QUANTIFIER ELIMINATION FOR A CLASS OF INTUITIONISTIC THEORIES

By

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Abstract

From classical, Fraïssé-homogeneous, ($\leq \omega$)-categorical theories over finite relational languages (which we refer to as JRS theories), we construct intuitionistic theories that are complete, prove negations of classical tautologies, and admit quantifier elimination. The technique we use considers Kripke models as functors from a small category to the category of $\mathcal{L}$-structures with morphisms, rather than the usual interpretation wherein the frame of a Kripke model is a partial order. While one can always "unravel" a functor Kripke model to obtain a partial order Kripke model with the same intuitionistic theory, our technique is perhaps an easier way to consider a Kripke model that includes a single classical node structure and all of the endomorphisms of that classical JRS structure. We also determine the intuitionistic universal fragments of these theories, in accordance with the hierarchy of intuitionistic formulas put forth in [9] and expounded on by Fleischmann in [11]. This portion of the thesis (Chapter 1) is the result of joint work with Ben Ellison, Jonathan Fleischmann, and Wim Ruitenburg, as published (up to minor structural changes) in [10].

Given a classical JRS theory, we determine axiomatizations of the corresponding intuitionistic theory in Chapter 2. We first do so by axiomatizing properties apparent from the behavior of the model, and discuss improvements to that axiom system. We then present another axiomatization, this time by axiomatizing the properties of quantifier elimination. We discuss improvements to this system, and show how this system and various subsystems thereof are equivalent to our first axiomatization and corresponding subsystems thereof.

In our original construction, the Kripke models contain every endomorphism of the underlying classical JRS structure and these theories admit quantifier elimination. The classical structure itself can be viewed as a Kripke model as well; one wherein the only morphism is the identity morphism. This intuitionistic theory (that also happens to be classical) also admits quantifier elimination. In Chapter 3, we determine whether other monoids of endomorphisms of JRS models give rise to single-node Kripke models whose theories admit quantifier elimination. We first show that if two monoids of endomorphisms have the same collection of finite subgraphs, then the intuitionistic theories of the corresponding Kripke models are the same. In so doing, we introduce a generalization of "bisimilarity", the standard notion of equivalence between Kripke models. We then give sufficient conditions on a monoid so that the intuitionistic theory of the corresponding Kripke model admits quantifier elimination.

Finally, in Chapter 4, we investigate the ramifications of adding nullary predicates to the language. Syntactically, this gives us a way to combine a classical JRS theory and its corresponding intuitionistic theory. Semantically, this gives us an opportunity to generalize our work to Kripke models with more than one node structure; specifically, where the node structures all satisfy the same "core" JRS theory, only varying in which nullary predicates they satisfy. We show that the theories of these multi-node models again admit quantifier elimination. For a given JRS theory in a language with nullary predicates, we construct a model that is in some sense universal. That is, all other multi-node Kripke models meeting certain other conditions in some sense embed into our model. We briefly discuss a theory that incorporates both a classical JRS theory and the corresponding intuitionistic version.

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Chapter 1

Quantifier Elimination for a Class of Intuitionistic Theories

Up to minor structural changes, this chapter is joint work with Ben Ellison, Jonathan Fleischmann, and Wim Ruitenburg, published as [10].

Abstract: From classical, Fraïssé-homogeneous, ($\leq \omega$)-categorical theories over finite relational languages, we construct intuitionistic theories that are complete, prove negations of classical tautologies, and admit quantifier elimination. We also determine the intuitionistic universal fragments of these theories.

1.1 Introduction

It is often assumed that intuitionistic theories that admit quantifier elimination are either very close to the classical situation or are essentially non-existent. We show that this is not the case. We present a straightforward method that converts a broad class of classical theories that admit quantifier elimination into intuitionistic ones.

Intuitionistic quantifier elimination has been studied before, see [22], [20], and [1] for example. Smoryński in [22] and Bagheri in [1] focus on intuitionistic theories that are in some ways nearly classical. Instead, we expand on the work in [20] and, in general, eliminate quantifiers in *very intuitionistic* theories, which in our case are theories that prove the negation of certain classical tautologies. Specifically, we start with a well known class of classical theories over finite relational languages that admit quantifier elimination, are Fraïssé-homogeneous, and are $(\leq \omega)$-categorical. We call these theories JRS theories, after Jaśkowski, Rabin and Scott, as explained in the next subsection. We construct intuitionistic variations of the JRS theories and show these new theories retain the properties of completeness (Theorem 1.16) and quantifier elimination (Theorem 1.33), but in general are very intuitionistic. We show that if the morphism structure of the canonical Kripke model is sufficiently rich, then all formulas are equivalent to particularly simple quantifier-free formulas (Theorem 1.34). Our techniques for proving intuitionistic quantifier elimination are classical.

In Section 1.5, as part of a deeper investigation into the idea of an intuitionistic model complete theory, we use the techniques and definitions of [9] to find the intuitionistic universal fragment of an intuitionistic JRS theory (Theorem 1.43). In the general intuitionistic case, quantifier-free formulas need not be universal formulas, in a sense that will be explained in Section 1.5. In our case, however, we show that all formulas are equivalent to quantifier-free, (intuitionistic) universal formulas (Theorem 1.38).

The authors thank Asher Kach for his helpful suggestions.

1.2 Classical JRS Theories

We review a special family of classical theories that admit quantifier elimination. We use the single turnstile $\vdash$ for "intuitionistically proves"; when we wish to indicate a classical proof, we use the $\vdash_c$ notation. Similarly, we write $\mathrm{Th}(\cdot)$ for the intuitionistic theory generated by a set of formulas or a structure, and $\mathrm{Th}_c(\cdot)$ for the classical theory. We write $\Gamma_\forall$ to represent the (classical) universal fragment of the theory Γ. A theory Γ is **consistent** if $\bot \notin \Gamma$.

1.2.1 What is a JRS Theory?

We consider languages $\mathcal{L}$ that have only finitely many predicates $\{R_i\}_{i<r}$, all of positive arity. We use $\top$, $\bot$, $\wedge$, $\vee$, $\to$, $=$, $\exists$, and $\forall$ to form formulas of $\mathcal{L}$. The symbols $\top$ and $\bot$ are nullary logical operators as well as atoms. The negation symbol, $\neg\varphi$, abbreviates $\varphi \to \bot$.

Definition 1.1. *Given a tuple $\mathbf{x} = x_0, x_1, \ldots, x_{n-1}$ of variables, consider the following definitions.*

1. *The set $\mathcal{A}\mathrm{t}(\mathbf{x})$ is the collection of all atoms with all free variables from $\mathbf{x}$.*
2. *The set $\mathcal{A}\mathrm{t}^{\pm}(\mathbf{x})$ is the collection of all atoms and negated atoms in $\mathbf{x}$.*

(Note that both of these sets are finite.)

3. *An $\mathcal{A}\mathrm{t}^{\pm}(\mathbf{x})$-**type** is a subset $t \subseteq \mathcal{A}\mathrm{t}^{\pm}(\mathbf{x})$ such that its conjunction, $\bigwedge t$, also written π_t or $\pi_t(\mathbf{x})$, is consistent.*
4. *We write t^+ for the sub-collection of atoms in t.*
5. *We define the formula π_t^+ to be the conjunction of atoms of t^+, and σ_t^- to be the disjunction of atoms whose negations occur in t.*

(So $\pi_t \leftrightarrow (\pi_t^+ \wedge \neg\sigma_t^-)$ is a tautology.)

6. *The formula* π_t *is called an* $\mathcal{A}\mathrm{t}^{\pm}(\mathbf{x})$-**description**. *A maximal* $\mathcal{A}\mathrm{t}^{\pm}(\mathbf{x})$*-type is called a* **complete** $\mathcal{A}\mathrm{t}^{\pm}(\mathbf{x})$**-type**, *and its corresponding formula* π_t *a* **complete** $\mathcal{A}\mathrm{t}^{\pm}(\mathbf{x})$-**description**.

(Each atom of $\mathcal{A}\mathrm{t}(\mathbf{x})$ *or its negation occurs in a complete* $\mathcal{A}\mathrm{t}^{\pm}(\mathbf{x})$*-type.)*

7. *Given a model* $\mathfrak{A}$ *and a tuple* $\mathbf{a} \in A$, $\mathbf{a}$ *satisfies the complete* $\mathcal{A}\mathrm{t}^{\pm}(\mathbf{x})$*-type* $\mathrm{tp}_{\mathbf{a}} = (\mathrm{Th}_c(\mathfrak{A}) \cap \mathcal{A}\mathrm{t}^{\pm}(\mathbf{a}))[\mathbf{a}/\mathbf{x}]$, *where* $\mathrm{Th}_c(\mathfrak{A})$ *is the theory of* $\mathfrak{A}$ *over the language* $\mathcal{L}(A)$.

(So $\mathrm{tp}_{\mathbf{a}} = \{\delta(\mathbf{x}) \ : \ \delta \in \mathcal{A}\mathrm{t}^{\pm}(\mathbf{x})$ *and* $\mathfrak{A} \models \delta(\mathbf{a})\}$.*)*

8. *Suppose* $n \geq 0$ *(where* $n = |\mathbf{x}|$*). Up to isomorphism, a complete* $\mathcal{A}\mathrm{t}^{\pm}(\mathbf{x})$*-type* t *has a unique smallest model. Specifically, let* $\mathfrak{A}_t$ *be the model formed from the variables* $\{x_i\}_{i<n}$ *by taking equivalence classes modulo the equivalence relation* $x_i \sim x_j$ *defined by* $(x_i = x_j) \in t$. *We write* $\overline{x_i}$ *or* a_i *for the equivalence class of* x_i.

(So given $\mathbf{a} = a_0, \ldots, a_{n-1}$ *and an atom* $\delta(\mathbf{x})$, $\mathfrak{A}_t \models \delta(\mathbf{a})$ *if and only if* $\delta(\mathbf{x}) \in t$, *and* $\mathfrak{A}_t \models \pi_t(\mathbf{a})$.*)*

9. *The size* $|A_t|$ *of the model* $\mathfrak{A}_t$ *is called the* **level** *of* t. *We allow the empty structure.*

10. *Let* u *be an* $\mathcal{A}\mathrm{t}^{\pm}(\mathbf{x}x_n)$*-type. Define* $d(u) = u \cap \mathcal{A}\mathrm{t}^{\pm}(\mathbf{x}) = u \restriction \mathbf{x}$.

(Note that $d(u)$ *is an* $\mathcal{A}\mathrm{t}^{\pm}(\mathbf{x})$*-type. If* u *is a complete* $\mathcal{A}\mathrm{t}^{\pm}(\mathbf{x}x_n)$*-type, then* $d(u)$ *is a complete* $\mathcal{A}\mathrm{t}^{\pm}(\mathbf{x})$*-type.)*

11. *Given a complete* $\mathcal{A}\mathrm{t}^{\pm}(\mathbf{x}x_n)$*-type* u, *define* δ_u *to be the sentence*

$$\forall \mathbf{x}(\pi_{d(u)} \rightarrow \exists x_n \pi_u).$$

We call such a sentence a JRS **sentence**.

12. *A (consistent) theory* Γ *over* $\mathcal{L}$ *is called a* JRS **theory** *if for all* $\mathbf{x}x_n$ *and all complete* $\mathcal{A}\mathrm{t}^{\pm}(\mathbf{x}x_n)$*-types* u *that are consistent with* Γ *(that is,* $\Gamma \cup \{\exists \mathbf{x}x_n \pi_u\}$ *is consistent, or* $\Gamma_\forall \nvdash \forall \mathbf{x}x_n \neg \pi_u$*), we have* $\delta_u \in \Gamma$.

As indicated by Bankston [2, page 962], this is not the first time that JRS theories and sentences have been studied. Gaifman attributes these sentences to Rabin and Scott, see [12, page 15], while Lynch attributes them to Jaśkowski, see [16, page 94], hence our choice of name.

1.2.2 Classical Quantifier Elimination

The following are some well known facts about JRS theories.

Definition 1.2. *A structure* $\mathfrak{A}$ *is* **Fraïssé homogeneous** *if isomorphisms between finite submodels of* $\mathfrak{A}$ *extend to automorphisms of* $\mathfrak{A}$.

Theorem 1.3. *Let* Γ *be a* JRS *theory. Then, up to isomorphism,* Γ *has exactly one model of size* $\leq \omega$*. Additionally, this model is Fraïssé homogeneous.*

Proof. The proof uses the axioms δ_u to complete a standard back and forth construction to extend finite isomorphisms to automorphisms. □

Definition 1.4. *An existential formula is a* **primitive formula** *if its quantifier-free part is a conjunction of atoms and negated atoms.*

Theorem 1.5. *Let* Γ *be a* JRS *theory, and let* $\exists x_n \varphi(\mathbf{x}x_n)$ *be a primitive formula. Then* $\Gamma \vdash_c \exists x_n \varphi \leftrightarrow \bigvee_{s \in S} \pi_{d(s)}$*, where*

$$S = \{s \ : \ s \textit{ is a complete } \mathcal{A}\mathrm{t}^{\pm}(\mathbf{x}x_n)\textit{-type consistent with } \Gamma \textit{ and } \Gamma \vdash_c \pi_s \to \varphi\}.$$

In particular, JRS *theories admit quantifier elimination.*

Proof. The formula $\exists x_n \varphi$ is equivalent to $\bigvee_{s\in S} \exists x_n \pi_s$, where an empty disjunction is identified with $\bot$. Apply the JRS sentences of Γ: $\exists x_n \varphi$ is equivalent to $\bigvee_{s\in S} \pi_{d(s)}$. $\square$

By the techniques in [13], Henson shows that there are continuum many JRS theories, even if the language has only one binary predicate. The work [2] of Bankston and Ruitenburg offers other construction techniques for JRS theories. Countable JRS theories can be built via certain types of games, and can also be viewed as theories whose tree of finite substructures satisfies certain properties, see [2, Theorem 5.7]. That is, given a theory Γ, form the following rooted tree T_Γ of types: for each $\mathbf{x} = x_0, \ldots, x_{n-1}$, take all complete $\mathcal{A}\mathrm{t}^{\pm}(\mathbf{x})$-types of level n that are consistent with Γ (each such type essentially contains $\bigwedge_{i<j<n} x_i \neq x_j$). When we order these types by set inclusion, we get a tree with the minimal type $\{\top, \neg\bot\}$ as its root, and with finitely many nodes at each level. Obviously, T_Γ is uniquely determined by the universal fragment $\Gamma_\forall$ of Γ.

Definition 1.6. *Given a universal theory* Π*, we define the* JRS **extension** Γ *of* Π *as the theory axiomatizable by* Π *and all* JRS *sentences* δ_u *for which* $\Pi \nvdash \forall \mathbf{x} \neg \pi_u$.

For a given universal theory Π, the consistency of the JRS extension is nicely expressible as a model-theoretic property of the collection of finite substructures $\mathfrak{A}_t$ of Π.

Definition 1.7. *A class of models* K *has the* **amalgamation** *property if for all models* $\mathfrak{A}$*,* $\mathfrak{B}$*, and* $\mathfrak{C}$ *in* K *where* $\mathfrak{A}$ *embeds in* $\mathfrak{B}$ *and* $\mathfrak{A}$ *embeds in* $\mathfrak{C}$*, there is a model* $\mathfrak{D}$ *in* K *such that* $\mathfrak{B}$ *embeds in* $\mathfrak{D}$*,* $\mathfrak{C}$ *embeds in* $\mathfrak{D}$*, and this diagram commutes.*

If K includes the empty structure, then the amalgamation property immediately implies the joint embedding property. This particularly applies to Theorem 1.8.

Theorem 1.8. *The* JRS *extension* Γ *of a universal theory* Π *is consistent if and only if the collection of models of the form* $\mathfrak{A}_t$*, for* $t \in T_\Pi$*, has the amalgamation property. If* Γ *is consistent, then* $\Gamma_\forall = \Pi$*.*

Proof. First, suppose Γ is consistent. Let $\mathfrak{A}$ be the unique (up to isomorphism) model of Γ of size $\leq \omega$. Consider finite models $\mathfrak{A}_t$, $\mathfrak{A}_u$, and $\mathfrak{A}_v$ of $\Gamma_\forall$ and suppose that $\mathfrak{A}_t$ embeds in both $\mathfrak{A}_u$ and $\mathfrak{A}_v$. By an inductive argument, we may assume that u and v are complete $\mathcal{A}\mathrm{t}^{\pm}(\mathbf{x}x_n)$-types and that t is a complete $\mathcal{A}\mathrm{t}^{\pm}(\mathbf{x})$-type. For some $\mathbf{a} \in A$, $\mathfrak{A}$ satisfies $\pi_t(\mathbf{a})$, δ_u and δ_v, so we have $\mathfrak{A} \models \exists x \pi_u(\mathbf{a}x) \wedge \exists x \pi_v(\mathbf{a}x)$. Fix $\mathbf{a}$, b and c such that $\mathfrak{A} \models \pi_u(\mathbf{a}b) \wedge \pi_v(\mathbf{a}c)$. Let $w = \mathrm{tp}_{\mathbf{a}bc}$. Then $\mathfrak{A}_w$ is the amalgam of $\mathfrak{A}_u$ and $\mathfrak{A}_v$ over $\mathfrak{A}_t$.

Conversely, suppose that the collection of models of Π of the form $\mathfrak{A}_t$ has the amalgamation property. We sketch a construction of a model $\mathfrak{A}$ of Γ as the limit of an ω-chain of models of the form $\mathfrak{A}_t$. Suppose we have a model $\mathfrak{A}_t$ of size n. For each complete $\mathcal{A}\mathrm{t}^{\pm}(\mathbf{x}x_n)$-type u consistent with Π and for all $\mathbf{a} \in A_t$ such that $\mathfrak{A}_t \models \pi_{d(u)}(\mathbf{a})$ there is an amalgam $\mathfrak{A}_{(u,\mathbf{a})}$ of $\mathfrak{A}_t$ and $\mathfrak{A}_u$ over $\mathfrak{A}_{d(u)}$. As the next model in the ω-chain, take the amalgam of all $\mathfrak{A}_{(u,\mathbf{a})}$ over $\mathfrak{A}_t$. So Γ is consistent.

For the last claim, it suffices to show that every finite structure of Π embeds into $\mathfrak{A}$, the unique largest model of size $\leq \omega$. Proceed by induction on the number of free variables in complete types consistent with Π. If u is a complete $\mathcal{A}\mathrm{t}^{\pm}$-type consistent with Π, then so is $d(u)$. By the inductive hypothesis, $\mathfrak{A}_{d(u)}$ embeds into $\mathfrak{A}$. By the JRS axiom δ_u, $\mathfrak{A}_u$ also embeds into $\mathfrak{A}$. $\square$

1.2.3 Classical Examples

We present some examples of JRS theories, and construction methods of new JRS theories from old ones.

Example 1.9. *Let $\mathcal{L}$ be any language with finitely many predicate symbols of positive arity, and set Π to be the minimal "empty" theory. Since all finite structures are allowed, amalgamation is obvious. By Theorem 1.8, the* JRS *extension of Π is consistent. This is an example of Burris' "theory of everything" [4].*

Example 1.10. *Let $\mathcal{L}$ be the minimal language (equality is the only relation). The theory $\Gamma = \Gamma_{\mathrm{e}}$ is the theory of infinite sets, with $\Gamma_\forall$ the "empty" theory. The tree T_Γ has just one node $t \supseteq \{x_i = x_j \to \bot \ : \ i < j < n\}$ at each level n.*

Example 1.11. *Let $\mathcal{L}$ be the language based on a new predicate $x \neq y$ for inequality. The theory of infinite sets $\Gamma = \Gamma_{\mathrm{ne}}$ has universal fragment axiomatizable by $x \neq y \leftrightarrow (x = y \to \bot)$. This direct translation makes Γ_{ne} "as* JRS *as" Γ_{e}.*

Given a theory Γ, we write Γ_{UH} for the theory axiomatizable by its universal Horn fragment. Recall that models of Γ_{UH} are, up to isomorphism, submodels of products of models of Γ. If Γ is a JRS theory, then it is companionable with **few existential formulas**, that is, for each $\mathbf{x}$, there are only finitely many inequivalent (over Γ) existential formulas with variables from $\mathbf{x}$. So Γ_{UH} has a model companion $(\Gamma_{\mathrm{UH}})^*$ by Burris and Werner's work [5].

