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CIE 1960 UCS Diagram and the Müller Theory of Color Vision

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A close relationship has been shown to exist between the second stage of the Müller theory and the MacAdam 1937 (u, v)-diagram recommended in 1960 by the CIE for interim use as a chromaticity diagram having approximately uniform scales. By considering normal vision as a combination of protanopia and tritanopia as suggested by the Müller second stage, a more general measure of the perceived size of a chromaticity difference is developed than the length of the line connecting the two chromaticity points. The general measure is the square root of the sum of the squares of the angles subtended by the line at the convergence points of the chromaticity confusion lines for protanopia and tritanopia, respectively. By this measure the chromatic sensibility to wavelength change in the spectrum is accounted for quantitatively not only for protanopic and tritanopic vision, but also for normal vision including the secondary maximum of sensibility in the neighborhood of 420 nm.

Key words: Chromaticity; color; perception; protanopia; tritanopia; vision.

1. Introduction

In a paper on Recent Developments of Thomas Young's Color Theory, delivered in 1886 before the British Association in Birmingham, A. König [1] [1] stated that "it ought not to be difficult in the construction of a chromaticity diagram so to modify the adopted arbitrary assumptions that the separation of two points on it would give a measure for the difference in sensation between the colors corresponding to them, ..." In 1932 one of us [2] drew attention to the fact that the (r, g)-chromaticity diagram based on the so-called "OSA excitation curves" yields some approach to uniform chromaticity scales, and in 1935 a Maxwell triangle yielding approximately uniform chromaticity scales (known as the UCS diagram) was defined [3] by reference to the colorimetric coordinate system recommended by the International Commission on Illumination (CIE) [4].

In 1937 MacAdam [5] showed that essentially the same chromaticity spacing could be achieved by plotting the v-chromaticity coordinate against the u-coordinate in a rectangular coordinate system based on the simple transformation of tristimulus values, X, Y, Z in the 1931 CIE colorimetric coordinate system:

$$U=(2/3)X$$
$$V=Y$$
$$W=-0.5X+1.5Y+0.5Z. \qquad (1)$$

[1] Figures in brackets indicate the literature references at the end of this paper.

The V-function, like the Y-function, is equal to the standard spectral luminous efficiency function. The (u, v)-chromaticity coordinates are related to the 1931 CIE tristimulus values, X, Y, Z and to the chromaticity coordinates, x, y in this simple way:

$$u=4X/(X+15Y+3Z)=4x/(-2x+12y+3)$$
$$v=6Y/(X+15Y+3Z)=6y/(-2x+12y+3). \qquad (2)$$

In 1939 Breckenridge and Schaub [6] developed a variation of the UCS diagram in which the equi-energy point was made the origin of a rectangular coordinate system (known as the RUCS diagram), and applied it to the specification of the colors of signal lights (railway, highway, marine, and airplane traffic) [7]. In 1942 Scofield, Judd, and Hunter [8] published the definition of the "alpha-beta" diagram that first, like the RUCS diagram, placed achromatic colors at the origin, second could be approximated closely by simple functions of the readings, A, G, B obtained from a photoelectric tristimulus colorimeter through amber, green and blue tristimulus filters, respectively, and finally gave a chromaticity spacing more in accord with Wright's study [9] of the RUCS diagram than any previously formulated diagram.

In 1945 the Committee on Colorimetry [10] made use of a method devised by MacAdam [11] to derive a rectangular-coordinate UCS diagram having the same spacing as the Judd UCS diagram, and this diagram is included in the final report of the committee [12].

In 1944 Farnsworth [13] described a rectilinear uniform chromaticity scale diagram designed so as to make the Munsell value 5, chroma 10 locus [14] approximate a circle. In 1949 Neugebauer [15] made use of the MacAdam 1937 (u, v)-diagram in a form representing achromatic object colors at the origin by plotting $v-v_0$ against $u-u_0$, where u_0, v_0 are the chromaticity coordinates of the illuminant. This form of UCS diagram has been known as the MacAdam-Neugebauer diagram. In 1959 Sugiyama and Fukuda [16] defined still another UCS diagram by adjusting the spacing to accord maximally with that of the Munsell color system [14] and with the MacAdam ellipses [17].

In 1960, the CIE recommended the 1937 MacAdam (u, v)-diagram [18] for use whenever a projective transformation of the CIE (x, y)-diagram is desired to give chromaticity spacing perceptually more nearly uniform than the (x, y)-diagram itself. In the third edition of the International Lighting Vocabulary, the diagram defined by eq (2) is called the "CIE 1960 UCS diagram."

