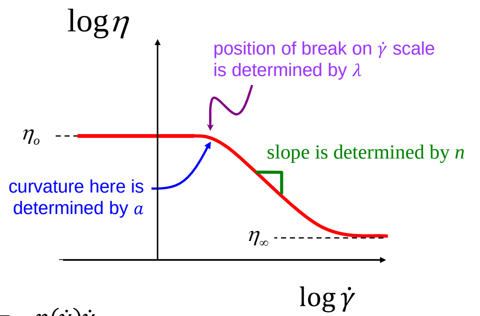
Chapter 7: Generalized Newtonian fluids

CM4650
Polymer Rheology
Michigan Tech

Carreau-Yassuda GNF



$$\underline{\underline{\tau}} = -\eta(\dot{\gamma})\underline{\underline{\dot{\gamma}}}$$

$$\eta(\dot{\gamma}) = \eta_\infty + (\eta_0 - \eta_\infty)[1 + (\dot{\gamma}\lambda)^a]^{\frac{n-1}{a}}$$

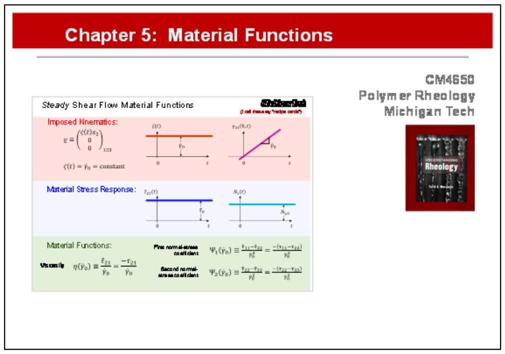
Develop Non-Newtonian Constitutive Equations

So Far:

- We seek constitutive equations for non-Newtonian fluids.
- We start with the Newtonian constitutive equation
- We "modify it" and make the Fake-O model.
- We checked out the model to see how we did.

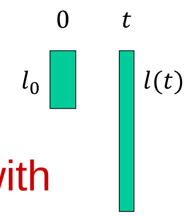
$$\underline{\underline{\tau}} = -M(\dot{\gamma}_0)[\nabla \underline{v} + (\nabla \underline{v})^T]$$

$$M(\dot{\gamma}_0) = \begin{cases} M_0 & \dot{\gamma}_0 > \dot{\gamma}_c \\ m\dot{\gamma}_0^{n-1} & \dot{\gamma}_0 \geq \dot{\gamma}_c \end{cases}$$



$$\underline{v} \equiv \begin{pmatrix} -\frac{1}{2}\dot{\varepsilon}(t)x_1 \\ -\frac{1}{2}\dot{\varepsilon}(t)x_2 \\ \dot{\varepsilon}(t)x_3 \end{pmatrix}_{123}$$

$$\dot{\varepsilon}(t) = \dot{\varepsilon}_0 = \text{constant}$$



When we tried the model out with elongational flow material functions, something odd happened …

3

Steady Elongational Flow Material Functions

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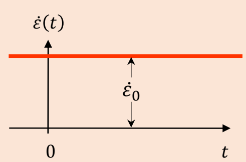
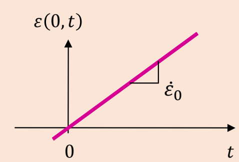
Imposed Kinematics:

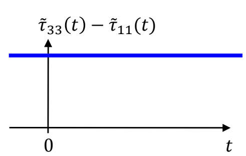
$$\underline{v} \equiv \begin{pmatrix} -\frac{1}{2}\dot{\varepsilon}(t)x_1 \\ -\frac{1}{2}\dot{\varepsilon}(t)x_2 \\ \dot{\varepsilon}(t)x_3 \end{pmatrix}_{123}$$

$$\dot{\varepsilon}(t) = \dot{\varepsilon}_0 = \text{constant}$$





Material Stress Response:



Material Functions:

Elongational Viscosity

$$\eta_e(\dot{\varepsilon}_0) \equiv \frac{\tilde{\tau}_{33} - \tilde{\tau}_{11}}{\dot{\varepsilon}_0} = \frac{-(\tau_{33} - \tau_{11})}{\dot{\varepsilon}_0}$$

Alternatively, $\bar{\eta}(\dot{\varepsilon}_0)$

4

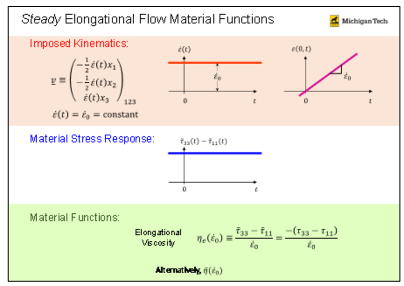
2) Predict what Newtonian fluids would do.

1. Choose a material function
2. Predict what Newtonian fluids would do
3. See what non-Newtonian fluids **do**
4. Hypothesize a $\underline{\underline{\tau}}(\underline{v})$
5. Predict the material function
6. Compare with what non-Newtonian fluids **do**
7. Reflect, **learn**, revise model, repeat.

2) Predict what Newtonian fluids would do.

?

$$\underline{\underline{\tau}} = -\mu \dot{\underline{\underline{\gamma}}}$$



5

2) Predict what Newtonian fluids would do.

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6. Compare with what non-Newtonian fluids **do**
7. Reflect, **learn**, revise model, repeat.

2) Predict what Newtonian fluids would do.

$$\eta_e = 3\mu$$

Trouton viscosity (Newtonian fluids)

- Constant
- Three times shear viscosity

6

Investigating **Stress/Deformation** Relationships (**Rheology**)

1. Choose a material function
2. Predict what Newtonian fluids would do
3. See what non-Newtonian fluids **do**
4. Hypothesize a $\underline{\underline{\tau}}(\underline{v})$
5. Predict the material function
6. Compare with what non-Newtonian fluids **do**
7. Reflect, **learn**, revise model, repeat.

5) Predict the material function (with new $\underline{\underline{\tau}}(\underline{v})$)

What does the Fake-O-model predict for steady elongational viscosity?

$$\underline{\underline{\tau}} = -M(\dot{\gamma}_0)[\nabla \underline{v} + (\nabla \underline{v})^T]$$

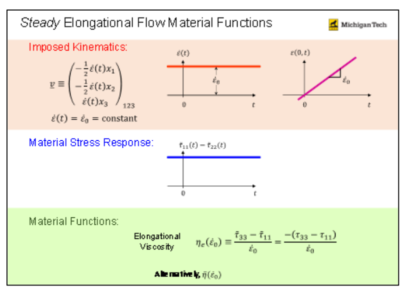
$$M(\dot{\gamma}_0) = \begin{cases} M_0 & \dot{\gamma}_0 > \dot{\gamma}_c \\ m\dot{\gamma}_0^{n-1} & \dot{\gamma}_0 \geq \dot{\gamma}_c \end{cases}$$

7

5) Predict the material function

$$\underline{\underline{\tau}} = -M(\dot{\gamma}_0)[\nabla \underline{v} + (\nabla \underline{v})^T]$$

$$M(\dot{\gamma}_0) = \begin{cases} M_0 & \dot{\gamma}_0 > \dot{\gamma}_c \\ m\dot{\gamma}_0^{n-1} & \dot{\gamma}_0 \geq \dot{\gamma}_c \end{cases}$$



What does the **Fake-O-model** predict for steady elongational viscosity?