Example 1.12. *It is a simple exercise to show that the theory of the random graph Γ_{g} is a* JRS *theory such that $(\Gamma_{\mathrm{g}})_{\mathrm{UH}} = (\Gamma_{\mathrm{ne}})_{\mathrm{UH}}$ (where we identify the single binary predicate R with the binary predicate $\neq$). Since Γ_{g} is model complete, $\Gamma_{\mathrm{g}} = ((\Gamma_{\mathrm{ne}})_{\mathrm{UH}})^*$.*

Comparing this with $\Gamma_{\mathrm{e}} = ((\Gamma_{\mathrm{e}})_{\mathrm{UH}})^*$ *shows that seemingly trivial changes to language may significantly affect the derived universal Horn theories and their companions.*

Example 1.13. *Let* $\mathcal{L}$ *be the language based on* $x \leq y$*. The theory* Γ_{lo} *of dense linear order without endpoints is a well-known* JRS *theory.*

Example 1.14. *Let* $\mathcal{L}$ *be the language based on* $x \leq y$*. Let* Γ_{p} *be the theory of the random poset. Then it is a standard exercise to show* $\Gamma_{\mathrm{p}} = ((\Gamma_{\mathrm{lo}})_{\mathrm{UH}})^*$ *(see [8, page 132], for example). Additionally,* $\Gamma_{\mathrm{p}} = ((\Delta)_{\mathrm{UH}})^*$ *where* Δ *is the non-*JRS *but obviously model complete trivial theory of a two-node linear order.*

Note that $(\Gamma_{\mathrm{UH}})^*$ need not be a JRS theory, even if Γ is the JRS theory of a finite model.

1.3 Intuitionistic Theories from JRS Theories

Given a (classical) JRS theory Γ_{JRS} and its unique (up to isomorphism) model $\mathfrak{A}_{\mathrm{JRS}}$ of size $\leq \omega$, we construct the Kripke model $\mathfrak{A}_{\mathrm{M}}$ as follows.

Definition 1.15.

1. *Following notational conventions in [9], our Kripke models are functors from small categories to the category of* $\mathcal{L}$*-structures and morphisms.*
2. *Morphisms, called homomorphisms in [7], preserve the truth of atoms, but not necessarily of negated atoms. We write morphisms as upper left superscripts and compose on the left. So* ${}^{gf}\mathbf{a}$ *represents the result of first applying the morphism* f *and then the morphism* g *to the tuple* $\mathbf{a}$*.*
3. *The underlying category of* $\mathfrak{A}_{\mathrm{M}}$ *consists of a single node with associated node structure* $\mathfrak{A}_{\mathrm{JRS}}$*.*

4. We include all morphisms from $\mathfrak{A}_{\mathrm{JRS}}$ to $\mathfrak{A}_{\mathrm{JRS}}$ as arrows.

(Technically speaking, the functor $\mathfrak{A}$ takes the arrow f from node k to node m in the small category to the morphism $\mathfrak{A}f$ from classical structure $\mathfrak{A}_k$ to the classical structure $\mathfrak{A}_m$. Whenever possible, we will ignore this distinction and use f to denote the morphism from $\mathfrak{A}_k$ to $\mathfrak{A}_m$, as we are not usually concerned with the structure of the small category itself.)

5. Let Γ_{M} be the intuitionistic theory of $\mathfrak{A}_{\mathrm{M}}$.

We can choose $\mathfrak{A}_{\mathrm{M}}$ to be countable and get the same theory Γ_{M}. Instead of including all morphisms, let $\mathfrak{A}'_{\mathrm{M}}$ have a single node structure $\mathfrak{A}_{\mathrm{JRS}}$ and include only a collection of morphisms closed under composition such that every finite graph of an endomorphism of $\mathfrak{A}_{\mathrm{JRS}}$ has a complete endomorphism extension in the collection. A straightforward proof by induction on sentence complexity shows that $\mathfrak{A}_{\mathrm{M}}$ and $\mathfrak{A}'_{\mathrm{M}}$ have the same intuitionistic theory Γ_{M}. So our Kripke model can be chosen countable - take a category of countably many morphisms and a single countable object. See Corollary 3.13 and Section 3.2 for a more general discussion.

Theorem 1.16. Γ_{M} *is complete.*

Proof. Let φ be an $\mathcal{L}$-sentence. If $\mathfrak{A}_{\mathrm{M}} \Vdash \varphi$, then we are done. Otherwise, $\mathfrak{A}_{\mathrm{M}} \nVdash \varphi$. But we have only one node, so $\mathfrak{A}_{\mathrm{M}} \Vdash \neg\varphi$. $\square$

Theorem 1.16 in no way implies that Γ_{M} proves classical logic. For example, if there is an endomorphism of $\mathfrak{A}_{\mathrm{JRS}}$ which is not an embedding, then for some R_i and some $\mathbf{a}$ we have $\mathfrak{A}_{\mathrm{M}} \nVdash R_i(\mathbf{a}) \vee \neg R_i(\mathbf{a})$, so $\mathfrak{A}_{\mathrm{M}} \Vdash \neg\forall\mathbf{x}(R_i(\mathbf{x}) \vee \neg R_i(\mathbf{x}))$. In [20], Ruitenburg introduces one concept of a **very intuitionistic** theory to distinguish theories that are somehow even more “not classical”. The two theories in [20], involving equality and linear order, are both examples of very intuitionistic theories. In general, suppose that instead of just one non-embedding endomorphism, we have two endomorphisms f and g, tuples $\mathbf{a}$ and $\mathbf{b}$, and formulas φ and ψ such that $\mathfrak{A}_{\mathrm{M}} \Vdash \varphi({}^f\mathbf{a})$ and $\mathfrak{A}_{\mathrm{M}} \nVdash \psi({}^f\mathbf{b})$, as well as $\mathfrak{A}_{\mathrm{M}} \nVdash \varphi({}^g\mathbf{a})$ and $\mathfrak{A}_{\mathrm{M}} \Vdash \psi({}^g\mathbf{b})$, as holds for the two examples from [20]. Then $\Gamma_{\mathrm{M}} \vdash \neg\forall\mathbf{xy}((\varphi(\mathbf{x}) \to \psi(\mathbf{y})) \vee (\psi(\mathbf{y}) \to \varphi(\mathbf{x})))$, and therefore Γ_{M} is a very intuitionistic theory.

However, if $\mathfrak{A}_{\mathrm{JRS}}$ is such that every endomorphism is also an embedding, then the theory Γ_{M} is not of new interest to us, since:

Theorem 1.17. *If all endomorphisms of $\mathfrak{A}_{\mathrm{JRS}}$ are embeddings, then $\Gamma_{\mathrm{M}} = \Gamma_{\mathrm{JRS}}$, and so Γ_{M} is a classical theory.*

Proof. Since Γ_{JRS} admits quantifier elimination, it is model complete. Thus, all embeddings of Γ_{JRS} models are elementary embeddings. Apply Theorem 1.44 in the Appendix. □

The examples from [20], as well as the examples from Subsection 1.2.3 satisfy the following special condition.

Definition 1.18. 1. *We say that a model $\mathfrak{A}$ is* **morphism homogeneous** *if whenever $\mathbf{a}, \mathbf{b} \in A$ are such that $\mathrm{tp}^+_{\mathbf{a}} \subseteq \mathrm{tp}^+_{\mathbf{b}}$ then there is an endomorphism f of $\mathfrak{A}$ such that $f(\mathbf{a}) = \mathbf{b}$.*

2. *A classical* JRS *theory Γ_{JRS} is* **morphism homogeneous** *if its unique countable model $\mathfrak{A}_{\mathrm{JRS}}$ is.*

We show in Theorem 1.34 that if $\mathfrak{A}_{\mathrm{JRS}}$ is morphism homogeneous, then Γ_{M} admits a particularly elegant kind of quantifier elimination.

Example 1.19. *Not all $\mathfrak{A}_{\mathrm{JRS}}$ are morphism homogeneous. Let $\mathcal{L}$ be the language with a unary predicate $P(x)$ and a binary predicate $x < y$, and let Γ_{JRS} be the (classical) theory of the finite model $\mathfrak{A}_{\mathrm{JRS}}$ with domain $A_{\mathrm{JRS}} = \{a, b\}$ such that $\mathfrak{A}_{\mathrm{JRS}} \models \neg P(a) \wedge P(b) \wedge (a < b)$ and no other nontrivial atomic sentences. We have that $\mathrm{tp}_a^+ \subseteq \mathrm{tp}_b^+$ (in fact, $\mathrm{tp}_b^+ = \mathrm{tp}_a^+ \cup \{P(x)\}$). However, there is no morphism of $\mathfrak{A}_{\mathrm{JRS}}$ taking a to b. That is, assume f is a morphism such that $f(a) = b$. Then we must have $\mathfrak{A}_{\mathrm{JRS}} \models f(a) < f(b)$. But this is not true if $f(a) = b$, as $\mathfrak{A}_{\mathrm{JRS}} \models \forall x\ \neg(b < x)$.*

1.4 Intuitionistic Quantifier Elimination in Γ_{M}

Recall that a theory has few (quantifier-free) formulas if for all $\mathbf{x} = x_0, x_1, ..., x_{n-1}$ there are finitely many non-equivalent (quantifier-free) formulas with all free variables from among $\mathbf{x}$. All classical theories over the finite relational language $\mathcal{L}$ have few quantifier-free formulas. So by quantifier elimination, Γ_{JRS} has few formulas. We show that the intuitionistic theory Γ_{M} admits quantifier elimination and also has few formulas. Our methods are classical.

Given a finite list of variables $\mathbf{x} = x_0, x_1, \ldots, x_{n-1}$, we first consider the complexity over Γ_{M} of the collection of quantifier-free formulas with all free variables from $\mathbf{x}$.

Definition 1.20. *1. Let $\mathfrak{C}(\mathbf{x})$ be the following Kripke model. As nodes for the underlying category $\mathbf{C}(\mathbf{x})$ we take all complete $\mathcal{A}\mathrm{t}^{\pm}(\mathbf{x})$-types t that are (classically) consistent with Γ_{JRS}.*

2. *We turn* $\mathbf{C}(\mathbf{x})$ *into a poset category as follows. Given a pair of nodes* t *and* u*, we set* $t \leq u$ *exactly when there are* $\mathbf{a} \in A_{\mathrm{JRS}}$ *and an endomorphism* f *of* $\mathfrak{A}_{\mathrm{JRS}}$ *such that* $t = \mathrm{tp}_{\mathbf{a}}$ *and* $u = \mathrm{tp}_{^f\mathbf{a}}$*. That is,* $\mathfrak{A}_{\mathrm{JRS}} \models \pi_t(\mathbf{a}) \wedge \pi_u(^f\mathbf{a})$*.*

(Note that $t \leq u$ *implies* $t^+ \subseteq u^+$*.)*

3. *To each node* t *we associate the finite classical model* $\mathfrak{A}_t$*.*

4. *If* $t \leq u$*, then the morphism sends the equivalence class* $\overline{x_i}(t)$ *of* x_i *in* $\mathfrak{A}_t$ *to the equivalence class* $\overline{x_i}(u)$ *of* x_i *in* $\mathfrak{A}_u$*. We write* $\overline{x_i}$ *for the "global" element* $t \mapsto \overline{x_i}(t)$ *of* $\mathfrak{C}(\mathbf{x})$*.*

The collection of nodes $|\mathbf{C}(\mathbf{x})|$ is finite. Note that $\mathfrak{A}_{\mathrm{JRS}}$ is morphism homogeneous exactly when $t^+ \subseteq u^+$ implies $t \leq u$ for every t and u in $|\mathbf{C}(\mathbf{x})|$.

Lemma 1.21. *Let* $\varphi(\mathbf{x})$ *be quantifier-free, and* $\mathbf{a} \in A_{\mathrm{JRS}}$*. Then* $\mathfrak{A}_{\mathrm{M}} \Vdash \varphi(\mathbf{a})$ *if and only if* $\mathrm{tp}_{\mathbf{a}} \Vdash \varphi(\overline{\mathbf{x}}(\mathrm{tp}_{\mathbf{a}}))$*.*

Proof. We complete the proof by induction on the complexity of φ for all elements $\mathbf{a}$ simultaneously. The case for atoms and the induction steps for $\wedge$ and $\vee$ are easy. Let φ equal $\psi \to \theta$.

Suppose $\mathfrak{A}_{\mathrm{M}} \Vdash \psi(\mathbf{a}) \to \theta(\mathbf{a})$. Let $\mathrm{tp}_{\mathbf{a}} \leq u$ be such that $u \Vdash \psi(\overline{\mathbf{x}}(u))$. It suffices to show that $u \Vdash \theta(\overline{\mathbf{x}}(u))$. There is an endomorphism f such that $u = \mathrm{tp}_{^f\mathbf{a}}$. By the inductive hypothesis, $\mathfrak{A}_{\mathrm{M}} \Vdash \psi(^f\mathbf{a})$. By supposition, $\mathfrak{A}_{\mathrm{M}} \Vdash \theta(^f\mathbf{a})$. So again by the inductive hypothesis, $u \Vdash \theta(\overline{\mathbf{x}}(u))$.

Conversely, suppose $\mathrm{tp}_{\mathbf{a}} \Vdash \psi(\overline{\mathbf{x}}(\mathrm{tp}_{\mathbf{a}})) \to \theta(\overline{\mathbf{x}}(\mathrm{tp}_{\mathbf{a}}))$. Let f be an endomorphism such that $\mathfrak{A}_{\mathrm{M}} \Vdash \psi(^f\mathbf{a})$. It suffices to show $\mathfrak{A}_{\mathrm{M}} \Vdash \theta(^f\mathbf{a})$. By the inductive hypothesis, $\mathrm{tp}_{^f\mathbf{a}} \Vdash \psi(\overline{\mathbf{x}}(\mathrm{tp}_{^f\mathbf{a}}))$. By Definition 1.20.2, $\mathrm{tp}_{\mathbf{a}} \leq \mathrm{tp}_{^f\mathbf{a}}$ so, by supposition,

$\text{tp}_{f_{\mathbf{a}}} \Vdash \theta(\overline{\mathbf{x}}(\text{tp}_{f_{\mathbf{a}}}))$. Again by the inductive hypothesis, $\mathfrak{A}_{\text{M}} \Vdash \theta(^f\mathbf{a})$. $\square$

Definition 1.22. *For each quantifier-free formula $\varphi(\mathbf{x})$, define $[\![\varphi(\overline{\mathbf{x}})]\!]$ be the set $\{t \in |\mathbf{C}(\mathbf{x})| \ : \ t \Vdash \varphi(\overline{\mathbf{x}}(t))\}$.*

We can rewrite Lemma 1.21 above as: $\mathfrak{A}_{\text{M}} \Vdash \varphi(\mathbf{a})$ exactly when $\text{tp}_{\mathbf{a}} \in [\![\varphi(\overline{\mathbf{x}})]\!]$. The sets $[\![\varphi(\overline{\mathbf{x}})]\!]$ form a finite Heyting algebra of upward closed subsets of the poset $\mathbf{C}(\mathbf{x})$ given by:

$[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \cap [\![\psi]\!]$,

$[\![\varphi \vee \psi]\!] = [\![\varphi]\!] \cup [\![\psi]\!]$, and

$[\![\varphi]\!] \cap [\![\psi]\!] \subseteq [\![\theta]\!]$ if and only if $[\![\varphi]\!] \subseteq [\![\psi \to \theta]\!]$,

where we write $[\![\varphi]\!]$ as short for $[\![\varphi(\overline{\mathbf{x}})]\!]$, et cetera.

Definition 1.23.

1. *Subsets of the form $[\![\varphi(\overline{\mathbf{x}})]\!]$ are* **definable**.
2. *Upward closed subsets of $\mathbf{C}(\mathbf{x})$ form the* **open** *subsets of the usual poset topology.*

So definable subsets are open. Below we show that open subsets are definable.

Lemma 1.24. *For all quantifier-free formulas $\varphi(\mathbf{x})$ and $\psi(\mathbf{x})$ we have $\Gamma_{\text{M}} \vdash \forall \mathbf{x}(\varphi \to \psi)$ exactly when $[\![\varphi(\overline{\mathbf{x}})]\!] \subseteq [\![\psi(\overline{\mathbf{x}})]\!]$. Modulo provable equivalence over Γ_{M}, there are for each* $\mathbf{x}$ *only finitely many quantifier-free formulas with all free variables from* $\mathbf{x}$.

Proof. Suppose that $\mathfrak{A}_{\text{M}} \Vdash \forall \mathbf{x}(\varphi(\mathbf{x}) \to \psi(\mathbf{x}))$. Let $t \in [\![\varphi(\overline{\mathbf{x}})]\!]$. It suffices to show $t \in [\![\psi(\overline{\mathbf{x}})]\!]$. There is $\mathbf{a} \in A_{\text{JRS}}$ such that $t = \text{tp}_{\mathbf{a}}$. By Lemma 1.21, $\mathfrak{A}_{\text{M}} \Vdash \varphi(\mathbf{a})$. By supposition, $\mathfrak{A}_{\text{M}} \Vdash \psi(\mathbf{a})$. Again by Lemma 1.21, $\text{tp}_{\mathbf{a}} \in [\![\psi(\overline{\mathbf{x}})]\!]$.

Conversely, suppose $[\![\varphi(\overline{\mathbf{x}})]\!] \subseteq [\![\psi(\overline{\mathbf{x}})]\!]$. Let $\mathbf{a} \in A_{\mathrm{JRS}}$ be such that $\mathfrak{A}_{\mathrm{M}} \Vdash \varphi(\mathbf{a})$. It suffices to show $\mathfrak{A}_{\mathrm{M}} \Vdash \psi(\mathbf{a})$. By Lemma 1.21, $\mathrm{tp}_{\mathbf{a}} \in [\![\varphi(\overline{\mathbf{x}})]\!]$. By supposition, $\mathrm{tp}_{\mathbf{a}} \in [\![\psi(\overline{\mathbf{x}})]\!]$. By Lemma 1.21 we get $\mathfrak{A}_{\mathrm{M}} \Vdash \psi(\mathbf{a})$.

So $[\![\varphi]\!] = [\![\psi]\!]$ exactly when $\Gamma_{\mathrm{M}} \vdash \forall \mathbf{x}(\varphi \leftrightarrow \psi)$. The second claim now follows, as $|\mathbf{C}(\mathbf{x})|$ is finite. □

Given $t \in |\mathbf{C}(\mathbf{x})|$, we have the following definitions.

Definition 1.25. *1. Let $\hat{t}$ be the set $\{u \in |\mathbf{C}(\mathbf{x})| \,:\, t \leq u\}$.*

2. Let $\check{t}$ be the set $\{u \in |\mathbf{C}(\mathbf{x})| \,:\, u \not\leq t\}$.

So $\hat{t}$ is the smallest open subset containing t, and $\check{t}$ is the largest open subset not containing t. Clearly, $\hat{t} \subseteq [\![\pi_t^+(\overline{\mathbf{x}})]\!]$

Lemma 1.26. *Let $t \in |\mathbf{C}(\mathbf{x})|$. Then $\check{t} = [\![\pi_t^+(\overline{\mathbf{x}}) \to \sigma_t^-(\overline{\mathbf{x}})]\!]$.*

Proof. Suppose $s \leq t$. Then there are $\mathbf{a} \in A_{\mathrm{JRS}}$ and an endomorphism f such that $s = \mathrm{tp}_{\mathbf{a}}$ and $t = \mathrm{tp}_{^f\mathbf{a}}$. So $\mathfrak{A}_{\mathrm{M}} \Vdash \pi_t^+(^f\mathbf{a})$ and $\mathfrak{A}_{\mathrm{M}} \nVdash \sigma_t^-(^f\mathbf{a})$. Thus, $\mathfrak{A}_{\mathrm{M}} \nVdash \pi_t^+(\mathbf{a}) \to \sigma_t^-(\mathbf{a})$. By Lemma 1.21, $s = \mathrm{tp}_{\mathbf{a}} \notin [\![\pi_t^+(\overline{\mathbf{x}}) \to \sigma_t^-(\overline{\mathbf{x}})]\!]$.