König's idea of a uniform-chromaticity-scale diagram has thus received wide-spread attention and has come to fruition in the form of an international recommendation after 75 years. It should not be supposed, however, that the separation of two points on the recommended UCS diagram is an exact measure of the perceived size of the difference between the chromaticities represented by the points; this separation is an approximate measure. The degree of approximation, however, has been shown by Wyszecki and Wright [19] to be sufficient for many practical purposes, and this degree is probably close to the highest achievable by projective transformation of the 1931 CIE (x, y)-diagram. Less restricted types of transformation should permit improved agreement with the experimental facts; and plane diagrams for this purpose derived by nonlinear transformations have achieved some success; see, for example, those derived by Sinden [20], Adams [21], Moon and Spencer [22], Farnsworth [23], Glasser et al. [24], Nickerson, Judd, and Wyszecki [25], and Saunderson and Milner [26]. The Munsell renotation system [14] itself is equivalent to a series of diagrams derived by nonlinear transformation of the (x, y)-diagram, one for each level of Munsell value, and if the Euclidian distance element derived by Godlove [27] is used, these are UCS diagrams plottable on planes.

Further refinements of these empirical models not subject to the restriction of being plottable on a plane have been derived; see, for example, Nickerson [28], MacAdam [29], Friele [30], and Chickering [31].

In 1949 one of us [32] derived response functions for the three stages of the Müller theory of color vision and showed how the chromatic response functions of the second stage could be combined with the standard spectral luminous-efficiency function, $\bar{y}(\lambda)$, to account for the minimum perceptible luminance purity determined by Priest and Brickwedde [33], and by Purdy [34]. The two chromatic response functions of the second stage correspond to a yellowish red (yR) process whose negative is bluish green (bG) and a greenish yellow (gY) process whose negative is reddish blue (rB). The failure of the (yR−bG)-process accounts in the Müller theory for protanopia; that of the (gY−rB)-process, for tritanopia. These response functions are defined in terms of the tristimulus values, $\bar{x}(\lambda)$, $\bar{y}(\lambda)$, $\bar{z}(\lambda)$, of the spectrum in the 1931 CIE system [4] by the equations:

$$\begin{aligned} yR(\lambda) &= -bG(\lambda) = 5.741\bar{x}(\lambda) - 4.601\bar{y}(\lambda) - 1.140\bar{z}(\lambda) \\ gY(\lambda) &= -rB(\lambda) = -0.932\bar{x}(\lambda) + 2.751\bar{y}(\lambda) - 1.819\bar{z}(\lambda). \end{aligned} \tag{3}$$

It was shown that the function of wavelength:

$$y(\lambda)/\{[(yR(\lambda)]^2 + [0.5gY(\lambda)]^2\}^{1/2}$$

agreed closely with the luminance purity of colors represented by points on a circle on the UCS diagram centered on the achromatic point. Thus, the same type of information derivable from the UCS triangle can be drawn from the yR and gY functions; but the colorimetric coordinate system proposed for the second stage of the Müller theory [32], and formed of these two chromatic response functions combined with the standard spectral luminous-efficiency function $\bar{y}(\lambda)$ yields a chromaticity diagram with scales departing maximally from perceputal uniformity because the diagram is degenerate. From eq (3) we see that the sum of the three functions chosen:

$$yR(\lambda) + gY(\lambda) + \bar{y}(\lambda) = 4.809\bar{x}(\lambda) - 0.850\bar{y}(\lambda) - 2.959\bar{z}(\lambda)$$

is positive for the long-wave part of the spectrum and negative for the short-wave portion; so for some intermediate wavelength this sum, the denominator of the fraction defining the chromaticity coordinates, is equal to zero, and the corresponding chromaticity point is located indefinitely far from points representing other chromaticities on this diagram.

This paper will show how to fit the yR(λ) and gY(λ) functions into a coordinate system yielding a UCS diagram, and will present a new distance element for the perceptibility of chromaticity differences based on the Müller theory of color vision.

2. Protanopic and Tritanopic Intersection Points as Primaries

Equation (3) defines the chromatic processes of the Müller theory in terms of the tristimulus values, $\bar{x}(\lambda)$, $\bar{y}(\lambda)$, $\bar{z}(\lambda)$, of the spectrum; that is, as functions of wavelength. If a heterogeneous stimulus of spectral distribution $S(\lambda)$ is viewed, the tristimulus values, X, Y, Z, correspond to:

$$X = \int_0^\infty S(\lambda)\bar{x}(\lambda)d\lambda,$$

$$Y = \int_0^\infty S(\lambda)\bar{y}(\lambda)d\lambda,$$

$$Z = \int_0^\infty S(\lambda)\bar{z}(\lambda)d\lambda$$

and it follows from eq (3) that the chromatic processes corresponding to this stimulus may be evaluated as functions of X, Y, Z by substituting in eq (3), X, Y, Z, respectively for $\bar{x}(\lambda)$, $\bar{y}(\lambda)$, $\bar{z}(\lambda)$.

