Worksheet 24

Math 607, Homological Algebra

Wednesday, June 3, 2020

Let R be a Noetherian local ring, let $I \subsetneq R$ be a proper ideal, and consider the radical of I:

$$\sqrt{I} = \{r \in R : r^n \in I \text{ for some } n \geq 1\}.$$

Recall that the *depth* of I is the length of some (and hence any) maximal regular sequence in I. Understand the following proof that

$$\operatorname{depth} I = \operatorname{depth} \sqrt{I},$$

which follows Eisenbud Corollary 17.8.

1. First, $I \subset \sqrt{I}$, so $\operatorname{depth} I \leq \operatorname{depth} \sqrt{I}$
2. Let $r_1, \ldots, r_d \in \sqrt{I}$. There is an integer $n \geq 1$ such that $r_1^n, \ldots, r_d^n \in I$.
3. If $r_1, \ldots, r_d$ is a regular sequence then $r_1, \ldots, r_{d-1}, r_d^n$ is a regular sequence.
4. And then $r_d^n, r_1, \ldots, r_{d-1}$ is a regular sequence, because R is local.
5. By induction, $r_d^n, \ldots, r_1^n$ is a regular sequence in I.
6. Thus $\operatorname{depth} \sqrt{I} \leq \operatorname{depth} I$.

If you phrase 3–5 in terms of Koszul complexes, it sounds quite surprising: if the Koszul complex of $r_1, \ldots, r_d$ has $H_{>0} = 0$, then the Koszul complex of $r_1^n, \ldots, r_d^n$ also has $H_{>0} = 0$.