Conversely, suppose $s \not\leq t$. There is $\mathbf{a} \in A_{\mathrm{JRS}}$ such that $s = \mathrm{tp}_{\mathbf{a}}$. It suffices to show that $\mathfrak{A}_{\mathrm{M}} \Vdash \pi_t^+(\mathbf{a}) \to \sigma_t^-(\mathbf{a})$. Let $s \leq u$ and let f be an endomorphism such that $u = \mathrm{tp}_{^f\mathbf{a}}$ and $\mathfrak{A}_{\mathrm{M}} \Vdash \pi_t^+(^f\mathbf{a})$. Then by supposition, $u \neq t$ and therefore there is an atomic formula δ such that $(\neg\delta) \in t$ and $\mathfrak{A}_{\mathrm{M}} \Vdash \delta(f(\mathbf{a}))$. So $\mathfrak{A}_{\mathrm{M}} \Vdash \sigma_t^-(^f\mathbf{a})$. □

For $t \in |\mathbf{C}(\mathbf{x})|$, the following formulas play a crucial role in quantifier elimination.

Definition 1.27. *1. Let $\bigwedge\{\pi_u^+ \to \sigma_u^- \,:\, u \in |\mathbf{C}(\mathbf{x})|, t^+ \subseteq u^+ \text{ and } t \not\leq u\}$ be the formula ρ_t^-.*

2. Let $\pi_t^+ \wedge \rho_t^-$ *be the formula* ρ_t^+*.*

In some sense, the formula ρ_t^+ keeps track of the "global" morphism behavior of $\mathfrak{A}_{\mathrm{M}}$. Consider a tuple $\mathbf{a}$ such that $\mathfrak{A}_{\mathrm{M}} \Vdash \rho_t^+(\mathbf{a})$. Since $\pi_t^+(\mathbf{a})$ is forced, it is possible that $t = \mathrm{tp}_{\mathbf{a}}$. It is also possible that there is more positive information true of $\mathbf{a}$, that is, that $u = \mathrm{tp}_{\mathbf{a}}$ where $t^+ \subseteq u^+$. However, not every u such that $t^+ \subseteq u^+$ is a viable candidate for $\mathrm{tp}_{\mathbf{a}}$. In non-morphism homogeneous theories, there can be types u such that $t^+ \subseteq u^+$, but $t \not\leq u$. ("Locally" there is a morphism from the finite model $\mathfrak{A}_t$ to the finite model $\mathfrak{A}_u$, but this morphism does not extend to a global endomorphism of $\mathfrak{A}$. See Lemma 3.18 for a broader discussion of these concepts.) The formula ρ_t^+ excludes such u's as candidates for $\mathrm{tp}_{\mathbf{a}}$. That is, for such a u, if $\mathfrak{A}_{\mathrm{M}} \Vdash \rho_t^+(\mathbf{a}) \wedge \pi_u^+(\mathbf{a})$, then $\mathfrak{A}_{\mathrm{M}} \Vdash \sigma_u^-(\mathbf{a})$, whereby $u \neq \mathrm{tp}_{\mathbf{a}}$.

Note that over a morphism homogeneous theory, ρ_t^+ is equivalent to π_t^+. The following lemma shows that ρ_t^+ defines the upward closure of t in $\mathbf{C}(\mathbf{x})$.

Lemma 1.28. *Let* $t \in |\mathbf{C}(\mathbf{x})|$*. Then* $\hat{t} = [\![\rho_t^+(\overline{\mathbf{x}})]\!]$*. So all open subsets of* $\mathbf{C}(\mathbf{x})$ *are definable.*

Proof. To show $\hat{t} \subseteq [\![\rho_t^+(\overline{\mathbf{x}})]\!]$, it suffices to show $t \in [\![\rho_t^+(\overline{\mathbf{x}})]\!]$. Obviously, $t \in [\![\pi_t^+(\overline{\mathbf{x}})]\!]$. Let u be such that $t^+ \subseteq u^+$ and $t \not\leq u$. Then, by Lemma 1.26, $t \in [\![\pi_u^+(\overline{\mathbf{x}}) \to \sigma_u^-(\overline{\mathbf{x}})]\!]$. And thus $t \in [\![\rho_t^+(\overline{\mathbf{x}})]\!]$.

Conversely, suppose $v \in [\![\rho_t^+(\overline{\mathbf{x}})]\!]$. There is $\mathbf{a} \in A_{\mathrm{JRS}}$ such that $v = \mathrm{tp}_{\mathbf{a}}$. Then $\mathfrak{A}_{\mathrm{M}} \Vdash \rho_t^+(\mathbf{a})$. So $\mathfrak{A}_{\mathrm{M}} \Vdash \pi_t^+(\mathbf{a})$ and $t^+ \subseteq \mathrm{tp}_{\mathbf{a}}^+$. Let u be such that $t^+ \subseteq u^+$ and $t \not\leq u$. Then $\mathfrak{A}_{\mathrm{M}} \Vdash \pi_u^+(\mathbf{a}) \to \sigma_u^-(\mathbf{a})$. By Lemma 1.26, $\mathrm{tp}_{\mathbf{a}} \neq u$. Thus $t \leq \mathrm{tp}_{\mathbf{a}} = v$.

The second claim follows from the fact that all open sets are finite unions of $\hat{t}$'s. $\square$

Definition 1.29. *1. An open subset U is called* **prime** *if whenever U is the union $U = V \cup W$ of two open subsets, then $U = V$ or $U = W$.*

2. A prime open subset has **depth** *n if there is a sequence of prime open subsets $U_0 \subseteq U_1 \subseteq \ldots \subseteq U_n$ such that $U_i \neq U_{i+1}$ for all i and $U_n = U$, but there is no longer sequence with these properties.*

So the empty subset has depth 0. The following is now obvious.

Lemma 1.30. *In $\mathbf{C}(\mathbf{x})$, each open subset equals a finite union of prime open subsets. A nonempty open subset is prime if and only if it is of the form $\hat{t}$, for some $t \in |\mathbf{C}(\mathbf{x})|$.*

Proof. All open subsets in the poset topology are finite unions of sets of the form $\hat{t}$, so it suffices to prove that sets $\hat{t}$ are prime. This is immediate since $\hat{t} \subseteq U$ is equivalent to $t \in U$. $\square$

Corollary 1.31. *Over Γ_{M}, every quantifier-free formula φ is equivalent to the formula $\bigvee\{\rho_t^+ : t \in [\![\varphi]\!]\}$.*

Proof. Immediate from Lemmas 1.30 and 1.28. $\square$

Lemma 1.32. *For all formulas $\varphi(\mathbf{x}x_n)$, and for all $t \in \mathbf{C}(\mathbf{x}x_n)$, Γ_{M} includes the sentence:*

$$\forall \mathbf{x} x_n(\varphi \wedge \rho_t^+ \to (\sigma_t^- \vee \forall x_n(\rho_t^+ \to \varphi))).$$

Proof. Fix φ, $t \in \mathbf{C}(\mathbf{x}x_n)$ and $\mathbf{a}, b \in A_{\mathrm{JRS}}$ and suppose $\mathfrak{A}_{\mathrm{M}} \Vdash \varphi(\mathbf{a}b) \wedge \rho_t^+(\mathbf{a}b)$. If $\mathfrak{A}_{\mathrm{M}} \Vdash \sigma_t^-(\mathbf{a}b)$ then we are done, so suppose not. Then $t = \mathrm{tp}_{\mathbf{a}b}$. We need to show that for an arbitrary $c \in A_{\mathrm{JRS}}$ and an arbitrary endomorphism f, if $\mathfrak{A}_{\mathrm{M}} \Vdash \rho_t^+({}^f(\mathbf{a})c)$ then $\mathfrak{A}_{\mathrm{M}} \Vdash \varphi({}^f(\mathbf{a})c)$. Fix such an element c and endomorphism f. Then $\mathrm{tp}_{{}^f(\mathbf{a})c} \in \hat{t}$ by

Lemma 1.28. So $\mathrm{tp}_{\mathbf{a}b} \leq \mathrm{tp}_{{}^f(\mathbf{a})c}$ and there is a morphism g such that $\mathrm{tp}_{{}^g(\mathbf{a}b)} = \mathrm{tp}_{{}^f(\mathbf{a})c}$. By the first supposition, $\mathfrak{A}_{\mathrm{M}} \Vdash \varphi({}^g(\mathbf{a}b))$. By Fraïssé homogeneity, there is an automorphism h such that ${}^{hg}(\mathbf{a}b) = {}^f(\mathbf{a})c$, so $\mathfrak{A}_{\mathrm{M}} \Vdash \varphi({}^f(\mathbf{a})c)$. $\square$

We are now ready to prove our main result:

Theorem 1.33. *The theory Γ_{M} admits quantifier elimination.*

Proof. We eliminate quantifiers from formulas of the form $\varphi \wedge \theta$ where θ is quantifier-free (we recover all formulas by letting θ be $\top$). By Corollary 1.31, θ is equivalent to a formula of the form $\bigvee_{t \in S}\{\rho_t^+\}$ for some set $S \subseteq |\mathbf{C}(\mathbf{x})|$. Thus, each $\varphi \wedge \theta$ is equivalent to $\bigvee_{t \in S}\{\varphi \wedge \rho_t^+\}$. So it suffices to eliminate quantifiers from formulas of the form $\varphi \wedge \rho_t^+$, where $t \in S$. Fix such a formula, and proceed by induction on the depth of $[\![\rho_t^+]\!]$ and the number of free variables of φ.

Given $\varphi \wedge \rho_t^+$, if we have no free variables in φ, then by Theorem 1.16, $\varphi \wedge \rho_t^+$ is equivalent to a quantifier-free formula (namely ρ_t^+ or $\bot$). Otherwise, apply Lemma 1.32. There are two cases.

In the first case, we get $\varphi \wedge \rho_t^+ \wedge \sigma_t^-$. As above, we use Corollary 1.31 to rewrite $\varphi \wedge (\rho_t^+ \wedge \sigma_t^-)$ as $\bigvee_{u \in R}(\varphi \wedge \rho_u^+)$ for some set $R \subseteq |\mathbf{C}(\mathbf{x})|$. Since $\bigvee_{u \in R} \rho_u^+ \to (\rho_t^+ \wedge \sigma_t^-)$, each ρ_u^+ implies ρ_t^+. By Lemma 1.24, for each $u \in R$, $[\![\rho_u^+]\!] \subseteq [\![\rho_t^+]\!]$. Likewise, since each ρ_u^+ implies σ_t^-, $[\![\rho_u^+]\!] \subseteq [\![\sigma_t^-]\!]$. By Lemma 1.30, each $[\![\rho_u^+]\!]$ is prime, and therefore $[\![\rho_u^+]\!] \subseteq [\![\delta]\!]$ for some atom δ found in σ_t^-. So $[\![\rho_u^+]\!] \neq [\![\rho_t^+]\!]$. By our inductive hypothesis on depth, each $\varphi \wedge \rho_u^+$ is equivalent to a quantifier-free formula, and therefore $\varphi \wedge \rho_t^+$ is equivalent to a quantifier-free formula.

In the second case, we get $\varphi \wedge \rho_t^+ \wedge \forall x_n(\rho_t^+ \to \varphi)$, which is equivalent to the formula $\forall x_n(\rho_t^+ \to \varphi) \wedge \rho_t^+$. By the inductive hypothesis on free variables, this is equivalent to a quantifier-free formula. □

As a corollary we obtain the following.

Theorem 1.34. *Let* $\varphi(\mathbf{x})$ *be a formula. Over* Γ_{M}*,* φ *is equivalent to a disjunction of formulas of the form* ρ_t^+ *with* $t \in |\mathbf{C}(\mathbf{x})|$*. If* Γ_{JRS} *is morphism homogeneous, then* φ *is equivalent to a disjunction of conjunctions of atoms, specifically,* φ *is equivalent to a disjunction of formulas of the form* π_t^+*, with* $t \in |\mathbf{C}(\mathbf{x})|$*.*

Proof. The first claim is immediate from Corollary 1.31 and Theorem 1.33. If Γ_{JRS} is morphism homogeneous, then for each t, $\Gamma_{\mathrm{M}} \vdash \pi_t^+ \leftrightarrow \rho_t^+$. So every quantifier-free formula φ is equivalent to $\bigvee\{\pi_t^+ : t \in [\![\varphi]\!]\}$, and therefore to a disjunction of conjunctions of atoms. □

As an illustration of Theorem 1.34 in the presence of morphism homogeneity, see the quantifier elimination results about the two theories in [20].

1.5 The Universal Fragment of Γ_{M}

Every classical model complete theory is uniquely determined by its universal fragment. Given the universal fragment, one can then recover the model companion as the largest inductive theory preserving this universal fragment (a theory is inductive if its class of models is closed under unions of chains). As a start to a generalization of this process to intuitionistic theories, we find the universal fragments of our intuitionistic theories that admit quantifier elimination. We first need to explain what we mean by an intuitionistic universal sentence. The definition is motivated by Theorem 1.37 below, see also [9].

Recall that a Kripke model is essentially a functor $\mathfrak{A}$ from a (small) category $\mathbf{A}$ to the category of classical $\mathcal{L}$-structures and morphisms. That is, to each i in $|\mathbf{A}|$ we assign a classical structure $\mathfrak{A}_i$, and to each arrow $f : i \to j$ in $\mathbf{A}$ we assign a morphism $\mathfrak{A}f : \mathfrak{A}_i \to \mathfrak{A}_j$.

Definition 1.35. *$\mathfrak{A}$ is a* **Kripke submodel** *of $\mathfrak{B}$, written $\mathfrak{A} \subseteq \mathfrak{B}$, if and only if $\mathbf{A} \subseteq \mathbf{B}$ as categories, and all morphisms and node structures of $\mathfrak{A}$ are restrictions of the corresponding morphisms and node structures of $\mathfrak{B}$.*

Definition 1.36. *A sentence is* **universal** *if it can be built from the atoms using the operations $\wedge$, $\vee$, $\to$ and $\forall$, with the restriction that no implications or universal quantifications occur in negative places.*

Theorem 1.37. *An intuitionistic theory Δ is axiomatizable by universal sentences if and only if its class of Kripke models is closed under Kripke submodels.*

Proof. Immediate from [9, Theorem 4.1]. □

Note that in the absence of Excluded Middle, not every quantifier-free formula is equivalent to a universal formula. Therefore, the following is an addition to Theorem 1.34:

Theorem 1.38. *Let $\varphi(\mathbf{x})$ be a formula. Over Γ_{M}, φ is equivalent to a quantifier-free universal formula.*

Proof. This easily follows from Theorem 1.34 since each ρ_t^+ is a universal formula. □

Next, we axiomatize the universal fragment of Γ_{M}.

Lemma 1.39. *Let* $t \in |\mathbf{C}(\mathbf{x})|$. *Then the theory* Γ_{M} *includes the universal sentence* $\forall \mathbf{x}(\pi_t^+ \rightarrow (\sigma_t^- \vee \rho_t^-))$.

Proof. Fix $\mathbf{a} \in A_{\mathrm{JRS}}$ and suppose that $\mathfrak{A}_{\mathrm{M}} \Vdash \pi_t^+(\mathbf{a})$. If $\mathfrak{A}_{\mathrm{M}} \Vdash \sigma_t^-(\mathbf{a})$, we are done, so suppose $\mathfrak{A}_{\mathrm{M}} \nVdash \sigma_t^-(\mathbf{a})$. Then $t = \mathrm{tp}_{\mathbf{a}}$. Suppose we have an endomorphism f and $u \in \mathbf{C}(\mathbf{x})$ such that $t^+ \subseteq u^+$, $t \nleq u$, and $\mathfrak{A}_{\mathrm{M}} \Vdash \pi_u^+(f(\mathbf{a}))$. Since $t \nleq u$, $u \neq \mathrm{tp}_{f(\mathbf{a})}$. So $\mathfrak{A}_{\mathrm{M}} \Vdash \sigma_u^-(f(\mathbf{a}))$. □

Lemma 1.40. *Let* $t \notin |\mathbf{C}(\mathbf{x})|$. *Then* Γ_{M} *includes the universal sentence* $\forall \mathbf{x}(\pi_t^+ \rightarrow \sigma_t^-)$.

Proof. Fix $\mathbf{a} \in A_{\mathrm{JRS}}$ and suppose that $\mathfrak{A}_{\mathrm{M}} \Vdash \pi_t^+(\mathbf{a})$. Since $\mathfrak{A}_{\mathrm{JRS}} \not\models \pi_t(\mathbf{a})$, we have $\mathfrak{A}_{\mathrm{JRS}} \models \sigma_t^-(\mathbf{a})$. So $\mathfrak{A}_{\mathrm{M}} \Vdash \sigma_t^-(\mathbf{a})$. □

Note that the formulas $\forall \mathbf{x}(\pi_t^+ \rightarrow \sigma_t^-)$ from Lemma 1.40 axiomatize the universal fragment of the classical theory Γ_{JRS}.

Definition 1.41.

1. *A formula is* **positive existential** *if it is built from the atoms and the connectives* $\wedge$, $\vee$ *and* $\exists$.
2. *A sentence is* **geometric** *if it is the universal closure of a formula of the form a positive existential formula implies another positive existential formula.*

Geometric sentences in some sense are the largest class of sentences that are treated the same both classically and intuitionistically. This idea is made more precise in the following well known lemma.

Lemma 1.42. *Let* $\mathfrak{B}$ *be a Kripke model and* φ *a geometric sentence. Then* $\mathfrak{B} \Vdash \varphi$ *if and only if for each node* $k \in |\mathbf{B}|$, *the node structure* $\mathfrak{B}_k \models \varphi$.

The schemas from Lemmas 1.39 and 1.40 suffice to axiomatize the universal fragment of Γ_{M}.

Theorem 1.43. *The axiom schemas*

$$\forall \mathbf{x}(\pi_t^+ \to \sigma_t^-) \qquad \text{for all } \mathbf{x} \text{ and } t \notin |\mathbf{C}(\mathbf{x})|, \quad \text{and}$$

$$\forall \mathbf{x}(\pi_t^+ \to (\sigma_t^- \vee \rho_t^-)) \quad \text{for all } \mathbf{x} \text{ and } t \in |\mathbf{C}(\mathbf{x})|$$

together axiomatize the universal fragment of Γ_{M}.

Proof. Let Δ be the set of all universal sentences described above. Let $\mathfrak{B} \Vdash \Delta$ be a Kripke model. By the completeness of intuitionistic logic for rooted Kripke models (see [23, Theorem 2.6.8], for example) and because $\mathcal{L}$ is countable, we may suppose that $\mathbf{B}$ is a tree (poset) of height ω, and for all $i \in |\mathbf{B}|$ the domain of the node structure $\mathfrak{B}_i$ is at most countable. Let $r \in |\mathbf{B}|$ be the root of $\mathbf{B}$. We construct a rooted Kripke model $\mathfrak{D}$ with root r such that $\mathfrak{B} \subseteq \mathfrak{D}$ and $\mathfrak{D} \Vdash \Gamma_{\mathrm{M}}$.

