressed in terms of the scalar observer variable, z, and the accessible scalar state variables as follows.

$$\hat{x}_8 = \frac{1}{t_8}[z - \sum_{i=1}^{7} t_i x_i] \tag{4–40}$$

For x_8 inaccessible, it is assumed that:

$$\mathbf{C} = \left[\begin{array}{c|c} & 0 \\ \mathbf{I}_7 & \vdots \\ & 0 \end{array} \right], \tag{4–41}$$

where $\mathbf{I}_7 = 7 \times 7$ identity matrix.

From $\mathbf{F}$=$\mathbf{GC}$,

$$\mathbf{G} = [f_1 \quad f_2 \quad f_3 \quad f_4 \quad f_5 \quad f_6 \quad f_7] \tag{4–42}$$

From $\mathbf{E} = \mathbf{TB}$,

$$\mathbf{E} = [t_2 \quad t_4 \quad t_6 \quad t_8] \qquad \text{for } r = 4 \text{ (control torques on all 4 bodies)} \tag{4–43}$$

4.4.3 Observers of Intermediate Order ($p = 2, 3, 4, 5$ or 6)

For those cases in which an intermediate number of the eight scalar states of the four-body single-axis model is inaccessible, the minimum order of the reduced state linear observer required to reconstruct these inaccessible states is given by p. In each case the number of null columns in the measurement or observation matrix, $\mathbf{C}$, and the $\mathbf{F}$ matrix also is equal to p. The general forms of the $\mathbf{E}$, $\mathbf{F}$ and $\mathbf{T}$ matrices are given in equations (4–11), (4–12) and (4–13) for p =2, 3, 4, 5 or 6 where p represents the number of inaccessible scalar state variables of the model.

Example

Suppose the scalar states, x_7 and x_8, which represent the angular position and rate of body 4, are inaccessible. Then $f_{i7} = f_{i8} = 0$ for i =1, 2 and the observer synthesis equations reduce to the form of equations (4–14) through (4–28) with the preceding conditions. From equation (2–11) the synthesized scalar states, $\hat{x}_7$ and $\hat{x}_8$, are expressed in terms of the scalar observer variables, z and z_2 and the accessiable scalar state variables as follows.

$$\hat{x}_7 = \frac{\sum_{i=1}^{2}(-1)^{i+1}(\Delta_2)_{i,1}(z_i - \sum_{j=1}^{6} t_{ij}x_j)}{\Delta_2} \tag{4–44}$$

$$\hat{x}_8 = \frac{\sum_{i=1}^{2}(-1)^{i+1}(\Delta_2)_{i,2}(z_i - \sum_{j=1}^{6} t_{ij}x_j)}{\Delta_2} \tag{4–45}$$

for

$$\Delta_2 = \begin{vmatrix} t_{17} & t_{18} \\ t_{27} & t_{28} \end{vmatrix} = t_{17}t_{28} - t_{18}t_{27} \neq 0 \tag{4–46}$$

where $(\Delta_8)_{i,j} = \Delta_8$ without the elements of the i^{th} row and j^{th} column.

For x_7 and x_8 inaccessible, it is assumed that:

$$\mathbf{C} = \left[\begin{array}{c|cc} & 0 & 0 \\ \mathbf{I}_6 & \vdots & \vdots \\ & 0 & 0 \end{array} \right] \tag{4-47}$$

where $\mathbf{I}_6 = 6 \times 6$ identity matrix.

Since $\mathbf{F} = \mathbf{GC}$,

$$\mathbf{G} = \begin{bmatrix} f_{11} & f_{12} & f_{13} & f_{14} & f_{15} & f_{16} \\ f_{21} & f_{22} & f_{23} & f_{24} & f_{25} & f_{26} \end{bmatrix} \tag{4-48}$$

From $\mathbf{E} = \mathbf{TB}$,

$$\mathbf{E} = \begin{bmatrix} t_{12} & t_{14} & t_{16} & t_{18} \\ t_{22} & t_{24} & t_{26} & t_{28} \end{bmatrix} \quad \text{for } r = 4 \text{ (control torques applied to all four bodies)} \tag{4-49}$$

$$\mathbf{E} = \begin{bmatrix} t_{12} & t_{14} & t_{16} & 0 \\ t_{22} & t_{24} & t_{26} & 0 \end{bmatrix} \quad \text{for } r = 3 \text{ (control torques applied to bodies 1, 2 and 3)} \tag{4-50}$$

$$\mathbf{E} = \begin{bmatrix} t_{12} & t_{14} & 0 & 0 \\ t_{22} & t_{24} & 0 & 0 \end{bmatrix} \quad \text{for } r = 2 \text{ (control torques applied to bodies 1 and 2)} \tag{4-51}$$

$$\mathbf{E} = \begin{bmatrix} t_{12} & 0 & 0 & 0 \\ t_{22} & 0 & 0 & 0 \end{bmatrix} \quad \text{for } r = 1 \text{ (control torque applied to body 1)} \tag{4-52}$$

4.4.4 Seventh Order Observers ($p = 7$)

When any seven of the eight scalar state variables of the four body model are inaccessible, a linear observer of at least order seven is required. The observer synthesis equations are as presented in equations (4–14) through (4–28) with $i = 1, 2, \ldots, 7$. Since a seventh order observer corresponds to seven of the scalar states being inaccessible, $f_{1j} = f_{2j} = \ldots = f_{7j} = 0$ for seven of the eight values of the subscript, j.

<u>Example</u>

Suppose only the scalar state variable representing the angular position of body 1, x_1, is accessible. Then the remaining scalar states, $x_2, x_3, \ldots, x_8$ are inaccessible, $f_{i2} = f_{i3} = \ldots = f_{i8} = 0$ for i =1, 2, 3, 4, 5, 6 and 7 and the observer synthesis equations reduce to the form of equations (4–14) through (4–28) with $f_{i2} = f_{i3} = \ldots = f_{i8} = 0$ and $i = 1, 2, \ldots, 7$. The synthesized scalar state variables, $\hat{x}_2$ through $\hat{x}_8$, are expressed in terms of the observer variables, z_1 through z_7, and the accessible state variable, x_1, by utilizing equation (2–11) in the following form.

$$\hat{x}_{k+1} = \frac{\sum_{i=1}^{7} (-1)^{i+1} (\Delta_7)_{i,k} (z_i - t_{i1} x_1)}{\Delta_7} \qquad k = 1, 2, \ldots, 7 \tag{4-53}$$

$$\begin{aligned} \Delta_7 &= \begin{vmatrix} t_{12} & \cdots & t_{18} \\ \vdots & & \vdots \\ t_{72} & \cdots & t_{78} \end{vmatrix} \\ &= t_{12}(\Delta_7)_{1,1} - t_{22}(\Delta_7)_{2,1} + t_{32}(\Delta_7)_{3,1} - t_{42}(\Delta_7)_{4,1} \end{aligned}$$

$$+t_{52}(\Delta_7)_{5,1} - t_{62}(\Delta_7)_{6,1} + t_{72}(\Delta_7)_{7,1} \tag{4-54}$$

where $(\Delta_7)_{i,j} = \Delta_7$, without the elements of the i^{th} row and the j^{th} column.
For only x_1 accessible, it is assumed that:

$$\mathbf{C} = [1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0] \tag{4-55}$$

From $\mathbf{F} = \mathbf{GC}$,

$$\mathbf{G} = \begin{bmatrix} f_{11} \\ \vdots \\ f_{71} \end{bmatrix} \tag{4-56}$$

From $\mathbf{E} = \mathbf{TB}$,

$$\mathbf{E} = \begin{bmatrix} t_{12} & t_{14} & t_{16} & t_{18} \\ \vdots & \vdots & \vdots & \vdots \\ t_{72} & t_{74} & t_{76} & t_{78} \end{bmatrix} \quad \text{for } r = 4 \text{ (control torques applied to all four bodies)} \tag{4-57}$$

4.5 REFERENCES

4-1 Luenberger, D.G., "Determining the State of a Linear System with Observers of Low Dynamic Order", Ph.D. dissertation, Stanford University, 1963.

4-2 Luenberger, D.G., "Observers for Multivariable Systems", *IEEE Transactions on Automatic Control*, Vol. AC-11, No. 2, April 1966, pp. 190–197.

4-3 Luenberger, D.G., "An Introduction to Observers", *IEEE Transactions on Automatic Control*, Vol. AC-16, No. 6, December 1971, pp. 596–602.

4-4 Sage, A.P., *Optimum Systems Control*. Englewood Cliffs, N.J.: Prentice–Hall, Inc. 1968, pp. 306–312.

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SECTION 5

DEVELOPMENT OF THE FIVE-BODY SINGLE-AXIS MODEL AND ITS REDUCED STATE LINEAR OBSERVERS

5.1 ORIGINAL DAMPED MODEL

In earlier work, Guidance Systems Division (5–1), it was shown that one axis of the three-axis five-body approximation of a prototype flexible spacecraft can be decoupled from the other two axes. The four-body single-axis models of a flexible spacecraft developed in the previous section were therefore extended to corresponding five body models to represent the decoupled axis of the three-axis five-body model.

The rotational dynamics of the five-body single-axis model of a flexible spacecraft with damping shown in Fig. 5–1 may be represented by the following set of equations.

$$I_1\ddot{\theta}_1 = -c_1(\dot{\theta}_1 - \dot{\theta}_2) - k_1(\theta_1 - \theta_2) + q_1 \tag{5-1}$$
$$I_2\ddot{\theta}_2 = c_1(\dot{\theta}_1 - \dot{\theta}_2) + k_1(\theta_1 - \theta_2) + c_2(\dot{\theta}_3 - \dot{\theta}_2) + k_2(\theta_2 - \theta_2) + q_2 \tag{5-2}$$
$$I_3\ddot{\theta}_3 = c_2(\dot{\theta}_2 - \dot{\theta}_3) + k_2(\theta_2 - \theta_3) + c_3(\dot{\theta}_4 - \dot{\theta}_3) + k_3(\theta_4 - \theta_3) + q_3 \tag{5-3}$$
$$I_4\ddot{\theta}_4 = c_3(\dot{\theta}_3 - \dot{\theta}_4) + k_3(\theta_3 - \theta_4) + c_4(\dot{\theta}_5 - \dot{\theta}_4) + k_4(\theta_5 - \theta_4) + q_4 \tag{5-4}$$
$$I_5\ddot{\theta}_5 = c_4(\dot{\theta}_4 - \dot{\theta}_5) + k_4(\theta_4 - \theta_5) + q_5 \tag{5-5}$$

where:

I_i = rotational inertia of body i, $\quad i = 1, 2, \ldots, 5$

θ_i = angular displacement of body i

$\dot{\theta}_i$ = angular rate of body i

q_i = torque applied to body i

k_j = rotational spring coefficient at interface $j \quad j = 1, 2, 3, 4$

c_j = rotational damping coefficient at interface j

5.2 STATE VARIABLE MODEL

The state variable form of the five-body single-axis model of a flexible spacecraft depicted in Fig. 5–1 was written in the following form.

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{5-6}$$
$$\mathbf{x}_A = \mathbf{C}\mathbf{x} \tag{5-7}$$

where:

$\mathbf{x} = [x_1 \ \cdots \ x_{10}]^T = [\mathbf{x}_A^T \ \ \mathbf{x}_I^T]^T = [\theta_1 \ \ \dot{\theta}_1 \ \ \theta_2 \ \ \dot{\theta}_2 \ \ \theta_3 \ \ \dot{\theta}_3 \ \ \theta_4 \ \ \dot{\theta}_4 \ \ \theta_5 \ \ \dot{\theta}_5]^T$ = state vector

$\mathbf{x}_A$ = m vector of accessible scalar states

$\mathbf{x}_I$ = p vector of inaccessible scalar states

$\mathbf{u} = [u_1 \ \cdots \ u_r]^T = \left[\frac{q_1}{I_1} \ \cdots \ \frac{q_r}{I_r}\right]^T \qquad r = 1, 2, \ldots, 5$

$\mathbf{C}$ = $m \times 10$ measurement or observation matrix.

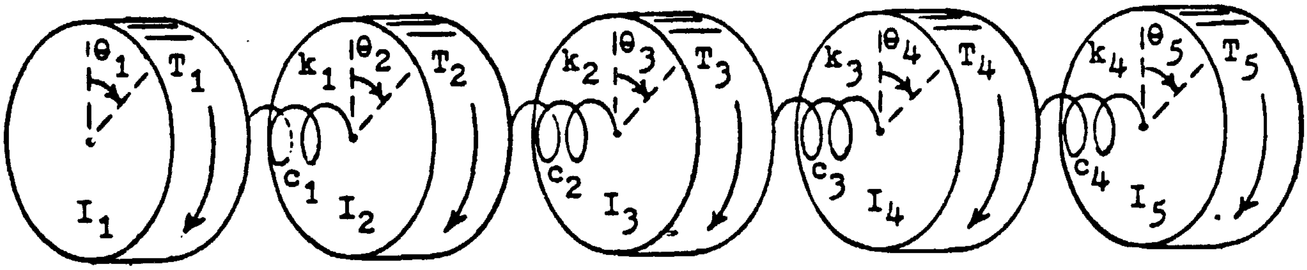

FIGURE 5-1

FIVE-BODY SINGLE-AXIS MODEL WITH DAMPING AT ALL FOUR INTERFACES

$$
\mathbf{A} = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-a_{23} & -a_{23}r_1 & a_{23} & a_{23}r_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
a_{41} & a_{41}r_1 & a_{43} & a_{44} & a_{45} & a_{45}r_2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & a_{63} & a_{63}r_2 & a_{65} & a_{66} & a_{67} & a_{67}r_3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & a_{85} & a_{85}r_3 & a_{87} & a_{88} & a_{89} & a_{89}r_4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & a_{10,7} & a_{10,7}r_4 & -a_{10,7} & -a_{10,7}r_4
\end{bmatrix} \tag{5-8}
$$

$$
\begin{aligned}
a_{23} &= \frac{k_1}{I_1} \\
a_{41} &= \frac{k_1}{I_2}, \quad a_{45} = \frac{k_2}{I_2}, \quad a_{43} = -(a_{41}+a_{45}), \quad a_{44} = -(a_{41}r_1 + a_{45}r_2) \\
a_{63} &= \frac{k_2}{I_3}, \quad a_{67} = -(a_{63}+a_{67}), \quad a_{65} = -(a_{63}+a_{67}), \quad a_{66} = -(a_{63}r_2 + a_{67}r_3) \\
a_{85} &= \frac{k_3}{I_4} \quad a_{89} = \frac{k_4}{I_4}, \quad a_{87} = -(a_{85}+a_{89}), \quad a_{88} = -(a_{85}r_3 + a_{89}r_4) \\
a_{10,7} &= \frac{k_4}{I_5}
\end{aligned} \tag{5-9}
$$

$$
r_j = \frac{c_j}{k_j}; \qquad j = 1,2,3,4 \tag{5-10}
$$

$$
\mathbf{B} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix} \text{ for } r = 5 \tag{5-11}
$$

