Model Theory

Contents

1 Predicate calculus

1.1 Languages and structures

Definition 1 (First-order language)**.** A first-order language $\mathcal{L}$ is defined by

- The logical symbols
  - Variables $\mathcal{V}$
  - Equality $\doteq$
  - Logical connectives $\neg$, $\wedge$
  - Existential quantifier $\exists$
- The signature $\sigma^{\mathcal{L}}$ of $\mathcal{L}$:
  - Constants $\mathcal{C}^{\mathcal{L}}$
  - Function symbols $\mathcal{F}^{\mathcal{L}}$
  - Function relations $\mathcal{R}^{\mathcal{L}}$

See [Tent and Ziegler, 2012, Definition 1.1.1].

Example 1.

- $\mathcal{L}_{\emptyset} \triangleq \emptyset$
- $\mathcal{L}_{\mathrm{Mon}} \triangleq \{e, \cdot\}$
- $\mathcal{L}_{\mathrm{Gp}} \triangleq \{e, \cdot, -^{-1}\}$
- $\mathcal{L}_{\mathrm{Ord}} \triangleq \{<\}$
- $\mathcal{L}_{\mathrm{Ens}} \triangleq \{\in\}$
- $\mathcal{L}_{\mathrm{Ring}} \triangleq \{0, 1, +, \times, -\}$

Definition 2 (Sublanguage)**.** A subset κ of $\sigma^{\mathcal{L}}$ defines a *sublanguage* of $\mathcal{L}$.

Example 2. $\mathcal{L}_{\mathrm{Mon}}$ is a sublanguage of $\mathcal{L}_{\mathrm{Gp}}$.

Definition 3 ($\mathcal{L}$-structure)**.** An $\mathcal{L}$-structure $\mathcal{A}$ is composed of

- A non-empty set A, called the *universe* of $\mathcal{A}$
- for every $c \in \mathcal{C}^{\mathcal{L}}$, a $c^{\mathcal{A}} \in A$
- for every $f \in \mathcal{F}^{\mathcal{L}}$, a $f^{\mathcal{A}} \in A^n \to A$
- for every $R \in \mathcal{R}^{\mathcal{L}}$, a $R^{\mathcal{A}} \in A^n$

We write $\mathcal{A} \triangleq \langle A, \left(z^{\mathcal{A}}\right)_{z \in \sigma^{\mathcal{L}}} \rangle$.

See [Tent and Ziegler, 2012, Definition 1.1.2].

Example 3.

- $\langle \mathbb{Z}, 0, + \rangle$ is an $\mathcal{L}_{\text{Mon}}$-structure
- $\langle \mathbb{N}, 0, + \rangle$ is an $\mathcal{L}_{\text{Mon}}$-structure
- $\langle \mathbb{Z}, 0, +, \lambda n. - n \rangle$ is an $\mathcal{L}_{\text{Gp}}$-structure
- $\langle \mathbb{N}, < \rangle$ is an $\mathcal{L}_{\text{Ord}}$-structure
- $\langle \mathbb{Z}, < \rangle$ is an $\mathcal{L}_{\text{Ord}}$-structure
- $\langle \mathbb{N}, < \rangle$ is an $\mathcal{L}_{\text{Ens}}$-structure
- $\langle \text{Sets}_{\text{Fin}}, \in \rangle$ is an $\mathcal{L}_{\text{Ens}}$-structure
- $\langle \text{Sets}, \in \rangle$ is an $\mathcal{L}_{\text{Ens}}$-structure
- $\langle Pow(S), \emptyset, S, \Delta, \cap, \lambda x.S - x \rangle$ is an $\mathcal{L}_{\text{Ring}}$-structure (a Boolean ring)

1.1.1 Homomorphisms of structures

Let $\mathcal{M}, \mathcal{N}$ be two $\mathcal{L}$-structures.