First we construct an intermediate rooted Kripke model $\mathfrak{C}$ with $\mathbf{C} = \mathbf{B}$, $\mathfrak{B}_i \subseteq \mathfrak{C}_i \cong \mathfrak{A}_{\mathrm{JRS}}$ for every $i \in |\mathbf{C}|$, and $\mathfrak{C}f \restriction B_i = \mathfrak{B}f$ for every $f : i \to j$ in $\mathbf{C}$. The construction is by induction on the height of $\mathbf{C}$. Let $\mathfrak{C}_r = \mathfrak{A}_{\mathrm{JRS}}$. By Lemmas 1.40 and 1.42, every node structure $\mathfrak{B}_i$ is a model of $(\Gamma_{\mathrm{JRS}})_\forall$. So up to isomorphism, $\mathfrak{B}_i \subseteq \mathfrak{A}_{\mathrm{JRS}}$ for every $i \in |\mathbf{B}|$. So without loss of generality, we may suppose that $\mathfrak{B}_r \subseteq \mathfrak{C}_r$. Now suppose that $\mathfrak{C}_i$ is defined for some $i \in |\mathbf{C}|$, with $\mathfrak{B}_i \subseteq \mathfrak{C}_i \cong \mathfrak{A}_{\mathrm{JRS}}$. Let $j \in |\mathbf{C}|$ be any immediate successor of i, and let $f : i \to j$ be the unique arrow from i to j in $\mathbf{C}$. Without loss of generality, we may suppose that $\mathfrak{B}_j \subseteq \mathfrak{C}_i$. We claim that there exists a $\mathfrak{C}_j \cong \mathfrak{A}_{\mathrm{JRS}}$ such that $\mathfrak{B}_j \subseteq \mathfrak{C}_j$, and a morphism $\mathfrak{C}f : \mathfrak{C}_i \to \mathfrak{C}_j$ such that $\mathfrak{C}f \restriction B_i = \mathfrak{B}f$. Let $\mathcal{L}^*$ be the language $\mathcal{L}$ extended by a new function symbol f^*, and let

$$\Theta = \mathrm{Th}_c(\mathfrak{C}_i) \cup \{f^*(b) = \mathfrak{B}f(b) : b \in B_i\} \cup (\mathrm{Th}_c(\mathfrak{C}_i) \cap \mathcal{A}\mathrm{t}(C_i))[c/f^*(c), c \in C_i],$$

where $\mathrm{Th}_c(\mathfrak{C}_i)$ is the theory of the classical model $\mathfrak{C}_i$ over the language $\mathcal{L}(C_i)$. Let Θ_0 be any finite subset of Θ. Then

$$\Theta_0 \subseteq \mathrm{Th}_c(\mathfrak{C}_i) \cup \{f^*(b) = \mathfrak{B}f(b) : b \in \mathbf{b}\} \cup (\mathrm{Th}_c(\mathfrak{C}_i) \cap \mathcal{A}\mathrm{t}(C_i))[c/f^*(c), c \in C_i],$$

for some finite $\mathbf{b} \subseteq B_i$. Obviously, $t = \mathrm{tp}_{\mathbf{b}}$ is consistent with Γ_{JRS}. Let $u = \mathrm{tp}_{\mathfrak{B}f(\mathbf{b})}$. Then, since $\mathfrak{B}f$ is a morphism, we have $t^+ \subseteq u^+$. Assume that $t \not\leq u$. Then $\mathfrak{B} \Vdash \forall \mathbf{x}(\pi_t^+ \to (\sigma_t^- \vee (\pi_u^+ \to \sigma_u^-)))$. Since $i \Vdash^{\mathfrak{B}} \pi_t^+(\mathbf{b})$, $i \Vdash^{\mathfrak{B}} \sigma_t^-(\mathbf{b}) \vee (\pi_u^+(\mathbf{b}) \to \sigma_u^-(\mathbf{b}))$. Since $\mathfrak{B}_i \models \pi_t(\mathbf{b})$, we have $i \nVdash^{\mathfrak{B}} \delta(\mathbf{b})$, for every $\neg\delta \in t$. So $i \nVdash^{\mathfrak{B}} \sigma_t^-(\mathbf{b})$. So we must have $i \Vdash^{\mathfrak{B}} \pi_u^+(\mathbf{b}) \to \sigma_u^-(\mathbf{b})$. Since $\mathfrak{B}f$ is a morphism, we have $j \Vdash^{\mathfrak{B}} \pi_u^+(\mathfrak{B}f(\mathbf{b}))$. By the definition of forcing, $j \Vdash^{\mathfrak{B}} \sigma_u^-(\mathfrak{B}f(\mathbf{b}))$. So $j \Vdash^{\mathfrak{B}} \delta(\mathfrak{B}f(\mathbf{b}))$ for some $\neg\delta \in u$. So $\mathfrak{B}_j \models \delta(\mathfrak{B}f(\mathbf{b}))$ for some $\neg\delta \in u$. Contradiction. So $t \leq u$. So there is an endomorphism $f^* : \mathfrak{C}_i \to \mathfrak{C}_i$ such that $f^* \upharpoonright \mathbf{b} = \mathfrak{B}f \upharpoonright \mathbf{b}$. Let $\mathfrak{C}_i^*$ be the expansion of $\mathfrak{C}_i$ to $\mathcal{L}^*$ where f^* is interpreted as this endomorphism. Then $\mathfrak{C}_i^* \models \Theta_0$. So by compactness, Θ is consistent.

Let $\mathfrak{C}_j^*$ be a countable model of Θ, and let $\mathfrak{C}_j$ be the $\mathcal{L}$-reduct of $\mathfrak{C}_j^*$. Then $\mathfrak{C}_i \preceq \mathfrak{C}_j$, and $f^* : \mathfrak{C}_i \to \mathfrak{C}_j$ is a morphism such that $f^* \upharpoonright B_i = \mathfrak{B}f$. (Note that f^* is a total function on $\mathfrak{C}_j$, but it is only a morphism on $\mathfrak{C}_i \subseteq \mathfrak{C}_j$.) Set $\mathfrak{C}f = f^*$. Since $\mathfrak{A}_{\mathrm{JRS}}$ is the unique model of Γ_{JRS} of size less than or equal to ω, we have $\mathfrak{C}_j \cong \mathfrak{A}_{\mathrm{JRS}}$. So the claim is proven. This completes the construction of $\mathfrak{C}$. Clearly, $\mathfrak{B} \subseteq \mathfrak{C}$.

Let $\mathfrak{D}$ be the extension of $\mathfrak{C}$ generated by adding for each $i \in |\mathbf{C}|$ all possible morphisms from $\mathfrak{C}_i$ to itself. Then for all $\varphi \in \mathcal{L}(A_{\mathrm{JRS}})$ we have $\mathfrak{D} \Vdash \varphi$ if and only if $\mathfrak{A}_{\mathrm{M}} \Vdash \varphi$, by a straightforward induction on the complexity of φ. So $\mathfrak{D} \Vdash \Gamma_{\mathrm{M}}$. Also $\mathfrak{B} \subseteq \mathfrak{D}$. So by

Theorem 1.37, $\mathfrak{B}$ forces the universal fragment of Γ_{M}. So Δ axiomatizes the universal fragment of Γ_{M}. $\square$

1.6 Appendix: Kripke Models of Classical Logic

It is well known that Kripke models satisfy classical logic exactly when all morphisms between node structures are elementary embeddings. See [22, page 110] for one direction. For the reader's convenience, we include a full proof. Recall that classical predicate logic CQC is axiomatizable over intuitionistic logic by the schema $\forall \mathbf{x}(\varphi(\mathbf{x}) \vee \neg\varphi(\mathbf{x}))$.

Theorem 1.44. *Let $\mathfrak{A}$ be a Kripke model. Then the following are equivalent:*

1. *Every morphism f in $\mathfrak{A}$ is an elementary embedding. That is, for all morphisms f from the structure at node k to the structure at node m and all $\mathcal{L}(A_k)$-sentences $\varphi(\mathbf{a})$:*

   $\mathfrak{A}_k \models \varphi(\mathbf{a})$ *if and only if* $\mathfrak{A}_m \models {}^{f}(\varphi(\mathbf{a}))$.

2. *For all nodes $k \in |\mathbf{A}|$, and every $\mathcal{L}$-sentence φ:*

   $\mathrm{CQC} \vdash_c \varphi$ *implies* $k \Vdash \varphi$.

3. *For every node k and for every $\mathcal{L}(A_k)$-sentence $\varphi(\mathbf{a})$, we have:*

   $\mathfrak{A}_k \models \varphi(\mathbf{a})$ *if and only if* $k \Vdash \varphi(\mathbf{a})$.

Proof. $2 \Rightarrow 3$: We proceed by induction on the complexity of sentences. 3 holds for all atomic sentences, while the induction steps for existential statements, conjunctions, and disjunctions all follow directly from the definitions.

Given a node k, suppose $\mathfrak{A}_k \models \psi \to \theta$, where 3 holds for ψ and θ. If $\mathfrak{A}_k \models \psi$, then $\mathfrak{A}_k \models \theta$. By the inductive hypothesis, $k \Vdash \theta$, and so $k \Vdash \psi \to \theta$. Otherwise, $\mathfrak{A}_k \not\models \psi$. Then by the inductive hypothesis, $k \nVdash \psi$. By 2, $k \Vdash \psi \vee \neg\psi$, so $k \Vdash \neg\psi$. So $k \Vdash \psi \to \theta$.

Now suppose that $k \Vdash \psi \to \theta$, where 3 holds for ψ and θ. If $\mathfrak{A}_k \models \neg\psi$, then $\mathfrak{A}_k \models \psi \to \theta$ trivially. Otherwise, $\mathfrak{A}_k \models \psi$. By the inductive hypothesis, $k \Vdash \psi$, so $k \Vdash \theta$. By the inductive hypothesis again, $\mathfrak{A}_k \models \theta$. So $\mathfrak{A}_k \models \psi \to \theta$.

Suppose $\mathfrak{A}_k \models \forall x\psi(x)$, where 3 holds for $\psi(a)$, for all $a \in A_k$. Then, $\mathfrak{A}_k \models \psi(a)$ for all $a \in A_k$. By the inductive hypothesis, $k \Vdash \psi(a)$ for all $a \in A_k$. Assume $k \nVdash \forall x\psi(x)$. Then there exists $f : k \to m$ where $m \nVdash {}^f\psi(b)$, for some $b \in A_m$. By 2, $m \Vdash {}^f\psi(b) \vee \neg{}^f\psi(b)$, so $m \Vdash \neg{}^f\psi(b)$. Therefore $m \Vdash \exists x\neg{}^f\psi(x)$. Now, $k \Vdash \exists x\neg\psi(x)$ or $k \Vdash \neg\exists x\neg\psi(x)$ (again by 2). The latter cannot hold, since $m \Vdash \exists x\neg{}^f\psi(x)$, so $k \Vdash \exists x\neg\psi(x)$. So, $k \Vdash \neg\psi(a)$ for some $a \in A_k$, a contradiction. Thus, $k \Vdash \forall x\psi(x)$.

Finally, suppose $k \Vdash \forall x\psi(x)$. So $k \Vdash \psi(a)$ for all $a \in A_k$. By the inductive hypothesis, $\mathfrak{A}_k \models \psi(a)$ for all $a \in A_k$. So $\mathfrak{A}_k \models \forall x\psi(x)$.

$3 \Rightarrow 2$: If CQC $\vdash_c \varphi$, then $\mathfrak{B} \models \varphi$ for all classical models $\mathfrak{B}$. Thus, given a node k, and a sentence φ proven by CQC, we have $\mathfrak{A}_k \models \varphi$. By 3, $k \Vdash \varphi$, proving 2.

$3 \Rightarrow 1$: Let f be a morphism from the structure at node k to the structure at node m, and suppose $\mathfrak{A}_k \models \varphi(\mathbf{a})$. By 3, $k \Vdash \varphi(\mathbf{a})$, and so $m \Vdash {}^f(\varphi(\mathbf{a}))$. By 3 again, $\mathfrak{A}_m \models {}^f(\varphi(\mathbf{a}))$.

$1 \Rightarrow 3$: We again proceed by induction on the complexity of sentences. By the definition of forcing, 3 always holds for atomic sentences, and the inductive steps for conjunctions, disjunctions, and existential statements are easy.

Suppose $\mathfrak{A}_k \models \psi \to \theta$. Let f be a morphism from the structure at node k to the structure at node m such that $m \Vdash {}^f\psi$. By the inductive hypothesis, $\mathfrak{A}_m \models {}^f\psi$. By 1,

$\mathfrak{A}_m \models {}^f\psi \to {}^f\theta$, hence $\mathfrak{A}_m \models {}^f\theta$. By the inductive hypothesis, $m \Vdash {}^f\theta$, so $k \Vdash \psi \to \theta$.

Suppose $k \Vdash \psi \to \theta$. If $\mathfrak{A}_k \models \psi$ then, by the inductive hypothesis, $k \Vdash \psi$. Then $k \Vdash \theta$, so by the inductive hypothesis again, $\mathfrak{A}_k \models \theta$. Thus, $\mathfrak{A}_k \models \psi \to \theta$.

Suppose $\mathfrak{A}_k \models \forall x \psi(x)$, with 3 holding for ${}^f\psi(b)$, for all $b \in A_m$, where m is a node and f is a morphism from the structure at node k to the structure at node m. Given such an f, by 1 we have $\mathfrak{A}_m \models \forall x {}^f\psi(x)$. Then, for all $a \in A_m$, $\mathfrak{A}_m \models {}^f\psi(a)$. By the inductive hypothesis, for every $a \in A_m$ we have $m \Vdash {}^f\psi(a)$. As f is arbitrary, we have that $k \Vdash \forall x \psi(x)$.

Finally, suppose $k \Vdash \forall x \psi(x)$. Then for all $a \in A_k$ we have $k \Vdash \psi(a)$. By the inductive hypothesis, $\mathfrak{A}_k \models \psi(a)$ for all $a \in A_k$. So $\mathfrak{A}_k \models \forall x \psi(x)$. $\square$

Chapter 2

Axiomatizations of the Intuitionistic Theory

The goal is, given a classical JRS theory Γ_{JRS}, to build an axiomatization of the corresponding intuitionistic theory Γ_{M}. We do this in different ways below. One axiomatization approach is motivated by studying the structure $\mathfrak{A}_{\mathrm{M}}$ and its morphism properties. Our second approach is motivated by examining the quantifier elimination properties of Γ_{M}. For both approaches, we show that our schemas axiomatize Γ_{M} (Theorems 2.16 and 2.24), and we go on to analyze axiom systems equivalent to the main schemas (Theorems 2.17, 2.20, and 2.39).

2.1 An Axiomatization of Γ_{M}

We begin by examining the properties of the canonical Kripke model $\mathfrak{A}_{\mathrm{M}}$ of Γ_{M}. Since $\mathfrak{A}_{\mathrm{M}}$ is such that all node structures are classical models of the same theory Γ_{JRS}, we look to the geometric fragment of Γ_{JRS} (see Definition 1.42 and Lemma 1.42). The first two axiom schemas below are geometric.

By Theorem 1.8, we may think of the classical JRS model $\mathfrak{A}_{\mathrm{JRS}}$ as the amalgamation of its finite submodels. Each such submodel is of the form $\mathfrak{A}_t$ for some complete $\mathcal{A}\mathrm{t}^{\pm}(\mathbf{x})$-type t where t is classically consistent with Γ_{JRS}. If we consider the tree of all finite $\mathcal{L}$-structures, the following axiom schema essentially removes all those nodes inconsistent with Γ_{JRS}.

$\forall \mathbf{x}(\pi_t^+ \to \sigma_t^-)$ for all complete $\mathcal{A}\mathrm{t}^{\pm}(\mathbf{x})$-types t not classically consistent with Γ_{JRS}

This schema classically axiomatizes the universal fragment of Γ_{JRS}, and is geometric. Since $\mathfrak{A}_{\mathrm{M}}$ is a Γ_{JRS}-local Kripke model, this schema holds in Γ_{M} as well, see Lemma 1.40.

However, this schema does not suffice to axiomatize the intuitionistic universal fragment (recall Definition 1.36) of Γ_{M}, as it does not fully describe the morphism relationship between finite submodels of $\mathfrak{A}_{\mathrm{JRS}}$ in JRS theories that are not morphism homogeneous. In order to keep track of the morphism behavior, we add the following schema.

$\forall \mathbf{x}(\pi_t^+ \to (\sigma_t^- \vee \rho_t^-))$ for all complete $\mathcal{A}\mathrm{t}^{\pm}(\mathbf{x})$-types t classically consistent with Γ_{JRS}

To be more specific, for any $\mathbf{a} \in A_{\mathrm{JRS}}$ such that $\mathfrak{A}_{\mathrm{M}} \Vdash \pi_t^+(\mathbf{a})$, it is possible that $\mathfrak{A}_{\mathrm{JRS}} \models \pi_u(\mathbf{a})$ for some complete $\mathcal{A}\mathrm{t}^{\pm}(\mathbf{x})$-type u with $t^+ \subsetneq u^+$, in which case $\mathfrak{A}_{\mathrm{M}} \Vdash \sigma_t^-(\mathbf{a})$. On the other hand, if $t = \mathrm{tp}_{\mathbf{a}}$, then we get that $\mathfrak{A}_{\mathrm{M}}$ forces not only $\pi_t^+(\mathbf{a})$, but also $\rho_t^-(\mathbf{a})$, which lists local morphisms from $\mathfrak{A}_t$ that do not lift to global endomorphisms of $\mathfrak{A}_{\mathrm{JRS}}$. See the discussion following Definition 1.27.

These two schemas together axiomatize the intuitionistic universal fragment of Γ_{M}; see Theorem 1.43.

We also include an intuitionistic version of the JRS axioms themselves. For each JRS sentence $\forall \mathbf{x}[(\pi^+_{d(u)} \wedge \neg\sigma^-_{d(u)}) \to \exists x_n(\pi^+_u \wedge \neg\sigma^-_u)]$ with $u \in |\mathbf{C}(\mathbf{x}x_n)|$, we include the following schema, which is essentially the Buss translation of this JRS sentence. In [6], Buss shows that the intuitionistic theory of the class of all Kripke models wherein every node satisfies a classical theory Γ is axiomatized by certain translations of all sentences in Γ. We suppress further detail, as we do not quite meet all of Buss' hypotheses. We show directly any properties we need. Note that the following schema is classically equivalent to the JRS axiom δ_u if we let φ be $\bot$.

$\forall \mathbf{xy}[(\pi^+_{d(u)} \wedge (\sigma^-_{d(u)} \to \varphi) \wedge \forall x_n(\pi^+_u \to (\sigma^-_u \vee \varphi))) \to \varphi]$ for all formulas $\varphi(\mathbf{xy})$ in which x_n is not free

Lastly, we axiomatize Fraïssé homogeneity. A useful property of the classical JRS theories is that the formulas π_t isolate the corresponding types. That is, if $\mathfrak{A}_{\mathrm{JRS}} \models \pi_t(\mathbf{a}) \wedge \psi(\mathbf{a}) \wedge \pi_t(\mathbf{b})$, then $\mathfrak{A}_{\mathrm{JRS}} \models \psi(\mathbf{b})$. We axiomatize the corresponding intuitionistic behavior with the following schema.

$\forall \mathbf{xy}x_n[\psi \wedge \rho^+_u \to (\sigma^-_u \vee \forall x_n(\rho^+_u \to \psi))]$ for $u \in |\mathbf{C}(\mathbf{x}x_n)|$ and all formulas $\psi(\mathbf{xy}x_n)$

To be more specific, if we have $\mathfrak{A}_{\mathrm{M}} \Vdash \psi(\mathbf{ab}c) \wedge \rho^+_u(\mathbf{a}c)$, then it is possible that $\mathfrak{A}_{\mathrm{JRS}} \models \pi_v(\mathbf{a}c)$ for some complete $\mathcal{A}\mathrm{t}^{\pm}(\mathbf{x}x_n)$-type v with $u^+ \subsetneq v^+$, in which case $\mathfrak{A}_{\mathrm{M}} \Vdash \sigma^-_u(\mathbf{a}c)$. Otherwise, $u = \mathrm{tp}_{\mathbf{a}c}$, so by the argument in Lemma 1.32 we get that for any $d \in A_{\mathrm{JRS}}$ such that $\mathfrak{A}_{\mathrm{M}} \Vdash \rho^+_u(\mathbf{a}d)$, we have $\mathfrak{A}_{\mathrm{M}} \Vdash \psi(\mathbf{ab}d)$.