5.3 REDUCED STATE LINEAR OBSERVERS

5.3.1 Introduction

The minimum order of a reduced state linear observer required to reconstruct the $10 - m$ inaccessible scalar state variables of the five-body single-axis model of a flexible spacecraft represented by equations (5-6) through (5–10) is $p = 10 - m$ where $m = 1, 2, 3, 4, 5, 6, 7, 8$ or 9. All of the reduced state linear observers for this five body model may be written in the form of equations (2–10) and (2–11) under the assumption that the observer coefficient matrix, D, is diagonal and of dimensions $p \times p$. The corresponding observer weighting matrix is of the following form

$$\mathbf{T} = \begin{bmatrix} t_{11} & \cdots & t_{1,10} \\ \vdots & & \vdots \\ t_{p,1} & \cdots & t_{p,10} \end{bmatrix} \tag{5–12}$$

From equations (2–12), (5–10) and (5–11),

$$\mathbf{E} = \begin{bmatrix} t_{12} & t_{14} & t_{16} & t_{18} & t_{1,10} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ t_{p,2} & t_{p,4} & t_{p,6} & t_{p,8} & t_{p,10} \end{bmatrix} \tag{5–13}$$

$$\mathbf{F} = \begin{bmatrix} f_{11} & \cdots & f_{1,10} \\ \vdots & & \vdots \\ f_{p,1} & \cdots & f_{p,10} \end{bmatrix} \tag{5–14}$$

The corresponding block diagram appears in Fig. 2–4.

5.3.2 Observer Synthesis Equations

From Luenberger (5–2), (5–3) and (5–4) and Sage (5–5) the equations for synthesizing the reduced state linear observers for the five-body single-axis model represented by equations (5–6) through (5–10) are given by equations (2–13) and (2–14). With coefficient matrices of the form listed in 5.3.1 this set of equations for generating the elements of the $\mathbf{T}$ matrix reduces to the following.

$$\begin{aligned} t_{i2} = {} & \left\{ \frac{\left[a_{41}a_{63}p_1p_2 + a_{41}a_{67}p_1p_3 + a_{45}a_{67}p_2p_3 + (a_{41}p_1 + a_{45}p_2 + a_{63}p_2 + a_{67}p_3)d_{ii}^2 + d_{ii}^4\right]}{a_{23}a_{45}p_1p_2\Delta'_{i5}}(\Delta'_{i5})_{1,3} \right. \\ & \left. - \frac{a_{67}a_{85}p_3^2(a_{41}p_1 + a_{45}p_2 + d_{ii}^2)(a_{10,7}p_4 + d_{ii}^2)}{\Delta'_{i5}} \right\}(f_{i1} + d_{ii}f_{i2}) \\ & + a_{41}\left[\frac{(a_{63}p_2 + a_{67}p_3 + d_{ii}^2)(\Delta'_{i5})_{1,3}}{a_{23}a_{45}p_2\Delta'_{i5}} - \frac{a_{67}a_{85}p_1p_3^2(a_{10,7}p_4 + d_{ii}^2)}{\Delta'_{i5}}\right](f_{i3} + d_{ii}f_{i4}) \\ & + \frac{a_{41}a_{63}(\Delta'_{i5})_{1,3}}{a_{23}a_{45}\Delta'_{i5}}(f_{i5} + d_{ii}f_{i6}) \\ & + \frac{a_{41}a_{63}a_{85}p_1p_2p_3}{\Delta'_{i5}}\left[(a_{10,7}p_4 + d_{ii}^2)(f_{i7} + d_{ii}f_{i8}) + a_{10,7}p_4(f_{i9} + d_{ii}f_{i10})\right]; \quad i = 1, 2, \ldots, p \end{aligned} \tag{5–15}$$

$$\begin{aligned} t_{i4} = {} & \left[\frac{(a_{63}p_2 + a_{67}p_3 + d_{ii}^2)(\Delta'_{i5})_{1,3}}{a_{23}a_{45}p_1p_2\Delta'_{i5}} - \frac{a_{67}a_{85}p_3^2(a_{10,7}p_4 + d_{ii}^2)}{\Delta'_{i5}}\right] \\ & \times \left[a_{23}p_1(f_{i1} + d_{ii}f_{i2}) + (a_{23}p_1 + d_{ii}^2)(f_{i3} + d_{ii}f_{i4})\right] + \frac{(a_{23}p_1 + d_{ii}^2)a_{63}(\Delta'_{i5})_{1,3}}{a_{23}a_{45}p_1\Delta'_{i5}}(f_{i5} + d_{ii}f_{i6}) \\ & + \frac{a_{63}a_{85}p_2p_3(a_{23}p_1 + d_{ii}^2)}{\Delta'_{i5}}\left[(a_{10,7}p_4 + d_{ii}^2)(f_{i7} + d_{ii}f_{i8}) + a_{10,7}p_4(f_{i9} + d_{ii}f_{i10})\right] \end{aligned} \tag{5–16}$$

$$
\begin{aligned}
t_{i6} &= \frac{(\Delta'_{i5})_{1,3}}{a_{23}p_1\Delta'_{i5}}\Big[a_{23}p_1(f_{i1}+d_{ii}f_{i2})+(a_{23}p_1+d_{ii}^2)(f_{i3}+d_{ii}f_{i4})\Big] \\
&+\frac{(\Delta'_{i5})_{1,3}(\Delta'_{i5})_{5,3}}{a_{23}a_{45}a_{65}a_{10,7}p_1p_2p_3p_4\Delta'_{i5}}(f_{i5}+d_{ii}f_{i6}) \\
&+\frac{(\Delta'_{i5})_{5,3}}{a_{10,7}p_4\Delta'_{i5}}\Big[(a_{10,7}p_4+d_{ii}^2)(f_{i7}+d_{ii}f_{i8})+a_{10,7}p_4(f_{i9}+d_{ii}f_{i10})\Big]
\end{aligned}
\tag{5-17}
$$

$$
\begin{aligned}
t_{i8} &= \frac{a_{45}a_{67}p_2p_3(a_{10,7}p_4+d_{ii}^2)}{\Delta'_{i5}}\Big[a_{23}p_1(f_{i1}+d_{ii}f_{i2})+(a_{23}p_1+d_{ii}^2)(f_{i3}+d_{ii}f_{i4})\Big] \\
&+\frac{a_{67}(\Delta'_{i5})_{5,3}(a_{10,7}p_4+d_{ii}^2)}{a_{65}a_{10,7}p_4\Delta'_{i5}}(f_{i5}+d_{ii}f_{i6}) \\
&+\left[\frac{(a_{63}p_2+a_{67}p_3+d_{ii}^2)(\Delta'_{i5})_{5,3}}{a_{65}a_{10,7}p_3p_4\Delta'_{i5}}-\frac{(a_{23}p_1+d_{ii}^2)a_{45}a_{63}p_2^2}{\Delta'_{i5}}\right] \\
&\qquad\times\Big[(a_{10,7}p_4+d_{ii}^2)(f_{i7}+d_{ii}f_{i8})+a_{10,7}p_4(f_{i9}+d_{ii}f_{i10})\Big]
\end{aligned}
\tag{5-18}
$$

$$
\begin{aligned}
t_{i10} &= \frac{a_{45}a_{67}a_{89}p_2p_3p_4\Big[a_{23}p_1(f_{i1}+d_{ii}f_{i2})+(a_{23}p_1+d_{ii}^2)(f_{i3}+d_{ii}f_{i4})\Big]}{\Delta'_{i5}} \\
&+\frac{a_{67}a_{89}(\Delta'_{i5})_{5,3}}{a_{65}a_{10,7}\Delta'_{i5}}(f_{i5}+d_{ii}f_{i6}) \\
&+a_{89}\left[\frac{(a_{63}p_2+a_{67}p_3+d_{ii}^2)(\Delta'_{i5})_{5,3}}{a_{65}a_{10,7}p_3\Delta'_{i5}}-\frac{(a_{23}p_1+d_{ii}^2)a_{45}a_{63}p_2^2p_4}{\Delta'_{i5}}\right](f_{i7}+d_{ii}f_{i8}) \\
&+\left\{\frac{\Big[a_{63}a_{65}p_2p_3+a_{63}a_{89}p_2p_4+a_{67}a_{89}p_3p_4+(a_{63}p_2+a_{67}p_3+a_{65}p_3+a_{89}p_4)d_{ii}^2+d_{ii}^4\Big]}{a_{65}a_{10,7}p_3p_4\Delta'_{i5}}(\Delta'_{i5})_{5,3}\right. \\
&\qquad\left.-\frac{(a_{23}p_1+d_{ii}^2)a_{45}a_{63}p_2^2(a_{65}p_3+a_{89}p_4+d_{ii}^2)}{\Delta'_{i5}}\right\}(f_{i9}+d_{ii}f_{i10})
\end{aligned}
\tag{5-19}
$$

$$
\begin{aligned}
t_{i1} &= \left[\left\{\frac{d_{ii}\Big[(a_{41}p_1+d_{ii}^2)(a_{63}p_2+a_{67}p_3+d_{ii}^2)+a_{45}p_2(a_{67}p_3+d_{ii}^2)\Big]}{a_{23}a_{45}p_1p_2\Delta'_{i5}}\right.\right. \\
&\qquad\left.+\frac{a_{23}r_1\Big[a_{63}p_2d_{ii}^2+(a_{45}p_2+d_{ii}^2)(a_{67}p_3+d_{ii}^2)\Big]}{a_{23}a_{45}p_1p_2\Delta'_{i5}}\right\}(\Delta'_{i5})_{1,3} \\
&\qquad\left.-\frac{a_{67}a_{65}p_3^2(a_{10,7}p_4+d_{ii}^2)\Big[d_{ii}(a_{41}p_1+a_{45}p_2+d_{ii}^2)+a_{23}r_1(a_{45}p_2+d_{ii}^2)\Big]}{\Delta'_{i5}}\right](f_{i1}+d_{ii}f_{i2}) \\
&\quad+f_{i3}+\frac{a_{41}d_{ii}}{a_{23}a_{45}p_1p_2\Delta'_{i5}}\Big\{\Big[(a_{63}p_2+a_{67}p_3+d_{ii}^2)(\Delta'_{i5})_{1,3}
\end{aligned}
$$

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$$
\begin{aligned}
&- a_{23}a_{45}a_{67}a_{85}p_1p_2p_3^2\left(a_{10,7}p_4 + d_{ii}^2\right)\Big]\left(f_{i3} + d_{ii}f_{i4}\right) \\
&+ a_{63}p_2\left(\Delta'_{i5}\right)_{1,3}\left(f_{i5} + d_{ii}f_{i6}\right) \\
&+ a_{23}a_{45}a_{63}a_{85}p_1p_2^2p_3\left[\left(a_{10,7}p_4 + d_{ii}^2\right)\left(f_{i7} + d_{ii}f_{i8}\right) + a_{10,7}p_4\left(f_{i9} + d_{ii}f_{i10}\right)\right]\Bigg\}
\end{aligned}
\tag{5-20}
$$

$$
\begin{aligned}
t_{i3} = &- \Bigg[\frac{\left\{\left[a_{45}\left(r_1 - r_2\right) + d_{ii}\right]\left(a_{67}p_3 + d_{ii}^2\right) + a_{63}p_2d_{ii}\right\}\left(\Delta'_{i5}\right)_{1,3}}{a_{45}p_1p_2\Delta'_{i5}} \\
&+ \frac{a_{23}a_{67}a_{85}p_3^2\left(a_{10,7}p_4 + d_{ii}^2\right)\left[a_{45}\left(r_1 - r_2\right) - d_{ii}\right]}{\Delta'_{i5}}\Bigg]\left(f_{i1} + d_{ii}f_{i2}\right) \\
&+ \Bigg[\frac{\left\{\left(a_{67}p_3 + d_{ii}^2\right)\left[\left(a_{23}p_1 + d_{ii}^2\right)\left(a_{45}r_2 + d_{ii}\right) + a_{41}r_1d_{ii}^2\right]\right.}{a_{23}a_{45}p_1p_2\Delta'_{i5}} \\
&+ \frac{\left. a_{63}d_{ii}p_2\left(a_{23}p_1 + a_{41}r_1d_{ii} + d_{ii}^2\right)\right\}\left(\Delta'_{i5}\right)_{1,3}}{a_{23}a_{45}p_1p_2\Delta'_{i5}} \\
&+ \frac{a_{67}a_{85}p_3^2\left(a_{10,7}p_4 + d_{ii}^2\right)\left[\left(a_{23}p_1 + d_{ii}^2\right)\left(a_{45}r_2 + d_{ii}\right) + a_{41}r_1d_{ii}^2\right]}{\Delta'_{i5}}\Bigg]\left(f_{i3} + d_{ii}f_{i4}\right) \\
&+ f_{i4} + \frac{a_{63}d_{ii}\left[a_{23}p_1 + a_{41}d_{ii}\left(r_1 - r_2\right) + d_{ii}^2\right]\left(\Delta'_{i5}\right)_{1,3}}{a_{23}a_{45}p_1p_2\Delta'_{i5}}\left(f_{i5} + d_{ii}f_{i6}\right) \\
&+ a_{63}\left\{\frac{a_{85}p_2p_3\left[a_{23}p_1\left(a_{45}r_2 + d_{ii}\right) + d_{ii}^2d_{ii2}\right]}{\Delta'_{i5}} - \frac{r_2\left(\Delta'_{i5}\right)_{5,3}}{a_{10,7}p_4\Delta'_{i5}}\right\} \\
&\quad \times \left[\left(a_{10,7}p_4 + d_{ii}^2\right)\left(f_{i7} + d_{ii}f_{i8}\right) + a_{10,7}p_4\left(f_{i9} + d_{ii}f_{i10}\right)\right]
\end{aligned}
\tag{5-21}
$$