Garbage.

$$\underline{\underline{\tau}} = -M(\dot{\gamma}_0)[\nabla \underline{v} + (\nabla \underline{v})^T]$$

$$M(\dot{\gamma}_0) = \begin{cases} M_0 & \dot{\gamma}_0 > \dot{\gamma}_c \\ m\dot{\gamma}_0^{n-1} & \dot{\gamma}_0 \geq \dot{\gamma}_c \end{cases}$$

$\dot{\gamma}_0$ is tied to shear flow only.

8

Investigating **Stress/Deformation** Relationships (**Rheology**)

1. Choose a material function
2. Predict what Newtonian fluids would do
3. See what non-Newtonian fluids **do**
4. Hypothesize a $\underline{\underline{\tau}}(\underline{v})$
5. Predict the material function
6. Compare with what non-Newtonian fluids **do**
7. Reflect, **learn**, revise model, repeat.

What if we make the following replacement?

$$\dot{\gamma}_0 \rightarrow \frac{\partial v_1}{\partial x_2}$$

Maybe this "guess" will work in more types of flows.

This at least can be written for any flow and it is equal to the shear rate in shear flow.

But it assumes a coordinate system.

9

4) Hypothesize a constitutive equation

Fake-O-model

$$\underline{\underline{\tau}} = -M(\dot{\gamma}_0)\left[\nabla \underline{v} + (\nabla \underline{v})^T\right]$$

$$M(\dot{\gamma}_0) = \begin{cases} M_0 & \dot{\gamma}_0 > \dot{\gamma}_c \\ m\dot{\gamma}_0^{n-1} & \dot{\gamma}_0 \geq \dot{\gamma}_c \end{cases}$$

Observations

- The model contains parameters that are specific to shear flow – makes it impossible to adapt for elongational or mixed flows
- Also, the model should only contain quantities that are independent of coordinate system (i.e. invariant)

We are identifying constraints on our freedom to "make up" constitutive models.

10

4) Hypothesize a constitutive equation

In Chapter 7, we find a solution:

- take out the shear rate $\dot{\gamma}_0$ and
- replace with the magnitude of the rate-of-deformation tensor $\left|\underline{\underline{\dot{\gamma}}}\right|$ (which is related to the second invariant of that tensor).

$$\underline{\underline{\tau}} = -f\left(\left|\underline{\underline{\dot{\gamma}}}\right|\right)\left[\nabla \underline{v} + (\nabla \underline{v})^T\right]$$

This is going to work.

11

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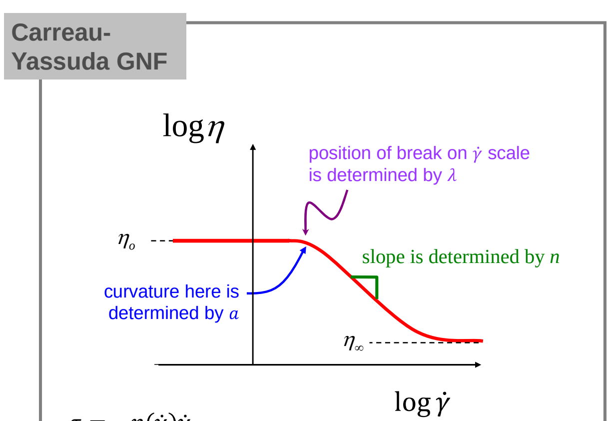
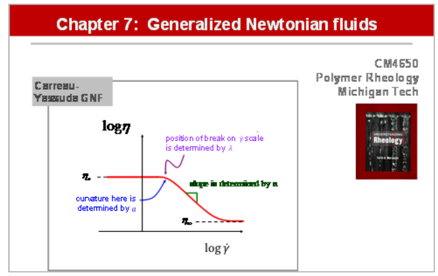


$$\underline{\underline{\tau}} = -\eta(\dot{\gamma})\underline{\underline{\dot{\gamma}}}$$

$$\eta(\dot{\gamma}) = \eta_\infty + (\eta_0 - \eta_\infty)\left[1 + (\dot{\gamma}\lambda)^a\right]^{\frac{n-1}{a}}$$

12



Current Goal:

Develop new constitutive models by trial and error:

- *Account for rate-dependence*
- *Accommodate known constraints*

Known constraints:

- no parameters specific to flow
- only use quantities that are independent of coordinate system

13

Develop new constitutive models by trial and error: Rate-Dependence

Back to our main goal:
Constitutive Equation – **an accounting for all stresses, all flows**

Newtonian fluids: (all flows)

stress tensor

$$\underline{\underline{\tau}} = \mu \underline{\underline{\dot{\gamma}}}$$

Rate-of-deformation tensor

In general:

$$\underline{\underline{\tau}} = f(\underline{v})\underline{\underline{\dot{\gamma}}}$$

In the general case, *f* needs to be a non-linear function (in time and position)

What should we choose for the function *f*?

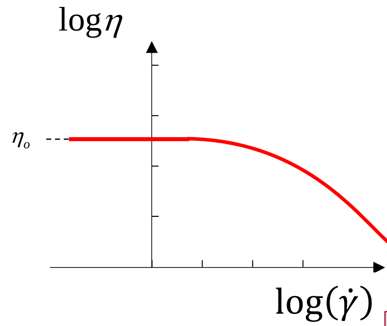
Answer: something that matches the data. Let's start with the steady shear data.

14

Develop new constitutive models by trial and error: Rate-Dependence

Non-Newtonian, Inelastic Fluids

First, we concentrate on the observation that **shear viscosity depends on shear rate**.



$$\eta \equiv \frac{-\tau_{21}}{\dot{\gamma}}$$

Non-Newtonian viscosity, η

$$\dot{\gamma} \equiv \left| \frac{\partial v_1}{\partial x_2} \right|$$

shear rate

We will design a constitutive equation that predicts this behavior in shear flow

15

Develop new constitutive models by trial and error: Rate-Dependence

Newtonian Constitutive Equation

$$\underline{\underline{\tau}} = -\mu \dot{\underline{\underline{\gamma}}}$$

For Newton's experiment (**shear flow**):

$$\underline{v} = \begin{pmatrix} v_1 \\ 0 \\ 0 \end{pmatrix}_{123}$$

$$\underline{\underline{\tau}} = \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix}_{123} = -\mu \begin{pmatrix} 0 & \frac{\partial v_1}{\partial x_2} & 0 \\ \frac{\partial v_1}{\partial x_2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{123}$$

We could make this equation give the right answer (shear thinning) in steady shear flow if we substituted ***a function of shear rate*** for the constant viscosity.