We collect the above schemas into the Ha axiom system. The choice of name denotes our first axiom system reflecting the homogeneity properties of $\mathfrak{A}_{\mathrm{M}}$; we introduce the axiom systems Hb and Hc in Section 2.2. The axiom systems Qa and Qb of Sections 2.3 and 2.4 reflect the process of eliminating quantifiers from Γ_{M}.

Ha1 $\forall \mathbf{x}(\pi_t^+ \to \sigma_t^-)$ for all $t \notin |\mathbf{C}(\mathbf{x})|$

Ha2 $\forall \mathbf{x}(\pi_t^+ \to (\sigma_t^- \vee \rho_t^-))$ for all $t \in |\mathbf{C}(\mathbf{x})|$

Ha3 $\forall \mathbf{xy}[(\pi_{d(u)}^+ \wedge (\sigma_{d(u)}^- \to \varphi) \wedge \forall x_n(\pi_u^+ \to (\sigma_u^- \vee \varphi))) \to \varphi]$ for all $u \in |\mathbf{C}(\mathbf{x}x_n)|$ and all formulas $\varphi(\mathbf{xy})$ in which x_n is not free

Ha4 $\forall \mathbf{xy}x_n[\psi \wedge \rho_u^+ \to (\sigma_u^- \vee \forall x_n(\rho_u^+ \to \psi))]$ for all $u \in |\mathbf{C}(\mathbf{x}x_n)|$ and all formulas $\psi(\mathbf{xy}x_n)$

We write Γ_{Ha} for the intuitionistic theory axiomatized by Ha1 through Ha4.

Below, we show that these four schemas suffice to axiomatize Γ_{M}, that is, we show that $\Gamma_{\mathrm{Ha}} = \Gamma_{\mathrm{M}}$. Our strategy is to show that $\Gamma_{\mathrm{M}} \vdash \Gamma_{\mathrm{Ha}}$. Then we show that Γ_{Ha} admits quantifier elimination. Since $\top$ and $\bot$ are the only quantifier-free sentences, Γ_{Ha} must be complete and therefore $\Gamma_{\mathrm{Ha}} = \Gamma_{\mathrm{M}}$.

Lemma 2.1. $\Gamma_{\mathrm{M}} \vdash \Gamma_{\mathrm{Ha}}$.

Proof. Together, the schemas Ha1 and Ha2 form the universal fragment of Γ_{M}; see Lemmas 1.39 and 1.40. That Γ_{M} proves Ha4 is just Lemma 1.32. It remains to show $\Gamma_{\mathrm{M}} \vdash$ Ha3.

Assume that $\mathfrak{A}_{\mathrm{M}}$ forces the left hand side of Ha3, and fix tuples $\mathbf{a}$ and $\mathbf{b}$ in A_{JRS}. If $\mathfrak{A}_{\mathrm{M}} \Vdash \sigma_{d(u)}^-(\mathbf{a})$, then $\mathfrak{A}_{\mathrm{M}} \Vdash \varphi(\mathbf{ab})$ and we are done. So suppose $\mathfrak{A}_{\mathrm{M}} \nVdash \sigma_{d(u)}^-(\mathbf{a})$. Since $\mathfrak{A}_{\mathrm{M}} \Vdash \pi_{d(u)}^+(\mathbf{a})$ and $\mathfrak{A}_{\mathrm{M}} \nVdash \sigma_{d(u)}^-(\mathbf{a})$, $d(u) = \mathrm{tp}_{\mathbf{a}}$. Since u is consistent with Γ_{JRS}, we can find a $c \in A_{\mathrm{JRS}}$ such that $u = \mathrm{tp}_{\mathbf{a}c}$. So $\mathfrak{A}_{\mathrm{M}} \Vdash \pi_u^+(\mathbf{a}c)$ and $\mathfrak{A}_{\mathrm{M}} \nVdash \sigma_u^-(\mathbf{a}c)$ and we must have $\mathfrak{A}_{\mathrm{M}} \Vdash \varphi(\mathbf{ab})$. So $\mathfrak{A}_{\mathrm{M}}$ forces Ha3, and therefore $\Gamma_{\mathrm{M}} \vdash$ Ha3. □

Lemma 2.2. *Let t be a complete $\mathcal{A}\mathrm{t}^{\pm}(\mathbf{x}x_n)$-type. Then*

$$\vdash \pi_t^+ \to \pi_{d(t)}^+ \quad \textit{and} \quad \vdash \sigma_{d(t)}^- \to \sigma_t^-.$$

Proof. Both tautologies immediately follow from $d(t) \subseteq t$. □

We use the term *p-morphism* in the next theorem. We direct the unfamiliar reader to Segerberg's paper [21].

Theorem 2.3. *For all $\mathbf{x}x_n$, the map d from $|\mathbf{C}(\mathbf{x}x_n)|$ to $|\mathbf{C}(\mathbf{x})|$ is onto, and extends to a poset morphism $d : \mathbf{C}(\mathbf{x}x_n) \to \mathbf{C}(\mathbf{x})$. Additionally, d is a so-called p-morphism, that is, for any $u \in |\mathbf{C}(\mathbf{x}x_n)|$, if $d(u) \leq t$, then there is $v \in |\mathbf{C}(\mathbf{x}x_n)|$ such that $u \leq v$ and $d(v) = t$.*

Proof. The claims are trivial when $n = 0$. Suppose $n > 0$. Onto is immediate by taking, for each $t \in |\mathbf{C}(\mathbf{x})|$, the unique $u \in |\mathbf{C}(\mathbf{x}x_n)|$ containing $t \cup \{x_n = x_0\}$. The poset morphism claim that $u \leq v$ implies $d(u) \leq d(v)$ follows from Definition 1.20.2. Finally, suppose $t \geq d(u)$. There are $\mathbf{a} \in A_{\mathrm{JRS}}$ and an endomorphism f of $\mathfrak{A}_{\mathrm{JRS}}$ such that $t = \mathrm{tp}_{f(\mathbf{a})}$ and $d(u) = \mathrm{tp}_{\mathbf{a}}$. There is $b \in A_{\mathrm{JRS}}$ such that $u = \mathrm{tp}_{\mathbf{a}b}$. Set $v = \mathrm{tp}_{f(\mathbf{a}b)}$. □

Lemma 2.4. *For $u, v \in |\mathbf{C}(\mathbf{x})|$ with $u^+ \subseteq v^+$ we have*

$$\vdash \sigma_v^- \to \sigma_u^-, \quad \textit{and} \quad \vdash \pi_v^+ \to \pi_u^+.$$

If additionally we have $u \leq v$, then $\vdash \rho_v^+ \to \rho_u^+$.

Proof. All follow tautologically from the definitions. □

Given $s \subseteq \mathcal{A}\mathrm{t}(\mathbf{x})$ and a theory Δ, let $\langle s\rangle_\Delta = \mathrm{Th}(\Delta \cup s) \cap \mathcal{A}\mathrm{t}(\mathbf{x})$. That is, $\langle s\rangle_\Delta$ is the closure of $\Delta \cup s$ under intuitionistic derivation restricted to atoms in $\mathbf{x}$.

Lemma 2.5. *Let $s \subseteq \mathcal{A}\mathrm{t}(\mathbf{x}) \setminus \{\bot\}$. Then there is a unique complete $\mathcal{A}\mathrm{t}^{\pm}(\mathbf{x})$-type t such that $\langle s\rangle_\emptyset = t^+$.*

Proof. Given s, let $u = \langle s\rangle_\emptyset$ and let $t = u \cup \{\neg\delta \ : \ \delta \in \mathcal{A}\mathrm{t}(\mathbf{x}) \setminus u\}$. As $\bot \notin s$, u is consistent. We must show that t is a complete $\mathcal{A}\mathrm{t}^{\pm}(\mathbf{x})$-type. So assume that t is classically inconsistent. That is, assume $\pi_u \wedge \bigwedge_{\delta\in t^-}(\neg\delta) \vdash_c \bot$. Fix $\delta \in t^-$. By the completeness of intuitionistic logic for rooted Kripke models (see [23, Theorem 2.6.8], for example), we get a rooted Kripke model $\mathfrak{A}_\delta$ such that $\mathfrak{A}_\delta \Vdash \pi_u(\mathbf{a})$ and $\mathfrak{A}_\delta \nVdash \delta(\mathbf{a})$ (such a model exists, otherwise $\delta \in u$). The classical root structure of $\mathfrak{A}_\delta$ satisfies $\pi_u(\mathbf{a}) \wedge \neg\delta(\mathbf{a})$. Let $\mathfrak{A}$ be the product of these root structures over all $\delta \in t^-$. Then $\mathfrak{A} \models \pi_u \wedge \bigwedge_{\delta\in t^-} \neg\delta$, which contradicts our assumption. □

In a general intuitionistic context, we rarely have that every quantifier-free formula can be written in conjunctive or disjunctive normal form. We show in Theorem 2.14 that over Γ_{Ha}, every quantifier-free formula has both a conjunctive and a disjunctive equivalent. We use these forms to eliminate the $\forall$ and the $\exists$ quantifiers, respectively. In the next lemma, we use $\bigwedge s$ as shorthand for $\bigwedge\{\delta \ : \ \delta \in s\}$.

Lemma 2.6. *For any $\mathbf{x}$ and any $s \subseteq \mathcal{A}\mathrm{t}(\mathbf{x}) \setminus \{\bot\}$, we have*

$$\mathrm{Ha1} \vdash \bigwedge s \leftrightarrow \bigvee\{\pi_u^+ \ : \ u \in |\mathbf{C}(\mathbf{x})| \textit{ and } \langle s\rangle_\emptyset^+ \subseteq u\}$$

and

$$\mathrm{Ha1} \cup \mathrm{Ha2} \vdash \bigwedge s \leftrightarrow \bigvee\{\rho_u^+ \ : \ u \in |\mathbf{C}(\mathbf{x})| \textit{ and } \langle s\rangle_\emptyset^+ \subseteq u\}.$$

Proof. By Lemma 2.5, we choose $t \in |\mathbf{C}(\mathbf{x})|$ such that $\langle s \rangle_\emptyset = t^+$, and replace $\bigwedge s$ with π_t^+ in both claims. We write ψ for the formula $\bigvee\{\pi_u^+ \ : \ u \in |\mathbf{C}(\mathbf{x})| \text{ and } t^+ \subseteq u\}$, and θ for $\bigvee\{\rho_u^+ \ : \ u \in |\mathbf{C}(\mathbf{x})| \text{ and } t^+ \subseteq u\}$.

For the first claim, it suffices to prove $\text{Ha1} \vdash \pi_t^+ \to \psi$. We prove the claim by reverse (strong) induction on the number of atoms in t^+. If $t \in |\mathbf{C}(\mathbf{x})|$, then $\vdash \pi_t^+ \to \psi$ and we are done, so consider $t \notin |\mathbf{C}(\mathbf{x})|$. If t^+ has the maximal number of atoms, we get $\text{Ha1} \vdash \pi_t^+ \to (\pi_t^+ \wedge \sigma_t^-)$. But by the maximality of t^+, σ_t^- is $\bot$, so $\text{Ha1} \vdash \pi_t^+ \to \bot$ and therefore $\text{Ha1} \vdash \pi_t^+ \to \psi$. For a non-maximal $t \notin |\mathbf{C}(\mathbf{x})|$, $\text{Ha1} \vdash \pi_t^+ \to (\pi_t^+ \wedge \sigma_t^-)$, which is equivalent to $\text{Ha1} \vdash \pi_u^+ \to \bigvee_{\delta \in t^-}(\pi_t^+ \wedge \delta)$. For each $\delta \in t^-$, $\langle t^+ \cup \{\delta\} \rangle_\emptyset$ has more atoms than t^+, so by Lemma 2.5, we can apply the inductive hypothesis. In all cases, we get $\text{Ha1} \vdash \pi_t^+ \to \psi$.

For the second claim, it suffices to show $\text{Ha1} \cup \text{Ha2} \vdash \pi_t^+ \to \theta$. We proceed via reverse (strong) induction on the number of atoms in t^+. If t^+ has the maximal number of atoms, then either $t \in |\mathbf{C}(\mathbf{x})|$ so $\vdash \pi_t^+ \leftrightarrow \rho_t^+$, or $t \notin |\mathbf{C}(\mathbf{x})|$ so $\text{Ha1} \vdash \pi_t^+ \to \bot$ as above. For all other t's, we have two cases. If $t \in |\mathbf{C}(\mathbf{x})|$, then by Ha2 we get $\pi_t^+ \leftrightarrow (\rho_t^+ \vee (\pi_t^+ \wedge \sigma_t^-))$. If $t \notin |\mathbf{C}(\mathbf{x})|$, then $\pi_t^+ \leftrightarrow (\pi_t^+ \wedge \sigma_t^-)$ by Ha1. For either case, apply Lemma 2.5 and the inductive hypothesis. This proves the claim. □

Lemma 2.7. *Let $s \in |\mathbf{C}(\mathbf{x})|$ and let t be an $\mathcal{A}\mathrm{t}^\pm(\mathbf{x})$-type. Then*

$$\text{Ha1} \cup \text{Ha2} \vdash (\rho_s^+ \wedge \pi_t^+) \leftrightarrow \bigvee\{\rho_v^+ \ : \ s \leq v \textit{ and } t^+ \subseteq v\}.$$

Proof. By Lemma 2.4, it suffices to prove the left to right direction. We claim that for $s \in |\mathbf{C}(\mathbf{x})|$ and t an $\mathcal{A}\mathrm{t}^\pm(\mathbf{x})$-type,

$$\text{Ha1} \cup \text{Ha2} \vdash (\rho_s^+ \wedge \pi_t^+) \leftrightarrow (\rho_s^+ \wedge \bigvee\{\pi_v^+ \ : \ s \leq v \text{ and } t^+ \subseteq v\}).$$

It suffices to show the left to right direction of this claim. Without loss of generality, we may assume that $s^+ \subseteq t^+$. We proceed via reverse (strong) induction on the number of atoms in t^+. Lemma 2.6 gives us

$$(\rho_s^+ \wedge \pi_t^+) \to (\rho_s^+ \wedge \bigvee\{\pi_v^+ \ : \ v \in |\mathbf{C}(\mathbf{x})| \text{ and } t^+ \subseteq v\}).$$

If $t \in |\mathbf{C}(\mathbf{x})|$ and $s \leq t$ then we are done. If $s \nleq t$, then ρ_s^+ tautologically implies $\pi_t^+ \to \sigma_t^-$, and we get $(\rho_s^+ \wedge \pi_t^+) \to (\rho_s^+ \wedge \bigvee_{\delta \in t^-}(\pi_t^+ \wedge \delta))$. Apply Lemma 2.5 and the inductive hypothesis. If $t \notin |\mathbf{C}(\mathbf{x})|$, then we get $\pi_t^+ \to (\pi_t^+ \wedge \sigma_t^-)$ by Ha1 and thus $(\rho_s^+ \wedge \pi_t^+) \to (\rho_s^+ \wedge \bigvee_{\delta \in t^-}(\pi_t^+ \wedge \delta))$. Again, apply Lemma 2.5 and the inductive hypothesis. If t^+ has the maximum number of atoms, then σ_t^- is the empty disjunction, which is just $\perp$. This proves the claim.

To prove the lemma, it suffices to show that

$$\text{Ha1} \cup \text{Ha2} \vdash (\rho_s^+ \wedge \pi_t^+) \to (\rho_s^+ \wedge \bigvee\{\rho_v^+ \ : \ s \leq v \text{ and } t^+ \subseteq v\}).$$

We again proceed via reverse (strong) induction on the number of atoms in t^+. Without loss of generality, we may assume that $s^+ \subseteq t^+$. First, suppose t^+ has the maximal number of atoms. If $s \leq t$, then $(\rho_s^+ \wedge \pi_t^+) \to \rho_t^+$. If $s \nleq t$, then because σ_t^- is equivalent to $\perp$, we have either $\text{Ha1} \vdash \pi_t^+ \to \perp$ or $\text{Ha2} \vdash (\rho_s^+ \wedge \pi_t^+) \to \perp$. Now suppose that t^+ does not have the maximal number of atoms. We may assume that $s \leq t$ by the claim. By Lemma 2.6, we rewrite $\rho_s^+ \wedge \pi_t^+$ as $\bigvee\{\rho_s^+ \wedge \rho_u^+ \ : \ u \in |\mathbf{C}(\mathbf{x})| \text{ and } t^+ \subseteq u\}$. Fix such a u. If $s \leq u$, we are done. If not, then u^+ is a proper extension of t^+, so we may apply the inductive hypothesis to $\rho_s^+ \wedge \pi_u^+$. This concludes the proof. □

Lemma 2.8. *For* $u \in |\mathbf{C}(\mathbf{x}x_n)|$, $\text{Ha1} \cup \text{Ha2} \vdash \rho_u^+ \to \rho_{d(u)}^+$.

Proof. Obviously, $\rho_u^+ \to \pi_{d(u)}^+$ is a tautology. Suppose that for some $s \in |\mathbf{C}(\mathbf{x})|$, $\pi_s^+ \to \sigma_s^-$ occurs in $\rho_{d(u)}^+$. Then $d(u)^+ \subseteq s^+$ and $d(u) \nleq s$. By Lemma 2.7 we have

$$\mathrm{Ha1} \cup \mathrm{Ha2} \vdash (\rho_u^+ \wedge \pi_s^+) \leftrightarrow \bigvee \{\rho_v^+ \ : \ u \leq v \text{ and } s^+ \subseteq v\}.$$

Pick such a v. Then $d(u) \leq d(v)$, while $s^+ \subseteq d(v)^+$. If $s^+ = d(v)^+$, then $s = d(v)$ and so $d(u) \leq s$, a contradiction. Thus, $\vdash \pi_{d(v)}^+ \to \sigma_s^-$. By Lemma 2.2, $\vdash \rho_v^+ \to \pi_{d(v)}^+$ and so $\vdash \rho_v^+ \to \sigma_s^-$. So $\mathrm{Ha1} \cup \mathrm{Ha2} \vdash \rho_u^+ \to (\pi_s^+ \to \sigma_s^-)$, and therefore $\mathrm{Ha1} \cup \mathrm{Ha2} \vdash \rho_u^+ \to \rho_{d(u)}^+$. □

The following lemma allows us to move between our two canonical forms.

Lemma 2.9. *Let U be any open subset of $|\mathbf{C}(\mathbf{x})|$. Then*

$$\mathrm{Ha1} \cup \mathrm{Ha2} \vdash \bigvee_{u \in U} \rho_u^+ \leftrightarrow \bigwedge_{s \notin U} (\pi_s^+ \to \sigma_s^-).$$

Proof. For the right to left direction, we claim that for $v \in |\mathbf{C}(\mathbf{x})|$,

$$\mathrm{Ha1} \cup \mathrm{Ha2} \cup \left\{ \bigwedge_{s \notin U} (\pi_s^+ \to \sigma_s^-) \right\} \vdash \rho_v^+ \to \bigvee_{u \in U} \rho_u^+.$$

We prove the claim via reverse (strong) induction on the number of atoms in v^+. If $v \in U$, then $\rho_v^+ \to \bigvee_{u \in U} \rho_u^+$, so suppose that $v \notin U$. If v^+ has the maximum number of atoms, then we have $\pi_v^+ \to \sigma_v^-$. Since σ_v^- is equivalent to $\perp$, the implication is vacuously satisfied. For any other v^+, we have $\pi_v^+ \to \sigma_v^-$ and thus $\rho_v^+ \wedge \bigvee_{\delta \in v^-} \delta$. Lemma 2.7 gives us

$$\bigvee_{\delta \in v^-} \left(\bigvee \{\rho_w^+ \ : \ v \leq w \text{ and } (v^+ \cup \{\delta\}) \subseteq w^+\} \right).$$

Each such w is such that w^+ has more atoms than v^+, so we may apply the inductive hypothesis. This proves the claim.