$$
\begin{aligned}
t_{i6} = {} & \left\{ \frac{\left[d_{ii} - a_{67}(r_2 - r_3)\right](\Delta'_{i5})_{1,3}}{a_{23}p_1p_2\Delta'_{i5}} + \frac{a_{45}a_{67}a_{85}p_3(a_{10,7}p_4 + d_{ii}^2)(r_2 - r_3)}{\Delta'_{i5}} \right\} \\
& \times \left[a_{23}p_1(f_{i1} + d_{ii}f_{i2}) + (a_{23}p_1 + d_{ii}^2)(f_{i3} + d_{ii}f_{i4})\right] \\
& - \left[\frac{(a_{23}p_1 + d_{ii}^2)a_{63}r_2(\Delta'_{i5})_{1,3}}{a_{23}p_1\Delta'_{i5}} - \frac{d_{ii3}(\Delta'_{i5})_{1,3}(\Delta'_{i5})_{5,3}}{a_{23}a_{45}a_{85}a_{10,7}p_1p_2p_3p_4\Delta'_{i5}} \right. \\
& \left. + \frac{a_{67}r_3(a_{10,7}p_4 + d_{ii}^2)(\Delta'_{i5})_{5,3}}{a_{10,7}p_4\Delta'_{i5}} \right] (f_{i5} + d_{ii}f_{i6}) + f_{i6} \\
& + \left\{ \frac{\left[d_{ii} + a_{63}(r_2 - r_3)\right](\Delta'_{i5})_{5,3}}{a_{10,7}p_3p_4\Delta'_{i5}} - \frac{a_{45}a_{63}a_{85}p_2(a_{23}p_1 + d_{ii}^2)(r_2 - r_3)}{\Delta'_{i5}} \right\} \\
& \times \left[(a_{10,7}p_4 - d_{ii}^2)(f_{i7} + d_{ii}f_{i8}) + a_{10,7}p_4(f_{i9} + d_{ii}f_{i10})\right]
\end{aligned}
\tag{5-22}
$$

$$
\begin{aligned}
t_{i7} = {} & -a_{67} \left\{ \frac{r_3(\Delta'_{i5})_{1,3}}{a_{23}p_1\Delta'_{i5}} + \frac{a_{45}p_2p_3\left[d_{ii}^2 d_{ii4} + a_{10,7}p_4(a_{85}r_3 + d_{ii})\right]}{\Delta'_{i5}} \right\} \\
& \times \left[a_{23}p_1(f_{i1} + d_{ii}f_{i2}) + (a_{23}p_1 + d_{ii}^2)(f_{i3} + d_{ii}f_{i4})\right] \\
& + a_{67}d_{ii} \frac{\left[a_{10,7}p_4 - a_{89}d_{ii}(r_3 - r_4) + d_{ii}^2\right](\Delta'_{i5})_{5,3}}{a_{85}a_{10,7}p_3p_4\Delta'_{i5}} (f_{i5} + d_{ii}f_{i6}) \\
& + \left[\left\{ \frac{(a_{63}p_2 + d_{ii}^2)\left[(a_{85}r_3 + d_{ii})(a_{10,7}p_4 + d_{ii}^2) + a_{89}r_4d_{ii}^2\right]}{a_{85}a_{10,7}p_3p_4\Delta'_{i5}} \right. \right. \\
& \left. + \frac{a_{67}p_3d_{ii}(a_{10,7}p_4 + a_{89}r_4d_{ii} + d_{ii}^2)}{a_{85}a_{10,7}p_3p_4\Delta'_{i5}} \right\} (\Delta'_{i5})_{5,3} \\
& \left. - \frac{(a_{23}p_1 + d_{ii}^2)a_{45}a_{63}p_2^2\left[d_{ii}^2 d_{ii4} + a_{10,7}p_4(a_{85}r_3 + d_{ii})\right]}{\Delta'_{i5}} \right] (f_{i7} + d_{ii}f_{i8}) + f_{i8} \\
& + \left\{ \frac{\left[a_{85}(a_{63}p_2 + d_{ii}^2)(r_4 - r_2) - (a_{63}p_2 + a_{67}p_3 + d_{ii}^2)d_{ii}\right](\Delta'_{i5})_{5,3}}{a_{85}p_3p_4\Delta'_{i5}} \right. \\
& \left. - \frac{(a_{23}p_1 + d_{ii}^2)a_{45}a_{63}a_{10,7}p_2^2\left[a_{85}(r_3 - r_4) + d_{ii}\right]}{\Delta'_{i5}} \right\} (f_{i9} + d_{ii}f_{i10})
\end{aligned}
\tag{5-23}
$$

$$
\begin{aligned}
t_{i9} = {} & \frac{a_{89}d_{ii}}{a_{85}a_{10,7}p_3p_4\Delta'_{i5}}\Big\{a_{45}a_{67}a_{85}a_{10,7}p_2p_3^2p_4\Big[a_{23}p_1(f_{i1}+d_{ii}f_{i2}) \\
& + (a_{23}p_1+d_{ii}^2)(f_{i3}+d_{ii}f_{i4})\Big] + a_{67}p_3(\Delta'_{i5})_{5,3}(f_{i5}+d_{ii}f_{i6}) \\
& + \Big[(a_{63}p_2+a_{67}p_3+d_{ii}^2)(\Delta'_{i5})_{5,3} - (a_{23}p_1+d_{ii}^2)a_{45}a_{63}a_{85}a_{10,7}p_2^2p_3p_4\Big](f_{i7}+d_{ii}f_{i8})\Big\} \\
& + \Bigg[\Bigg\{\frac{d_{ii5}\Big[a_{63}a_{85}p_2p_3+a_{63}a_{89}p_2p_4+a_{67}a_{89}p_3p_4+(a_{63}p_2+a_{67}p_3+a_{85}p_3+a_{89}p_4)d_{ii}^2+d_{ii}^4\Big]}{a_{85}a_{10,7}p_3p_4\Delta'_{i5}} \\
& \qquad - \frac{a_{89}p_4a_{10,7}r_4(a_{63}p_2+a_{67}p_3+d_{ii}^2)}{a_{85}a_{10,7}p_3p_4\Delta'_{i5}}\Bigg\}(\Delta'_{i5})_{5,3} \\
& \qquad + \frac{(a_{23}p_1+d_{ii}^2)a_{45}a_{63}p_2^2\Big[a_{89}p_4a_{10,7}r_4 - d_{ii5}(a_{85}p_3+a_{89}p_4+d_{ii}^2)\Big]}{\Delta'_{i5}}\Bigg] \\
& \qquad \times (f_{i9}+d_{ii}f_{i,10}) + f_{i,10}
\end{aligned}
\tag{5-24}
$$

where:

$$r_j = \frac{c_j}{k_j}; \qquad j = 1,2,3,4 \tag{5-10}$$

$$p_j = 1 + r_jd_{ii}; \qquad i = 1,2,\ldots,p \qquad j = 1,2,3,4 \tag{5-25}$$

$$d_{ii1} = a_{23}r_1 + d_{ii} \tag{5-26}$$

$$d_{ii2} = a_{41}r_1 + a_{45}r_2 + d_{ii} \tag{5-27}$$

$$d_{ii3} = a_{63}r_2 + a_{67}r_3 + d_{ii} \tag{5-28}$$

$$d_{ii4} = a_{85}r_3 + a_{89}r_4 + d_{ii} \tag{5-29}$$

$$d_{ii5} = a_{10,7}r_4 + d_{ii} \tag{5-30}$$

$$
\begin{aligned}
\Delta'_{i5} = {} & \begin{vmatrix}
-(a_{23}p_1+d_{ii}^2) & a_{41}p_1 & 0 & 0 & 0 \\
a_{23}p_1 & -(a_{41}p_1+a_{45}p_2+d_{ii}^2) & a_{63}p_2 & 0 & 0 \\
0 & a_{45}p_2 & -(a_{63}p_2+a_{67}p_3+d_{ii}^2) & a_{85}r_3 & 0 \\
0 & 0 & a_{67}r_3 & -(a_{85}p_3+a_{89}p_4+d_{ii}^2) & a_{10,7}p_4 \\
0 & 0 & 0 & a_{89}p_4 & -(a_{10,7}p_4 + \cdots
\end{vmatrix} \\
= {} & -(a_{23}p_1+d_{ii}^2)(\Delta'_{i5})_{1,1} - a_{23}p_1(\Delta'_{i5})_{2,1} \\
= {} & -(a_{23}p_1+d_{ii}^2)(\Delta'_{i5})_{1,1} - a_{23}p_1(\Delta'_{i5})_{1,2}
\end{aligned}
\tag{5-31}
$$

5.3.3 Elimination of Damping From Model Interfaces

Elimination of damping at interface j ($j = 1,2,3,4$ for the five body model) corresponds to setting $r_j = 0$ and $p_j = 1$ in the equations for generating the elements of the **T** matrix, equations (5-16) through (5-31).

Example All Interface Damping Eliminated

If damping is removed from all four interfaces of the five body model, $r_j \to 0$, $p_j \to 1$, $d_{ii1} \to d_{ii}$, $d_{ii2} \to d_{ii}$, $d_{ii3} \to d_{ii}$, $d_{ii4} \to d_{ii}$, $d_{ii5} \to d_{ii}$ and $\Delta'_{i5} \to \Delta_{i5}$.

$$\begin{aligned} t_{i2} &= \Bigg\{ \frac{\left[a_{41}a_{63} + a_{41}a_{67} + a_{45}a_{67} + (a_{41} + a_{45} + a_{63} + a_{67})d_{ii}^2 + d_{ii}^4\right]}{a_{23}a_{45}\Delta_{i5}}(\Delta_{i5})_{1,3} \\ &\qquad - \frac{a_{67}a_{85}(a_{41} + a_{45} + d_{ii}^2)(a_{10,7} + d_{ii}^2)}{\Delta_{i5}} \Bigg\} (f_{i1} + d_{ii}f_{i2}) \\ &\quad + a_{41}\left[\frac{(a_{63} + a_{67} + d_{ii}^2)(\Delta_{i5})_{1,3}}{a_{23}a_{45}\Delta_{i5}} - \frac{a_{67}a_{85}(a_{10,7} + d_{ii}^2)}{\Delta_{i5}}\right](f_{i3} + d_{ii}f_{i4}) \\ &\quad + \frac{a_{41}a_{63}(\Delta_{i5})_{1,2}}{a_{23}a_{45}\Delta_{i5}}(f_{i5} + d_{ii}f_{i6}) \\ &\quad + \frac{a_{41}a_{63}a_{85}}{\Delta_{i5}}\left[(a_{10,7} + d_{ii}^2)(f_{i7} + d_{ii}f_{i8}) + a_{10,7}(f_{i9} + d_{ii}f_{i10})\right]; \qquad i = 1, 2, \ldots, p \qquad (5\text{–}32) \end{aligned}$$

$$\begin{aligned} t_{i4} &= \left[\frac{(a_{63} + a_{67} + d_{ii}^2)(\Delta_{i5})_{1,3}}{a_{23}a_{45}\Delta_{i5}} - \frac{a_{67}a_{85}(a_{10,7} + d_{ii}^2)}{\Delta_{i5}}\right] \\ &\qquad \times \left[a_{23}(f_{i1} + d_{ii}f_{i2}) + (a_{23} + d_{ii}^2)(f_{i3} + d_{ii}f_{i4})\right] + \frac{(a_{23} + d_{ii}^2)a_{63}(\Delta_{i5})_{1,3}}{a_{23}a_{45}\Delta_{i5}}(f_{i5} + d_{ii}f_{i6}) \\ &\quad + \frac{a_{63}a_{85}(a_{23} + d_{ii}^2)}{\Delta_{i5}}\left[(a_{10,7} + d_{ii}^2)(f_{i7} + d_{ii}f_{i8}) + a_{10,7}(f_{i9} + d_{ii}f_{i10})\right] \qquad (5\text{–}33) \end{aligned}$$

$$\begin{aligned} t_{i6} &= \frac{(\Delta_{i5})_{1,3}}{a_{23}\Delta_{i5}}\left[a_{23}(f_{i1} + d_{ii}f_{i2}) + (a_{23} + d_{ii}^2)(f_{i3} + d_{ii}f_{i4})\right] \\ &\quad + \frac{(\Delta_{i5})_{1,3}(\Delta_{i5})_{5,3}}{a_{23}a_{45}a_{85}a_{10,7}\Delta_{i5}}(f_{i5} + d_{ii}f_{i6}) \\ &\quad + \frac{(\Delta_{i5})_{5,3}}{a_{10,7}\Delta_{i5}}\left[(a_{10,7} + d_{ii}^2)(f_{i7} + d_{ii}f_{i8}) + a_{10,7}(f_{i9} + d_{ii}f_{i10})\right] \qquad (5\text{–}34) \end{aligned}$$

$$\begin{aligned} t_{i8} &= \frac{a_{45}a_{67}(a_{10,7} + d_{ii}^2)}{\Delta_{i5}}\left[a_{23}(f_{i1} + d_{ii}f_{i2}) + (a_{23} + d_{ii}^2)(f_{i3} + d_{ii}f_{i4})\right] \\ &\quad + \frac{a_{67}(\Delta_{i5})_{5,3}(a_{10,7} + d_{ii}^2)}{a_{85}a_{10,7}\Delta_{i5}}(f_{i5} + d_{ii}f_{i6}) \\ &\quad + \left[\frac{(a_{63} + a_{67} + d_{ii}^2)(\Delta_{i5})_{5,3}}{a_{85}a_{10,7}\Delta_{i5}} - \frac{(a_{23} + d_{ii}^2)a_{45}a_{63}}{\Delta_{i5}}\right] \\ &\qquad \times \left[(a_{10,7} + d_{ii}^2)(f_{i7} + d_{ii}f_{i8}) + a_{10,7}(f_{i9} + d_{ii}f_{i10})\right] \qquad (5\text{–}35) \end{aligned}$$