16

Develop new constitutive models by trial and error: Rate-Dependence

Generalized Newtonian Fluid (GNF)
constitutive equation

$$\underline{\underline{\tau}} = \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{21} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix}_{123} = \eta(\dot{\gamma}) \begin{pmatrix} 0 & \frac{\partial v_1}{\partial x_2} & 0 \\ \frac{\partial v_1}{\partial x_2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{123}$$

SHEAR FLOW

$$\underline{v} = \begin{pmatrix} v_1 \\ 0 \\ 0 \end{pmatrix}_{123} \qquad \dot{\gamma} = \left| \frac{\partial v_1}{\partial x_2} \right|$$

GNF

$$\underline{\underline{\tau}} = \eta(\dot{\gamma}) \begin{pmatrix} 2\frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} & \frac{\partial v_1}{\partial x_3} + \frac{\partial v_3}{\partial x_1} \\ \frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} & 2\frac{\partial v_2}{\partial x_2} & \frac{\partial v_3}{\partial x_2} + \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_1}{\partial x_3} + \frac{\partial v_3}{\partial x_1} & \frac{\partial v_3}{\partial x_2} + \frac{\partial v_2}{\partial x_3} & 2\frac{\partial v_3}{\partial x_3} \end{pmatrix}_{123}$$

ALL FLOWS

$$\underline{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}_{123} \qquad \dot{\gamma} = \left| \underline{\underline{\dot{\gamma}}} \right|$$

GNF is designed in shear flow; fingers crossed it'll work in all flows.

17

Develop new constitutive models by trial and error: Rate-Dependence

Constitutive Equation – **an accounting for <u>all</u> stresses, all flows**

$$\underline{\underline{\tau}} = -f(\underline{v})\underline{\underline{\dot{\gamma}}}$$

A simple choice for f:

$$\underline{\underline{\tau}} = -\eta(\dot{\gamma})\underline{\underline{\dot{\gamma}}}$$

Generalized Newtonian Fluids (GNF)

$$\dot{\gamma} \equiv \left|\underline{\underline{\dot{\gamma}}}\right| = \sqrt{\frac{1}{2}\left(\underline{\underline{\dot{\gamma}}}:\underline{\underline{\dot{\gamma}}}\right)}$$

GNF is designed in shear flow; fingers crossed it'll work in all flows.

18

Develop new constitutive models by trial and error: Rate-Dependence

$$\underline{\underline{\tau}} = -\eta(\dot{\gamma})\dot{\underline{\underline{\gamma}}}$$

Generalized Newtonian Fluids (GNF)

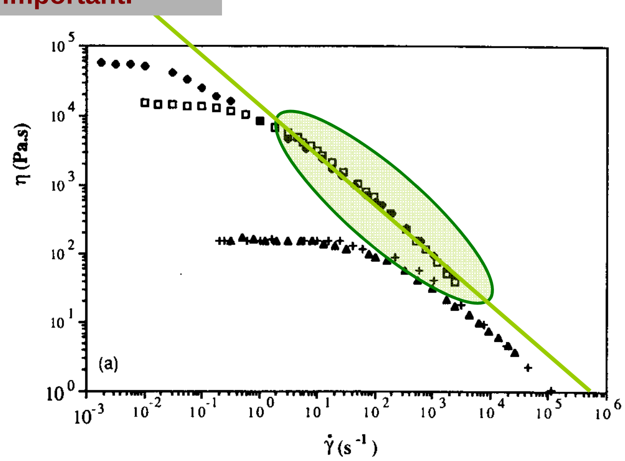
What do we pick for $\eta(\dot{\gamma})$?

- Something that matches the data;
- Something simple, so that the calculations are easy

19

Develop new constitutive models by trial and error: Rate-Dependence

In processing, the high-shear-rate behavior is the most important.



+ linear 131 kg/mole
▲ branched 156 kg/mole
□ linear 418 kg/mol
◆ branched 428 kg/mol

Figure 6.3, p. 172 Piau et al., linear and branched PDMS

20

Develop new constitutive models by trial and error: Rate-Dependence

Power-law model for viscosity

$$\eta(\dot{\gamma}) = m\dot{\gamma}^{n-1}$$

In shear flow $\dot{\gamma} \equiv \left|\frac{dv_1}{dx_2}\right|$

$$\eta = m\left|\frac{dv_1}{dx_2}\right|^{n-1}$$ (in shear flow)

On a log-log plot, this would give a straight line:

$$\underbrace{\log\eta}_{Y} = \underbrace{\log m}_{B} + \underbrace{(n-1)}_{M}\underbrace{\log\left|\frac{dv_1}{dx_2}\right|}_{X}$$

21

Develop new constitutive models by trial and error: Rate-Dependence

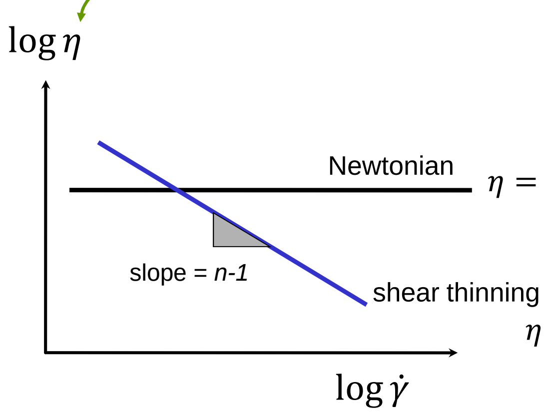
Power-law model for viscosity (GNF)

steady shear flow

Non-Newtonian shear viscosity

$$\eta \equiv \frac{-\tau_{21}}{\dot{\gamma}}$$

$$\dot{\gamma} \equiv \left|\frac{dv_1}{dx_2}\right|$$



Newtonian $\eta = m\dot{\gamma}^{n-1}, n = 1$

shear thinning $\eta = m\dot{\gamma}^{n-1}, n < 1$

22

Develop new constitutive models by trial and error: Rate-Dependence

Power-Law Generalized Newtonian Fluid (PL-GNF)

$$\underline{\underline{\tau}} = -\eta(\dot{\gamma})\underline{\underline{\dot{\gamma}}}$$

$$\eta(\dot{\gamma}) = m\dot{\gamma}^{n-1}$$

m or *K* = consistency index ($m = \mu$ for Newtonian)
n = power-law index (*n* = 1 for Newtonian)

$$\dot{\gamma} \equiv \left|\underline{\underline{\dot{\gamma}}}\right|$$

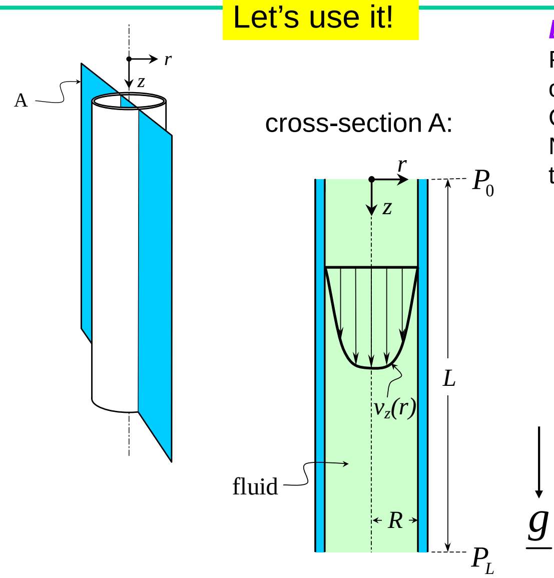
(Usually 0.5≤ *n* ≤1)

23

Calculate flow fields with GNF constitutive equations

Let's use it!