By applying Lemma 2.6 with $s = \emptyset$, we get that $\mathrm{Ha1} \cup \mathrm{Ha2} \vdash \bigvee \{\rho_v^+ \ : \ v \in |\mathbf{C}(\mathbf{x})|\}$. So Lemma 2.6 together with the claim gives us

$$\mathrm{Ha1} \cup \mathrm{Ha2} \vdash \bigwedge_{s \notin U} (\pi_s^+ \to \sigma_s^-) \to \bigvee_{u \in U} \rho_u^+.$$

For the left to right direction, fix $u \in U$ and $s \notin U$. Since $s \notin U$, we know $u \not\leq s$. If $u^+ \subseteq s^+$, then ρ_u^+ tautologically implies $\pi_s^+ \to \sigma_s^-$. If $u^+ \nsubseteq s^+$, then there is an atom $\delta \in (u^+ \cap s^-)$. Then $\rho_u^+ \to \delta$, so $\rho_u^+ \to \sigma_s^-$ and thus $\rho_u^+ \to (\pi_s^+ \to \sigma_s^-)$. So, for each $u \in U$ and each $s \notin U$, we get $\rho_u^+ \to (\pi_s^+ \to \sigma_s^-)$. Therefore, $\bigvee_{u \in U} \rho_u^+ \to \bigwedge_{s \notin U}(\pi_s^+ \to \sigma_s^-)$. $\square$

The next two lemmas enable us to eliminate a quantifier from each of our canonical forms.

Lemma 2.10. *For* $u \in |\mathbf{C}(\mathbf{x}x_n)|$, $\mathrm{Ha1} \cup \mathrm{Ha2} \cup \mathrm{Ha3} \vdash \exists x_n \rho_u^+ \leftrightarrow \rho_{d(u)}^+$.

Proof. The left to right direction immediately follows from Lemma 2.8.

We prove the right to left direction using the axiom schema Ha3 and reverse (strong) induction on the number of atoms in $d(u)^+$. Suppose $\rho_{d(u)}^+$ and consider the schema Ha3 where φ is the formula $\exists x_n \rho_u^+$. We immediately satisfy the first and third clause of the antecedent of Ha3 as follows. We have $\rho_{d(u)}^+$, and thus $\pi_{d(u)}^+$. Also, for any x_n such that π_u^+ holds, apply the schema Ha2. We have σ_u^- or ρ_u^+, the latter implying φ. Thus, we have the first and the third clause of the antecedent of Ha3; it remains to show that the second clause of the antecedent of Ha3 holds. We claim that $(\rho_{d(u)}^+ \wedge \sigma_{d(u)}^-) \to \varphi$.

If $d(u)^+$ has the maximum number of atoms, then $\sigma_{d(u)}^-$ together with $\rho_{d(u)}^+$ gives us $\perp$, so the claim is satisfied. Otherwise, we get $\bigvee\{\rho_{d(u)}^+ \wedge \delta \ : \ \delta \in d(u)^-\}$. Fix such a δ. By Lemma 2.7, we have

$$(\rho_{d(u)}^+ \wedge \delta) \to \bigvee\{\rho_t^+ \ : \ d(u)^+ \cup \{\delta\} \subseteq t^+\}.$$

Fix such a t. Theorem 2.3 gives us a $v \in |\mathbf{C}(\mathbf{x}x_n)|$ such that $t = d(v)$ and $u \leq v$. By our inductive hypothesis, we get $\exists x_n \rho_v^+$. By Lemma 2.4, the second clause of the antecedent of Ha3 holds.

So in all cases, we satisfy all three clauses of the antecedent of Ha3, and therefore from $\rho^+_{d(u)}$ we deduce $\exists x_n \rho^+_u$. □

Lemma 2.11. *For* $u \in |\mathbf{C}(\mathbf{x}x_n)|$, $\mathrm{Ha3} \vdash \forall x_n(\pi^+_u \to \sigma^-_u) \leftrightarrow (\pi^+_{d(u)} \to \sigma^-_{d(u)})$.

Proof. From right to left follows from Lemma 2.2. For the left to right direction, suppose we have $\forall x_n(\pi^+_u \to \sigma^-_u)$ and $\pi^+_{d(u)}$ and apply Ha3 with $\sigma^-_{d(u)}$ as φ. Then the first clause of Ha3 is immediately satisfied, the second clause is a tautology, and the third clause follows from our hypothesis. So $\sigma^-_{d(u)}$ follows by Ha3. □

Lemma 2.12. *For* $u, v \in |\mathbf{C}(\mathbf{x})|$ *such that* $u \leq v$*, we have*

$$\Gamma_{\mathrm{Ha}} \vdash (\pi^+_v \to \sigma^-_v) \to (\pi^+_u \to \sigma^-_u).$$

Proof. We proceed by induction on the size of $\mathbf{x}$. For the case with no free variables, $(\pi^+_v \to \sigma^-_v) \equiv \top \to \bot$, as $v \in |\mathbf{C}(\emptyset)|$. In such a case, the result vacuously holds. For the inductive step, now suppose we have $(\pi^+_v \to \sigma^-_v) \wedge \pi^+_u$. We must show σ^-_u.

By Lemma 2.6, our supposition is equivalent to $\bigvee\{(\pi^+_v \to \sigma^-_v) \wedge \rho^+_w \ : \ u^+ \subseteq w\}$. If $u^+ \subsetneq w^+$, then we have $\pi^+_w \vdash \sigma^-_u$. So all that remains is to show that $(\pi^+_v \to \sigma^-_v) \wedge \rho^+_u$ gives us σ^-_u. Apply Ha4, with $(\pi^+_v \to \sigma^-_v)$ as ψ. We get

$$\forall \mathbf{x} x_n[(\pi^+_v \to \sigma^-_v) \wedge \rho^+_u \to (\sigma^-_u \vee \forall x_n(\rho^+_u \to (\pi^+_v \to \sigma^-_v)))].$$

It is now enough to show that $\forall x_n(\rho^+_u \to (\pi^+_v \to \sigma^-_v))$ gives us σ^-_u. By Lemma 2.7, $\forall x_n(\rho^+_u \to (\pi^+_v \to \sigma^-_v))$ is equivalent to

$$\forall x_n \left(\bigvee\{\rho^+_w \ : \ u \leq w \text{ and } v^+ \subseteq w\} \to \sigma^-_v\right).$$

This can be rewritten as

$$\bigwedge \{\forall x_n(\rho_w^+ \to \sigma_v^-) \ : \ u \leq w \text{ and } v^+ \subseteq w\},$$

from which we deduce $\forall x_n(\rho_v^+ \to \sigma_v^-)$. Ha2 shows that $\forall x_n(\rho_v^+ \to \sigma_v^-) \leftrightarrow \forall x_n(\pi_v^+ \to \sigma_v^-)$. By Lemma 2.11, $\forall x_n(\pi_v^+ \to \sigma_v^-)$ is equivalent to $\pi_{d(v)}^+ \to \sigma_{d(v)}^-$. From our inductive hypothesis, we deduce $\pi_{d(u)}^+ \to \sigma_{d(u)}^-$. By Lemma 2.2, we conclude σ_u^-. □

Lemma 2.13. *For all $t, u \in |\mathbf{C}(\mathbf{x})|$, we have*

$$\Gamma_{\text{Ha}} \vdash (\rho_t^+ \to (\pi_u^+ \to \sigma_u^-)) \to \bigwedge\{\pi_v^+ \to \sigma_v^- \ : \ v \leq u \text{ and } t \leq u\}.$$

Proof. Fix t and u and suppose $\rho_t^+ \to (\pi_u^+ \to \sigma_u^-)$. We may assume $t \leq u$. Fix $v \leq u$. We must show $\pi_v^+ \to \sigma_v^-$. By Ha2, we may replace $\rho_t^+ \to (\pi_u^+ \to \sigma_u^-)$ with $(\rho_t^+ \wedge \rho_u^+) \to \sigma_u^-$. With Lemma 2.4, we get $((\rho_t^+ \wedge \rho_u^+) \to \sigma_u^-) \to (\rho_u^+ \to \sigma_u^-)$ since $t \leq u$. Again by Ha2, $\rho_u^+ \to \sigma_u^-$ is equivalent to $\pi_u^+ \to \sigma_u^-$. Apply Lemma 2.12. □

We are now ready to show that Γ_{Ha} admits quantifier elimination. First, we show that over Γ_{Ha}, quantifier-free formulas are equivalent to formulas of certain forms. We call these forms "disjunctive normal form" and "conjunctive normal form" - these terms are perhaps overused, but seem to serve our purpose here. Recall that in general over intuitionistic logic, quantifier-free formulas have no such canonical forms.

Theorem 2.14. *For every quantifier-free formula $\varphi(\mathbf{x})$, there is an open subset U of $|\mathbf{C}(\mathbf{x})|$ such that*

$$\Gamma_{\text{Ha}} \vdash \varphi \leftrightarrow \bigvee_{u \in U} \rho_u^+ \leftrightarrow \bigwedge_{s \notin U} (\pi_s^+ \to \sigma_s^-).$$

So every quantifier-free formula has both a disjunctive and a conjunctive normal form over Γ_{Ha}.

Proof. The proof is by induction on the complexity of φ. If φ is atomic, then Lemma 2.7 tells us that $\varphi \leftrightarrow \bigvee\{\rho_v^+ \,:\, \varphi \in v\}$. Then Lemma 2.9 gives us a conjunctive normal form for φ. If $\varphi \equiv \psi \wedge \theta$, then we can write $\psi \leftrightarrow \bigwedge_{s \notin U}(\pi_s^+ \to \sigma_s^-)$ and $\theta \leftrightarrow \bigwedge_{v \notin W}(\pi_v^+ \to \sigma_v^-)$. So $\varphi \leftrightarrow \bigwedge_{z \notin U \cap W}(\pi_z^+ \to \sigma_z^-)$, and again we can use Lemma 2.9 to write φ in disjunctive normal form. The case for $\varphi \equiv \psi \vee \theta$ works similarly by writing each component in disjunctive normal form.

For the case where $\varphi \equiv \psi \to \theta$, we can write $\psi \leftrightarrow \bigvee_{u \in U} \rho_u^+$ and $\theta \leftrightarrow \bigwedge_{v \notin W}(\pi_v^+ \to \sigma_v^-)$. Then

$$\varphi \leftrightarrow \bigwedge_{u \in U} \bigwedge_{v \notin W}(\rho_u^+ \to (\pi_v^+ \to \sigma_v^-)).$$

By Lemma 2.13, this is equivalent to

$$\bigwedge_{u \in U} \bigwedge_{v \notin W} \bigwedge\{\pi_w^+ \to \sigma_w^- \,:\, u \leq v \text{ and } w \leq v\}.$$

Therefore, again by Lemma 2.9, φ can be rewritten in disjunctive and conjunctive normal form. □

In Chapter 1, we show semantically that every quantifier-free formula has a disjunctive normal form over Γ_{M} (Corollary 1.31). This equivalence played a strong role in proving that Γ_{M} admits quantifier elimination (Theorem 1.33). That every quantifier-free formula has a conjunctive normal form can be deduced from Lemma 1.26 using some basic (finite) topology.

Theorem 2.15. Γ_{Ha} *admits quantifier elimination.*

Proof. We induct on formula complexity. By Lemma 2.14, we rewrite formulas of the form $\exists x \varphi$, where φ is quantifier-free, as $\exists x \bigvee_{u \in U} \rho_u^+$, which is equivalent to $\bigvee_{u \in U} \rho_{d(u)}^+$ by

Lemma 2.10. By Lemma 2.14, we rewrite formulas of the form $\forall x\varphi$ as $\forall x \bigwedge_{u\notin U}(\pi_u^+ \to \sigma_u^-)$, which, by Lemma 2.11, is equivalent to $\bigwedge_{u\notin U}(\pi_{d(u)}^+ \to \sigma_{d(u)}^-)$. □

We achieve our goal as a corollary.

Theorem 2.16. *The axiom system* Ha *axiomatizes* Γ_{M}.

Proof. By Lemma 2.1, $\Gamma_{\mathrm{Ha}} \subseteq \Gamma_{\mathrm{M}}$. Since $\mathcal{L}$ has only $\top$ and $\bot$ as nullary predicates, over Γ_{M}, all quantifier-free sentences are equivalent to combinations of $\top$ and $\bot$. So Γ_{Ha} is complete by Theorem 2.15. Therefore, $\Gamma_{\mathrm{Ha}} = \Gamma_{\mathrm{M}}$. □

2.2 Sharpening the Ha Axiom System

The Ha axiom system was chosen to reflect properties of the Kripke model $\mathfrak{A}_{\mathrm{M}}$, so in some sense the Ha system can be viewed as a natural axiomatization choice. However, in the preceding section, and to some extent in Section 1.4, those properties are not used directly to prove quantifier elimination. In both instances, we more specifically use the properties that every quantifier-free formula can be put in one of two canonical forms, and that we can eliminate quantifiers from formulas with matrices of those forms. Now that we have an axiomatization that works, we can examine other axiom systems that will perhaps generalize in different ways.

We did not use the full power of the Ha axiom system in Section 2.1. The axiom schemas Ha3 and Ha4 were used but a handful of times, and thus those schemas are stated in broader generality than necessary. The axiom schema Ha3 was used twice; φ was replaced by $\exists x_n \rho_u^+$ in Lemma 2.10, and φ was replaced by $\sigma_{d(u)}^-$ in Lemma 2.11.

The schema Ha4 was used only once, with ψ replaced by $\pi_t^+ \to \sigma_t^-$ (where $u \leq t$) in Lemma 2.12.

In classical logic, one often classifies formulas by their place in the hierarchy of formulas, e.g., Ha3 might read "...for all Π_2^0 formulas $\varphi(\mathbf{x})$ with x_n not free". There is no prenex normal form theorem for intuitionistic logic, however, some authors have proposed hierarchies of intuitionistic formulas; see [3], [18], and [25] for example. Fleischmann, building on work in [9], proposes in [11] several intuitionistic hierarchies of formulas and uses them to prove some analogs of classical, model-theoretic preservation theorems. We use his hierarchies and notation to mention the axiom system Hb, before narrowing the scope of the axiom schemas Ha3 and Ha4 even further in the axiom system Hc.

Hb1 $\forall \mathbf{x}(\pi_t^+ \to \sigma_t^-)$ for all $t \notin |\mathbf{C}(\mathbf{x})|$

Hb2 $\forall \mathbf{x}(\pi_t^+ \to (\sigma_t^- \vee \rho_t^-))$ for all $t \in |\mathbf{C}(\mathbf{x})|$

Hb3 $\forall \mathbf{x}[(\pi_{d(u)}^+ \wedge (\sigma_{d(u)}^- \to \varphi) \wedge \forall x_n(\pi_u^+ \to (\sigma_u^- \vee \varphi))) \to \varphi]$ for all $u \in |\mathbf{C}(\mathbf{x}x_n)|$ and all formulas $\varphi(\mathbf{x}) \in \mathcal{E}_1$ in which x_n is not free

Hb4 $\forall \mathbf{x}x_n[\psi \wedge \rho_u^+ \to (\sigma_u^- \vee \forall x_n(\rho_u^+ \to \psi))]$ for all $u \in |\mathbf{C}(\mathbf{x}x_n)|$ and all formulas $\psi(\mathbf{x}x_n) \in \mathcal{U}_1$

We write Γ_{Hb} for the intuitionistic theory axiomatized by Hb1 through Hb4.

If we prefer to use one of Fleischmann's other hierarchies, we may replace the $\mathcal{E}_1$ in Hb3 and $\mathcal{U}_1$ in Hb4 with, respectively, $\mathcal{E}_1^*$ and $\mathcal{U}_1^*$; or $\mathcal{E}_1^{\mathcal{S}}$ and $\mathcal{U}_1^{\mathcal{S}}$. While $\mathcal{U}_1 = \mathcal{U}_1^* = \mathcal{U}_1^{\mathcal{S}}$, we have that $\mathcal{E}_1^* = \mathcal{E}^+ \neq \mathcal{E}_1$ and $\mathcal{E}_1^{\mathcal{S}} = \mathcal{E}_1 \cap \mathcal{S}$ where $\mathcal{S}$ is the set of semi-positive formulas.

Clearly, $\Gamma_{\mathrm{Hb}} \subseteq \Gamma_{\mathrm{Ha}}$. By examining the structure of Lemmas 2.10, 2.11 and 2.12 and noting that the axiom schemas Ha3 and Ha4 are not called upon elsewhere in the proof to Theorem 2.15, we get the following corresponding result.

Theorem 2.17. Γ_{Hb} *admits quantifier elimination.*

And again, we get the following as a corollary.

Theorem 2.18. *The axiom system* Hb *axiomatizes* Γ_{M}.

While limiting the scope of the schemas Ha3 and Ha4 to the appropriate level in an intuitionistic hierarchy creates a more specific axiomatization than the Ha system, we can be even more specific. With Ha3 and Ha4 used so infrequently in the proof of Theorem 2.15, we create the axiom system Hc by using the schemas in the system Hb with formulas of the specific form used in the proofs in Section 2.1 already substituted in.

Hc1 $\forall \mathbf{x}(\pi_t^+ \to \sigma_t^-)$ for all $t \notin |\mathbf{C}(\mathbf{x})|$

Hc2 $\forall \mathbf{x}(\pi_t^+ \to (\sigma_t^- \vee \rho_t^-))$ for all $t \in |\mathbf{C}(\mathbf{x})|$

Hc3
1. $\forall \mathbf{x}[(\pi_{d(u)}^+ \wedge (\sigma_{d(u)}^- \to \exists x_n \rho_u^+) \wedge \forall x_n(\pi_u^+ \to (\sigma_u^- \vee \exists x_n \rho_u^+))) \to \exists x_n \rho_u^+]$ for all $u \in |\mathbf{C}(\mathbf{x}x_n)|$
2. $\forall \mathbf{x}[(\pi_{d(u)}^+ \wedge \forall x_n(\pi_u^+ \to \sigma_u^-)) \to \sigma_{d(u)}]$ for all $u \in |\mathbf{C}(\mathbf{x}x_n)|$

Hc4 $\forall \mathbf{x}x_n[(\pi_t^+ \to \sigma_t^-) \wedge \rho_u^+ \to (\sigma_u^- \vee \forall x_n(\rho_u^+ \to (\pi_t^+ \to \sigma_t^-)))]$ for all $t, u \in |\mathbf{C}(\mathbf{x}x_n)|$ with $u \leq t$

We write Γ_{Hc} for the intuitionistic theory axiomatized by Hc1 through Hc4.

The axiom schema Hc3.1 is the schema Ha3 with φ replaced by $\exists x_n \rho_u^+$; compare with the usage of Ha3 in Lemma 2.10. By Lemma 2.2, the axiom schema Hc3.2 is equivalent to the schema Ha3 with φ replaced by $\sigma_{d(u)}^-$; compare with the usage of Ha3 in Lemma 2.11. Lastly, the axiom schema Hc4 is the schema Ha4 with ψ replaced by $\pi_t^+ \to \sigma_t^-$ where $t \in |\mathbf{C}(\mathbf{x}x_n)|$ and $u \leq t$; compare to the usage of Ha4 in Lemma 2.12. It is again clear that $\Gamma_{\mathrm{Hb}} \subseteq \Gamma_{\mathrm{Ha}}$. As above, we can immediately prove quantifier elimination; in retracing the proof of Theorem 2.15 we may skip Lemma 2.11 and prove Lemma 2.12 from the schema Hc3.2.