$$
\begin{aligned}
t_{i10} = & -\frac{a_{45}a_{67}a_{89}\left[a_{23}(f_{i1}+d_{ii}f_{i2})+(a_{23}+d_{ii}^2)(f_{i3}+d_{ii}f_{i4})\right]}{\Delta_{i5}} \\
& + \frac{a_{67}a_{89}(\Delta_{i5})_{5,3}}{a_{85}a_{10,7}\Delta_{i5}}(f_{i5}+d_{ii}f_{i6}) \\
& + a_{89}\left[\frac{(a_{63}+a_{67}+d_{ii}^2)(\Delta_{i5})_{5,3}}{a_{85}a_{10,7}\Delta_{i5}} - \frac{(a_{23}+d_{ii}^2)a_{45}a_{63}}{\Delta_{i5}}\right](f_{i7}+d_{ii}f_{i8}) \\
& + \Bigg\{\frac{\left[a_{63}a_{85}+a_{63}a_{89}+a_{67}a_{89}+(a_{63}+a_{67}+a_{85}+a_{89})d_{ii}^2+d_{ii}^4\right]}{a_{85}a_{10,7}\Delta_{i5}}(\Delta_{i5})_{5,3} \\
& \qquad - \frac{(a_{23}+d_{ii}^2)a_{45}a_{63}(a_{85}+a_{89}+d_{ii}^2)}{\Delta_{i5}}\Bigg\}(f_{i9}+d_{ii}f_{i10})
\end{aligned}
\tag{5-36}
$$

$$
\begin{aligned}
t_{i1} = d_{ii} & \Bigg[\Bigg\{\frac{\left[(a_{41}+d_{ii}^2)(a_{63}+a_{67}+d_{ii}^2)+a_{45}(a_{67}+d_{ii}^2)\right]}{a_{23}a_{45}\Delta_{i5}}(\Delta_{i5})_{1,3} \\
& \qquad - \frac{a_{67}a_{85}(a_{10,7}+d_{ii}^2)(a_{41}+a_{45}+d_{ii}^2)}{\Delta_{i5}}\Bigg\}(f_{i1}+d_{ii}f_{i2}) \\
& + \frac{a_{41}}{a_{23}a_{45}\Delta_{i5}}\Bigg\{\left[(a_{63}+a_{67}+d_{ii}^2)(\Delta_{i5})_{1,3} - a_{23}a_{45}a_{67}a_{85}(a_{10,7}+d_{ii}^2)\right](f_{i3}+d_{ii}f_{i4}) \\
& \qquad + a_{63}(\Delta_{i5})_{1,3}(f_{i5}+d_{ii}f_{i6}) \\
& \qquad + a_{23}a_{45}a_{63}a_{85}\left[(a_{10,7}+d_{ii}^2)(f_{i7}+d_{ii}f_{i8})+a_{10,7}(f_{i9}+d_{ii}f_{i10})\right]\Bigg\}\Bigg] + f_{i2}
\end{aligned}
\tag{5-37}
$$

$$
\begin{aligned}
t_{i3} = -d_{ii} & \Bigg[\left[\frac{(a_{63}+a_{67}+d_{ii}^2)(\Delta_{i5})_{1,3}}{a_{45}\Delta_{i5}} - \frac{a_{23}a_{67}a_{85}(a_{10,7}+d_{ii}^2)}{\Delta_{i5}}\right](f_{i1}+d_{ii}f_{i2}) \\
& - \Bigg\{\frac{\left[(a_{23}+d_{ii}^2)(a_{67}+d_{ii}^2)+a_{63}(a_{23}+d_{ii}^2)\right](\Delta_{i5})_{1,3}}{a_{23}a_{45}\Delta_{i5}} \\
& \qquad + \frac{a_{67}a_{85}(a_{23}+d_{ii}^2)(a_{10,7}+d_{ii}^2)}{\Delta_{i5}}\Bigg\}(f_{i3}+d_{ii}f_{i4}) \\
& - \frac{a_{63}(a_{23}+d_{ii}^2)(\Delta_{i5})_{1,3}}{a_{23}a_{45}\Delta_{i5}}(f_{i5}+d_{ii}f_{i6}) \\
& - \frac{a_{63}a_{85}(a_{23}+d_{ii}^2)}{\Delta_{i5}}\left[(a_{10,7}+d_{ii}^2)(f_{i7}+d_{ii}f_{i8})+a_{10,7}(f_{i9}+d_{ii}f_{i10})\right]\Bigg] + f_{i4}
\end{aligned}
\tag{5-38}
$$

$$
\begin{aligned}
t_{i5} = d_{ii} \Bigg\{ & \frac{(\Delta_{i5})_{1,3}}{a_{23}\Delta_{i5}} \Big[a_{23}(f_{i1}+d_{ii}f_{i2}) + (a_{23}+d_{ii}^2)(f_{i3}+d_{ii}f_{i4})\Big] \\
& + \frac{(\Delta_{i5})_{1,3}(\Delta_{i5})_{5,3}}{a_{23}a_{45}a_{85}a_{10,7}\Delta_{i5}}(f_{i5}+d_{ii}f_{i6}) \\
& + \frac{(\Delta_{i5})_{5,3}}{a_{10,7}\Delta_{i5}} \Big[(a_{10,7}-d_{ii}^2)(f_{i7}+d_{ii}f_{i8}) + a_{10,7}(f_{i9}+d_{ii}f_{i10})\Big] \Bigg\} + f_{i6}
\end{aligned}
\tag{5-39}
$$

$$
\begin{aligned}
t_{i7} = -d_{ii} \Bigg[& \frac{a_{45}a_{67}(a_{10,7}+d_{ii}^2)}{a_{23}\Delta_{i5}} \Big[a_{23}(f_{i1}+d_{ii}f_{i2}) + (a_{23}+d_{ii}^2)(f_{i3}+d_{ii}f_{i4})\Big] \\
& - \frac{a_{67}(a_{10,7}+d_{ii}^2)(\Delta_{i5})_{5,3}}{a_{85}a_{10,7}\Delta_{i5}}(f_{i5}+d_{ii}f_{i6}) \\
& - (a_{10,7}+d_{ii}^2) \left\{ \frac{\Big[(a_{63}+d_{ii}^2)(a_{85}+d_{ii}^2)+a_{67}\Big]}{a_{85}a_{10,7}\Delta_{i5}}(\Delta_{i5})_{5,3} - \frac{(a_{23}+d_{ii}^2)a_{45}a_{63}}{\Delta_{i5}} \right\} (f_{i7}+d_{ii}f_{i8}) \\
& + \left[\frac{(a_{63}+a_{67}+d_{ii}^2)(\Delta_{i5})_{5,3}}{a_{85}\Delta_{i5}} + \frac{(a_{23}+d_{ii}^2)a_{45}a_{63}a_{10,7}}{\Delta_{i5}} \right] (f_{i9}+d_{ii}f_{i10}) \Bigg] + f_{i8}
\end{aligned}
\tag{5-40}
$$

$$
\begin{aligned}
t_{i9} = d_{ii} \Bigg[& \frac{a_{89}}{a_{85}a_{10,7}\Delta_{i5}} \Big\{ a_{45}a_{67}a_{85}a_{10,7}\Big[a_{23}(f_{i1}+d_{ii}f_{i2}) \\
& + (a_{23}+d_{ii}^2)(f_{i3}+d_{ii}f_{i4})\Big] + a_{67}(\Delta_{i5})_{5,3}(f_{i5}+d_{ii}f_{i6}) \\
& + \Big[(a_{63}+a_{67}+d_{ii}^2)(\Delta_{i5})_{5,3} - (a_{23}+d_{ii}^2)a_{45}a_{63}a_{85}a_{10,7}\Big](f_{i7}+d_{ii}f_{i8}) \Big\} \\
& + \Bigg\{ \frac{\Big[a_{63}a_{85}+a_{63}a_{89}+a_{67}a_{89}+(a_{63}+a_{67}+a_{85}+a_{89})d_{ii}^2+d_{ii}^4\Big](\Delta_{i5})_{5,3}}{a_{85}a_{10,7}\Delta_{i5}} \\
& - \frac{(a_{23}+d_{ii}^2)a_{45}a_{63}(a_{85}+a_{89}+d_{ii}^2)}{\Delta_{i5}} \Bigg\} (f_{i9}+d_{ii}f_{i,10}) \Bigg] + f_{i,10}
\end{aligned}
\tag{5-41}
$$

5.4 SOLUTION FOR SYNTHESIZED STATE VARIABLES

5.4.1 Introduction

Inaccessibility of a scalar state variable in equations (5–6) and (5–7) is reflected by a corresponding null column in the **C** and **F** matrices as implied in equation (2–14). For the generation of reduced state observers for the five body model, the number of inaccessible states can vary between one and nine.

5.4.2 First Order Observers ($p = 1$)

An observer of order at least one is required when any one of the ten scalar state variables of the five body model is inaccessible. The first order form of the linear observer equation is as follows:

$$\dot{z} = dz + \mathbf{E}\mathbf{u} + \mathbf{G}\mathbf{y} \tag{5-42}$$

The $\mathbf{F}$ and $\mathbf{T}$ matrices associated with a first order observer for the five body model then reduce to the following row forms:

$$\mathbf{F} = [f_1 \quad f_2 \quad \cdots \quad f_{10}] \tag{5-43}$$

$$\mathbf{T} = [t_1 \quad t_2 \quad \cdots \quad t_{10}] \tag{5-44}$$

The observer synthesis equations are then of the form of equations (5–15) through (5–31) with $i = 1$. Since a first order observer corresponds to one of the scalar state variables being inaccessible, one of the f_i $(i = 1, 2, \ldots, 10) = 0$.

Example

Suppose the scalar state representing the angular rate of body 5, x_{10}, is inaccessible. Then $f_{10} = 0$ and the observer synthesis equations reduce to the form of equations (5–15) through (5–31) with the subscript, i, omitted and $f_{10} = 0$. From equation (2–11), the synthesized scalar state, $\hat{x}_{10}$, is expressed in terms of the scalar observer state variable z, and the accessible model scalar state variables as follows.

$$\hat{x}_{10} = \frac{1}{t_{10}}\left[z - \sum_{i=1}^{9} t_i x_i\right] \tag{5-45}$$

For x_{10} inaccessible, it is assumed that:

$$\mathbf{C} = \left[\begin{array}{c|c} & 0 \\ \mathbf{I}_9 & \vdots \\ & 0 \end{array}\right] \tag{5-46}$$

where $\mathbf{I}_9 = 9 \times 9$ identity matrix.

From $\mathbf{F} = \mathbf{GC}$,

$$\mathbf{G} = [f_1 \quad f_2 \quad \cdots \quad f_9] \tag{5-47}$$

From $\mathbf{E} = \mathbf{TB}$,

$$\mathbf{E} = [t_2 \quad t_4 \quad t_6 \quad t_8 \quad t_{10}] \text{ for } r = 5 \text{ (control torques applied to all 5 bodies)} \tag{5-48}$$

5.4.3 Observers of Order Greater Than One $(1 < p < 10)$

For those cases in which more than one of the ten scalar states of the five-body single-axis model are inaccessible, the minimum order of the reduced state linear observer required to reconstruct these inaccessible states is given by p. In each case the number of null columns in the measurement or observation matrix, $\mathbf{C}$, and the $\mathbf{F}$ matrix also is equal to p. The general forms of the $\mathbf{E}$, $\mathbf{F}$ and $\mathbf{T}$ matrices are given in equations (5–12) through (5–14) for $p = 2, 3, 4, 5, 6, 7, 8$ or 9.

Example

Suppose the scalar states, x_9 and x_{10}, which represent the angular position and rate of body 5, are inaccessible. Then $f_{i9} = f_{i,10} = 0$ for $i = 1, 2$ and the observer synthesis equations reduce to the form of equations (5–14) through (5–31) with $i = 1, 2$ and $f_{i9} = f_{i,10} = 0$. From equation (2–11) the synthesized scalar states, $\hat{x}_9$ and $\hat{x}_{10}$, are expressed in terms of the scalar observer states, z_1 and z_2 and the accessible model scalar state variables as follows.

$$\hat{x}_9 = \frac{\sum_{i=1}^{2}(-1)^{i+1}(\Delta_2)_{i,1}\left(z_i - \sum_{j=1}^{8} t_{ij}x_j\right)}{\Delta_2} \tag{5-49}$$

$$\hat{x}_{10} = \frac{\sum_{i=1}^{2}(-1)^{i+1}(\Delta_2)_{i,2}\left(z_i - \sum_{j=1}^{8} t_{ij}x_j\right)}{\Delta_2} \tag{5-50}$$

for

$$\Delta_2 = \begin{vmatrix} t_{19} & t_{1,10} \\ t_{29} & t_{2,10} \end{vmatrix} = t_{19}t_{2,10} - t_{1,10}t_{29} \neq 0 \tag{5-51}$$

Where $(\Delta_2)_{i,j} = \Delta_2$ without the elements of the i^{th} row and j^{th} column.

For x_9 and x_{10} inaccessible, it is assumed that:

$$\mathbf{C} = \left[\begin{array}{c|cc} & 0 & 0 \\ \mathbf{I}_8 & \vdots & \vdots \\ & 0 & 0 \end{array}\right] \tag{5-52}$$

where $\mathbf{I}_8$ is an 8×8 identity matrix.

Since $\mathbf{F} = \mathbf{GC}$,

$$\mathbf{G} = \begin{bmatrix} f_{11} & \cdots & f_{18} \\ f_{21} & \cdots & f_{28} \end{bmatrix} \tag{5-53}$$

From $\mathbf{E} = \mathbf{TB}$,

$$\mathbf{E} = \begin{bmatrix} t_{12} & t_{14} & t_{16} & t_{18} & t_{1,10} \\ t_{22} & t_{24} & t_{26} & t_{28} & t_{2,10} \end{bmatrix} \text{ for } r = 5 \text{ (control torques applied to all five bodies)} \tag{5-54}$$

$$\mathbf{E} = \begin{bmatrix} t_{12} & t_{14} & t_{16} & t_{18} & 0 \\ t_{22} & t_{24} & t_{26} & t_{28} & 0 \end{bmatrix} \text{ for control torques applied to bodies 1,2,3 and 4} \tag{5-55}$$

5.5 REFERENCES

5-1 Guidance Systems Division, The Bendix Corporation, "Modular Design Attitude Control System",Final Report, Contract NAS8-33979 for George C. Marshall Space Flight Center, March 8, 1982.