EXAMPLE: Pressure-driven flow of a Power-Law Generalized Newtonian fluid in a tube



- steady state
- well developed
- long tube

24

Calculate flow fields with GNF constitutive equations

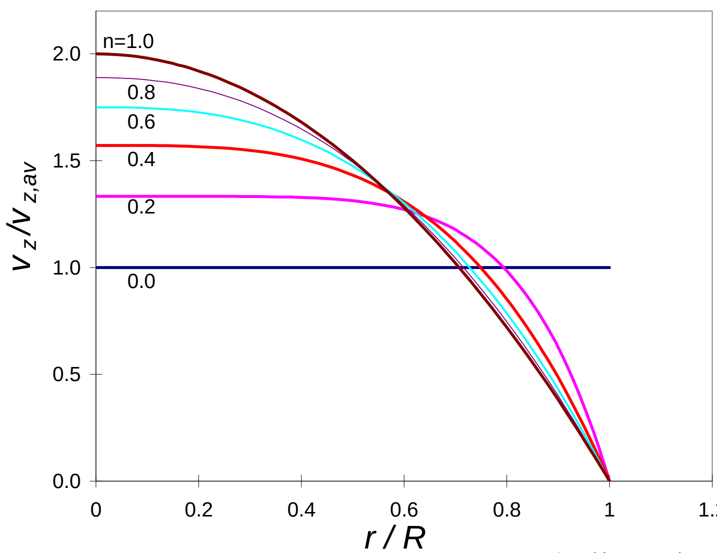
Velocity field
Poiseuille flow of a power-law fluid:

$$v_z(r) = \left(\frac{R(L\rho g + P_o - P_L)}{2Lm}\right)^{\frac{1}{n}} \left(\frac{R}{\frac{1}{n}+1}\right)\left(1-\left(\frac{r}{R}\right)^{\frac{1}{n}+1}\right)$$

25

Calculate flow fields with GNF constitutive equations

Solution to Poiseuille flow in a tube
incompressible, power-law fluid

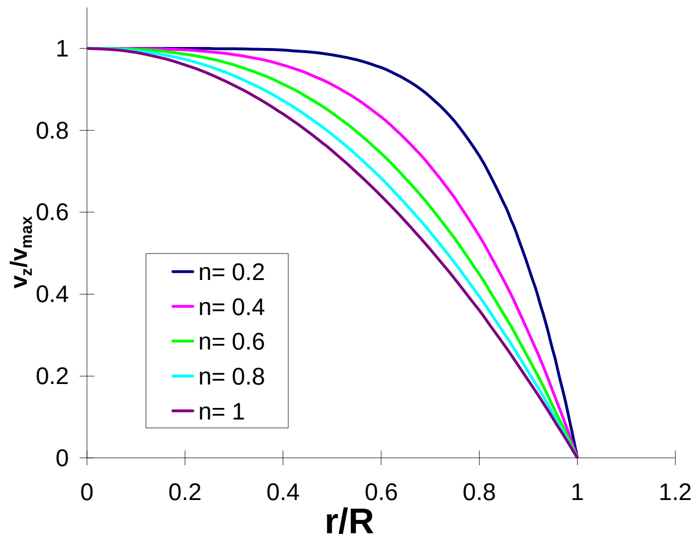


26

Calculate flow fields with GNF constitutive equations

Solution to Poiseuille flow in a tube

incompressible, power-law fluid

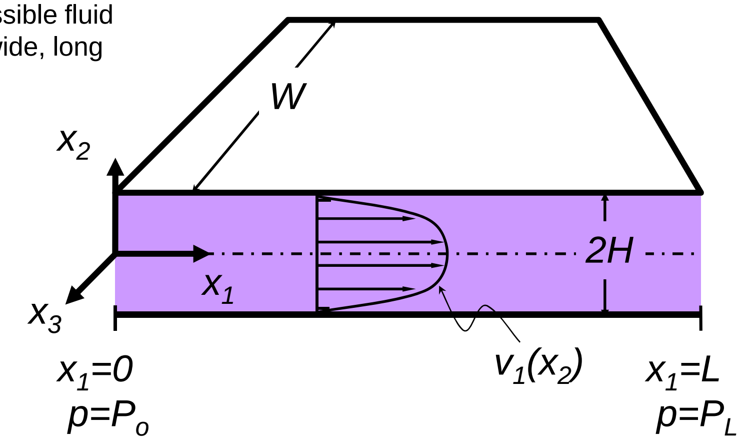


27

Calculate flow fields with GNF constitutive equations

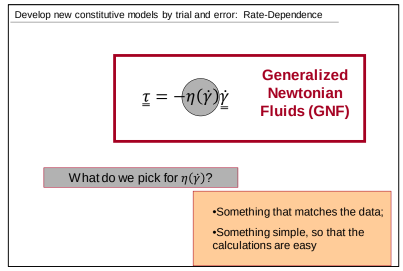
EXAMPLE: **Pressure-driven flow of a Power-Law GNF between infinite parallel plates**

- steady state
- incompressible fluid
- infinitely wide, long

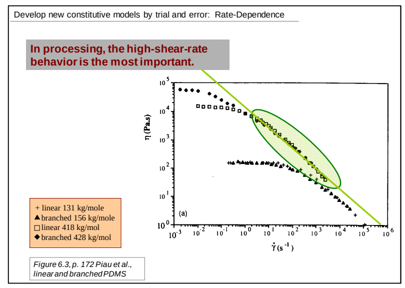


28

We can pick any function we wish for $\eta(\dot{\gamma})$.



When we are focused on the shear-thinning regime, we picked the power-law model.



There are other choices.

29

$$\underline{\underline{\tau}} = -\eta(\dot{\gamma})\underline{\underline{\dot{\gamma}}}$$

Other GNF viscosity models

Carreau-Yasuda

$$\eta(\dot{\gamma}) = \eta_\infty + (\eta_0 - \eta_\infty)[1 + (\dot{\gamma}\lambda)^a]^{\frac{n-1}{a}}$$

Ellis Model

$$\eta = \frac{\eta_0}{1 + \left|\frac{\tau}{\tau_0}\right|^{\alpha-1}} \qquad \tau = \left|\underline{\underline{\tau}}\right|$$

4-Parameter Carreau Model (same as CY with $a = 2$)

Cross-Williamson Model (same as CY with $a = 1, \eta_\infty = 0$)

DeKee Model

$$\eta = \eta_1 e^{-\lambda\dot{\gamma}} + \eta_2 e^{-0.1\lambda\dot{\gamma}} + \eta_\infty$$

Casson Model

$$\sqrt{\tau} = \sqrt{\tau_0} + \sqrt{\eta_0 \gamma} \qquad \tau = \left|\underline{\underline{\tau}}\right|$$

Herschel-Bulkley Model

$$\eta = \frac{\tau_0}{\dot{\gamma}} + m\dot{\gamma}^{n-1}$$

DeKee-Turcotte Model

$$\eta = \frac{\tau_0}{\dot{\gamma}} + \eta_1 e^{-\lambda\dot{\gamma}}$$

See Carreau, DeKee, and Chhabra for complete discussion (*Rheology of Polymeric Systems*, Hanser, 1997)

