Downey, Kach, Lempp, Lewis-Pye, Montalbán, and Turetsky (2015) proved that the index set of computably categorical structures is Π^1_1-complete.

On the other hand, *on-a-cone* there is a clean structural characterization.

For $X \in 2^\omega$, an X-computable structure $\mathcal{S}$ is *X-computably categorical* if for any X-computable isomorphic copies $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{S}$, there is an X-computable isomorphism $f\colon \mathcal{A} \cong \mathcal{B}$.

A countable structure $\mathcal{S}$ is *computably categorical on a cone* if there is $Y \in 2^\omega$ such that $\mathcal{S}$ is X-computably categorical for all $X \geq_T Y$.

Theorem (Montalbán 2015)

A countable structure $\mathcal{S}$ is computably categorical on a cone if and only if $\mathcal{S}$ has a Σ^{in}_3 Scott sentence.

An X-computable structure $\mathcal{S}$ is *X-computably bi-embeddably categorical* if for any X-computable structure $\mathcal{A}$ bi-embeddable with $\mathcal{S}$, there exist X-computable isomorphic embeddings

$$f\colon \mathcal{A} \hookrightarrow \mathcal{S} \ \text{ and } \ g\colon \mathcal{S} \hookrightarrow \mathcal{A}.$$

For the case $X = \emptyset$, Bazhenov showed that the index set of computably bi-embeddably categorical graphs is Π^1_1-complete.

A countable structure $\mathcal{S}$ is *computably bi-embeddably categorical on a cone* if there is $Y \in 2^\omega$ such that $\mathcal{S}$ is X-computably bi-embeddably categorical for all $X \geq_T Y$.

Problem

Can one obtain a nice syntactic characterization for computable bi-embeddable categoricity on a cone?

Side Problem

Can we simplify the syntactic characterization for *punctual categoricity on a cone*:

- $\mathcal{S}$ has a Σ_3^{in} Scott sentence;
- $\mathcal{S}$ is ω-categorical;
- up to some finitely many constants in $\mathcal{S}$ for every every finite tuple $\vec{x}$ in $\mathcal{S}$ there is an n such that $\vec{x}$ together with any disctinct new elements $y_1, \ldots, y_n \notin \vec{x}$ generates a substructure intersecting all automorphism orbits over $\vec{x}$.

Example.

The empty graph; the abelian group $\bigoplus_\infty \mathbb{Z}_p$ for a prime p.