5-2 Luenberger, D.G., "Determining the State of a Linear System with Observers of Low Dynamic Order", Ph.D. dissertation, Stanford University, 1963.

5-3 Luenberger, D.G., "Observers for Multivariable Systems", *IEEE Transactions on Automatic Control*, Vol. AC-11, No. 2, April 1966, pp. 190–197.

5-4 Luenberger, D.G., "An Introduction to Observers", *IEEE Transactions on Automatic Control*, Vol. AC-16, No. 6, December 1971, pp. 596–602.

5-5 Sage, A.P., *Optimum Systems Control*. Englewood Cliffs, N.J.: Prentice-Hall, Inc. 1968, pp. 306–312.

SECTION 6

APPLICATION OF MODAL MODELING AND DIRECT MATRIX PRODUCTS

6.1 INTRODUCTION

In the development of reduced state observers for the class of single-axis models of a flexible spacecraft presented in sections 2 through 5 of this report, it was assumed that the state vector coefficient matrix of the observer model was diagonal in order to reduce the amount of computation involved n solving the observer synthesis equation,

$$\mathbf{TA} - \mathbf{DT} = \mathbf{F}, \tag{6-1}$$

for the elements of the **T** matrix as a function of the elements of the **F** matrix where

$$\mathbf{F} = \mathbf{GC} \tag{6-2}$$

Despite this rather arbitrary assumption, the computational effort involved in this solution grew with alarming rapidity as the number of flexibly connected rigid bodies incorporated in the single-axis model was increased. Furthermore, especiallly for the models incorporating both larger numbers of rigid bodies and damping, the assumption of a diagonal **D** matrix seemed a rather poor approximation in view of the considerable departure from diagonal form of the state vector coefficient matrices (**A** matrices) of these models. In view of these problems, Dr. Henry Waites (6–1) of Marshall Space Flight Center suggested that a more fruitful approach to synthesizing observers for this class of single-axis models of a flexible spacecraft treated in the preceding sections of this report would be based upon the following sequence of steps.

1. Recast the state variable forms of each undamped single-axis model into modal form.
2. Add modai damping to each modal model.
3. Recast the observer synthesis equation in terms of direct matrix products.

The advantages cited for this approach include the following.

1. The state vector coefficient matrix, **A**, in each modal single-axis model appears in 2×2 block diagonal form implying that the state vector coefficient matrix, **D**, of the corresponding observer required to accurately synthesize the inaccessible states would be no less sparse than 2×2 block diagonal.
2. The modal model is more amenable to truncation of less significant oscillatory modes.
3. The coefficient of damping associated with each vibrational mode can be specified at the outset of the analysis.

6.2 TRANSFORMATION OF THE TWO-BODY SINGLE-AXIS MODEL TO MODAL FORM

The approach utilized in transforming the undamped two-body single-axis model of Section 2 to damped modal form follows that presented in Thomson (6–2). It consists of the following steps.

1. Write original single-axis model in undamped form.
2. Write undamped single-axis model in terms of inertia and stiffness matrices.
3. Solve extended eigenvalue problem for eigenvalues and corresponding eigenvectors.
4. Normalize the eigenvectors.
5. Construct the modal matrix from the normalized eigenvectors.
6. Transform the model to principal coordinates (modal form) utilizing the modal matrix.
7. Add modal damping to the model in modal form.

8. Write the modal model with damping in state variable form.

6.2.1 Original Undamped Two-Body Single-Axis Model

The undamped form of equations (2–1) and (2–2) is the following.

$$I_1\ddot{\theta}_1 = -k_1\theta_1 + k_1\theta_2 = q_1 \tag{6-3}$$
$$I_2\ddot{\theta}_2 = -k_1\theta_2 + k_1\theta_1 = q_2 \tag{6-4}$$

where the coefficients and variables appearing in this set of equations are defined in Fig. 2-1.

6.2.2 Undamped Two-Body Model in Terms of Inertia and Stiffness Matrices

$$\mathbf{I}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{q} \tag{6-5}$$

where:

$$\mathbf{x} = [\theta_1 \quad \theta_2]^T = [x_1 \quad x_2]^T$$
$$\mathbf{I} = \begin{bmatrix} I_1 & 0 \\ 0 & I_2 \end{bmatrix} = \text{rotational inertia matrix}$$
$$\mathbf{K} = k_1 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \text{rotational stiffnes matrix}$$
$$\mathbf{q} = [q_1 \quad q_2]^T$$

6.2.3 Determination of Eigenvalues and Eigenvectors

The eigenvalues for equation (6–5) are obtained by solving the extended eigenvalue problem which is equivalent to solving the following equation for λ.

$$\lambda\mathbf{I}\mathbf{x} = \mathbf{K}\mathbf{x} \tag{6-6}$$

An equivalent form of the above equation is:

$$[\lambda\mathbf{I} - \mathbf{K}]\mathbf{x} = 0 \tag{6-7}$$

A non-trivial solution of the extended eigenvalue problem exists if the following holds with the expanded forms of the rotational inertia and stiffness matrices for the two-body model expressed immediately following equation (6–5)

$$\begin{aligned} |\lambda\mathbf{I} - \mathbf{K}| &= \begin{vmatrix} \lambda I_1 - k_1 & k_1 \\ k_1 & \lambda I_2 - k_1 \end{vmatrix} \\ &= I_1 I_2 \lambda \left(\lambda - k_1 \frac{I_1 + I_2}{I_1 I_2}\right) = 0 \end{aligned} \tag{6-8}$$

The solutions for this extended eigenvalue problem are:

$$\lambda_1 = 0, \qquad \lambda_2 = \frac{k_1}{I_1} + \frac{k_1}{I_2} \tag{6-9}$$

The eigenvectors corresponding to λ_i are obtained by solving equations of the following form for $\mathbf{v}_i$ where $i = 1,2$.

$$[\lambda_i \mathbf{I} - \mathbf{K}]\mathbf{v}_i = 0 \tag{6-10}$$

The eigenvectors are normalised by solving the following equation for c_i ($i = 1,2$

$$\mathbf{v}_i^T \mathbf{I} \mathbf{v}_i = 1 \tag{6-11}$$

The eigenvalues, eigenvectors and normalised eigenvector coefficients are displayed in Table 6-1

6.2.4 Construction of the Modal Matrix

$$\mathbf{P} = [\mathbf{v}_1 \mid \mathbf{v}_2] = c_1 \begin{bmatrix} 1 & \left(\frac{I_1}{I_2}\right)^{-1/2} \\ 1 & -\left(\frac{I_1}{I_2}\right)^{1/2} \end{bmatrix} \tag{6-12}$$

6.2.5 Transformation to Principal Coordinates

$$\mathbf{y} = \mathbf{p}^T \mathbf{x} \tag{6-13}$$

$$y_1 = c_1 x_1 + c_1 x_2 = c_1 \theta_1 + c_1 \theta_2 \tag{6-14}$$

$$y_2 = c_1 \left(\frac{I_1}{I_2}\right)^{-1/2} x_1 - c_1 \left(\frac{I_1}{I_2}\right)^{1/2} x_2 = c_1 \left(\frac{I_1}{I_2}\right)^{-1/2} \theta_1 - c_1 \left(\frac{I_1}{I_2}\right)^{1/2} \theta_2 \tag{6-15}$$

$$\mathbf{q}' = \mathbf{p}^T \mathbf{q} \tag{6-16}$$

$$q_1' = c_1 q_1 + c_1 q_2 \tag{6-17}$$

$$q_2' = c_1 \left(\frac{I_1}{I_2}\right)^{-1/2} q_1 - c_1 \left(\frac{I_1}{I_2}\right)^{1/2} q_2 \tag{6-18}$$

Model in Principal Coordinates (Modal Model)

$$\ddot{y}_1 = q_1' \tag{6-19}$$

$$\ddot{y}_2 = -\omega_1^2 y_2 + q_2' \tag{6-20}$$

$$\omega_1 = \left(\frac{k_1}{I_1} + \frac{k_1}{I_2}\right)^{1/2} \tag{6-21}$$

6.2.6 Two Body Modal Model With Damping

Modal damping is added to the modal model described by equations (6-19) and (6-20) by adding a damping term to the equation with which the modal frequency, ω_1, is associated. The two body model with damping in modal form then may be written as follows.

$$\ddot{y}_1 = q_1' \tag{6-22}$$

$$\ddot{y}_2 = -2\varsigma_1 \omega_1 \dot{y}_2 - \omega_1^2 y_2 + q_2' \tag{6-23}$$

where ς_1 is the damping ratio associated with ω_1.

Table 6–1

Eigenvalues and Eigenvectors for Each Mode of Two Body Model

Mode No.	Eigenvalue $\lambda_i = \omega_i^2$	Eigenvector $\mathbf{v}_i$	Normalized Eigenvector Coefficient, c_i
1	0	$c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\pm(I_1 + I_2)^{-1/2}$
2	$\frac{k_1}{I_1} + \frac{k_1}{I_2}$	$c_2 \begin{bmatrix} 1 \\ \frac{-I_1}{I_2} \end{bmatrix}$	$c_1 \left(\frac{I_2}{I_1}\right)^{-1/2}$

6.2.7 State Variable form of the Two-Body Modal Model with Damping

The subscript on y_2 has been changed to "3" so that the following relationships can be used in constructing the state variable form of the modal model.

$$y_2 = \dot{y}_1 \tag{6–24}$$
$$y_4 = \dot{y}_3 \tag{6–25}$$

State Variable Modal Model

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \\ \dot{y}_3 \\ \dot{y}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\omega_1^2 & -2\zeta_1\omega_1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \\ y_3 \\ y_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} q_1' \\ \\ q_2' \end{bmatrix} \tag{6–26}$$

6.3 TRANSFORMATION OF THREE-BODY SINGLE-AXIS MODEL TO MODAL FORM

6.3.1 Original Undamped Three-Body Single-Axis Model

The undamped form of equations (3–1) through (3–3) is the following.

$$I_1\ddot{\theta}_1 = -k_1\theta_1 + k_1\theta_2 + q_1 \tag{6–27}$$
$$I_2\ddot{\theta}_2 = k_1\theta_1 + (k_1 + k_2)\theta_2 + k_2\theta_3 + q_2 \tag{6–28}$$
$$I_3\ddot{\theta}_3 = k_2\theta_2 - k_2\theta_3 + q_3 \tag{6–29}$$

where the coefficients and variables appearing in this set of equations are defined in Fig. 3–1.

6.3.2 Undamped Three-Body Model in Terms of Inertia and Stiffness Matrices

$$\mathbf{I}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{q} \tag{6–30}$$

where:

$$\mathbf{x} = [\theta_1 \quad \theta_2 \quad \theta_3]^T = [x_1 \quad x_2 \quad x_3]^T$$

$$\mathbf{I} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix} = \text{rotational inertia matrix}$$

$$\mathbf{K} = \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} = \text{rotational stiffness matrix}$$

$$\mathbf{q} = [q_1 \quad q_2 \quad q_3]^T$$

6.3.3 Determination of Eigenvalues and Modal Frequencies

Corresponding to equation (6–30) an extended eigenvalue problem can be defined which consists of solving the following equation for λ.

$$\lambda \mathbf{I}\mathbf{x} = \mathbf{K}\mathbf{x} \tag{6–31}$$

This is equivalent to setting the following determinant equal to zero

$$|\lambda \mathbf{I} - \mathbf{K}| = \begin{vmatrix} \lambda I_1 - k_1 & k_1 & 0 \\ k_1 & \lambda I_2 - (k_1 + k_2) & k_2 \\ 0 & k_2 & \lambda I_3 - k_2 \end{vmatrix}$$

$$= I_1 I_2 I_3 \lambda \left[\lambda^2 - \left(\frac{k_1}{I_1} + \frac{k_1 + k_2}{I_2} + \frac{k_2}{I_3} \right) \lambda + \frac{k_1 k_2}{I_1 I_2} + \frac{k_1 k_2}{I_1 I_3} + \frac{k_1 k_2}{I_2 I_3} \right] = 0 \tag{6–32}$$

Since each solution for λ corresponds to the square of a modal frequency, equation (6–32) may also be written as follows.

$$\lambda(\lambda - \omega_1^2)(\lambda - \omega_2^2) = 0 \tag{6–33}$$

for which the solutions are: $\lambda_1 = 0$, $\lambda_2 = \omega_1^2$ and $\lambda_3 = \omega_2^2$.
The eigenvectors corresponding to λ_i are obtained by solving equations of the form presented in equation (6–10) for $\mathbf{v}_i$ where $i = 1, 2, 3$. The resulting pairs of eigenvalues and eigenvectors are displayed in Table 6–2.

Application of the remaining steps in the approach utilized in Section 6.2 yields the following state variable modal form for the three-body single-axis model of a flexible spacecraft.