Definition 4 ($\mathcal{L}$-homomorphism)**.** $F : M \to N$ is an *$\mathcal{L}$-homomorphism* if

- for every $c \in \mathcal{C}^{\mathcal{L}}$, $F(c^{\mathcal{M}}) = c^{\mathcal{N}}$;
- for every $f \in \mathcal{F}^{\mathcal{L}}$, for every $\overline{a} \in M$, $F(f^{\mathcal{M}}(\overline{a})) = f^{\mathcal{N}}(F(\overline{a}))$;
- for every $R \in \mathcal{R}^{\mathcal{L}}$, for every $\overline{a} \in M$, if $\overline{a} \in R^{\mathcal{M}}$ then $F(\overline{a}) \in R^{\mathcal{N}}$.

See [Tent and Ziegler, 2012, Definition 1.1.3].

Example 4. $\lambda A.\text{card}(A) : \text{Sets}_{\text{Fin}} \to \mathbb{N}$ is an $\mathcal{L}_{\text{Ens}}$-homomorphism

Definition 5 (Embedding)**.** $F : M \to N$ is an *embedding*, written $F : \mathcal{M} \hookrightarrow \mathcal{N}$, if F is injective and

- for every $c \in \mathcal{C}^{\mathcal{L}}$, $F(c^{\mathcal{M}}) = c^{\mathcal{N}}$;
- for every $f \in \mathcal{F}^{\mathcal{L}}$, for every $\overline{a} \in M$, $F(f^{\mathcal{M}}(\overline{a})) = f^{\mathcal{N}}(F(\overline{a}))$;
- for every $R \in \mathcal{R}^{\mathcal{L}}$, for every $\overline{a} \in M$, $\overline{a} \in R^{\mathcal{M}}$ iff $F(\overline{a}) \in R^{\mathcal{N}}$.

Example 5. $\mathbb{N} \hookrightarrow \mathbb{Z}$ is an $\mathcal{L}_{\text{Ord}}$-embedding.

Definition 6 (Isomorphism)**.** An *isomorphism* of structures, written $\mathcal{M} \cong \mathcal{N}$, is a surjective embedding, *i.e.* a bijection $F : M \to N$ that defines an embedding $F : \mathcal{M} \hookrightarrow \mathcal{N}$.

See [Tent and Ziegler, 2012, Isomorphism (p.2)].

Example 6. $\mathbb{N} - \{0\} \cong \mathbb{N}$, by the embedding $\lambda n.n - 1 : \mathbb{N} - \{0\} \hookrightarrow \mathbb{N}$.

1.1.2 Reduct and expansion

Definition 7 (Reduct)**.** Let $\mathcal{K}$ be a sublanguage of $\mathcal{L}$. Let $\mathcal{A}$ be an $\mathcal{L}$-structure.

The *reduct* of $\mathcal{A}$ to $\mathcal{K}$, written $\mathcal{A} \upharpoonright \mathcal{K}$, is defined by forgetting the interpretation of the symbols from $L - K$:

$$\mathcal{A} \upharpoonright \mathcal{K} \triangleq \langle A, (z^{\mathcal{A}})_{z \in \sigma^{\mathcal{K}}} \rangle$$

Example 7. $\mathbb{Z}$ is an $\mathcal{L}_{\mathrm{Gp}}$-structure. Its reduct $\mathbb{Z} \upharpoonright \mathcal{L}_{\mathrm{Mon}}$ is an $\mathcal{L}_{\mathrm{Mon}}$-structure.

Definition 8 (Expansion)**.** Conversely, $\mathcal{A}$ is an *expansion* of $\mathcal{A} \upharpoonright \mathcal{K}$.

Definition 9 (Expansion by a subset)**.** Let $\mathcal{A}$ be an $\mathcal{L}$-structure. Let B be a subset of A.