Theorem 2.19. Γ_{Hc} *admits quantifier elimination.*

Theorem 2.20. *The axiom system* Hc *axiomatizes* Γ_{M}.

Again using Fleischmann's intuitionistic hierarchies (see [11]), we still have that Hc1 and Hc2 are both universal ($\mathcal{U}_1$). Furthermore, the schema Hc3.1 $\subseteq \mathcal{U}(\mathcal{E}_1, \mathcal{U}_2)$, the schema Hc3.2 $\subseteq \mathcal{U}(\mathcal{E}_0, \mathcal{U}_1)$, and the schema Hc4 $\subseteq \mathcal{U}(\mathcal{U}(\mathcal{U}_1, \mathcal{U}_0), \mathcal{U}_0)$.

2.3 An Axiom System Motivated by Quantifier Elimination

The axiom system Ha was constructed to reflect important properties of the Kripke model $\mathfrak{A}_{\mathrm{M}}$. The systems Hb and Hc are simply more specific versions of Ha. However, by examining Theorem 2.16, we construct a qualitatively different axiom schema. Consider the axiom system Qa:

Qa1 $\forall \mathbf{x}(\delta \leftrightarrow \bigvee\{\rho_t^+ \ : \ t \in |\mathbf{C}(\mathbf{x})| \text{ and } \delta \in t\})$ for all $\delta \in \mathcal{A}\mathrm{t}(\mathbf{x})$

Qa2 $\forall \mathbf{x}(\bigvee_{t \in U} \rho_t^+ \leftrightarrow \bigwedge_{t \notin U}(\pi_t^+ \to \sigma_t^-))$ for all open $U \subseteq |\mathbf{C}(\mathbf{x})|$

Qa3 $\forall \mathbf{x}[(\rho_t^+ \to (\pi_u^+ \to \sigma_u^-)) \leftrightarrow \bigwedge\{\pi_v^+ \to \sigma_v^- \ : \ t \leq u \text{ and } v \leq u\}]$ for all $t, u \in |\mathbf{C}(\mathbf{x})|$

Qa4 $\forall \mathbf{x}(\exists x_n \rho_u^+ \leftrightarrow \rho_{d(u)}^+)$ for all $u \in |\mathbf{C}(\mathbf{x}x_n)|$

Qa5 $\forall \mathbf{x}(\forall x_n(\pi_u^+ \leftrightarrow \sigma_u^-) \leftrightarrow (\pi_{d(u)}^+ \to \sigma_{d(u)}^-))$ for all $u \in |\mathbf{C}(\mathbf{x}x_n)|$

Let Γ_{Qa} be the theory axiomatizable by Qa1 through Qa5.

As we show below, the axiom schema Qa1 allows us to put each atom in disjunctive normal form. The schema Qa2 gives the equivalence of our disjunctive and conjunctive normal forms. The schema Qa3 is required for the implication step of the inductive argument showing that all quantifier-free formulas can be put in both normal forms. And the schemas Qa4 and Qa5 explicitly allow the elimination of quantifiers from these normal forms. Compare these schemas to the structure of the proof of Theorem 2.16.

Lemma 2.21. $\Gamma_{\mathrm{M}} \vdash \Gamma_{\mathrm{Qa}}$.

Proof. Given Lemma 2.1, the cases for the schemas Qa2, Qa4 and Qa5 are already directly proven in Lemmas 2.14, 2.10, and 2.11, respectively. The non-obvious direction of Qa1 follows from Lemma 2.6.

It remains to show that $\Gamma_{\mathrm{M}} \vdash$ Qa3. First, the left to right direction. Fix $t \leq u$ and $v \leq u$ and suppose that for some $\mathbf{a} \in A_{\mathrm{JRS}}$ we have

$$\mathfrak{A}_{\mathrm{M}} \Vdash \pi_v^+(\mathbf{a}) \wedge (\rho_t^+(\mathbf{a}) \to (\pi_u^+(\mathbf{a}) \to \sigma_u^-(\mathbf{a}))).$$

We must show that $\mathfrak{A}_{\mathrm{M}} \Vdash \sigma_v^-(\mathbf{a})$. Assume not. Then, $\mathfrak{A}_{\mathrm{JRS}} \models \pi_v(\mathbf{a})$. Since $v \leq u$, $\mathfrak{A}_{\mathrm{JRS}} \models \pi_u(f(\mathbf{a}))$ for some endomorphism f of $\mathfrak{A}_{\mathrm{JRS}}$. So $\mathfrak{A}_{\mathrm{M}} \Vdash \rho_u^+(f(\mathbf{a}))$ (by Lemma 1.28, for example). Since $t \leq u$, we have $\mathfrak{A}_{\mathrm{M}} \Vdash \rho_t^+(f(\mathbf{a}))$ and so by supposition, $\mathfrak{A}_{\mathrm{M}} \Vdash \sigma_u^-(f(\mathbf{a}))$. But then $\mathfrak{A}_{\mathrm{JRS}} \not\models \pi_u(f(\mathbf{a}))$, a contradiction. So $\mathfrak{A}_{\mathrm{M}} \Vdash \sigma_v^-(\mathbf{a})$.

For the right to left direction, fix $t, u \in |\mathbf{C}(\mathbf{x})|$ and suppose that for some $\mathbf{a} \in A_{\mathrm{JRS}}$ we have

$$\mathfrak{A}_{\mathrm{M}} \Vdash \rho_t(\mathbf{a}) \wedge \pi_u^+(\mathbf{a}) \wedge \bigwedge\{\pi_v^+(\mathbf{a}) \rightarrow \sigma_v^-(\mathbf{a}) \;:\; t \leq u \text{ and } v \leq u\}.$$

We must show $\mathfrak{A}_{\mathrm{M}} \Vdash \sigma_u^-(\mathbf{a})$. If $t \leq u$, then we are done as

$$\bigwedge\{\pi_v^+ \rightarrow \sigma_v^- \;:\; t \leq u \text{ and } v \leq u\} \vdash \pi_u^+ \rightarrow \sigma_u^-.$$

So we may suppose that $t \nleq u$. If $t^+ \cap u^- \neq \emptyset$, then again we are done, as $\pi_t^+ \vdash \sigma_u^-$. So we may suppose that $t^+ \subsetneq u^+$. But then $\rho_t^+ \vdash \pi_u^+ \rightarrow \sigma_u^-$. Therefore $\Gamma_{\mathrm{M}} \vdash$ Qa3. □

Lemma 2.22. *For every* $\mathbf{x}$ *and every quantifier-free formula* $\varphi(\mathbf{x})$*, there is an open subset* $U \subseteq |\mathbf{C}(\mathbf{x})|$ *such that*

$$\mathrm{Qa1} \cup \mathrm{Qa2} \cup \mathrm{Qa3} \vdash \varphi \leftrightarrow \bigvee_{u \in U} \rho_u^+ \leftrightarrow \bigwedge_{s \notin U} (\pi_s^+ \rightarrow \sigma_s^-).$$

Proof. We complete the proof by induction on the complexity of φ. The case for atoms and the induction steps for disjunction and conjunction all easily follow with schemas Qa1 and Qa2. Suppose $\varphi \equiv \psi \rightarrow \theta$, where the claim holds for ψ and θ. That is, there are open subsets $U, V \subseteq |\mathbf{C}(\mathbf{x})|$ such that $\psi \leftrightarrow \bigvee_{u \in U} \rho_u^+$ and $\theta \leftrightarrow \bigwedge_{t \notin V} (\pi_t^+ \rightarrow \sigma_t^-)$. So $\varphi \leftrightarrow \bigwedge_{u \in U} \bigwedge_{t \notin V} (\rho_u^+ \rightarrow (\pi_t^+ \rightarrow \sigma_t^-))$. By Qa3, we get $\varphi \leftrightarrow \bigwedge_{w \in W} (\pi_w^+ \rightarrow \sigma_w^-)$ where

$$W = \{w \in |\mathbf{C}(\mathbf{x})| \;:\; \exists u \in U \exists t \notin V (u \leq t \wedge w \leq t)\}.$$

Since W is downward closed, φ is equivalent to a formula in conjunctive normal form. By Qa2, φ is also equivalent to a formula in disjunctive normal form. This completes the induction. □

Theorem 2.23. Γ_{Qa} *admits quantifier elimination.*

Proof. Fix an $\mathcal{L}$-formula φ and proceed by induction on the number of quantifiers in φ. By the inductive hypothesis, we have two cases; either $\varphi \equiv \exists x_n \psi$ or $\varphi \equiv \forall x_n \psi$ where ψ is quantifier-free. In the first case, we may write ψ as $\exists x_n \bigvee_{t \in U} \rho_t^+$ for some open $U \subseteq |\mathbf{C}(\mathbf{x}x_n)|$ by Lemma 2.22. Then φ is equivalent to $\bigvee_{t \in U} \rho_{d(t)}^+$ by Qa4. In the second case, we may write ψ as $\forall x_n \bigwedge_{t \notin U} (\pi_t^+ \to \sigma_t^-)$ for some open $U \subseteq |\mathbf{C}(\mathbf{x}x_n)|$ by Lemma 2.22. Then φ is equivalent to $\bigwedge_{t \notin U} (\pi_{d(t)}^+ \to \sigma_{d(t)}^-)$ by Qa5. □

As above, we again get the following as a corollary.

Theorem 2.24. *The axiom system* Qa *axiomatizes* Γ_{M}.

2.4 Sharpening the Qa Axiom System

As with the Ha system in Section 2.2, we consider ways to improve the Qa system. To this end, consider the Qb axiom system:

Qb1 $\forall \mathbf{x}(\delta \to \bigvee\{\rho_t^+ \;:\; t \in |\mathbf{C}(\mathbf{x})| \text{ and } \delta \in t\})$ for all atoms $\delta \in \mathcal{A}\mathrm{t}(\mathbf{x})$

Qb2 $\forall \mathbf{x}(\bigwedge_{t \notin U} (\pi_t^+ \to \sigma_t^-) \to \bigvee_{t \in U} \rho_t^+)$ for all open $U \subseteq |\mathbf{C}(\mathbf{x})|$

Qb3 $\forall \mathbf{x}[(\pi_u^+ \to \sigma_u^-) \to (\pi_v^+ \to \sigma_v^-)]$ for all $v \leq u \in |\mathbf{C}(\mathbf{x})|$

Qb4 $\forall \mathbf{x}(\rho_{d(u)}^+ \to \exists x_n \rho_u^+)$ for all $u \in |\mathbf{C}(\mathbf{x}x_n)|$

Qb5 $\forall \mathbf{x}(\forall x_n(\pi^+_u \to \sigma^-_u) \to (\pi^+_{d(u)} \to \sigma^-_{d(u)}))$ for all $u \in |\mathbf{C}(\mathbf{x}x_n)|$

Let Γ_{Qb} be the theory axiomatizable by Qb1 through Qb5.

Lemma 2.25. $\Gamma_{\mathrm{M}} \vdash \Gamma_{\mathrm{Qb}}$.

Proof. Given Lemma 2.1, the schemas Qb1 through Qb5 follow directly from Lemmas 2.6, 2.14, 2.12, 2.10, and 2.11, respectively. (More broadly, all but Qb3 follow from Lemma 2.21.) □

We could follow the pattern above and directly prove that Γ_{Qb} eliminates quantifiers, thereby showing the system Qb axiomatizes Γ_{M}. Instead, we will arrive at the same result by drawing connections between the systems Qa and Qb.

Lemma 2.26. *Let U be an open subset of $|\mathbf{C}(\mathbf{x})|$. Then*

$$\vdash \bigvee_{u \in U} \rho^+_u \to \bigwedge_{v \notin U} (\pi^+_v \to \sigma^-_v).$$

Proof. Fix such a u and v. It then suffices to prove $\vdash \rho^+_u \to (\pi^+_v \to \sigma^-_v)$. If $u^+ \subseteq v^+$ then ρ^-_u tautologically implies $\pi^+_v \to \sigma^-_v$. Otherwise, $u^+ \not\subseteq v^+$, so there is an atom $\delta \in u^+ \cap v^-$. Then the following implications are tautologies: $\rho^+_u \to \pi^+_u$, $\pi^+_u \to \delta$, $\delta \to \sigma^-_v$, and $\sigma^-_v \to (\pi^+_v \to \sigma^-_v)$. □

Lemma 2.27. *The axiom subsystem* Qa1 $\cup$ Qa2 *is equivalent to the axiom subsystem* Qb1 $\cup$ Qb2.

Proof. The implication present in Qa1 but absent in Qb1 is a tautology. The implication present in Qa2 but absent in Qb2 is Lemma 2.26. □

Definition 2.28. *For an open subset U of $|\mathbf{C}(\mathbf{x})|$, we write $\rho_U^+(\mathbf{x})$ or just ρ_U^+ to represent the formula $\bigvee_{u\in U}\rho_u^+(\mathbf{x})$.*

Lemma 2.29. *For open subsets U and V of $|\mathbf{C}(\mathbf{x})|$, we have*

$$\vdash (\rho_U^+ \vee \rho_V^+) \leftrightarrow \rho_{U\cup V}^+$$

and

$$\mathrm{Qb2} \vdash (\rho_U^+ \wedge \rho_V^+) \leftrightarrow \rho_{U\cap V}^+.$$

Proof. The first claim is a tautology. For the second claim, over Qb2, $\rho_{U\cap V}^+$ is equivalent to $\bigwedge_{t\notin U\cup V}(\pi_t^+ \to \sigma_t^-)$, which is tautologically equivalent to

$$\bigwedge_{t\notin U}(\pi_t^+ \to \sigma_t^-) \wedge \bigwedge_{t\notin V}(\pi_t^+ \to \sigma_t^-),$$

which is equivalent to $\rho_U^+ \wedge \rho_V^+$ by Qb2. □

Lemma 2.30. Qb1 ∪ Qb2 ∪ Qb3 ⊢ Qa3.

Proof. For $t, u \in |\mathbf{C}(\mathbf{x})|$, we first claim

$$\vdash \bigwedge\{\pi_v^+ \to \sigma_v^- \ : \ t \leq u \text{ and } v \leq u\} \to (\rho_t^+ \to (\pi_u^+ \to \sigma_u^-)).$$

If $t \leq u$, then we are done as $\bigwedge\{\pi_v^+ \to \sigma_v^- \ : \ t \leq u \text{ and } v \leq u\} \vdash \pi_u^+ \to \sigma_u^-$. So we may suppose that $t \nleq u$. If $t^+ \cap u^- \neq \emptyset$, then again we are done, as $\pi_t^+ \vdash \sigma_u^-$. So we may suppose that $t^+ \subsetneq u^+$. But then $\rho_t^+ \vdash \pi_u^+ \to \sigma_u^-$.

It is left to show

$$\mathrm{Qb1} \cup \mathrm{Qb2} \cup \mathrm{Qb3} \vdash (\rho_t^+ \to (\pi_u^+ \to \sigma_u^-)) \to \bigwedge\{\pi_v^+ \to \sigma_v^- \ : \ t \leq u \text{ and } v \leq u\}.$$

Suppose the conjunction is not empty, and fix such a v. We now claim that

$$\text{Qb1} \cup \text{Qb2} \vdash ((\pi_u^+ \to \sigma_u^-) \to (\pi_v^+ \to \sigma_v^-)) \leftrightarrow ((\rho_u^+ \to \sigma_u^-) \to (\pi_v^+ \to \sigma_v^-)).$$

By Qb1 and Lemma 2.29, π_u^+ is equivalent to $\bigvee\{\rho_w^+ : u^+ \subseteq w^+\}$, so $\pi_u^+ \to \sigma_u^-$ is equivalent to $\bigwedge\{\rho_w^+ \to \sigma_u^- : u^+ \subseteq w^+\}$. For each w such that $u^+ \subsetneq w^+$, we have $\pi_w^+ \vdash \sigma_u^-$, leaving only the case when $w = u$. So over Qb1, $\pi_u^+ \to \sigma_u^-$ is equivalent to $\rho_u^+ \to \sigma_u^-$, and the claim is proved.

To prove the lemma, it suffices to prove our final claim:

$$((\rho_u^+ \to \sigma_u^-) \to (\pi_v^+ \to \sigma_v^-)) \vdash ((\rho_t^+ \to (\pi_u^+ \to \sigma_u^-)) \to (\pi_v^+ \to \sigma_v^-)).$$

We have $\vdash (\rho_u^+ \to \sigma_u^-) \leftrightarrow (\rho_u^+ \to (\pi_u^+ \to \sigma_u^-))$. Also, since $t \leq u$, we have $(\rho_t^+ \to (\pi_u^+ \to \sigma_u^-)) \vdash (\rho_u^+ \to (\pi_u^+ \to \sigma_u^-))$ tautologically (with Lemma 2.4). This proves the final claim. □

Lemma 2.31. *The axiom subsystem* Qa1 ∪ Qa2 ∪ Qa3 *is equivalent to the axiom subsystem* Qb1 ∪ Qb2 ∪ Qb3.

Proof. Clearly, Qa3 ⊢ Qb3. Apply Lemmas 2.27 and 2.30. □

The relationship between Qa4 and Qb4 is a bit more complicated. We now introduce notation and machinery to help show that they are equivalent over Qb1 ∪ Qb2 ∪ Qb3.

Definition 2.32. *For $t \in |\mathbf{C}(\mathbf{x})|$ and $\mathcal{L}(\mathbf{x})$-formula φ, we write $t \Vdash_3 \varphi$ as short for* $\text{Qb1} \cup \text{Qb2} \cup \text{Qb3} \vdash \rho_t^+ \to \varphi$.

Lemma 2.33. *For open subset U of $|\mathbf{C}(\mathbf{x})|$ and $t \in |\mathbf{C}(\mathbf{x})|$ we have $t \Vdash_3 \rho_U^+$ if and only if $t \in U$.*

Proof. If $t \in U$, then clearly $t \Vdash_3 \rho_U^+$. If $t \notin U$, then notice that $\Gamma_{\mathrm{M}} \nvdash \rho_t^+ \to \rho_U^+$ by Lemma 1.24. Therefore, $t \nVdash_3 \rho_U^+$ by Lemma 2.25. □

Lemma 2.34. *For all $t \in |\mathbf{C}(\mathbf{x})|$, all $\delta \in \mathcal{A}\mathrm{t}(\mathbf{x})$, and all quantifier-free $\mathcal{L}(\mathbf{x})$ formulas φ and ψ, we have the following.*

1. *$t \Vdash_3 \delta$ if and only if $\delta \in t^+$.*
2. *$t \Vdash_3 \varphi \wedge \psi$ if and only if $t \Vdash_3 \varphi$ and $t \Vdash_3 \psi$.*
3. *$t \Vdash_3 \varphi \vee \psi$ if and only if $t \Vdash_3 \varphi$ or $t \Vdash_3 \psi$.*
4. *$t \Vdash_3 \varphi \to \psi$ if and only if for all $u \geq t$, if $u \Vdash_3 \varphi$, then $u \Vdash_3 \psi$.*

Proof. Claim 1 follows from Qb1 with Lemma 2.33. Claim 2 is a tautology.

For Claims 3 and 4, by Lemma 2.31, we may apply Lemma 2.22 to get open sets U and V such that, over $\mathrm{Qb1} \cup \mathrm{Qb2} \cup \mathrm{Qb3}$, φ is equivalent to ρ_U^+ and ψ is equivalent to ρ_V^+. Now Claim 3 follows from Lemmas 2.29 and 2.33.