6.3.4 State Variable Form of Three-Body Modal Model with Damping

$$\begin{bmatrix} \dot y_1 \\ \dot y_2 \\ \dot y_3 \\ \dot y_4 \\ \dot y_5 \\ \dot y_6 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\omega_1^2 & -2\zeta_1\omega_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -\omega_2^2 & -2\zeta_2\omega_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} q'_1 \\ q'_2 \\ q'_3 \end{bmatrix} \tag{6–34}$$

Table 6-3

Eigenvalues and Eigenvectors for Each Mode of Three Body Model

Mode No.	Eigenvalue $\lambda_i = \omega_i^2$	Normalized Eigenvector $\mathbf{v}_i$	Eigenvector Coefficient, c_i
1	0	$c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$	$\pm(I_1 + I_2 + I_3)^{-1/2}$
2	ω_1^2	$c_2 \begin{bmatrix} 1 \\ \dfrac{k_1 - \omega_1^2 I_1}{k_1} \\ \dfrac{k_1 - \omega_1^2 I_1}{k_2 - \omega_1^2 I_3}\dfrac{k_2}{k_1} \end{bmatrix}$	$\pm\left[I_1 + \left(\dfrac{k_1 - r_1 I_1}{k_1}\right)^2 I_2 + \left(\dfrac{k_1 - r_1 I_1}{k_2 - r_1 I_3}\dfrac{k_2}{k_1}\right)^2 I_3\right]^{-1/2}$
3	ω_2^2	$c_3 \begin{bmatrix} 1 \\ \dfrac{k_1 - \omega_2^2 I_1}{k_1} \\ \dfrac{k_1 - \omega_2^2 I_1}{k_2 - \omega_2^2 I_3}\dfrac{k_2}{k_1} \end{bmatrix}$	$\pm\left[I_1 + \left(\dfrac{k_1 - r_1 I_1}{k_1}\right)^2 I_2 + \left(\dfrac{k_1 - r_1 I_1}{k_2 - r_1 I_3}\dfrac{k_1}{k_2}\right)^2 I_3\right]^{-1/2}$

6.4 EXTENSION OF RESULTS TO SINGLE-AXIS MODELS WITH FOUR OR MORE RIGID BODIES

Inspection of the state variable modal form of the three-body single-axis model with modal damping in equation (6–34) and the corresponding modal two-body model in equation (6–26) reveals that the state vector coefficient matrix, $\mathbf{A}$, in these modal models could be written in the following 2×2 block diagonal forms.

Two-Body Model:

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & 0 \\ 0 & \mathbf{A}_{22} \end{bmatrix} \tag{6–35}$$

Three-Body Model:

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & 0 & 0 \\ 0 & \mathbf{A}_{22} & 0 \\ 0 & 0 & \mathbf{A}_{33} \end{bmatrix} \tag{6–36}$$

where $\mathbf{A}_{ii}$ and $\mathbf{0}$ are 2×2 submatrices.

Furthermore,

$$\mathbf{A}_{11} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{6–37}$$

and

$$\mathbf{A}_{ii} = \begin{bmatrix} 0 & 1 \\ -\omega_{i-1}^2 & -2\zeta_{i-1}\omega_i \end{bmatrix} \text{ for } i > 1 \tag{6-38}$$

It also should be noted that the dimensions of each **A** matrix are equal to twice the number of rigid bodies in the model. Application of the approach utilized in transforming the two-body and three-body single-axis models to state variable modal form with damping yields a set of models that extend the patterns for these models. In particular, a single-axis model involving r rigid bodies can be transformed to a modal state variable model with a state vector coefficient matrix of the following form.

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & & \bigcirc \\ & \mathbf{A}_{22} & \\ \bigcirc & \ddots & \\ & & \mathbf{A}_{n/2,n/2} \end{bmatrix} \tag{6-39}$$

which is a 2×2 block diagonal matrix of overall dimension $n \times n$ where $n = 2r$. The forms of the 2×2 submatrices along the principal diagonal of this coefficient matrix are given in equations (6-37) and (6-38) for $i = 1, 2, \ldots, n/2$. The remaining elements in the **A** matrix are zero.

6.5 OBSERVER SYNTHESIS EQUATIONS EXPRESSED IN TERMS OF DIRECT MATRIX PRODUCTS

The observer synthesis equations are expressed in the following form in Section 2 of this report

$$\mathbf{TA} - \mathbf{DT} = \mathbf{F} = \mathbf{GC} \tag{6-40}$$

for the state variable form of a single-axis model of a flexible spacecraft, with some scalar states inaccessible,

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu} \tag{6-41}$$

$$\mathbf{x}_A = \mathbf{Cx} \tag{6-42}$$

where the corresponding reduced state observer is given by:

$$\dot{\mathbf{s}} = \mathbf{Ds} + \mathbf{Eu} + \mathbf{Gx}_A \tag{6-43}$$

$$\mathbf{s} = \mathbf{Tx} \tag{6-44}$$

The coefficient matrices and vectors appearing in equations (6-40) through (6-44) are defined for a linear model of dimension n with m accessible scalar states and p inaccessible scalar states as follows.

A $= n \times n$ model state vector coefficient matrix

B $= n \times r$ model control vector coeficient matrix

C $= m \times n$ model measurement or observation matrix

D $= p \times p$ observer state vector coefficient matrix

E $= p \times r$ observer control vector coefficient matrix

G $= p \times m$ observer observed vector coefficient matrix

T $= p \times n$ transformation matrix from model state vector to observer state vector

$\mathbf{x} = [\mathbf{x}_A^T \;\; \mathbf{x}_I^T]^T = n$-vector of model scalar states

$\mathbf{x}_A = m$-vector of accessible scalar states

$\mathbf{x}_I = p$-vector of inaccessible scalar states

$\mathbf{u}$ = r-vector of normalized control torques

$\mathbf{z}$ = p-vector of observer scalar states

From these definitions, each of the matrix products appearing in equation (6–40) has the dimensions $p \times n$. If $\mathbf{I}_p$ is defined as the identity matrix of dimensions $p \times p$ and $\mathbf{I}_n$ is defined correspondingly, then each of the matrix products, $\mathbf{I}_p\mathbf{TA}$ and $\mathbf{DTI}_n$ also has the dimensions $p \times n$. The observer synthesis equations may now be written in the following form.

$$\mathbf{I}_p\mathbf{TA} - \mathbf{DTI}_n = \mathbf{GC} = \mathbf{F} \tag{6–45}$$

However, the definition of a direct matrix product given in Lancaster (6–3) may be used to write the observer synthesis equations in the following equivalent form.

$$[\mathbf{I}_p \otimes \mathbf{A}^T - \mathbf{D} \otimes \mathbf{I}_n]\tilde{\mathbf{T}} = \tilde{\mathbf{F}} \tag{6–46}$$

where:

$$\tilde{\mathbf{T}} = \begin{bmatrix} \mathbf{T}_1^T \\ \mathbf{T}_2^T \\ \vdots \\ \mathbf{T}_p^T \end{bmatrix}$$

$\mathbf{T}_i^T$ = vector comprised of the elements of the i^{th} row of the $\mathbf{T}$ matrix

and $\tilde{\mathbf{F}}$ is related to $\mathbf{F}$ in the same way.

From Lancaster (6–3), the direct matrix products appearing in equation (6–40) may be expanded as follows.

$$\mathbf{I}_p \otimes \mathbf{A}^T = \begin{bmatrix} \mathbf{A}^T & & & 0 \\ & \mathbf{A}^T & & \\ & & \ddots & \\ 0 & & & \mathbf{A}^T \end{bmatrix}_{np \times np} \tag{6–47}$$

For

$$\mathbf{D} = \begin{bmatrix} d_{11} & \cdots & d_{1p} \\ \vdots & & \vdots \\ d_{p1} & \cdots & d_{pp} \end{bmatrix} \tag{6–48}$$

$$\mathbf{D} \otimes \mathbf{I}_n = \begin{bmatrix} d_{11}\mathbf{I}_n & \cdots & d_{1p}\mathbf{I}_n \\ \vdots & & \vdots \\ d_{p1}\mathbf{I}_n & \cdots & d_{pp}\mathbf{I}_n \end{bmatrix}_{np \times np} \tag{6–49}$$

Solving for $\tilde{\mathbf{T}}$ yields

$$\tilde{\mathbf{T}} = [\mathbf{I}_p \otimes \mathbf{A}^T - \mathbf{D} \otimes \mathbf{I}_n]^{-1}\widetilde{\mathbf{GC}} \tag{6–50}$$

In general, this solution would require inversion of a matrix of dimension $np \times np$.

6.6 OBSERVER SYNTHESIS EQUATIONS FOR SINGLE-AXIS MODELS IN MODAL FORM

If **A** is the state vector coefficient matrix of a single-axis model in modal form, it was shown in Subsection 5.4 that it assumes the form given in equation (6–39). The transpose of such a 2×2 block diagonal matrix also is block diagonal and of the following form.

$$\mathbf{A}^T = \begin{bmatrix} \mathbf{A}_{11}^T & & & \\ & \mathbf{A}_{22}^T & & \bigcirc \\ & \bigcirc & \ddots & \\ & & & \mathbf{A}_{n/2,n/2}^T \end{bmatrix} \tag{6–51}$$

where each submatrix, $\mathbf{A}_{ii}$, is 2×2 and the remaining elements in $\mathbf{A}^T$ are zero.

For an observer of even order, p, of a state variable single-axis model in modal form with an **A** matrix of the 2×2 block diagonal form appearing in equation (6–39) the observer state vector coefficient matrix is of the following 2×2 block diagonal form.

$$\mathbf{D} = \begin{bmatrix} \mathbf{D}_{11} & & & \\ & \mathbf{D}_{22} & & \bigcirc \\ & \bigcirc & \ddots & \\ & & & \mathbf{D}_{n/2,n/2} \end{bmatrix} \tag{6–52}$$

where $\mathbf{D}_{ii} = 2 \times 2$ submatrix on the principal diagonal.

$$\mathbf{D} \otimes \mathbf{I}_n = \begin{bmatrix} \mathbf{D}_{11} \otimes \mathbf{I}_n & & & \\ & \mathbf{D}_{22} \otimes \mathbf{I}_n & & \bigcirc \\ & \bigcirc & \ddots & \\ & & & \mathbf{D}_{p/2,p/2} \otimes \mathbf{I}_n \end{bmatrix} \tag{6–53}$$

where:

$$\mathbf{D}_{11} = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} \tag{6–54}$$

$$\mathbf{D}_{22} = \begin{bmatrix} d_{33} & d_{34} \\ d_{43} & d_{44} \end{bmatrix} \tag{6–55}$$

$$\vdots$$

$$\mathbf{D}_{\frac{p-1}{2},\frac{p-1}{2}} = \begin{bmatrix} d_{p-2,p-2} & d_{p-2,p-1} \\ d_{p-1,p-2} & d_{p-1,p-1} \end{bmatrix} \tag{6–56}$$

$$\mathbf{D}_{\frac{p}{2},\frac{p}{2}} = \begin{bmatrix} d_{p-1,p-1} & d_{p-1,p} \\ d_{p,p-1} & d_{p,p} \end{bmatrix} \tag{6–57}$$

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The equation for generating the elements of the $\mathbf{T}$ matrix may now be written in the followng form

$$\begin{bmatrix} \mathbf{A}_{jj}^T - d_{11}\mathbf{I}_2 & -d_{12}\mathbf{I}_2 & & & \\ -d_{21}\mathbf{I}_2 & \mathbf{A}_{jj}^T - d_{22}\mathbf{I}_2 & & \bigcirc & \\ & & \ddots & & \\ & \bigcirc & & \mathbf{A}_{jj}^T - d_{p-1,p-1}\mathbf{I}_2 & -d_{p-1,p}\mathbf{I}_2 \\ & & & -d_{p,p-1}\mathbf{I}_2 & \mathbf{A}_{jj}^T - d_{p,p}\mathbf{I}_2 \end{bmatrix} \begin{bmatrix} \mathbf{T}_{1j} \\ \mathbf{T}_{2j} \\ \vdots \\ \mathbf{T}_{p-1,j} \\ \mathbf{T}_{p,j} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_{1j} \\ \mathbf{F}_{2j} \\ \vdots \\ \mathbf{F}_{p-1,j} \\ \mathbf{F}_{p,j} \end{bmatrix}; \quad j = 1,2,\ldots,n/2 \tag{6-58}$$

where:

$$\mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{T}_{ij} = [t_{i,2j-1} \quad t_{i,2j}]^T \qquad i = 1,2,\ldots,p \tag{6-59}$$

$$\mathbf{F}_{ij} = [f_{i,2j-1} \quad f_{i,2j}]^T \tag{6-60}$$

It should be noted that all of the 2×2 submatrices appearing in the coefficient matrices of equation set (6–58) are commutative under multiplication because this property is useful in the solution for the elements of the $\mathbf{T}$ matrix for $p > 1$. Since all diagonal matrices commute, it is necessary only to show that the matrices, $\mathbf{A}_{jj}^T - d_{ik}I_2$ and $\mathbf{A}_{jj}^T - d_{rs}I_2$, commute for $j = 1,2,\ldots,n/2$.

$$[\mathbf{A}_{11}^T - d_{kl}\mathbf{I}_2][\mathbf{A}_{11}^T - d_{rs}\mathbf{I}_2] = \begin{bmatrix} -d_{kl} & 0 \\ 1 & -d_{kl} \end{bmatrix} \begin{bmatrix} -d_{kl} & 0 \\ 1 & -d_{kl} \end{bmatrix}$$

$$= \begin{bmatrix} d_{kl}d_{rs} & 0 \\ -(d_{kl}+d_{rs}) & d_{kl}d_{rs} \end{bmatrix} = [\mathbf{A}_{11}^T - d_{rs}\mathbf{I}_2][\mathbf{A}_{11}^T - d_{kl}\mathbf{I}_2] \tag{6-61}$$

$$\begin{aligned} [\mathbf{A}_{jj}^T - d_{kl}\mathbf{I}_2][\mathbf{A}_{jj}^T - d_{rs}\mathbf{I}_2] &= \begin{bmatrix} -d_{kl} & -\omega_{j-1}^2 \\ 1 & -(d_{kl} + 2\zeta_{j-1}\omega_{j-1}) \end{bmatrix} \begin{bmatrix} -d_{rs} & -\omega_{j-1}^2 \\ 1 & -(d_{rs} + 2\zeta_{j-1}\omega_{j-1}) \end{bmatrix} \\ &= \begin{bmatrix} d_{kl}d_{rs} - \omega_{j-1}^2 & \omega_{j-1}^2 + 2\zeta_{j-1}\omega_{j-1} \\ -(d_{kl} + d_{rs} + 2\zeta_{j-1}\omega_{j-1}) & d_{kl}d_{rs} + 2\zeta_{j-1}\omega_{j-1}(d_{kl}+d_{rs}) + (4\zeta_{j-1}^2 - 1)\omega_{j-1}^2 \end{bmatrix} \\ &= [\mathbf{A}_{jj}^T - d_{rs}\mathbf{I}_2][\mathbf{A}_{jj}^T - d_{kl}\mathbf{I}_2] \qquad j = 2,3,\ldots,n/2 \end{aligned} \tag{6-62}$$

Hence, the 2×2 submatrices, $\mathbf{A}_{jj} - d_{ik}I_2$ and $\mathbf{A}_{jj} - d_{rs}I_2$ are commutative under matrix multiplication for $j = 1,2,\ldots,n/2$.

