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Entire Functions of Exponential Type *

Fred Gross * *

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Let f be an entire function satisfying for some integer p and some constant C

$$\sum_{j=0}^{N}\left(\int_0^{2\pi} / f^{(j)}(re^{i\theta}) /^p d\theta\right)^{1/p} \geqslant C \sum_{j=N+1}^{\infty}\left(\int_0^{2\pi} / f^{(j)}(re^{i\theta}) /^p\right)^{1/p} \tag{1}$$

for sufficiently large r. Then f is of exponential type. Conversely, (1) is satisfied whenever f is periodic of exponential type. Similar conditions on the maximum moduli $M_f(j)(r)$ yield the same result. The analogous condition on $/f^{(j)}/$ is also discussed.

Key words: Bounded index; convexity; entire function; exponential type; maximum modulus.

1. Introduction

An entire function, $f(z)$, has a Taylor expansion about any point a in the complex plane of the form

$$f(z)=\sum_{i=0}^{\infty} a_n(z-a)^n$$

Since this series is absolutely convergent everywhere in the plane, $|a_n|$ must approach zero as n approaches infinity. Consequently, there exists for each a, an index $n(a)$ for which $|a_n|$ is a maximal coefficient. B. Lepson [3][1] raised the question of characterizing entire functions for which $n(a)$ is bounded in a.[2] In the sequel we shall consider certain interesting variations of Lepson's problem. Though some of the results of this paper can also be obtained from the Wiman-Valiron theory, we shall use a more elementary and direct method which seems of interest in itself.

2. Preliminaries

We begin by defining the notion of bounded index and the related properties that we shall consider in the sequel.

DEFINITION 1: An entire function is said to be of bounded index if and only if, there exists an integer N, such that for all z

$$\max_{i=0, 1, 2, \ldots N} \{|f^i(z)|/i!\} > |f^j(z)|/j!; \tag{1}$$

$j=0, 1, 2, 3 \ldots$, where $f^{(0)}(z)$ denotes $f(z)$.

The bounded index condition is closely related to the condition that for some integer $N \geqslant 0$, some $C > 0$,

$$\sum_{i=0}^{N} |f^{(i)}(z)|/i! \geqslant C|f^{(j)}(z)|/j!;\ j=N+1, N+2, \ldots. \tag{2}$$

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**Present address: U.S. Naval Research Laboratory, Washington, D. C. 20390; and University of Maryland, Baltimore County.

[1] Figures in brackets indicate the literature references at the end of this paper.

[2] Since the completion of this paper a number of papers have appeared on this problem. e.g., [4, 5]

We concern ourselves with a slight variation of (2); namely that for some integer $N \geqslant 0$, some $C > 0$, some $r_0 > 0$ and all z with $|z| > r_0$,

$$\sum_{i=0}^{N} |f^{(i)}(z)|/i! > C \sum_{j=N+1}^{\infty} |f^{(j)}(z)|/j!. \tag{3}$$

We also consider those variations of (3) obtained by replacing $|f^{(i)}(z)|$ by

$$\left(\int_0^{2\pi} |f^{(i)}(\mathrm{r}e^{i\theta})|^p d\theta \right)^{1/p}$$

p any positive integer, and by $M_f(i)^{(r)}$, the maximum modulus of $f^{(i)}$ on $|z| = r$, respectively. In order to simplify notation we let $0(|f|, a, b)$, $0\left(\int_f, p, a, b \right)$ and $0(\mathrm{M}_f, a, b)$ denote

$$\sum_{j=a}^{b} |f^{(j)}(z)|/j!, \quad \sum_{j=a}^{b} \frac{\left(\int_0^{2\pi} |f^{(j)}(\mathrm{r}e^{i\theta})|^p d\theta \right)^{1/p}}{j!}$$

and

$$\sum_{j=a}^{b} M_f^{(j)}(r)/j!$$

respectively.

We shall show in the sequel that entire functions satisfying condition (3) or any of the conditions obtained from it by the substitutions suggested above are functions of exponential type.