In the second stage of the Müller theory, yR corresponds to the process whose failure reduces the normal trichromatic mechanism to the protanopic variety of dichromatic mechanism, and rB corresponds to the process whose failure produces tritanopia. Luminosity, w-s, corresponds to a direct contribution $L(P_i)$ from the primary processes P_1, P_2, P_3, of the first stage plus contributions associated with the yR and rB chromatic processes. From eq (3a) of the previous paper [32] there may be written:

$$w\text{-}s = 0.0075P_1 + 0.1912P_2 + 0.0013P_3 + 0.0810\text{yR} - 0.0024\text{rB} \qquad (4)$$

whence it follows that the direct contribution $L(P_i)$ to luminosity, w-s, from the primary processes $P_i (i=1, 2, 3,)$ of the first stage is:

$$L(P_i) = 0.0075P_1 + 0.1912P_2 + 0.0013P_3. \qquad (5)$$

The definition of $L(P_i)$ may be written in terms of the tristimulus values, X, Y, Z, of the color in the 1931 CIE system from the definitions of P_i given in eq (1) of the previous paper [32]:

$$\begin{aligned} \mathrm{P}_1 &= 3.1956X + 2.4478Y - 0.6434Z \\ \mathrm{P}_2 &= -2.5455X + 7.0492Y + 0.4963Z \\ \mathrm{P}_3 &= 0.0000X + 0.0000Y + 5.0000Z \end{aligned} \qquad (6)$$

whence:

$$L(P_i) = -0.463X + 1.366Y + 0.097Z. \qquad (7)$$

A colorimetric coordinate system for the second stage, more in keeping with the Müller theory than that previously proposed [32], may be formed by combining $L(P_i)$ with the chromatic processes yR and rB, thus:

$$\begin{aligned} \text{yR} &= 5.741X - 4.601Y - 1.140Z \\ \text{rB} &= 0.932X - 2.751Y + 1.819Z \\ L(P_i) &= -0.463X + 1.366Y + 0.097Z \end{aligned} \qquad (8a)$$

$$\begin{aligned} X &= 0.2391\ \text{yR} + 0.0967\ \text{rB} + L(P_i) \\ Y &= 0.0810\ \text{yR} - 0.0024\ \text{rB} + L(P_i) \\ Z &= 0.4966\ \text{rB} + L(P_i). \end{aligned} \qquad (8b)$$

Consideration of eq (8b) for the reverse transformation will show that the variables, yR, rB, $L(P_i)$, defined in eq (8a), do indeed correspond to the Müller theory. The expression for the tristimulus value Y is seen to correspond to the expression for normal luminosity, w-s, given in eq (4) when the definition of $L(P_i)$ given in eq (5) is taken into account. By setting each pair of columns to zero, it may be shown that the chromaticity point for the yR-primary is at $x=0.2391/(0.2391+0.0810)=0.747$, $y=0.0810/(0.2391+0.0810)=0.253$. This point was used in the previous paper [32] to derive eq (10) of that paper. Similarly it is seen that the chromaticity point for the $L(P_i)$ primary is at $x=y=1/3$, the equi-energy point, which was used to derive eq (7) of that paper. Finally it is seen that the rB primary is at:

$$x = 0.0967/(0.0967 - 0.0024 + 0.4966) = 0.164$$

$$y = -0.0024/(0.0967 - 0.0024 + 0.4966) = -0.004$$

which is close to the point ($x=0.165$, $y=0.000$) used in the derivation of eq (11) of that paper. The reason for this slight discrepancy is that formerly the tritanopic intersection point was taken as the intersection of the line $P_1=P_2$ with the alychne, $y=0$; the present choice places it at the intersection of this line with what might be called the protanopic alychne, that is, the line corresponding to $L(P_i)=0$.

Equation (8) without detailed explanation has already been published [35] except that gY was substituted for its equivalent $-$rB, and the $L(P_i)$-function was labeled W_p for protanopic luminosity. This designation and notation is appropriate for eq (5), as a comparison with W_p from the König theory (see previous eq (14a)) and $(w-s)_p$ from the Müller theory (see previous eq (4a)) shows:

$$\begin{aligned} W_p &= 0.0075P_1 + 0.1902P_2 + 0.0023P_3 \\ L(P_i) &= 0.0075P_1 + 0.1912P_2 + 0.0013P_3 \\ (w-s)_p &= 0.0075P_1 + 0.1921P_2 + 0.0003P_3 \end{aligned}$$

that is, $L(P_i)$ is closely an average of W_p and $(w-s)_p$ both of which have been shown by figure 7 of the previous paper [32] to be satisfactory representatives of protanopic luminosity experimentally determined.