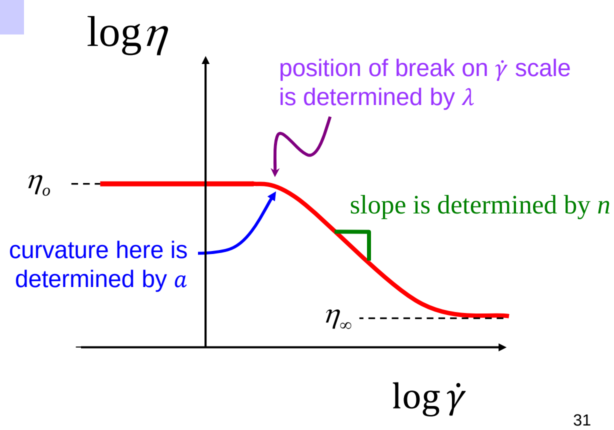
30

Develop new constitutive models by trial and error: Rate-Dependence

A model that captures the whole range of $\dot{\gamma}$ is the Carreau-Yasuda GNF.

$$\underline{\underline{\tau}} = -\eta(\dot{\gamma})\underline{\underline{\dot{\gamma}}}$$

$$\eta(\dot{\gamma}) = \eta_\infty + (\eta_0 - \eta_\infty)[1 + (\dot{\gamma}\lambda)^a]^{\frac{n-1}{a}}$$

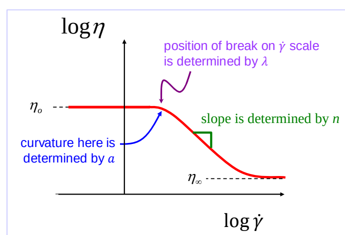


31

Develop new constitutive models by trial and error: Rate-Dependence

Carreau-Yassuda GNF

$$\underline{\underline{\tau}} = -\eta(\dot{\gamma})\underline{\underline{\dot{\gamma}}}$$



A model with 5 parameters

$$\eta(\dot{\gamma}) = \eta_\infty + (\eta_0 - \eta_\infty)[1 + (\dot{\gamma}\lambda)^a]^{\frac{n-1}{a}}$$

- The viscosity function approaches the constant value of η_∞ as deformation rate get large
- The viscosity function approaches the constant value η_0 as deformation rate gets small
- λ is the time constant for the fluid
- n determines the slope of the power-law region
- a modifies the sharpness of the transition from η_0 to thinning

32

Develop new constitutive models by trial and error: Rate-Dependence

What about shear thickening?

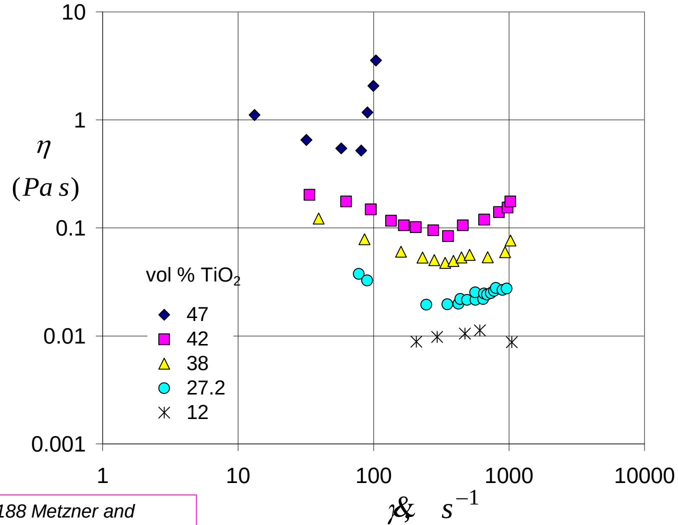


Figure 6.27, p. 188 Metzner and Whitlock; TiO_2/water suspensions

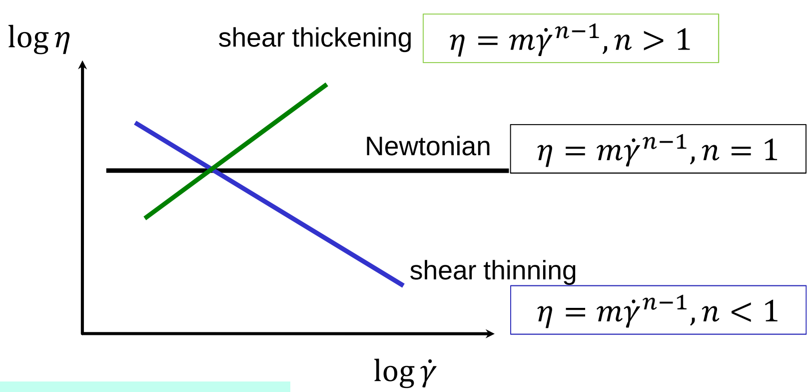
33

Develop new constitutive models by trial and error: Rate-Dependence

Power-Law GNF

steady shear flow

The Power-Law GNF model can capture shear thickening.



The GNF models are **inelastic** models.

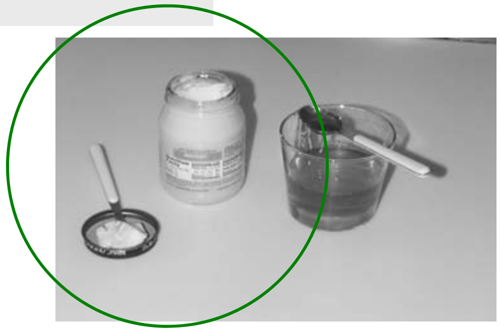
34

Develop new constitutive models by trial and error: Rate-Dependence

Other Inelastic Fluids

What about mayonnaise?

Mayonnaise and many other like fluids (paint, ketchup, most suspensions, asphalt) is able to sustain a **yield stress.**



Once the fluid begins to deform under an imposed stress, the viscosity may either be constant or may shear-thin. This type of steady shear viscosity behavior can be modeled with a GNF.

35

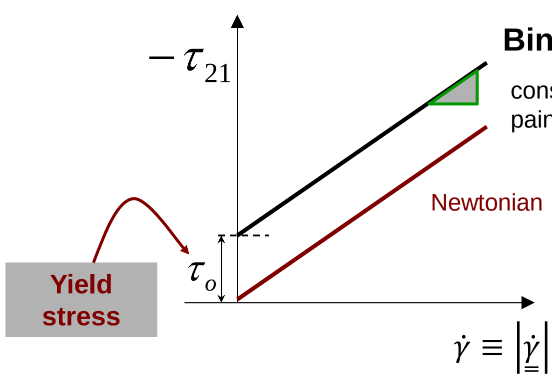
Develop new constitutive models by trial and error: Rate-Dependence

Non-Newtonian Fluids, inelastic

For some fluids, no flow occurs when moderate stresses are applied.

Bingham plastic

constant slope = m_o (mayo, paints, suspensions)



36

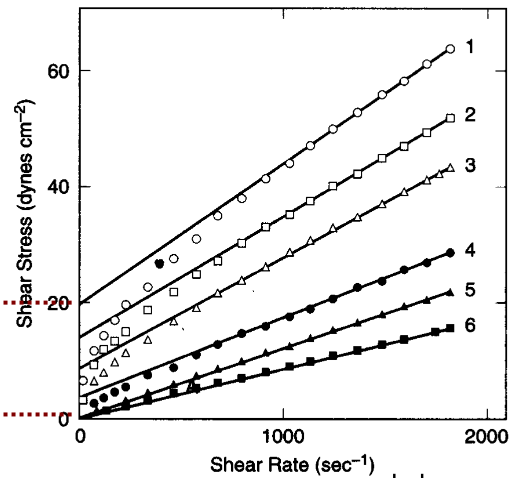
Develop new constitutive models by trial and error: Rate-Dependence

Non-Newtonian Fluids, inelastic **Bingham plastic**

PMMA in water

Yield stress

τ_o



$\dot{\gamma} \equiv \left|\dot{\underline{\underline{\gamma}}}\right|$

Friend and Hunter, 1971; dispersions of PMMA in water at various z-potentials; From Larson, p353.