3. Entire Functions Satisfying (3) and Related Properties

THEOREM 1. *Let* f *be entire and* C *be a positive constant. If for* i = 0, 1, 2, . . ., N, f *satisfies one of the following for all* z *with* |z| *sufficiently large:*

(a) $0(|\mathrm{f}^{(i)}|, 0, \mathrm{N}) > \mathrm{C}\,0(|\mathrm{f}^{(i)}|, \mathrm{N}+1, \infty)$

(b) $0\left(\int_{\mathrm{f}} (\mathrm{i}), \mathrm{p}, 0, \mathrm{N} \right) > \mathrm{C}\,0\left(\int_{\mathrm{f}} (\mathrm{i}), \mathrm{p}, \mathrm{N}+1, \infty \right)$

for some positive integer p,

(c) $0(\mathrm{M}_{\mathrm{f}^{(i)}}, 0, \mathrm{N}) > \mathrm{C}\,0(\mathrm{M}_{\mathrm{f}^{(i)}}, \mathrm{N}+1, \infty)$,

then f *is of exponential type.*

PROOF: For convenience we choose $C = 1$. The proof for other C is similar. For any arbitrary entire F and any complex number A, we have,

$$F(z) = \sum_{j=0}^{\infty} \frac{F^{(j)}(A)}{j!} (z-A)^j.$$

Let n be any integer, a and ξ complex numbers with $|\xi| = 1$. Choosing $A = (n-1)\xi + a$, $z = n\xi + a$ and $F = f^{(i)}$ we obtain

$$|f^{(i)}(a + n\xi)| \leqslant 0(|f^{(i)}(a + (n-1)\xi)|, 0, \infty), \tag{4}$$

for $i = 0, 1, 2, \ldots .$

Assume that hypothesis (a) holds. Then (4) yields

$$|f^{(i)}(a+n\xi)| \leqslant 2\, 0(|f^{(i)}(a+(n-1)\xi)|,\ 0, N); \qquad (5)$$
$$i=0, 1, 2, \ldots, N.$$

One observes that for $i \leqslant N$

$$\begin{aligned} 0(|f^{(i)}|, 0, N) &= \sum_{j=0}^{N} |f^{(i+j)}(z)|/j! \\ &= \sum_{j=0}^{N} \frac{|f^{(i+j)}(z)|}{(j+1)!} \frac{(j+i)!}{j!} \\ &\leqslant \sum_{l=0}^{2N} \frac{|f^{(l)}(z)|}{l!} (2N)^N \\ &\leqslant 2. \sum_{l=0}^{N} \frac{|f^{(l)}(z)|}{l!} (2N)^N = 2^{N+1}N^N 0(|f|, 0, N). \end{aligned}$$

Equations (5) and (6) yield

$$0(|f(a+n\xi)|, 0, N) \leqslant (N+1)2^{N+1}N^N 0\,(|f(a+(n-1)\xi)|, 0, N). \qquad (7)$$

Letting $\lambda=(N+1)2^{N+1}N^N$ and using (7) recursively we have

$$0(|f(a+n\xi)|, 0, N) < \lambda^n 0(|f(a)|, 0, N). \qquad (8)$$

For $|a|<1$, we get

$$0(|f(a+n\xi)|, 0, N) < C\lambda^n,\ C \text{ constant}.$$

Letting $z=a+n\xi$, $r=|z|$ we get

$$|f(z)| < C\lambda^{2r}.$$

Hence, f must be of exponential type.
If (c) is assumed instead of (a) the argument is almost identical to the one above.
If (b) is assumed instead of (a), then letting $M_p(r)$ (or $M_p(r, f)$) denote

$$\left(\frac{1}{2\pi}\int_0^{2\pi} |f(\mathrm{re}^{i\theta})|^p d\theta\right)^{1/p},$$

we have as before

$$M_p(r) < C\lambda^{2r}. \qquad (9)$$

One readily sees, by means of Cauchy's formula that for $R>r>0$

$$M_f(r) \leqslant \left(\frac{R}{R-r}\right)^{1/p} M_p(R)$$

Furthermore it is easy to verify that $M_p(R)$ (see Hardy [1]) is continuous and increases faster than any power of r whenever f is transcendental.

Letting $R=\frac{1}{\log M_p(R)}+r$ and applying a lemma of Bore [2 pp. 374–376], we get for some $\epsilon>0$ that

$$M_f\ (r) < (M_p(r))^{1+\epsilon}$$

outside a set of r of finite measure. One can easily show however that if there exists an infinite sequence r_n such that

$$M_f\ (r_n) > e^{C_n r_n},\ C_n \rightarrow \infty,$$

then there exists a set of r of infinite measure with the same property. Thus $M_f(r) < C\lambda^{2r}$ for sufficiently large r and our proof is complete.