3. UCS Diagram Derived from Müller Second Stage

The chromaticity diagram formed by plotting rB/S_a against yR/S_a, $S_a=\text{yR}+\text{rB}+L(P_i)$, in accord with eq (8a), yields chromaticity scales that are far from uniform. Note that $S_a=6.210X-5.986Y+0.776Z$, although greater than zero near the extremes of the visible spectrum, where Y is much smaller than X or Z, respectively, is less than zero near the middle of the spectrum. It follows that for two wavelengths $S_a(\lambda)$ will be zero, and that the chromaticity diagram is degenerate. To regenerate the diagram, however, it is necessary only to change the scale of $L(P_i)$ by multiplying by any constant larger than that required to make the minimum value of $S(\lambda)$ in the middle of the spectrum be precisely zero. If this factor is very large, the point representing the equi-energy point on the resulting chromaticity diagram will fall indefinitely close to the spectrum locus. We have found that the factor 11.70 causes the equi-energy point to fall between the point for 570 nm (greenish yellow) and that for 430 nm (violet) with closely the same spacing as that of the CIE 1960 UCS diagram; that is, the distance to the 570 nm point is about 0.22 times that to the point for 430 nm. The definition of the coordinate system with the regenerated and adjusted chromaticity diagram is simply:

$$\begin{aligned} \mathrm{yR} &= 5.741X - 4.601Y - 1.140Z \\ \mathrm{rB} &= 0.932X - 2.751Y + 1.819Z \\ 11.7L(P_i) &= -5.417X + 15.982Y + 1.135Z \end{aligned} \tag{8c}$$

$$\begin{aligned} X &= 0.2391\,\mathrm{yR} - 0.0967\,\mathrm{rB} + 0.0855[11.7L(P_i)] \\ Y &= 0.0810\,\mathrm{yR} - 0.0024\,\mathrm{rB} + 0.0855[11.7L(P_i)] \\ Z &= \qquad\qquad\quad 0.4966\,\mathrm{rB} + 0.0855[11.7L(P_i)]. \end{aligned} \tag{8d}$$

All statements previously made regarding eqs (8a) and (8b) apply to eqs (8c) and (8d). The sum $S = \mathrm{yR} + \mathrm{rB} + 11.7L(P_i)$, however, is obviously greater than zero for all parts of the spectrum because first $\bar{x}(\lambda)$, $\bar{y}(\lambda)$, $\bar{z}(\lambda)$ are nowhere in the spectrum less than zero, and second this sum S is seen to be equal to: $1.256X + 8.630Y + 1.814Z$, an expression that has coefficients all greater than zero. By plotting rB/S in rectangular coordinates against yR/S we obtain a chromaticity diagram in which the equi-energy point is at the origin (0, 0) of the coordinate system; the protanopic convergence point, (1, 0); and the tritanopic convergence point, (0, 1). This diagram already bears considerable resemblance to the CIE 1960 UCS diagram because that diagram shows the line connecting the equi-energy point with the protanopic convergence point to be approximately perpendicular to the line connecting the equi-energy point with the tritanopic convergence point. The length of the latter line, however, is only about three-fourths that of the former. To make the (yR/S, rB/S)-diagram agree in spacing more closely with the CIE 1960 UCS diagram, we have plotted 0.42yR/S as abscissa with -0.32rB/S as ordinate; see large circles on figure 1. The solid dots refer to the spectrum locus of the Neugebauer [15] form of the CIE 1960 UCS diagram obtained by plotting $u - u_0$ as abscissa and $v - v_0$ as ordinate, where u_0, v_0 are the chromaticity coordinates (0.2105, 0.3158) of the equi-energy point.

It will be noted that the degree of difference in spacing of the two diagrams is not large enough to have any practical significance. The diagrams have the equi-energy point in common. The diagram yielded by eq (8c) in accord with the second stage of the Müller theory has the protanopic convergence point (0.42, 0.00), and the tritanopic convergence point (0.00, -0.32); the corresponding points are shown as solid dots for the CIE 1960 UCS diagram. It is concluded that the CIE UCS diagram, developed empirically, yields spacing in accord with the second stage of the Müller theory of color vision. The accommodation of the CIE 1960 UCS diagram within the second stage