We extend the language $\mathcal{L}$ with every element of B as a new constant:

$$\mathcal{L}(B) \triangleq L \cup B$$

and define the expansion of $\mathcal{L}$ by B, written $\mathcal{A}_B$, with

$$\mathcal{A}_B \triangleq \langle A, (z^{\mathcal{A}})_{z \in \sigma^{\mathcal{L}}} \cup (b)_{b \in B} \rangle$$

Example 8. Find example.

1.2 Terms and formulas

Definition 10 ($\mathcal{L}$-terms)**.** The set of $\mathcal{L}$-terms $\mathcal{T}^{\mathcal{L}}$ is built according to the following rules:

- Every constant in $\mathcal{C}^{\mathcal{L}}$ is an $\mathcal{L}$-term.
- For every $f \in \mathcal{F}^{\mathcal{L}}$, and every $\mathcal{L}$-terms $\bar{t}$, $f(\bar{t})$ is an $\mathcal{L}$-term.

Definition 11 (Height of a term)**.** Written $\mathit{height}(-)$, denotes the number of occurence of function symbols

See [Tent and Ziegler, 2012, Complexity (p.6)].

Definition 12 (Atomic $\mathcal{L}$-formula)**.** formulas built from terms and relations on terms

Definition 13 ($\mathcal{L}$-formulas)**.** Written $\mathit{Fml}^{\mathcal{L}}$, formulas built from atomic formulas and logical operators.

Definition 14 (Height of a formula)**.** Written $\mathit{height}(-)$, denotes the number of logical operators.

Definition 15 (Free occurrence)**.** Variable that is not used at all in a formula

Definition 16 (Bound occurrence)**.** Opposite of free

Definition 17 (Free variables)**.** Written $\mathit{Free}(\varphi)$, variables occurring free

Definition 18 (Sentence)**.** A formula φ is called a *sentence* if it has not free variable, *i.e.* $\mathit{Free}(\varphi) = \emptyset$.

Definition 19 (Theory)**.** A *theory* T is a set of $\mathcal{L}$-sentences.

1.3 Semantics

Let $\mathcal{A}$ be an $\mathcal{L}$-structure.

Definition 20 (Assignment)**.** Function $\alpha : \mathcal{V} \to A$

Definition 21 (Intepretation)**.** Let α be an assignment.
Written $t^{\mathcal{L}}[\alpha]$, interprets the term in A.

Definition 22 (Satisfiability)**.** Written $\mathcal{A} \models t[\alpha]$, translates a formula into the structure $\mathcal{A}$.

Definition 23 (Satisfiability of a sentence)**.** $\mathcal{A}$ satisfies a sentence φ if $\mathcal{A} \models \varphi[\epsilon]$, written $\mathcal{A} \models \varphi$ for short.

Definition 24 (Model)**.** $\mathcal{A}$ is a model of T, written $\mathcal{A} \models T$, if for every $\varphi \in T$, we have $\mathcal{A} \models \varphi$.

1.4 Substitution

Definition 25 (Simultaneous substitution)**.** Written $t_{\overline{s}/\overline{v}}$, substitutes $\overline{s}$ for $\overline{v}$ in t.

1.5 Universally valid formulas

Definition 26 (Universally valid formula)**.** Written $\models \varphi$, a formula that is satisfiable in all $\mathcal{L}$-structures, for all valuations

Definition 27 (Universal closure)**.** Universally quantify the free variables of a formula: just as universally valid.

Definition 28 (Logical consequence)**.** Let T be a $\mathcal{L}$-theory. Let φ be a $\mathcal{L}$-sentence.
φ is a *(logical) consequence* of T, written $T \models \varphi$, if for every $\mathcal{L}$-structure $\mathcal{A}$, $\mathcal{A} \models T$ implies $\mathcal{A} \models \varphi$.

See [?, p.41].