4. The Converse of Theorem 1

In general the converse of Theorem 1 is not true, since one can easily find functions of order zero which do not satisfy (a). For example $f=\prod_{n=0}^{\infty}\left(1-\frac{z}{n^n}\right)^n$. We do, however, have the following related theorem.

THEOREM 2: *A periodic function of exponential type satisfies* (b) *and* (c) *for sufficiently large* r *and* i=0, 1, 2, . . ., *N*.

PROOF: We prove (b). The proof for (c) is similar. Any such function g may be written as

$$g(z)=\sum_{j=1}^{n}\left[a_j(e^{ijz}+e^{-ijz})+b_j(e^{ijz}-e^{-ijz})\right]+C. \tag{10}$$

Without any loss of generality we may assume that $C=0$.
For $g^{(K)}$ we have:

$$g^{(K)}(z)=\sum_{j=1}^{n}(ij)^K\left[a_j(e^{ijz}+(-1)^K e^{-ijz})+b_j(e^{ijz}+(-1)^{K+1}e^{-ijz})\right] \tag{11}$$

Now $\log M_p(r)$ is a convex function of $\log r$ and $M_p(r)$ grows faster than any power of r (see Hardy [1]). Furthermore, one may assume without any loss of generality that $M_p(1)=1$. Using these facts one can easily verify by means of the three circles theorem, that for any constant C and any $\epsilon>0$

$$M_p((1+\epsilon)r) > CM_p(r), \tag{12}$$

for sufficiently large $r > r(C, p, f)$. Let $g_j=a_j(e^{ijz}+e^{-ijz})+b_j(e^{ijz}e^{-ijz}))$ and let $\bar{g}_1=g_n$.

$$\text{Then } M_p(r, g(z)) \geqslant M_p(r, g_n(z))-M_p\left(r, \sum_{j=1}^{n-1} g_j(z)\right).$$

Thus, by (12), for any C, $M_p(r, g_n(z)) > CM_p(r, g_j(z))$ and consequently for any $\delta>0$.

$$M_p(r, g(z)) > (1-\delta)M_p(r, g_n(z))=(1-\delta)M_p(r, \bar{g},\ (z))$$

for sufficiently large r.

Similarly (11) and (12) yield

$$M_p(r, g^{(j)}(z)) \geq \frac{n^j}{j!}(1-\delta)M_p(r, \bar{g}_1)$$

for j even and for j odd

$$M_p(r, g^{(j)}(z)) \geq \frac{n^j}{j!}(1-\delta)M_p(r, \bar{g}_2),$$

where $$\bar{g}_2 = a_n(e^{inz} - e^{-inz}) + b_n(e^{inz} + e^{-inz}).$$

Using these inequalities, one computes for any integer N, which we may assume to be even without any loss of generality,

$$0\left(\int_g, p, 0, N\right) \geq (i-\delta)\left(1+\frac{n^2}{2!}+\frac{n^4}{4!}+\ldots+\frac{n^N}{N!}\right)\left(\int_0^{2\pi} |\bar{g}_1(z)|^p d\theta\right)^{1/p}$$
$$+(1-\delta)\left(n+\frac{n^3}{3!}+\frac{n^5}{5!}+\ldots+\frac{n^{N-1}}{(N-1)!}\right)\left(\int_0^{2\pi} |\bar{g}_2(z)|^p d\theta\right)^{1/p} \tag{13}$$

for sufficiently large $|z| = r$.

On the other hand (12) also yields

$$0\left(\int_g, p, 0, \infty\right) \leq (1+\delta)\left(\frac{n^{N+2}}{(N+2)!}+\frac{n^{N+3}}{(N+3)!}+\ldots\right)\left(\int_0^{2\pi} |\bar{g}_1|^p d\theta\right)^{1/p}$$
$$+(1+\delta)\left(\frac{n^{N+1}}{(N+1)!}+\frac{n^{N+3}}{(N+3)!}+\ldots\right)\left(\int_0^{2\pi} |\bar{g}_2|^p d\theta\right)^{1/p}$$

for sufficiently large $|z| = r$.

Thus, choosing N sufficiently larger than n our conclusion follows for g. It is clear that it follows for $g^{(i)}$, $i = 1, 2, \ldots$, as well with N fixed.

5. References

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