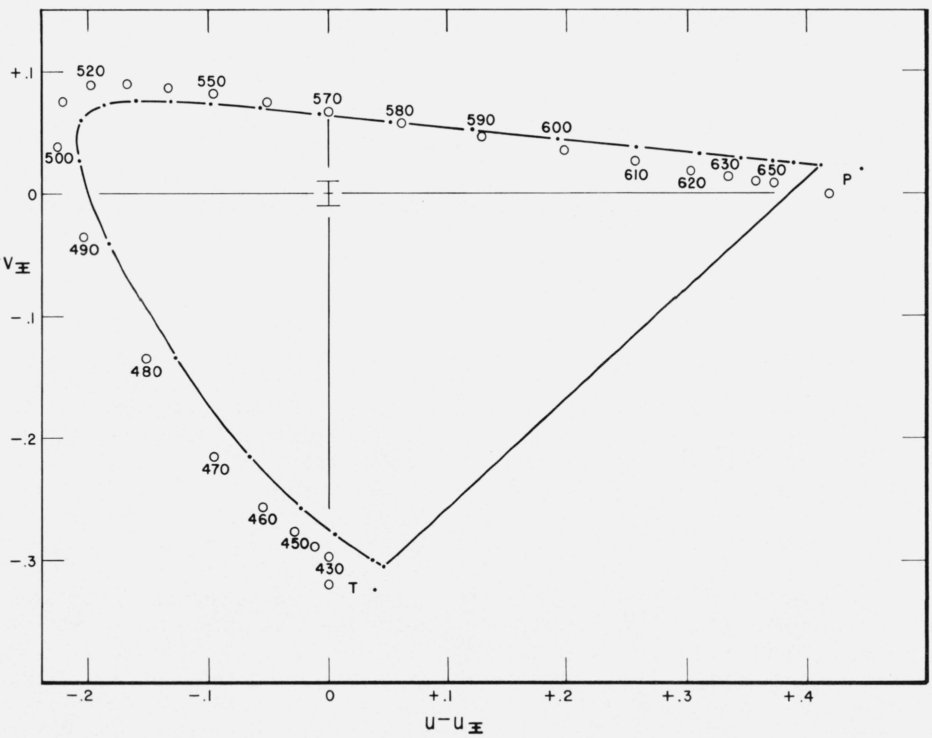

FIGURE 1. *The chromaticity diagram derived in accord with the second stage of the Müller theory (0.42yR/S and −0.32rB/S from eq (8c) plotted in rectangular coordinates) compared with the CIE 1960 UCS diagram in the Neugebauer [15] form obtained by plotting the* (u, v)*-differences from the equi-energy point.*

Note that the spectrum locus on the former diagram (large circles) has a spacing not significantly different from that on the CIE 1960 UCS diagram (dots joined by solid line).

suggests that for the Müller theory the processes determining discrimination of chromatic differences occur in the retina; the third stage must be thought of as the process by which the discriminations made in the retina are recoded for transmission along the optic nerve with no loss of information.

4. Proposed Measure of Perceived Size of Chromatic Differences for Dichromats

If the length of the line connecting two points on a uniform chromaticity diagram is an approximate measure of the size of the chromatic difference perceived for an observer with normal color vision between the chromaticities represented by the two points, what measure is appropriate for dichromatic vision? To a dichromat, the chromaticities represented by any point on any straight line passing through the dichromatic convergence point are all identical. The component of the vector on the UCS diagram along such lines contributes nothing to dichromatic discrimination; only the component orthogonal to the direction of the chromaticity-confusion line counts. It is natural, therefore, to take the angular separation of the confusion lines containing the chromaticity points as this measure; that is, if we have two chromaticities specified by chromaticity coordinates u_1, v_1, and u_2, v_2 on the UCS diagram, the perceived size ΔC_d of the chromaticity difference for a dichromat whose convergence point is at u_d, v_d is:

$$\Delta C_d = K_d \left\{ \tan^{-1}\left[\frac{v_1 - v_d}{u_1 - u_d}\right] - \tan^{-1}\left[\frac{v_2 - v_d}{u_2 - u_d}\right] \right\}. \quad (9)$$

The small open circles connected by dot-dash lines on figure 2 show the wavelength difference just detectable in protanopic vision computed by this measure for $K_d = 0.81$ with the angles expressed in degrees, and for $u_d = 0.658$, $v_d = 0.334$. This choice of $K_d = 0.81$ is derived from the assumption that a change in the direction of the chromaticity confusion line by 1.23° is required for a protanope to detect the chromaticity difference [36]. Note that the wavelength-sensibility curve predicted by this measure for protanopic vision shows a pronounced minimum of about 1.2 nm at about 481 nm and rises to 10 nm both at 434 and 517 nm.