1.6 Formal proofs and Gödel completeness theorem

Axioms: Tautologies (decidable by truth tables)

Equality:

$\forall x. x \doteq x$	(Sym)
$\forall xy. x \doteq y \to y \doteq x$	(Refl)
$\forall xyz. x \doteq y \to y \doteq z \to x \doteq z$	(Trans)
$\forall \overline{x}\overline{y}. \overline{x} \doteq \overline{y} \to f(\overline{x}) \doteq f(\overline{y})$	(Cong-Fun)
$\forall \overline{x}\overline{y}. \overline{x} \doteq \overline{y} \to R(\overline{x}) \leftrightarrow R(\overline{y})$	(Cong-Rel)

Existential: $\varphi_{t/x} \to \exists x. \varphi$ ($\exists$-Ax)

Inference rules: $\dfrac{\varphi \quad \varphi \to \psi}{\psi}$ (MP)

$$\frac{\varphi \to \psi}{\exists x.\varphi \to \psi}\ (x \notin \mathit{Free}(\varphi)) \qquad (\exists\text{-Intro})$$

Definition 29 (Formal proof)**.** Tree built from axioms or inferences.

Definition 30 (Provability)**.** Written $T \vdash_{\mathcal{L}} \varphi$, exists a formal proof of φ in T

Theorem 1 (Completeness (Gödel))**.** *Let T be an $\mathcal{L}$-theory. Let φ be an $\mathcal{L}$-formula. We have:*

$$T \models \varphi \text{ iff } T \vdash_{\mathcal{L}} \varphi$$

Let T be an $\mathcal{L}$-theory.

Definition 31 (Inconsistent theory)**.** T is *inconsistent* if there exists an $\mathcal{L}$-sentence φ such that

$$T \vdash_{\mathcal{L}} \varphi \text{ and } T \vdash_{\mathcal{L}} \neg\varphi$$

Definition 32 (Consistent theory)**.** T is *consistent* if it is not inconsistent.

Definition 33 (Complete theory)**.** T is complete if it is consistent and for every sentence φ, we either have $T \vdash \varphi$ or $T \vdash \neg\varphi$.

Remark 1 (Theory of a structure). Let $\mathcal{A}$ be an $\mathcal{L}$-structure.
The theory $Th(A) \triangleq \{\varphi \mid \mathcal{A} \models \varphi\}$ is a complete theory.

Theorem 2 (Existence of a model)**.** *Let T be a consistent theory. T necessarily has a model.*

Definition 34 (Henkin theory)**.** Let $\mathcal{L}$ be a language. Let $C \subseteq \mathcal{C}^{\mathcal{L}}$ be a set of constants.
An $\mathcal{L}$-theory T is called a *Henkin theory* (or is said to *contain (Henkin) witnesses* in C) if for all $\mathcal{L}$-formula $\varphi(x)$, there exists $c \in C$ such that $(\exists x.\varphi \to \varphi_{c/x}) \in T$

Remark 2. We are asking for the witness to be part of the theory (not deducible, for example), that's quite strong, isn't it?

2 Model Theory

2.1 Fundemental results

Theorem 3 (Compactness theorem)**.** *A theory T has a model if every finite subset of T has a model.*

See [Tent and Ziegler, 2012, §2.2].

Theorem 4 (Semantic characterisations of homomorphisms and embeddings)**.** *Let $\mathcal{M}$ and $\mathcal{N}$ be two $\mathcal{L}$-structures. Let $F : M \to N$ be a function on their universes.*

1. *f is a homomorphism iff for every atomic $\mathcal{L}$-formula $\varphi(\overline{x})$ and every tuple $\overline{a} : M$, we have*

$$\mathcal{M} \models \varphi\,[\overline{a}] \Rightarrow \mathcal{N} \models \varphi\,[F\,\overline{a}]$$

2. *f is an embedding iff for every atomic $\mathcal{L}$-formula $\varphi(\overline{x})$ and every tuple $\overline{a} : M$, we have*

$$\mathcal{M} \models \varphi\,[\overline{a}] \Leftrightarrow \mathcal{N} \models \varphi\,[F\,\overline{a}]$$