37

Develop new constitutive models by trial and error: Rate-Dependence

Bingham Plastic GNF
steady shear flow

$$-\tau_{21} \equiv \tau_0 + \mu\dot{\gamma}$$

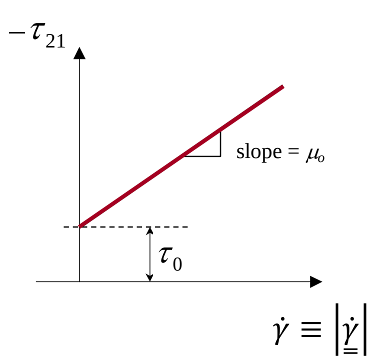
Non-Newtonian viscosity, η

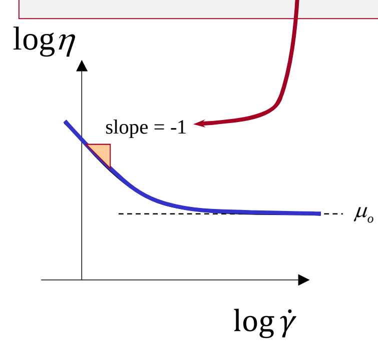
$$\eta \equiv \frac{-\tau_{21}}{\dot{\gamma}}$$

$$\eta = \frac{\tau_0}{\dot{\gamma}} + \mu_0$$

$$\lim_{\dot{\gamma}\to 0}(\eta) = \lim_{\dot{\gamma}\to 0}\left(\frac{\tau_0}{\dot{\gamma}} + \mu_0\right) = \frac{\tau_0}{\dot{\gamma}}$$

$$\log\eta = \log\tau_0 - \log\dot{\gamma}$$





38

Develop new constitutive models by trial and error: Rate-Dependence

Bingham GNF

$$\underline{\underline{\tau}} = -\eta(\dot{\gamma})\underline{\underline{\dot{\gamma}}}$$

A model with 2 parameters

$$\eta(\dot{\gamma}) = \begin{cases} \infty & \left|\underline{\underline{\tau}}\right| \le \tau_0 \\ \mu_0 + \dfrac{\tau_0}{\dot{\gamma}} & \left|\underline{\underline{\tau}}\right| > \tau_0 \end{cases}$$

μ_0= viscosity parameter
τ_0= yield stress

There is no flow until the shear stress exceeds a critical value τ_0 called the yield stress.

39

Other GNF viscosity models

$$\underline{\underline{\tau}} = -\eta(\dot{\gamma})\underline{\underline{\dot{\gamma}}}$$

Carreau-Yasuda

$$\eta(\dot{\gamma}) = \eta_\infty + (\eta_0 - \eta_\infty)[1 + (\dot{\gamma}\lambda)^a]^{\frac{n-1}{a}}$$

Ellis Model

$$\eta = \frac{\eta_0}{1 + \left|\frac{\tau}{\tau_0}\right|^{\alpha - 1}} \qquad \tau = \left|\underline{\underline{\tau}}\right|$$

4-Parameter Carreau Model (same as CY with $a = 2$)

Cross-Williamson Model (same as CY with $a = 1, \eta_\infty = 0$)

DeKee Model

$$\eta = \eta_1 e^{-\lambda\dot{\gamma}} + \eta_2 e^{-0.1\lambda\dot{\gamma}} + \eta_\infty$$

Casson Model

$$\sqrt{\tau} = \sqrt{\tau_0} + \sqrt{\eta_0\dot{\gamma}} \qquad \tau = \left|\underline{\underline{\tau}}\right|$$

Herschel-Bulkley Model

$$\eta = \frac{\tau_0}{\dot{\gamma}} + m\dot{\gamma}^{n-1}$$

Yield stress plus power-law viscosity behavior

DeKee-Turcotte Model

$$\eta = \frac{\tau_0}{\dot{\gamma}} + \eta_1 e^{-\lambda\dot{\gamma}}$$

See Carreau, DeKee, and Chhabra for complete discussion (*Rheology of Polymeric Systems*, Hanser, 1997)

40

Develop new constitutive models by trial and error: Rate-Dependence

What now?

Address other non-Newtonian phenomena?

•Predict material functions with the Generalized Newtonian Constitutive Equation.

Example: Elongational viscosity, etc.

•Calculate velocity and stress fields predicted by Generalized Newtonian Constitutive Equations

Example: Poiseuille flow, drag flow, etc.

Investigating **Stress/Deformation** Relationships (**Rheology**)

1. Choose a material function
2. Predict what Newtonian fluids would do
3. See what non-Newtonian fluids **do**
4. Hypothesize a $\underline{\underline{\tau}}(\underline{v})$
5. Predict the material function
6. Compare with what non-Newtonian fluids **do**
7. Reflect, **learn**, revise model, repeat.

5) Predict the material function (with new $\underline{\underline{\tau}}(\underline{v})$)

Develop new constitutive models by trial and error: Rate-Dependence

What now?

Address other non-Newtonian phenomena?

•Predict material functions with the Generalized Newtonian Constitutive Equation.

Example: Elongational viscosity, etc.

•Calculate velocity and stress fields predicted by Generalized Newtonian Constitutive Equations

Example: Poiseuille flow, drag flow, etc.

We can also calculate non-standard flow fields and see if the predictions are sensible.

Investigating **Stress/Deformation** Relationships (**Rheology**)

1. Choose a material function
2. Predict what Newtonian fluids would do
3. See what non-Newtonian fluids **do**
4. Hypothesize a $\underline{\underline{\tau}}(\underline{v})$
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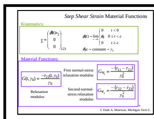


What kind of rheology calculation am I considering?

Predict a material function for a constitutive equation

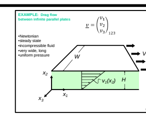
Find the recipe card (has kinematics, definitions of material functions; $\underline{v}(t)$ is known)

↓

For the given kinematics, calc. $\underline{\underline{\tau}}(t)$ from the constitutive equation

↓

From $\underline{\underline{\tau}}(t)$, calculate the material function from the definition on the recipe card



Calculate a flow field or stress field for a fluid in a situation

Sketch the situation and choose an appropriate coordinate system

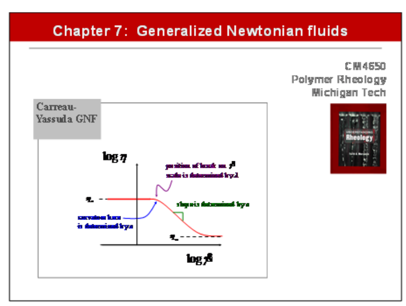
↓

Model the flow by asking yourself questions about the velocity field (are some components and derivatives zero? The constitutive equation must be known)

↓

Solve the differential equations (Solve EOM and continuity for $\underline{v}$ and $\underline{\underline{\tau}}$ and apply the boundary conditions)

43



Status:

Current Goal:

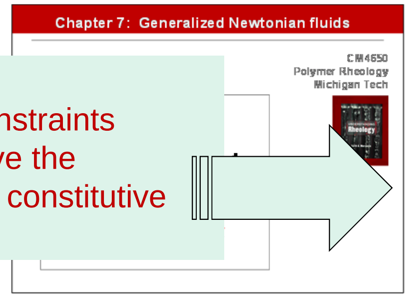
Develop new constitutive models by trial and error:

- ✓ *Account for Rate-Dependence*
- ✓ *Accommodate known constraints*

Known constraints:

- no parameters specific to flow
- only use quantities that are independent of coordinate system

44



Next?
Look for more constraints
Innovate—improve the predictions of the constitutive equation

Status:

Current Goal:
Develop new constitutive models by trial and error:
- ✓ *Account for Rate-Dependence*
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Known constraints:
- no parameters specific to flow
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45

Innovate and improve constitutive equations

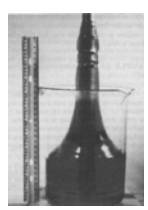
The steady shear viscosity function $\eta(\dot{\gamma})$ can be fit to experimental data to an arbitrarily high precision.

Does this mean that *Generalized Newtonian Fluid* models are okay to use in all situations?

Not necessarily. A constitutive model needs to be able to predict *all stresses* in *all flows,* not just shear stresses in steady shearing. We need to check predictions.

For example, does the GNF predict the shear normal stresses?

$$\underline{\underline{\tau}} = \begin{pmatrix} \tau_{11} & \tau_{12} & 0 \\ \tau_{21} & \tau_{22} & 0 \\ 0 & 0 & \tau_{33} \end{pmatrix}_{123}$$



46

Innovate and improve constitutive equations

Generalized Newtonian Fluid (GNF) constitutive equation

$$\underline{\underline{\tau}} = \eta(\dot{\gamma}) \begin{pmatrix} 2\frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} & \frac{\partial v_1}{\partial x_3} + \frac{\partial v_3}{\partial x_1} \\ \frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} & 2\frac{\partial v_2}{\partial x_2} & \frac{\partial v_2}{\partial x_3} + \frac{\partial v_3}{\partial x_2} \\ \frac{\partial v_1}{\partial x_3} + \frac{\partial v_3}{\partial x_1} & \frac{\partial v_2}{\partial x_3} + \frac{\partial v_3}{\partial x_2} & 2\frac{\partial v_3}{\partial x_3} \end{pmatrix}_{123}$$

In Shear Flow:

$$\underline{v} = \begin{pmatrix} v_1 \\ 0 \\ 0 \end{pmatrix}_{123} \qquad \dot{\gamma} = \left| \frac{\partial v_1}{\partial x_2} \right| \qquad \underline{\underline{\tau}} = \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{21} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix}_{123} = \eta(\dot{\gamma}) \begin{pmatrix} 0 & \frac{\partial v_1}{\partial x_2} & 0 \\ \frac{\partial v_1}{\partial x_2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{123}$$

No matter what we pick for the function $\eta(\dot{\gamma})$, **we cannot predict** shear normal stresses with a Generalized Newtonian Fluid.

47

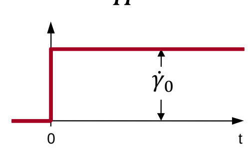
Innovate and improve constitutive equations

shear stress response

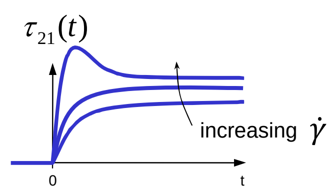
What does the GNF predict for start-up shear stresses?

imposed shear rate

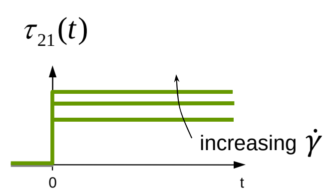
$$\dot{\gamma}_{21} = \frac{v_1(t)}{H}$$



What the data show:



What the GNF models predict:

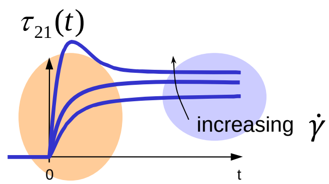


48

Innovate and improve constitutive equations

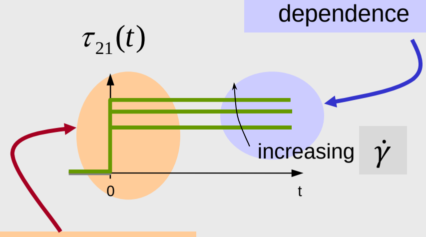
Start-up shear stresses

What the data show:



What the **GNF** models predict:

Correctly captures rate dependence



misses start-up effects

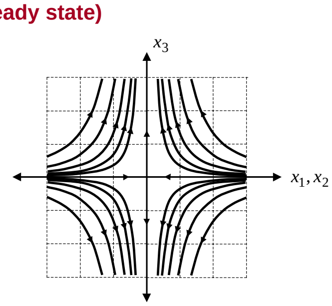
No matter what we pick for the function $\eta(\dot{\gamma})$, **we cannot predict** the time-dependence of shear start-up correctly with a GNF.

Innovate and improve constitutive equations

elongational stress response

What does the GNF predict in *steady elongational* flow?

imposed deformation
(steady state)



What the data show:

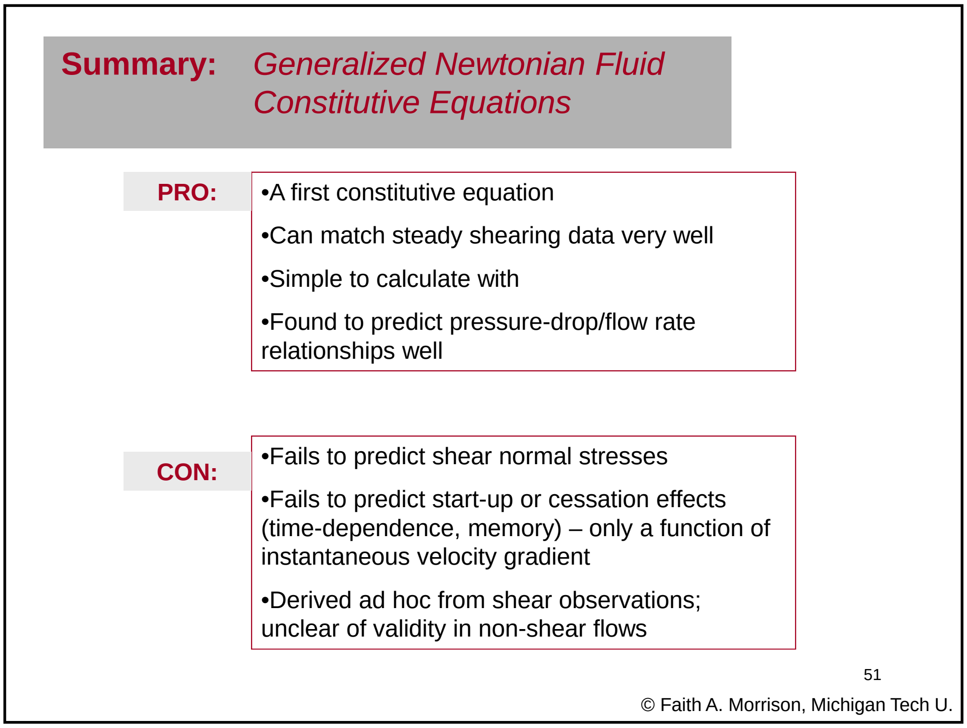
$$\lim_{\dot{\varepsilon}\to 0}\bar{\eta} = 3\eta_0$$ Trouton's Rule

(there is limited elongational viscosity data available)

What the GNF models predict:

$$\bar{\eta} = 3\eta$$ For all deformation rates

If a material shear-thins, GNF predicts it will tension-thin.
This is not observed.