The double circles connected by dashed lines on figure 2 show analogous predictions of the wavelength difference just detectable in tritanopic vision computed from eq (9) also for $K_d = 0.81$. The tritanopic intersection point has been taken as required by the Müller theory at $x = 0.164$, $y = -0.004$ by setting $u_d = 0.249$, $v_d = -0.009$. Note that this predicted curve of tritanopic sensibility to wavelength difference among spectrum colors is much more complicated than that for protanopic sensibility. It has three

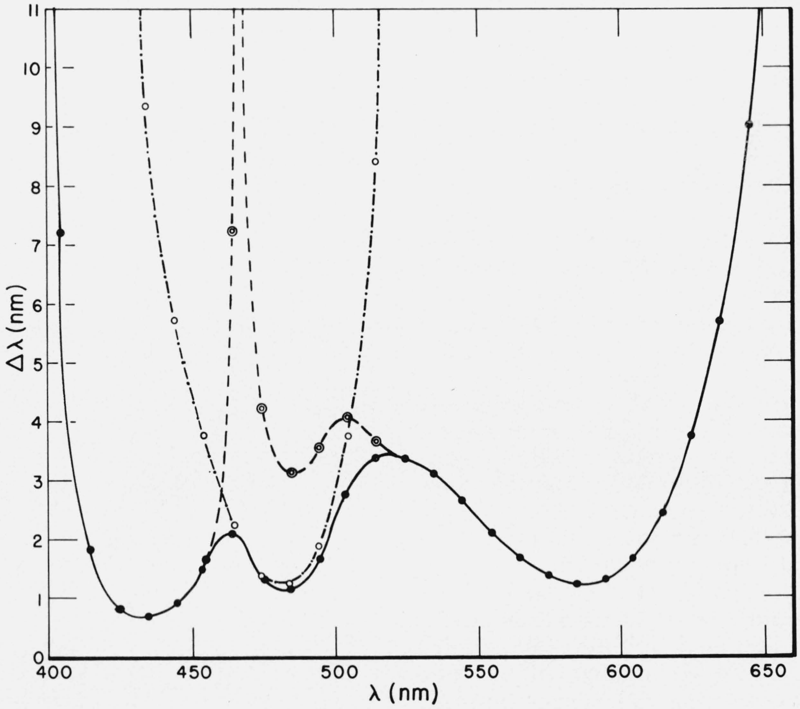

FIGURE 2. *Wavelength difference just detectable by chromaticity discrimination according to the Müller theory for protanopic (circles connected by dot-dash line), tritanopic (double circles connected by dashed line), and normal (solid circles connected by solid line) vision as a function of wavelength; see eqs (9) and (10),* $K_d = 0.81$.

Note that tritanopic wavelength discrimination by this theory differs significantly from normal wavelength discrimination only between 450 and 520 nm, the region where protanopic discrimination is relatively good.

minima (430, 487, and 585 nm), one sharp maximum (467 nm) and one relative maximum (504 nm) as well as two indefinitely high maxima at the spectrum extremes. We might say that the computations indicate "good discrimination in the yellow region of the spectrum, discontinuity in the blue-green wavelengths where wavelength discrimination virtually disappears, and, most striking of all, very keen wavelength discrimination in the far violet." These are, however, not our words, but those of Wright [37] describing wavelength discrimination found experimentally by him for four tritanopes. Detailed quantitative comparison will be shown presently.

5. Proposed General Measure of Perceived Size of Chromatic Differences

Three main types of dichromatic vision, all reduced forms of normal trichromatic vision, are generally recognized and dealt with theoretically both in the three-components theory, and in the Müller stage theory. These are protanopia, deuteranopia, and tritanopia. A fourth type of reduced color vision, tetartanopia, has mainly theoretical existence. It corresponds to failure of the yellow-blue process of the Hering opponent-colors theory which is the same as the third stage of the Müller theory. No uncomplicated case of tetartanopia has ever been reported.

In the Müller second stage, normal vision may be regarded as the result of combining the protanopic discriminative ability with that of the tritanope. The Müller second stage thus suggests the following general measure of perceived size of chromatic differences:

$$\Delta C = [(K_p \Delta\theta_p)^2 + (K_t \Delta\theta_t)^2]^{1/2} \qquad (10)$$

where $K_p\Delta\theta_p$ is eq (9) applied to protanopic vision, and $K_t\Delta\theta_t$ is eq (9) applied to tritanopic vision. The assumption on which this proposal is based is that whatever the protanope can discriminate, the normal observer, endowed with both protanopic and tritanopic discriminative ability, can see better. The solid circles connected by a solid curve on figure 2 correspond to a combination of the two dotted curves in accord with eq (10). It will be noted that, by this view, tritanopic vision accounts for nearly all of the ability of the normal observer to discriminate wavelength by chromaticity difference in the spectrum up to 430 nm at the short-wave end and from 520 nm on in the long-wave end; only between 430 and 520 nm does protanopic discriminative ability contribute appreciably.