*See [**?**, Theorem 1.3.1].*

Definition 35 (Elementary embedding)**.** $F : M \to N$ is an *elementary embedding*, written $F : \mathcal{M} \overset{\preccurlyeq}{\hookrightarrow} \mathcal{N}$, if for every $\mathcal{L}$-formulas $\varphi(\overline{x})$, for every $\overline{a} \in M$, we have

$$\mathcal{M} \models \varphi\,[\overline{a}] \Leftrightarrow \mathcal{N} \models \varphi\,[F\,\overline{a}]$$

Remark 3. An elementary embedding is, in particular, an embedding (by Theorem 4 (2))

Example 9. Find example? *See [**?**, Exercise 2.5.1].*

Definition 36 (Elementary equivalence)**.** $\mathcal{M}$ and $\mathcal{N}$ are *elementarily equivalent*, written $\mathcal{M} \equiv \mathcal{N}$, if $Th(\mathcal{M}) = Th(\mathcal{N})$.

See Remark 1.

Lemma 1. *If $\mathcal{M} \cong \mathcal{N}$, then $\mathcal{M} \equiv \mathcal{N}$.*

See Definition 6.

Proof. Using the bijection on the universes M and N, valid formulas in $Th(\mathcal{M})$ translates to (valid) formulas in $Th(\mathcal{N})$, and vice versa. □

Example 10. By the above lemma, we have: $\mathbb{N} - \{0\} \equiv \mathbb{N}$. For instance, to the formula $\varphi \triangleq \exists x, suc x \doteq 1$ satisfying $\mathbb{N} \models \varphi$ corresponds $\psi \triangleq \exists x, suc x \doteq 2$ satisfying $\mathbb{N} - \{0\} \models \psi$.

Definition 37 ((Elementary) substructure)**.** $\mathcal{M}$ is an (elementary) substructure of $\mathcal{N}$, written $\mathcal{M} \subseteq \mathcal{N}$ ($\mathcal{M} \preccurlyeq \mathcal{N}$), if $M \subseteq N$ and the inclusion is an (elementary) embedding.

Example 11. $\mathbb{N} - \{0\}$ is a substructure of $\mathbb{N}$, which is *not* elementary: the formula $\exists x, suc\ x \doteq 1$ is satisfied in the latter, but not in the former.

Definition 38 (Extension)**.** Conversely, $\mathcal{N}$ is an extension of $\mathcal{M}$ if $\mathcal{M}$ is a substructure of $\mathcal{N}$.

Lemma 2. *If $\mathcal{M} \subseteq \mathcal{N}$, then $\mathcal{M} \equiv \mathcal{N}$.*

Proof. □

Theorem 5 (Tarski-Vaught's test)**.** *Let $\mathcal{M}$ be an $\mathcal{L}$-structure. Let A be a subset of M.*
A is the universe of an elementary substructure of $\mathcal{M}$ iff for all $\mathcal{L}(A)$-formula $\varphi(x)$, if $\mathcal{M}_A \models \exists x.\varphi$, then there exists $a \in A$ such that $\mathcal{M}_A \models \varphi\,[a]$.

Proof. Fix and finish!

⟨1⟩1. ASSUME: 1. A is the universe of an elementary substructure of $\mathcal{M}$
2. $\varphi(x)$ is an $\mathcal{L}(A)$-formula
3. $\mathcal{M}_A \models \exists x.\varphi$

PROVE: There exists $a \in A$ such that $\mathcal{M}_A \models \varphi\,[a]$

⟨2⟩1. $\mathcal{A} \models \exists x.\varphi$

PROOF: By Assumption (3) and the fact that $\mathcal{A} \preccurlyeq \mathcal{M}$