For an even p the solution for the elements of the $\mathbf{T}$ matrix now involves $np/4$ inversions of the coefficient matrices of dimensions 4×4 partitioned into 2×2 submatrices. Since the 2×2 submatrices are commutative under multiplication, each of the $np/4$ vector-matrix equations of the set can be solved in terms of each $\mathbf{T}_{ij}$ which has the effect of reducing dimensions of the matrix inversion involved by a factor of two. The definitions of $\mathbf{T}_{ij}$ and $\mathbf{F}_{ij}$ in equations (6–59) and (6–60) in terms of the indevidual elements of the $\mathbf{T}$ and $\mathbf{F}$ matrices would then be invoked to complete the solution.

For the case in which p is an odd integer, the corresponding $\mathbf{D}$ matrix is 2×2 block diagonal except at one location along its principal diagonal where there occurs a degenerate "block" in the form of a single non-zero scalar element. With the assumption that the individual scalar state variables can be reordered, this isolated principal diagonal element can be placed at the lower right hand corner so that the $\mathbf{D}$ matrix assumes the following form.

$$\mathbf{D} = \begin{bmatrix} \mathbf{D}_{11} & & & & \\ & \mathbf{D}_{22} & & \bigcirc & \\ & & \ddots & & \\ & \bigcirc & & \mathbf{D}_{\frac{p-1}{2},\frac{p-1}{2}} & \\ & & & & d_{p,p} \end{bmatrix} \tag{6-63}$$

where $\mathbf{D}_{ii}$ are 2×2 submatrices defined in the same way as for p even for $i = 1, 2, \ldots, (p-1)/2$ and $d_{p,p}$ is a scalar on the principal diagonal of the $\mathbf{D}$ matrix.

$$\mathbf{D} \otimes \mathbf{I}_n = \begin{bmatrix} \mathbf{D}_{11} \otimes \mathbf{I}_n & & & & \\ & \mathbf{D}_{22} \otimes \mathbf{I}_n & & \bigcirc & \\ & & \ddots & & \\ & \bigcirc & & \mathbf{D}_{\frac{p-1}{2},\frac{p-1}{2}} \otimes \mathbf{I}_n & \\ & & & & d_{p,p} \otimes \mathbf{I}_n \end{bmatrix} \tag{6-64}$$

By the same procedure as utilized for even p the equations for generating the elements of the $\mathbf{T}$ matrix when p is odd may be expressed in the following form with $\mathbf{F}_{ij}$ and $\mathbf{T}_{ij}$ defined in equations (6–59) and (6–60).

$$\begin{bmatrix} \mathbf{A}_{jj}^T - d_{11}\mathbf{I}_2 & \mathbf{A}_{jj}^T - d_{12}\mathbf{I}_2 & & & & \\ \mathbf{A}_{jj}^T - d_{21}\mathbf{I}_2 & \mathbf{A}_{jj}^T - d_{22}\mathbf{I}_2 & & & \bigcirc & \\ & & \ddots & & & \\ & \bigcirc & & \mathbf{A}_{jj}^T - d_{p-2,p-2}\mathbf{I}_2 & -d_{p-2,p-1}\mathbf{I}_2 & \\ & & & -d_{p-1,p-2}\mathbf{I}_2 & \mathbf{A}_{jj}^T - d_{p-1,p-1}\mathbf{I}_2 & \\ & & & & & \mathbf{A}_{jj}^T - d_{p,p}\mathbf{I}_2 \end{bmatrix} \begin{bmatrix} \mathbf{T}_{1j} \\ \mathbf{T}_{2j} \\ \vdots \\ \mathbf{T}_{p-2,j} \\ \mathbf{T}_{p-1,j} \\ \mathbf{T}_{pj} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_{1j} \\ \mathbf{F}_{2j} \\ \vdots \\ \mathbf{F}_{p-2,j} \\ \mathbf{F}_{p-1,j} \\ \mathbf{F}_{pj} \end{bmatrix} \tag{6-65}$$

From equation set (6–65), it is evident that when p is an odd integer, $n/2$ of the equations in the set reduce to the form,

$$[\mathbf{A}_{jj}^T - d_{p,p}\mathbf{I}_2]\mathbf{T}_{p,j} = \mathbf{F}_{p,j} \qquad j = 1, 2, \ldots, \tfrac{n}{2} \tag{6-66}$$

where it has been assumed that the state variable model can be rearranged, if necessary, so that this vector-matrix equation appears last in each of the $\frac{n}{2}$ sets of equations.

From equation (6–37),

$$[\mathbf{A}_{11}^T - d_{kl}\mathbf{I}_2]^{-1} = \frac{\begin{bmatrix} -d_{kl} & 0 \\ -1 & -d_{kl} \end{bmatrix}}{d_{kl}^2} \tag{6-67}$$

From equation (6–38),

$$[\mathbf{A}_{jj}^T - d_{kl}\mathbf{I}_2]^{-1} = \frac{\begin{bmatrix} -(2\varsigma_{j-1}\omega_{j-1} + d_{kl}) & \omega_{j-1}^2 \\ -1 & -d_{kl} \end{bmatrix}}{\omega_{j-1}^2 + 2d_{kl}\varsigma_{j-1}\omega_{j-1} + d_{kl}^2} \tag{6-68}$$

Hence, for the case in which the order of the observer, p, is an odd integer the solution for the elements of the $\mathbf{T}$ matrix in terms of the elements of the $\mathbf{F}$ matrix reduces to $n(p-1)/4$ inversions of coefficient matrices of dimension 4×4 partitioned into 2×2 submatrices and $n/2$ inversions of coefficient matrices of dimensions 2×2. Since all of the 2×2 matrices of equation set (6–56) commute under multiplication, the dimensions of the $n(p-1)/4$ coefficient matrices to be inverted are in effect reduced by a factor of two by first solving for the $\mathbf{T}_{ij}$'s in terms of the $\mathbf{F}_{ij}$'s and then applying equations (6–59) and (6–60).

6.6.1 First Order Observers ($p = 1$)

Since p is an odd integer, equation (6–65) applies and reduces to the following form.

$$[\mathbf{A}_{jj}^T - d_{11}\mathbf{I}_2]\mathbf{T}_{1j} = \mathbf{F}_{1j}, \qquad j = 1, 2, \ldots, n/2 \tag{6-69}$$

where $\mathbf{F}_{1j}$ and $\mathbf{T}_{1j}$ are defined in equations (6–59) and (6–60) and the $\mathbf{F}$ matrix, which has a single row, may be written as follows:

$$\mathbf{F} = [\mathbf{F}_{11}^T \quad \mathbf{F}_{12}^T \quad \cdots \quad \mathbf{F}_{1,n/2}^T]^T \tag{6-70}$$

The corresponding single row $\mathbf{T}$ matrix may be written in the same form as the $\mathbf{F}$ matrix. Solving equation (6–69) for $\mathbf{T}_{ij}$ yields the following:

$$\mathbf{T}_{1j} = [\mathbf{A}_{jj}^T - d_{11}\mathbf{I}_2]^{-1}\mathbf{F}_{1j} \tag{6–71}$$

which, in view of equations (6–59) and (6–60), becomes:

$$\begin{bmatrix} t_{1,2j-1} \\ t_{1,2j} \end{bmatrix} = [\mathbf{A}_{jj}^T - d_{11}\mathbf{I}_2]^{-1} \begin{bmatrix} f_{2,2j-1} \\ f_{2,2j} \end{bmatrix} \tag{6–72}$$

where $[\mathbf{A}_{11}^T - d_{11}\mathbf{I}_2]^{-1}$ is given in equation (6–67) and $[\mathbf{A}_{jj}^T - d_{11}\mathbf{I}_2]^{-1}$ is given in equation set (6–68) for $j = 2, 3, \ldots, n/2$.

6.6.2 Second Order Observers ($p = 2$)

Since p is even, equation (6–58) applies. For $p = 2$ it reduces to:

$$\begin{bmatrix} \mathbf{A}_{jj}^T - d_{11}\mathbf{I}_2 & -d_{12}\mathbf{I}_2 \\ -d_{21}\mathbf{I}_2 & \mathbf{A}_{jj}^T - d_{22}\mathbf{I}_2 \end{bmatrix} \begin{bmatrix} \mathbf{T}_{1j} \\ \mathbf{T}_{2j} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_{1j} \\ \mathbf{F}_{2j} \end{bmatrix}; \qquad j = 1, 2, \ldots, \frac{n}{2} \tag{6–73}$$

where $\mathbf{T}_{ij}$ and $\mathbf{F}_{ij}$ are defined in equations (6–59) and (6–60).

$$\Delta_{2j} = \begin{bmatrix} \mathbf{A}_{jj}^T - d_{11}\mathbf{I}_2 & -d_{12}\mathbf{I}_2 \\ -d_{21}\mathbf{I}_2 & \mathbf{A}_{jj}^T - d_{22}\mathbf{I}_2 \end{bmatrix} \tag{6–74}$$

Since $\mathbf{A}_{jj}^T - d_{kl}\mathbf{I}_2$ commutes with $\mathbf{A}_{jj}^T - d_{rs}\mathbf{I}_2$ under matrix multiplication

$$|\Delta_{2j}| = |[\mathbf{A}_{jj}^T - d_{11}\mathbf{I}_2][\mathbf{A}_{jj}^T - d_{22}\mathbf{I}_2] - d_{12}d_{21}\mathbf{I}_2| \tag{6–75}$$

Then

$$\mathbf{T}_{1j} = \frac{\mathbf{A}_{jj}^T - d_{22}\mathbf{I}_2}{|\Delta_{2j}|}\mathbf{F}_{1j} + \frac{d_{12}\mathbf{I}_2}{|\Delta_{2j}|}\mathbf{F}_{2j}; \qquad j = 1, 2, \ldots, \frac{n}{2} \tag{6–76}$$

$$\mathbf{T}_{2j} = \frac{d_{21}\mathbf{I}_2}{|\Delta_{2j}|}\mathbf{F}_{1j} + \frac{\mathbf{A}_{jj}^T - d_{11}\mathbf{I}_2}{|\Delta_{2j}|}\mathbf{F}_{2j} \tag{6–77}$$

Solutions for the individual elements of the $\mathbf{T}$ matrix in terms of the elements of the $\mathbf{F}$ matrix are then obtained by application of the definitions of $\mathbf{T}_{ij}$ and $\mathbf{F}_{ij}$ in equations (6–59) and (6–60).

$$\begin{bmatrix} t_{1,2j-1} \\ t_{1,2j} \end{bmatrix} = \frac{[\mathbf{A}_{jj}^T - d_{22}\mathbf{I}_2}{|\Delta_{2j}|} \begin{bmatrix} f_{1,2j-1} \\ f_{1,2j} \end{bmatrix} + \frac{d_{12}}{|\Delta_{2j}|} \begin{bmatrix} f_{2,2j-1} \\ f_{2,2j} \end{bmatrix}$$

$$j = 1, 2, \ldots, \frac{n}{2} \tag{6–78}$$

$$\begin{bmatrix} t_{2,2j-1} \\ t_{2,2j} \end{bmatrix} = \frac{d_{12}}{|\Delta_{2j}|} \begin{bmatrix} f_{1,2j-1} \\ f_{1,2j} \end{bmatrix} + \frac{[\mathbf{A}_{jj}^T - d_{11}\mathbf{I}_2]}{|\Delta_{2j}|} \begin{bmatrix} f_{2,2j-1} \\ f_{2,2j} \end{bmatrix} \tag{6–79}$$

6.6.3 Observers of Higher Order

For $p = 3$ equation (6–65) reduces to the following.

$$\begin{bmatrix} \mathbf{A}_{jj}^T - d_{11}\mathbf{I}_2 & -d_{12}\mathbf{I}_2 \\ -d_{21}\mathbf{I}_2 & \mathbf{A}_{jj}^T - d_{22}\mathbf{I}_2 \end{bmatrix} \begin{bmatrix} \mathbf{T}_{1j} \\ \mathbf{T}_{2j} \end{bmatrix} = \begin{bmatrix} \mathbf{T}_{1j} \\ \mathbf{T}_{2j} \end{bmatrix} \tag{6–73}$$

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$$[\mathbf{A}_{jj}^{T} - d_{s3}\mathbf{I}_{3}]\mathbf{T}_{3j} = \mathbf{F}_{3j}; \qquad j = 1,2,\ldots,\frac{n}{2} \tag{6-80}$$

The observer synthesis equations for $p = 3$ differ from those for $p = 2$ by the addition of equation set (6–80) which is of the same form as the observer synthesis equations for $p = 1$ with the subscript, 3, substituted for the subscript, 1. Therefore, the solutions for the elements of the first two rows of the **T** matrix for $p = 3$ are identical with those for the two rows of the matrix for $p = 2$, equations sets (6–76) and (6–77). While the solutions for the elements of the third row are of the same form as those for **T** matrix for $p = 1$, equation set (6–71) with the numerical subscript, 3, substituted for the subscript, 1.

For $p = 4$ equation set (6–58) reduces to one equation set identical with the one for $p = 2$, and another equation set of the same form with each numerical subscript incremented by one. Hence, the solutions for the elements for the first two rows of the **T** matrix for $p = 4$ are identical with those for the two rows of the **T** matrix for $p = 2$, equation set (6–76) and (6–77). The solutions for the elements of the third and fourth rows are given by the same equations with each of the numerical subscripts incremented by one.

The solutions for the elements of the **T** matrix for larger values of p follow the same pattern. Thus, they can be constructed directly by using the solutions for $p = 1$ and $p = 2$ as "building blocks" as was demonstrated for $p = 3$ and $p = 4$.