The form of eq (10), square-root of the sum of two squares, is a usual assumption. It was made in eq (24) of the previous paper [32], and by Stiles [38], Sinden [20], Schrödinger [39], and Helmholtz [40] in their treatments of sensibility to color differences from the standpoint of the three-components theory. This assumption applied to angular measures becomes identical to that on which the UCS diagram is based for the special case of a projective transformation in which both protanopic and tritanopic convergence points are projected to infinity. Wyszecki [41] derived such a projective transformation, but the resulting chromaticity diagram proved to be spectacularly different from a UCS diagram.

The argument by which eq (10) is proposed may seem somewhat circular. The CIE 1960 UCS diagram is used to justify the measure for dichromatic discriminations given by eq (9); then a new measure, eq (10), based on eq (9) is proposed that implies that distance on the UCS diagram does not really accord precisely with the perceived size of chromaticity difference after all. The justification of this circular argument lies in whether a more successful model of chromatic discrimination is thereby discovered. The model need not necessarily be that suggested by the Müller second stage; it might be that suggested by the Müller third stage, a combination of deuteranopia and tetartanopia; it might be a combination of all three established forms of dichromatic vision as in the three-components theory; or it might be based on a view, suggested by Walraven [42] that normal color vision be considered a combination of tritanopia with deuteranopia.

6. Wavelength Discrimination by Theory and by Experiment

Figure 3 compares the wavelength difference just detectable in protanopic vision computed from eq (9) with wavelength discrimination curves of six protanopes studied by Pitt [43]. The same values of u_d, v_d used for

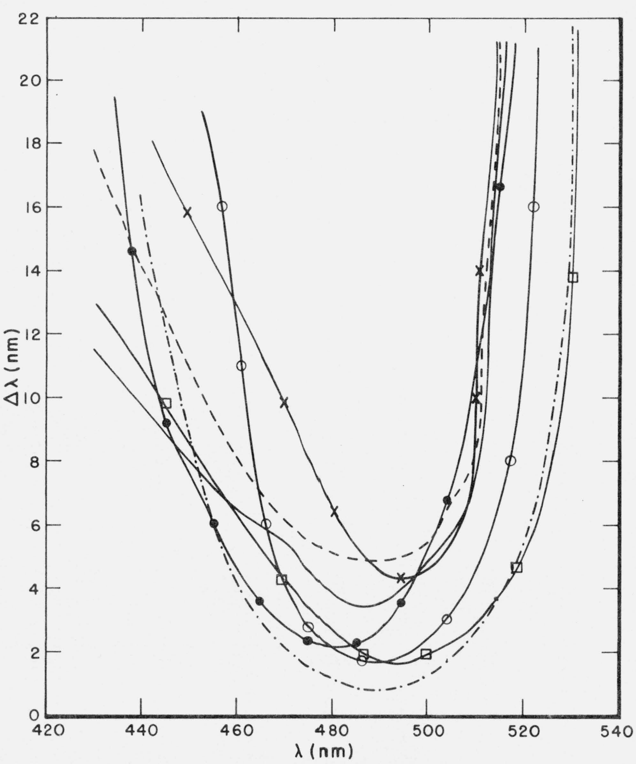

FIGURE 3. *Wavelength difference just detectable by chromaticity discrimination in protanopic vision.*

Solid dots connected by a solid line refer to values computed from eq (9) for $K_d = 0.47$; all other curves refer to values found experimentally by Pitt [41] for six protanopes.

figure 2 were used again, but K_d was set at 0.47. This choice of K_d corresponds to detectability of an angular change of 2.13° for protanopic vision. The agreement is seen to be good in the sense that the predictions are generally within the spread among the six observers, but not perfect. The computed curve has a minimum at about 481 nm; the minima of the experimental curves range between 486 and 496 nm.