⟨2⟩2. there exists $a \in A$ such that $\mathcal{A} \models \varphi\,[a]$

PROOF: By ⟨2⟩1 and definition of $\models$

⟨2⟩3. LET: $a \in A$ such that $\mathcal{A} \models \varphi\,[a]$

PROVE: $\mathcal{M}_A \models \varphi\,[a]$

PROOF: By ⟨2⟩2 and the fact that $\mathcal{A} \preccurlyeq \mathcal{M}$

⟨1⟩2. ASSUME: for all $\mathcal{L}(A)$-formula $\varphi(x)$, if $\mathcal{M}_A \models \exists x.\varphi$, then there exists $a \in A$ such that $\mathcal{M}_A \models \varphi\,[a]$

PROVE: A is the universe of an elementary substructure of $\mathcal{M}$

□

Definition 39 (Generated substructure)**.** Let A be any non-empty subset of M.
There exists a smallest substructure $\langle A\rangle^{\mathcal{M}}$ – the substructure *generated* by S – that contains S. If S is finite, then $\langle A\rangle^{\mathcal{M}}$ is said to be *finitely generated.*

Theorem 6 (Downward Löwenheim-Skolem)**.** *Let $\mathcal{M}$ be an $\mathcal{L}$-structure. Let A be a subset of A. Let κ be a cardinal (infinite).*
Assume that $\max(\mathrm{card}(A), \mathrm{card}(\mathcal{L})) \leq \kappa \leq \mathrm{card}(\mathcal{M})$.
We have that there exists $\mathcal{B} \preccurlyeq \mathcal{M}$ with $A \subseteq B$ and $\mathrm{card}(\mathcal{B}) = \kappa$.

Let $(I, <)$ be a totally ordered set.

Definition 40 ((Elementary) chain)**.** A family $(\mathcal{A}_i)_{i\in I}$ forms a *chain* (respectively, an *elementary chain*) if for every $i < j$, $\mathcal{A}_i \subseteq \mathcal{A}_j$ (respectively, $\mathcal{A}_i \preccurlyeq \mathcal{A}_j$).

Let $\mathcal{A}$ be an $\mathcal{L}$-structure.

Definition 41 (Elementary diagram)**.** The elementary diagram of $\mathcal{A}$, written $D(\mathcal{A})$, is the $\mathcal{L}(A)$-theory

$$D(\mathcal{A}) \triangleq Th(\mathcal{A}_A)$$

Definition 42 (Atomic diagram)**.** The atomic diagram of $\mathcal{A}$, written $\Delta(\mathcal{A})$, is the $\mathcal{L}(A)$-theory

$$\Delta(\mathcal{A}) \triangleq \{\varphi \text{ atomic } \mathcal{L}(A)\text{-sentence} \mid \mathcal{A}_A \models \varphi\,\}$$

Theorem 7 (Upward Lowenheim-Skolem). *Let $\mathcal{M}$ be an infinite $\mathcal{L}$-structure. Let κ be a cardinal such that* $\max(\text{card}(\mathcal{M}), \text{card}(\mathcal{L})) \leq \kappa$.

$\mathcal{M}$ has an elementary extension of cardinality κ.

Let $\mathcal{M}$ be an $\mathcal{L}$-structure. Let A be a subset of M. Let $\varphi(\overline{x})$ be an $\mathcal{L}(A)$-formula.

Definition 43 (A-definable set). The set $\varphi[\mathcal{M}]$, called *A-definable set*, is defined by

$$\varphi[\mathcal{M}] \triangleq \{\bar{b} \in M \mid \mathcal{M} \models \varphi\, \bar{b}\}$$

Definition 44 (0-definable set). A 0-definable set is a set definable without parameter.

Definition 45 (Definable set). A relation D of M is said *definable* if it is M-definable.

Definition 46 (Categorical theory). Let κ be an (infinite) cardinal. Let T be a theory.

T is a κ-categorical theory if T has a model of cardinality κ and if its models of cardinality κ are isomorphic.