Summary: Generalized Newtonian Fluid Constitutive Equations
PRO:
•A first constitutive equation
•Can match steady shearing data very well
•Simple to calculate with
•Found to predict pressure-drop/flow rate relationships well
CON:
•Fails to predict shear normal stresses
•Fails to predict start-up or cessation effects (time-dependence, memory) – only a function of instantaneous velocity gradient
•Derived ad hoc from shear observations; unclear of validity in non-shear flows
51
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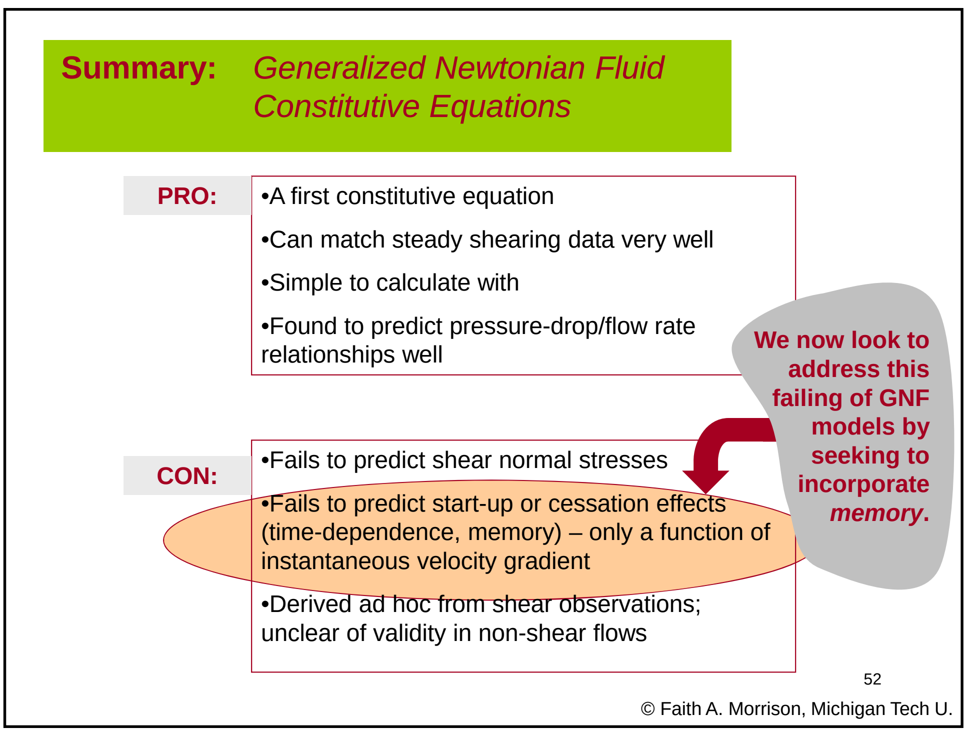
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•Derived ad hoc from shear observations; unclear of validity in non-shear flows
We now look to address this failing of GNF models by seeking to incorporate memory.
52
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Innovate and improve constitutive equations

What we know so far…

Rules for Constitutive Equations

$$\underline{\underline{\tau}}(t) = f\left(\underline{\underline{\dot{\gamma}}}, I_{\underline{\underline{\dot{\gamma}}}}, II_{\underline{\underline{\dot{\gamma}}}}, III_{\underline{\underline{\dot{\gamma}}}}, \text{material info}\right)$$

The stress expression:

- ***Must be of tensor order***
- ***Must be a tensor (independent of coordinate system)***
- ***Must be a symmetric tensor***
- ***Must make predictions that are independent of the observer***
- ***Should correctly predict observed flow/deformation behavior***

53

Innovate and improve constitutive equations

What we know so far…

Rules for Constitutive Equations

$$\underline{\underline{\tau}}(t) = f\left(\underline{\underline{\dot{\gamma}}}, I_{\underline{\underline{\dot{\gamma}}}}, II_{\underline{\underline{\dot{\gamma}}}}, III_{\underline{\underline{\dot{\gamma}}}}, \text{material info}\right)$$

Tensor invariants – scalars associated with a tensor that do not depend on coordinate system

The stress expression:

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54

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Innovate and improve constitutive equations

Tensor Invariants

$$I_{\underline{\underline{A}}} \equiv trace\underline{\underline{A}} = tr\underline{\underline{A}}$$

For the tensor written in Cartesian coordinates:

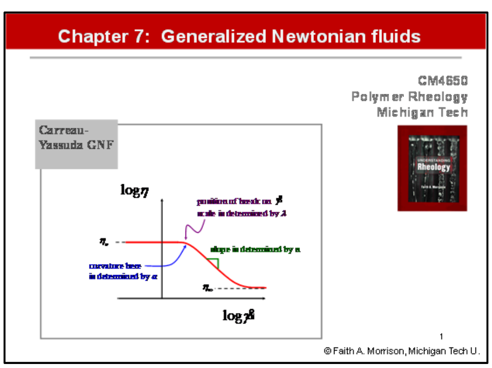
$$trace\underline{\underline{A}} = \sum_{p=1}^{3} A_{pp} = A_{11} + A_{22} + A_{33}$$

$$II_{\underline{\underline{A}}} \equiv trace\left(\underline{\underline{A}} \cdot \underline{\underline{A}}\right) = \underline{\underline{A}} : \underline{\underline{A}} = \sum_{p=1}^{3}\sum_{k=1}^{3} A_{pk}A_{kp}$$

$$III_{\underline{\underline{A}}} \equiv trace\left(\underline{\underline{A}} \cdot \underline{\underline{A}} \cdot \underline{\underline{A}}\right) = \sum_{p=1}^{3}\sum_{j=1}^{3}\sum_{h=1}^{3} A_{pj}A_{jh}A_{hp}$$

Note: the definitions of invariants written in terms of coefficients are only valid when the tensor is written in Cartesian coordinates.

Done with Inelastic models (GNF).



Let's move on to Linear Elastic models

Chapter 8: Memory Effects: GLVE

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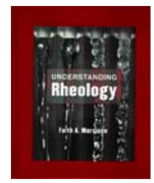
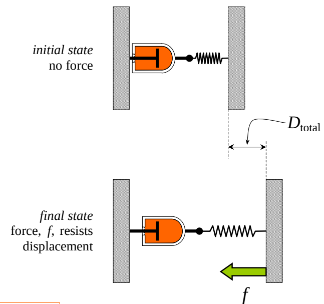


Maxwell's model combines viscous and elastic responses in series

Spring (elastic) and dashpot (viscous) in series:



Displacements are additive:

$$D_{total} = D_{spring} + D_{dashpot}$$

57