Figure 4 compares the wavelength difference just detectable in tritanopic vision computed from eq (9) with wavelength discrimination curves of four tritanopes studied by Wright [36]. K_d was set at 0.24 (corresponding to detectability of a 4.17° change for tritanopic vision), but the tritanopic convergence point implied by combining the Müller theory with the CIE standard observer ($x=0.164$, $y=-0.004$) was not used. Instead we have taken advantage of the fact that Wright [36] determined the color-matching functions of seven tritanopes including the four whose wavelength discrimination curves are shown in figure 4. By using the method previously used by Thomson and Wright [44] to evaluate the tritanopic convergence point within a colorimetric coordinate system tentatively proposed for the consideration of the CIE Committee on Colorimetry, we have found it in the standard CIE coordinate system [4] to be $x=0.171$, $y=-0.001_5$ with an uncertainty less than 0.002 at the 0.1 of 1% level of confidence. These values are rather close to those (0.164, −0.004) yielded by the Müller theory applied to that system, and are still closer to those (0.171, 0.000) given by Wyszecki and Stiles [45] without indicating the source. It will be noted by comparing figure 4 with figure 2 that the small change in tritanopic convergence point has caused the minimum in the violet region of the spectrum to be higher relative to that at 585 nm by about a factor of two and has changed the wavelength position from about 435 to 425 nm. The agreement with the experimentally determined wavelength differences detected by the four tritanopes (indicated by plotted points) is nearly perfect. The predicted curve falls outside the observer spread only for the spectral range 440 to 470 nm.

Figure 5 compares the wavelength difference just detectable in normal vision computed from eq (10) for $K_p=K_t=0.47$ with the wavelength discrimination curves found by Bedford and Wyszecki [46] for observing fields subtending 1° with luminance held constant throughout the spectrum. Note that the secondary minimum in the far violet, previously explained only by the Stiles [37] three-components theory, is more simply and precisely accounted for by eq (10), though it would seem that reasonable modifications of the Stiles theory would permit a similar accounting within the framework of the three-components theory. We think, however, that this degree of agreement, combined with the simplicity of the theory warrants comparison of the model formulated by eq (10) with other types of chromaticity discrimination data.

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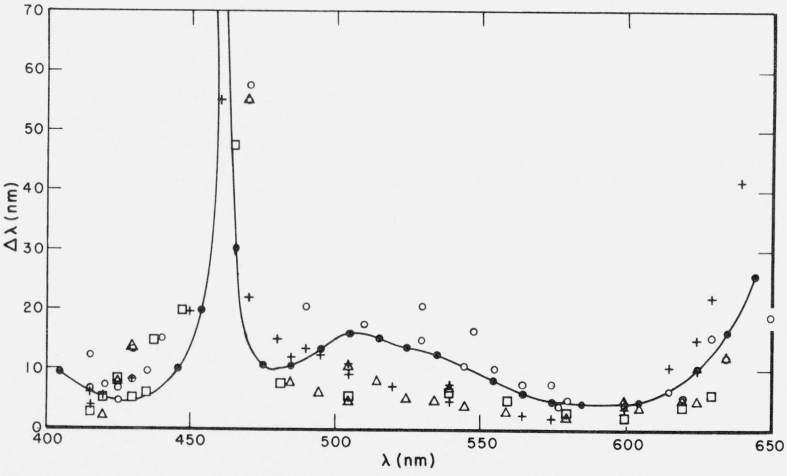

FIGURE 4. *Wavelength difference just detectable by chromaticity discrimination in tritanopic vision.*

Solid dots connected by a solid line refer to values computed from eq (9) for $K_d=0.24$; the plotted points refer to values found experimentally by Wright [36] for four tritanopes.

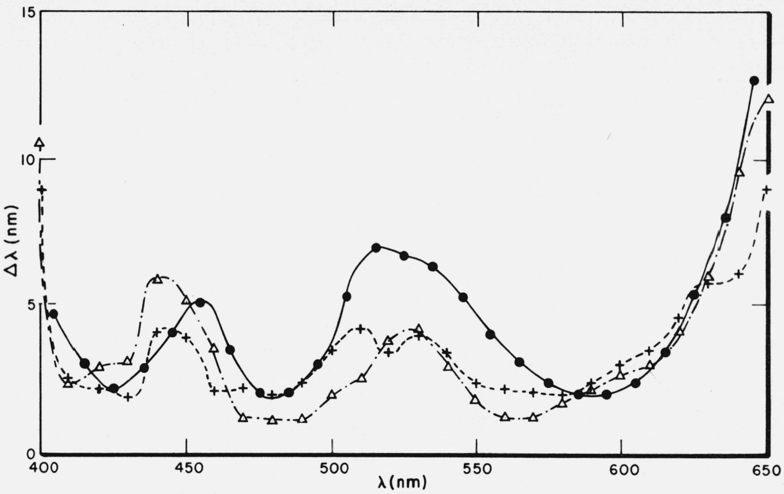

FIGURE 5. *Wavelength difference just detectable by chromaticity discrimination in normal vision.*

Solid dots connected by a solid line refer to values computed from eq (10) for $K_p = K_t = 0.47$; the other two curves refer to values obtained by Bedford and Wyszecki [44] for observation of a 1°-field by observers (themselves) having normal vision.

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