Theorem 8 (Vaught's criterion). *Let T be a theory.*

If T is κ-categorical for $\kappa \geq$ card$(\mathcal{L})$ and does not have a finite model, then T is complete.

See [Tent and Ziegler, 2012, Exercise 2.1.2].

2.2 Preservation theorems

Lemma 3 (Separation). *Let T_1 and T_2 be two consistent $\mathcal{L}$-theories. Let $\mathcal{H}$ be a set of $\mathcal{L}$-sentences closed by conjunction and disjunction.*

The following are equivalent:

Local separation: *for all $\mathcal{M}_1 \models T_1$ and $\mathcal{M}_2 \models T_2$, there exists $\varphi \in \mathcal{H}$ such that $\mathcal{M}_1 \models \varphi$ and $\mathcal{M}_\in \models \neg\varphi$*

Global separation: *there exists $\varphi \in \mathcal{H}$ such that $T_1 \models \varphi$ and $T_2 \models \neg\varphi$*

Remark 4 (Notation). Let ϕ be a set of $\mathcal{L}$-formulas. Let $\mathcal{M}, \mathcal{N}$ be two $\mathcal{L}$-structures. Let $f : M \to N$ be a function.

- We write $\phi_\mathcal{M}$ the set of sentences from ϕ that are true in $\mathcal{M}$:

$$\phi_\mathcal{M} \triangleq \{\varphi \in \phi \mid \mathcal{M} \models \varphi\}$$

- We write $f : \mathcal{M} \to_\phi \mathcal{N}$ if f preserves all the formulas in ϕ
- We write $f : \mathcal{M} \Rightarrow_\phi \mathcal{N}$ if $\phi_\mathcal{M} \subseteq \phi_\mathcal{N}$

Remark 5 (Terminology).

- A *universal formula* is a formula $\forall\overline{x}.\varphi$, with φ without quantifier.
- An *existential formula* is a formula $\exists\overline{x}.\varphi$, with φ without quantifier.

Theorem 9 (Preservation of the universal quantifier). *Let T be an $\mathcal{L}$-theory.*

Write statement.

2.3 Quantifier elimination

Definition 47 (Quantifier elimination)**.** A theory T has quantifier elimination if for all formula $\varphi(\overline{x})$, there exists a formula $\psi(\overline{x})$ without quantifiers that is equivalent modulo T to φ.

Definition 48 (Primitive existential formula)**.** State.

Definition 49 (Substructure completeness)**.** Let $\mathcal{M}, \mathcal{N}$ two models of T. Let $\mathcal{A}$ a common substructure of $\mathcal{M}$ and $\mathcal{N}$.

T is substructure-complete if

$$\mathcal{M}_A \equiv \mathcal{N}_A$$

Theorem 10. *Let T be an $\mathcal{L}$-theory. The following are equivalent:*

- *T has quantifier elimination ;*
- *T is substructure-complete ;*
- *For $\mathcal{M}, \mathcal{N}$ two models of T, a common substructure $\mathcal{A}$ of $\mathcal{M}$ and $\mathcal{N}$, an primitive existential $\mathcal{L}$-formula $\varphi(\overline{x})$, and $\overline{a} \in A$, we have $\mathcal{M} \models \varphi[\overline{a}]$ iff $\mathcal{N} \models \varphi[\overline{a}]$.*

Definition 50 (Model completeness)**.** Let T be an $\mathcal{L}$-theory.

T is *model-complete* if for all $\mathcal{M}, \mathcal{N} \models T$ with $\mathcal{M} \subseteq \mathcal{N}$, we have $\mathcal{M} \preccurlyeq \mathcal{N}$.

Proposition 1 (Robinson's test)**.** *State.*

2.4 Theories with quantifier elimination

2.5 Types and saturation

Definition 51 (Partial n-type)**.** State.

Definition 52 (Complete n-type)**.** State.

Definition 53 (Set of n-types)**.** State.

Definition 54 (Realisation of a type)**.** State.

References

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