6.7 SOLUTION FOR SYNTHESIZED STATE VARIABLES

In Subsection 6.4 it was shown that the single-axis modal models of a flexible spacecraft treated in this report can be written in the state variable form in terms of the modal state vector as follows.

$$\dot{\mathbf{y}} = \mathbf{A}\mathbf{y} + \mathbf{B}\mathbf{u} \tag{6-81}$$

$$\mathbf{y}_{A} = \mathbf{C}\mathbf{y} \tag{6-82}$$

where

A $= n \times n$ state vector coefficient matrix

B $= n \times r$ control vector coefficient matrix

C $= m \times n$ observation or measurement matrix

y $=$ modal state vector of dimension n

$\mathbf{y}_A$ $=$ vector of accessible modal state variables of dimension m

u $=$ control vector of dimension r

The block diagram corresponding to this model is the same as Fig. 2–2 except that the vectors, **x** and $\mathbf{x}_A$, are replaced by **y** and $\mathbf{y}_A$, respectively. The 2×2 block diagonal form of the **A** matrix is shown in equation (6–39).

If the number of inaccessible modal model scalar states is given by $p = n - r$ $(1 < p < n)$, then the corresponding reduced modal state linear observer model is the following.

$$\dot{\mathbf{z}} = \mathbf{D}\mathbf{z} + \mathbf{E}\mathbf{u} + \mathbf{G}\mathbf{y}_{A} \tag{6-83}$$

$$\mathbf{z} = \mathbf{T}\mathbf{y} \tag{6-84}$$

where

D $= p \times p$ observer state vector coefficient matrix

E $= p \times r$ observer control vector coefficient matrix

G $= p \times m$ observer observed vector coefficient matrix

T $= p \times n$ observer weighting matrix

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$\mathbf{z}$ = observer state vector

The block diagram for this observer is the same as that shown in Fig. 2–4 except that the vector, $\mathbf{y}$, is substituted for the vector $\mathbf{x}$.

After the observer synthesis equation given by equation (6–40) or one of its equivalent forms such as equation set (6–58) for even p and equation set (6–65) for odd p has been solved for the elements of the $\mathbf{T}$ matrix, t_{ij}, equation (6–84) can be solved to express the synthesized inaccessible modal model scalar states in terms of the accessible modal model scalar states. This last step generally will require the inversion of a $p \times p$ matrix. A block diagram of the modal model of a flexible spacecraft and its reduced state linear observer appears in Fig. 6–1.

Example: Solution for two synthesized modal states in the two body model.

Suppose that for the state variable modal two-body model the modal scalar states, y_3 and y_4 are inaccessible. Then $p = 2$, $n = 4$ and the $\mathbf{T}$ matrix is thus:

$$\mathbf{T} = \begin{bmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ t_{21} & t_{22} & t_{23} & t_{24} \end{bmatrix} \tag{6–85}$$

corresponding to:

$$\mathbf{y} = [y_1 \quad y_2 \quad y_3 \quad y_4]^T \tag{6–86}$$

$$\mathbf{z} = [z_1 \quad z_2]^T \tag{6–87}$$

If the remaining modal model scalar states, y_1 and y_2 and all of the elements of the $\mathbf{T}$ matrix are known then equation (6–84) can be solved to express the synthesized inaccessible modal model scalar states $\hat{y}_3$ and $\hat{y}_4$ as follows.

$$\hat{y}_3 = \frac{(\Delta_2)_{1,1}}{\Delta_2}(z_1 - t_{11}y_1 - t_{12}y_2) - \frac{(\Delta_2)_{2,1}}{\Delta_2}(z_2 - t_{21}y_1 - t_{22}y_2) \tag{6–88}$$

$$\hat{y}_4 = \frac{(\Delta_2)_{1,2}}{\Delta_2}(z_1 - t_{11}y_1 - t_{12}y_2) - \frac{(\Delta_2)_{2,2}}{\Delta_2}(z_2 - t_{21}y_1 - t_{22}y_2) \tag{6–89}$$

where

$$\Delta_2 = \begin{vmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{vmatrix} = t_{11}t_{22} - t_{12}t_{21} \neq 0 \tag{6–90}$$

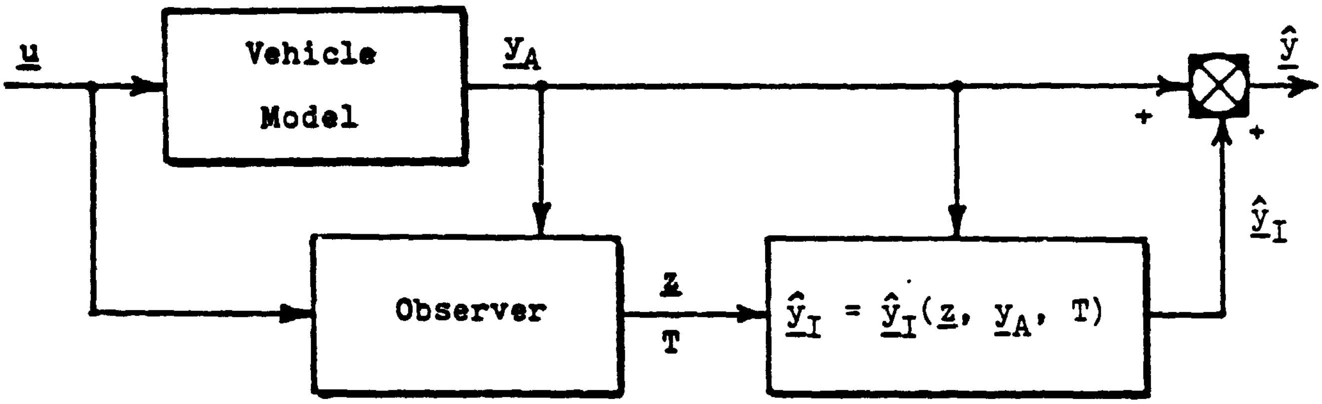
$(\Delta_2)_{i,j}$ = Δ_2 without the elements of the i^{th} row and the j^{th} column.

6.8 REFERENCES

6–1 Waites, H.B., Marshall Space Flignt Center, Huntsville, AL, telephone conversation of March 8, 1984.

6–2 Thomson, W.T., *Theory of Vibrations with Applications*. Englewood Cliffs, N.J.: Prentice–Hall, Inc., 1972, pp. 226–230.

6–3 Lancaster, Peter, *Theory of Matrices*. New York: Academic Press, 1959, p. 256.



$\underline{u}$ = vector of scalar inputs to vehicle modal model

$\underline{y}_A$ = vector of accessible states of modal model

$\underline{z}$ = vector of scalar states of observer

T = observer weighting matrix

$\hat{\underline{y}}_I$ = vector of reconstructed scalar states of model

$\hat{\underline{y}} = \left[\begin{array}{c} \underline{y}_A \\ \hline \hat{\underline{y}}_I \end{array}\right]$ = reconstructed vector of all scalar state variables of vehicle model

FIGURE 6-1

BLOCK DIAGRAM OF MODAL SPACECRAFT MODEL AND ITS REDUCED STATE LINEAR OBSERVER

SECTION 7

CONCLUSIONS AND RECOMMENDATIONS

During the period covered by this report, the class of single-axis state variable models with some inaccessible states was extended three ways.

1. The patterns involved in the prior development of the state variable forms of the two-body, three-body, and four-body single-axis models of a flexible spacecraft were extended to produce a five-body model that could represent the single axis that was found to be decoupled from the remaining axes of a five-body three axis model treated in earlier work.

2. A rotational damping coefficient was added to each flexible interconnection between the rigid bodies comprising each model.

3. Each undamped single axis model was transformed to a modal model with one or more inaccessible model state variables.

For each combination of single axis state variable model and inaccessible scalar state(s) a reduced state linear observer was generated to reconstruct those scalar states that were inaccessible. This was done because the application of linear quadratic regulator (LQR) and closely related time domain approaches to attitude control utilize all or nearly all of the scalar states of the model of the spacecraft to be controlled.

7.1 CONCLUSIONS

The following conclusions were drawn mainly from the development of the damped two-, three-, four- and five-body single-axis models with inaccessible scalar state variables of a prototype flexible spacecraft and the generation of the corresponding linear observers of minimum order required to reconstruct these inaccessible scalar states.

7.1.1 Observers Generated for Single-Axis Models Based on Angular Displacement and Rate State Variables

1. Since, of the four coefficient matrices appearing in the observer synthesis equation, **A**, **D**, **F** and **T**, only the state vector coefficient matrix in the single-axis model, **A**, is known a priori, the following approach was used to generate the elements of the coefficient matrix, **T**, for the transformation from the state vector of the model, **x**, to the state vector of the observer, **z**.

   a. The elements of **F** can be determined by utilizing the known values for the elements of **C**, the observation matrix in the single-axis model and the assumed values of the elements of **G** in conjunction with the equation, **F**=**GC**.

   b. Assuming that **D** is diagonal simplifies significantly the solution of the equations for determining the elements of **T**.

2. The minimum order required for a reduced state linear observer to reconstruct p inaccessible scalar states of a single-axis state variable model with a total of n scalar states is p where $p = 1, 2, \ldots, n-1$. Therefore, the number of elements in the **T** matrix to be determined equals np and solving for the p inaccessible synthesized scalar state variables requires the inversion of a $p \times p$ coefficient matrix.

3. The rigid-body flexible-joint single-axis models of a flexible spacecraft treated in this report are in a more general form when damping is added to each joint connecting the rigid bodies. Therefore it is far easier to develop the observer synthesis equations for the damped models than to begin with the equations for the undamped models and generalize them to account for the effects of added damping.

4. If $n-1$ of the n scalar state variables of the single-axis model are accessible, a reduced state observer of order at least one ($p = 1$) is required to synthesize the inaccessible state variable. The number of elements in the **T** matrix to be determined equals n and solving for the one inaccessible scalar state variable does not require the inversion of a matrix.

5. As the number of accessible scalar state variables decreases, the number of inaccessible scalar states, p, the number of elements in the T matrix to be determined, np and the dimensions of the coefficient matrix to be inverted in solving for the inaccessible synthesized scalar state variables, $p \times p$, increase, which increases the number of computations required.

6. At least one of the n state variables of the single axis model must be accessible in order for the inaccessible state variables to be synthesized by a reduced state observer.

7.1.3 Observers Generated for Single-Axis Models Based on Modal State Variables

1. The state vector coefficient matrix, A, in each modal single-axis model appears in 2×2 block diagonal form implying that the state vector coefficient matrix, D, in the corresponding reduced state observer is 2×2 block diagonal.

2. When the observer synthesis equation is expressed in terms of direct matrix products, solution for the elements of the T matrix generally requires inversion of an $np \times np$ coefficient matrix.

3. When the number of inaccessible states of the model, p, is even, use of A and D matrices in 2×2 block diagonal form in the observer synthesis equation reduces the solution for the elements of the T matrix to the inversion of $\frac{np}{4}$ 4×4 matrices partitioned into 2×2 submatrices all of which commute under multiplication.

4. When the number of inaccessible states of the model is odd, use of A and D matrices in 2×2 block diagonal form in the observer synthesis equation reduces the solution for the elements of the T matrix to the inversion of $\frac{n(p-1)}{4}$ 4×4 matrices partitioned into 2×2 commutative submatrices and $\frac{n}{2}$ 2×2 matrices.

5. The modal matrix operates on only the angular displacement state variables and thus each modal state variable generally is a weighted linear sum of all of the angular displacements.

   a. Reduced state linear observers predicated upon a modal single axis model generally require that at least one of the modal state variables be accessible which is equivalent to requiring that all of the angular displacement state variables of the original state variable model be accessible.

   b. Reduced state observers based on the modal model can be used to synthesize one or more inaccessible angular rate state variables of the original state model.

   c. If no modal state variable is accessible or, equivalently, if any one of the angular displacement state variables for the original state model is inaccessible, reduced state observers predicated upon the modal model cannot be used to synthesize any state variables.

6. Even if all of the necessary conditions required for synthesis of state variables by a reduced state observer predicated upon a modal single axis model are satisfied, two significant disadvantages of this approach are the following:

   a. Modal state variables that are weighted sums of angular displacements and rate state variables are difficult to interpret physically.

   b. Transformation from the modal state variables to the angular displacement and rate state variables may be very complicated.

7.2 RECOMMENDATIONS

The following directions are suggested for future study in the application of attitude control to state variable models of flexible spacecraft for which one or more scalar states are inaccessible.

1. The modular control techniques developed for the attitude control of models of flexible spacecraft for which all scalar state variables are accessible should be modified for application to series of single axis models and their associated reduced state linear observers developed in the work treated in this report.

2. Selected combinations of single axis model and its associated linear observer and modular attitude control system should be simulated on a digital computer to support investigation of effects of changes in the following single-axis model and observer characteristics.

a. Ratios between the masses (rotational inertias) of bodies comprising the single-axis model.

b. Magnitudes of spring and damping coefficients at the interfaces between the rigid bodies of the single-axis model.

3. The generation of reduced state observers to reconstruct inaccessible scalar states of a model of a flexible spacecraft should be extended to the three-axis five-body model of a prototype flexible spacecraft developed earlier.

4. The application of modular techniques to the attitude control of selected combinations of a single-axis model and its corresponding reduced state linear observer should be extended to the combinations of the single-axis and two-axis five body models representing the prototype flexible spacecraft and the corresponding reduced state observers.

5. The combination of single-axis and two-axis five body models and their linear observers and modular attitude control systems should be simulated on a digital computer.

6. Coefficients representing the sensitivity of the scalar states to parameters of the combination of single-axis and two-axis five body models and their linear observers and modular attitude control systems should be developed.

7. Since, for a model with n scalar state variables, the number of elements of the $\mathbf{T}$ matrix to be determined, np, and the dimensions of the coefficient matrices to be inverted in solving for the synthesized inaccessible variables, $p \times p$, increase as the number of inaccessible variables, p, increases, it would be desirable to determine whether there is a value for the ratio, $\frac{p}{n}$, at which a full state observer would be more readily implemented than a reduced state observer.

8. In view of the especially convenient forms of the modal state variable single-axis models of the flexible spacecraft and of the corresponding observer synthesis equation it appears worthwhile to investigate ways to mitigate the requirement that all rotational displacement state variables in the original model be accessible.