For Claim 4, $t \Vdash_3 \varphi \to \psi$ if and only if $t \Vdash_3 \rho_U^+ \to \rho_V^+$, if and only if

$$\mathrm{Qb1} \cup \mathrm{Qb2} \cup \mathrm{Qb3} \vdash \rho_t^+ \wedge \rho_U^+ \to \rho_V^+.$$

Let $\hat{t}$ be the smallest open set containing t. Then Lemmas 2.29 and 2.4 give us that $\vdash \rho_t^+ \leftrightarrow \rho_{\hat{t}}^+$. Now the statement displayed above is true if and only if

$$\mathrm{Qb1} \cup \mathrm{Qb2} \cup \mathrm{Qb3} \vdash \rho_{\hat{t} \cap U}^+ \to \rho_V^+,$$

if and only if

$$\mathrm{Qb1} \cup \mathrm{Qb2} \cup \mathrm{Qb3} \vdash \rho_u^+ \to \rho_V^+, \text{ for all } u \geq t \text{ such that } u \Vdash_3 \rho_U^+,$$

if and only if $u \Vdash_3 \psi$, for all $u \geq t$ such that $u \Vdash_3 \varphi$. □

Lemma 2.35. *For $u \in |\mathbf{C}(\mathbf{x}x_n)|$ and for any quantifier-free $\mathcal{L}(\mathbf{x})$-formula φ, $u \Vdash_3 \varphi$ if and only if $d(u) \Vdash_3 \varphi$.*

Proof. Induct on the complexity of φ. The base case as well as the cases where φ is a conjunction or a disjunction all follow from Lemma 2.34. Suppose $\varphi \equiv \psi \to \theta$.

Suppose $d(u) \Vdash_3 \varphi$ and choose any v such that $u \le v$ and $v \Vdash_3 \psi$. By the inductive hypothesis, $d(v) \Vdash_3 \psi$. Since $d(u) \le d(v)$, $d(v) \Vdash_3 \theta$. By the inductive hypothesis, $v \Vdash_3 \theta$. Therefore, by Claim 4 of Lemma 2.34, $u \Vdash_3 \varphi$.

Conversely, suppose $u \Vdash_3 \varphi$ and choose any t such that $d(u) \le t$ and $t \Vdash_3 \psi$. By Lemma 2.3, $t = d(v)$ for some $v \ge u$. By the inductive hypothesis, $v \Vdash_3 \psi$, so also $v \Vdash_3 \theta$. By the inductive hypothesis, $t = d(v) \Vdash_3 \theta$. Therefore, by Claim 4 of Lemma 2.34, $d(u) \Vdash_3 \varphi$. $\square$

Lemma 2.36. *For every open subset U of $|\mathbf{C}(\mathbf{x})|$ and every $t \in |\mathbf{C}(\mathbf{x})|$ we have*

$$\mathrm{Qb1} \cup \mathrm{Qb2} \cup \mathrm{Qb3} \vdash \rho_t^+ \leftrightarrow \bigvee\{\rho_u^+ \;:\; t \le d(u)\}$$

and

$$\mathrm{Qb1} \cup \mathrm{Qb2} \cup \mathrm{Qb3} \vdash \rho_U^+ \leftrightarrow \rho_{d^{-1}(U)}^+.$$

Proof. It suffices to prove the first claim. By viewing ρ_t^+ as an $\mathcal{L}(\mathbf{x}x_n)$-formula, Lemmas 2.31 and 2.22 give us on open subset V of $|\mathbf{C}(\mathbf{x}x_n)|$ such that

$$\mathrm{Qb1} \cup \mathrm{Qb2} \cup \mathrm{Qb3} \vdash \rho_t^+ \leftrightarrow \rho_V^+.$$

Let $U = \{u \in |\mathbf{C}(\mathbf{x}x_n)| \;:\; t \le d(u)\}$. For $u \in |\mathbf{C}(\mathbf{x}x_n)|$, Lemma 2.35 gives us $u \Vdash_3 \rho_t^+$ if and only if $d(u) \Vdash_3 \rho_t^+$ if and only if (by Lemma 2.33) $t \le d(u)$. So

$$\mathrm{Qb1} \cup \mathrm{Qb2} \cup \mathrm{Qb3} \vdash \rho_u^+ \rightarrow \rho_t^+ \text{ if and only if } t \leq d(u),$$

and so $V \subseteq U$. Therefore, we have that $\mathrm{Qb1} \cup \mathrm{Qb2} \cup \mathrm{Qb3}$ proves the following implications: $\rho_U^+ \rightarrow \rho_t^+$, $\rho_t^+ \rightarrow \rho_V^+$, and $\rho_V^+ \rightarrow \rho_U^+$. Therefore $\mathrm{Qb1} \cup \mathrm{Qb2} \cup \mathrm{Qb3} \vdash \rho_U^+ \leftrightarrow \rho_V^+$. □

We now have enough information to compare Qa4 to Qb4.

Lemma 2.37. $\mathrm{Qa1} \cup \mathrm{Qa2} \cup \mathrm{Qa3} \cup \mathrm{Qa4}$ *is equivalent to* $\mathrm{Qb1} \cup \mathrm{Qb2} \cup \mathrm{Qb3} \cup \mathrm{Qb4}$.

Proof. Clearly, $\mathrm{Qa4} \vdash \mathrm{Qb4}$. For each $u \in |\mathbf{C}(\mathbf{x}x_n)|$, Lemma 2.36 gives us

$$\mathrm{Qb1} \cup \mathrm{Qb2} \cup \mathrm{Qb3} \vdash \rho_u^+ \rightarrow \rho_{d(u)}^+.$$

Therefore,

$$\mathrm{Qb1} \cup \mathrm{Qb2} \cup \mathrm{Qb3} \cup \mathrm{Qb4} \vdash \mathrm{Qa4}.$$

The result now follows from Lemma 2.31. □

Lemma 2.38. *The axiom schema* Qa5 *is equivalent to the axiom schema* Qb5.

Proof. We must show that $\vdash (\pi_{d(u)}^+ \rightarrow \sigma_{d(u)}^-) \rightarrow \forall x_n(\pi_u^+ \rightarrow \sigma_u^-)$. Fix x_n and suppose that π_u^+ and $\pi_{d(u)}^+ \rightarrow \sigma_{d(u)}^-$ both hold. By Lemma 2.2, π_u^+ gives us $\pi_{d(u)}^+$, which by supposition gives us $\sigma_{d(u)}^-$, which gives us σ_u^- by Lemma 2.2. □

Combining Lemmas 2.37 and 2.38, we conclude the following.

Theorem 2.39. *The axiom system* Qa *is equivalent to the axiom system* Qb.

In terms of Fleischmann's hierarchies (see [11]), we can categorize our axiom schemas as follows: $\mathrm{Qb1} \subseteq \mathcal{U}_1$, $\mathrm{Qb2} \subseteq \mathcal{U}(\mathcal{U}_0, \mathcal{U}_0)$, $\mathrm{Qb3} \subseteq \mathcal{U}(\mathcal{U}_0, \mathcal{U}_0)$, $\mathrm{Qb4} \subseteq \mathcal{U}(\mathcal{E}_1, \mathcal{U}_0)$, and $\mathrm{Qb5} \subseteq \mathcal{U}(\mathcal{U}_0, \mathcal{U}_1)$.

Chapter 3

Monoids of Morphisms

In Chapter 1, we defined the Kripke model $\mathfrak{A}_{\mathrm{M}}$ to include a single node structure $\mathfrak{A}_{\mathrm{JRS}}$ and all endomorphisms of that structure. The intuitionistic theory of this Kripke model admits quantifier elimination. The classical structure $\mathfrak{A}_{\mathrm{JRS}}$ can also be viewed as a single-node Kripke model: one whose only morphism is the identity morphism. The (intuitionistic) theory of this Kripke model also admits quantifier elimination. Are there other monoids of morphisms of $\mathfrak{A}_{\mathrm{JRS}}$ that will yield single-node Kripke models whose theories admit quantifier elimination? In this chapter, we investigate the ramifications of changes to the monoid of morphisms in Kripke models with $\mathfrak{A}_{\mathrm{JRS}}$ as their only node.

3.1 A Motivating Example

Starting with a certain classical JRS model, we construct four different single-node Kripke models by varying the monoid of morphisms in the Kripke models. We will get four different, complete intuitionistic theories that all admit quantifier elimination. First, some definitions and a general result.

Definition 3.1. *1. Let $\mathfrak{A}$ be a classical* JRS *model in the language $\mathcal{L}$ and let $\mathfrak{B}$ be a classical* JRS *model in the language $\mathcal{M}$. Let R be a unary predicate new to $\mathcal{L} \cup \mathcal{M}$. In the obvious way, let $\mathfrak{A} \rtimes_{R(x)} \mathfrak{B}$ be the classical model with universe $A \dot{\cup} B$ where*

$\mathfrak{A} \rtimes_{R(x)} \mathfrak{B} \models R(x)$ *if and only if* $x \in A$.

2. *Let* $\Gamma_1 = \mathrm{Th}(\mathfrak{A}) \restriction \mathcal{L}$ *and* $\Gamma_2 = \mathrm{Th}(\mathfrak{B}) \restriction \mathcal{L}$*. Then we define* $\Gamma_1 \rtimes_{R(x)} \Gamma_2$ *as the* $\mathcal{L}$*-theory of the model* $\mathfrak{A} \rtimes_{R(x)} \mathfrak{B}$*.*

Lemma 3.2. $\mathfrak{A} \rtimes_{R(x)} \mathfrak{B}$ *is a* JRS *model.*

Proof. Let $\mathfrak{C} = \mathfrak{A} \rtimes_{R(x)} \mathfrak{B}$, and let $\Gamma = \mathrm{Th}_c(\mathfrak{C})$. Let t, u and v be $\mathcal{A}\mathrm{t}^{\pm}$-types consistent with Γ such that $\mathfrak{A}_t$ embeds into $\mathfrak{A}_u$ and $\mathfrak{A}_v$. We wish to use Theorem 1.8 and show that there is a model of $\Gamma_\forall$, which we will call $\mathfrak{A}_w$, that is an amalgam of $\mathfrak{A}_u$ and $\mathfrak{A}_v$ over $\mathfrak{A}_t$. It suffices to assume t is a complete $\mathcal{A}\mathrm{t}^{\pm}(\mathbf{xy})$-type and u and v are $\mathcal{A}\mathrm{t}^{\pm}(\mathbf{xy}x_n)$-types, where $t \vdash_c R(x)$ for each $x \in \mathbf{x}$ and $t \vdash_c \neg R(y)$ for each $y \in \mathbf{y}$.

Suppose $R(x_n) \in u$ and $R(x_n) \in v$. Then by Theorem 1.8 applied to $\mathrm{Th}(\mathfrak{A})$, over $\mathfrak{A}_{t\restriction\mathbf{x}}$, $\mathfrak{A}_{u\restriction\mathbf{x}}$ and $\mathfrak{A}_{v\restriction\mathbf{x}}$ have an amalgam $\mathfrak{A}_{w'}$. Then $w' \cup (t \restriction \mathbf{y})$ generates, over $\Gamma_\forall$, a complete $\mathcal{A}\mathrm{t}^{\pm}(\mathbf{xy}x_n)$-type w such that $\mathfrak{A}_w$ is an appropriate amalgam. A similar argument holds when $\neg R(x_n) \in u$ and $\neg R(x_n) \in v$. Finally, if instead $R(x_n) \in u$ and $\neg R(x_n) \in v$, then if w is the type generated by $(u \restriction \mathbf{x}x_n) \cup (v \restriction \mathbf{y})$, then $\mathfrak{A}_w$ is an appropriate amalgam.

Therefore, by Theorem 1.8, $\mathfrak{C}$ is a JRS model. □

We may now construct our example. Let $\mathfrak{B}$ be the JRS model of the theory of dense linear orders without endpoints in the language with $\{\leq\}$ (see Example 1.13). Let $\mathfrak{A} = \mathfrak{B} \rtimes_{R(x)} \mathfrak{B}$. We construct four single-node Kripke models, $\mathfrak{C}_1$ through $\mathfrak{C}_4$, where the classical node structure of each $\mathfrak{C}_i$ is $\mathfrak{A}$. The second column of the table in Figure 1 describes the morphisms present in each $\mathfrak{C}_i$. For example, the monoid of morphisms found in $\mathfrak{C}_2$ consists of all morphisms that preserve $\neg R$ (that is, for every morphism f found in $\mathfrak{C}_2$, if $\mathfrak{A} \models \neg R(a)$, then $\mathfrak{A} \models {}^{f}(\neg R(a))$). We take the monoid of morphisms in $\mathfrak{C}_4$ to be the monoid of all endomorphisms of $\mathfrak{A}$, so $\mathfrak{C}_4$ is just $\mathfrak{A}_{\mathrm{M}}$. Note that as $\mathfrak{C}_1$

Model (theory)	Negated atoms preserved	Decidability of R in $\mathfrak{C}_i$	Decidability of $\leq$ in $\mathfrak{C}_i$
$\mathfrak{C}_1$ $(\mathrm{Th}_c(\mathfrak{A}))$	$R, \leq$	$\forall x(R(x) \vee \neg R(x))$	$\forall xy(x \leq y \vee \neg(x \leq y))$
$\mathfrak{C}_2$	R	$\forall x(R(x) \vee \neg R(x))$	$\neg\forall xy(x \leq y \vee \neg(x \leq y))$
$\mathfrak{C}_3$	$\leq$	$\neg\forall x(R(x) \vee \neg R(x))$	$\forall xy(x \leq y \vee \neg(x \leq y))$
$\mathfrak{C}_4$ $(\mathrm{Th}(\mathfrak{A}_\mathrm{M}))$		$\neg\forall x(R(x) \vee \neg R(x))$	$\neg\forall xy(x \leq y \vee \neg(x \leq y))$

Figure 1: Properties of different Kripke models with the same node structure

contains all morphisms that preserve negations of atoms built from R and those built from $\leq$, the monoid of morphisms of $\mathfrak{C}_1$ is just $\mathrm{Aut}(\mathfrak{A})$. By Theorem 1.5, $\mathrm{Th}(\mathfrak{A})$ is model complete, so every automorphism of $\mathfrak{A}$ is an elementary embedding. By Theorem 1.44, $\mathrm{Th}(\mathfrak{C}_1) = \mathrm{Th}_c(\mathfrak{A})$. The results from the following sections will show that each of these Kripke models has a complete intuitionistic theory that admits quantifier elimination. With these assumptions, we fill in the table below with sentences forced by the $\mathfrak{C}_i$'s, showing that each of the theories is distinct. (In intuitionistic logic, a predicate $R(\mathbf{x})$ is **decidable** over a theory if the theory proves $\forall \mathbf{x}(R(\mathbf{x}) \vee \neg R(\mathbf{x}))$.)

3.2 Preserving Γ_M

To go about investigating changes to the morphism structure of single-node Kripke models with a JRS node structure, we first visit some general theorems about Kripke models.

Definition 3.3. *1. We write $F(\mathfrak{A})$ for the collection of all morphisms found in a*

Kripke model $\mathfrak{A}$.

2. $F(\mathfrak{A}, k)$ *represents the collection of all morphisms of the structure* $\mathfrak{A}_k$ *found at node* k *in a Kripke model* $\mathfrak{A}$.

3. $F(\mathfrak{A}, k, n)$ *represents the collection of all morphisms from the structure at node* k *to the structure at node* n.

(So $F(\mathfrak{A}, k, n) \subseteq F(\mathfrak{A}, k)$*.)*

4. *We write* $(\mathfrak{A}, k) \Vdash \varphi$ *to indicate that the sentence* φ *is forced at node* k *in a Kripke model* $\mathfrak{A}$.

5. *We write* $\mathrm{Th}(\mathfrak{A}, k)$ *for the collection of all* $\mathcal{L}(A_k)$ *sentences forced at the node* k *in* $\mathfrak{A}$.

Definition 3.4. 1. *For a classical* $\mathcal{L}$*-theory* Γ*, a Kripke model* $\mathfrak{A}$ *is* **locally-**Γ *if every node structure of* $\mathfrak{A}$ *is (classically) a model of* Γ.

2. *For a classical* $\mathcal{L}$*-structure* $\mathfrak{B}$*, a Kripke model* $\mathfrak{A}$ *is* **locally-**$\mathfrak{B}$ *if* $\mathfrak{B}$ *is the associated structure of each node of* $\mathfrak{A}$.

Note that for a locally-$\mathfrak{B}$ Kripke model $\mathfrak{A}$, the intuitionistic theory of $\mathfrak{A}$, $\mathrm{Th}(\mathfrak{A}) = \bigcap_{k \in |\mathbf{A}|} \mathrm{Th}(\mathfrak{A}, k)$ is an $\mathcal{L}(B)$ theory.

The model theory of intuitionistic logic lacks a clear analog of the classical notion of two models being isomorphic. In [24], Visser shows that defining two Kripke models as isomorphic when their underlying partial orders are isomorphic and the corresponding classical structures are classically isomorphic is problematic. Instead, model theorists

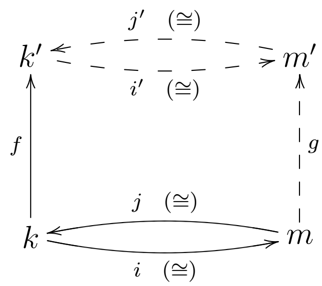

Figure 2: A bisimulation diagram

working in both intuitionistic logic and modal logic have settled on the concept of bisimulation.

Definition 3.5. *A* **bisimulation** $\mathcal{B}$ *from a Kripke model* $\mathfrak{A}$ *to a Kripke model* $\mathfrak{B}$ *(as defined in [24], translated to our notation) is a collection of quadruples* $\langle k, i, j, m\rangle$ *where* $k \in |A|$, $m \in |B|$, i *is an isomorphism from the classical structure* $\mathfrak{A}_k$ *at node* k *to the classical structure* $\mathfrak{B}_m$ *at node* m*, and* j *is the inverse of* i*.* $\mathcal{B}$ *must also meet the following two requirements, which Visser calls "Zig" and "Zag":*

Zig If $\langle k, i, j, m\rangle \in \mathcal{B}$ *and* $f \in F(\mathfrak{A}, k, k')$*, then there are* i'*,* j' *and* m' *and* $g \in F(\mathfrak{B}, m, m')$ *such that* $\langle k', i', j', m'\rangle \in \mathcal{B}$ *and* $gi = i'f$*.*

Zag If $\langle k, i, j, m\rangle \in \mathcal{B}$ *and* $g \in F(\mathfrak{B}, m, m')$*, then there are* i'*,* j' *and* k' *and* $f \in F(\mathfrak{A}, k, k')$ *such that* $\langle k', i', j', m'\rangle \in \mathcal{B}$ *and* $fj = j'g$*.*

The Criterion Zig can be represented in diagram form in Figure 2 (obviously Zag can be represented with a similar diagram).

Definition 3.6. $\mathcal{B}$ *is a* **full bisimulation** *from* $\mathfrak{A}$ *to* $\mathfrak{B}$ *if for every* $k \in |A|$ *there is an* $m \in |B|$ *such that* $\langle k, i, j, m\rangle \in \mathcal{B}$*, and vice versa.*

One then proves results like the following, linking bisimilarity to forcing.

Lemma 3.7. *Suppose $\mathcal{B}$ is a bisimulation from $\mathfrak{A}$ to $\mathfrak{B}$, $\langle k, i, j, m\rangle \in \mathcal{B}$, and φ is an $\mathcal{L}$-sentence. If $(\mathfrak{A}, k) \Vdash \varphi$, then $(\mathfrak{B}, m) \Vdash {}^i\varphi$. Conversely, if $(\mathfrak{B}, m) \Vdash \varphi$, then $(\mathfrak{A}, k) \Vdash {}^j\varphi$.*

Proof. See [24, Lemma A.4], for example. □

We instead choose to work with a slight generalization of the usual definition of bisimulation, somewhat in the spirit of Połacik's bounded bisimulation in [19].

Definition 3.8. *Given Kripke models $\mathfrak{A}$ and $\mathfrak{B}$, a* **graph bisimulation** *$\mathcal{B}$ from $\mathfrak{A}$ to $\mathfrak{B}$ is a set of quadruples $\langle k, p, q, m\rangle$ where $k \in |\mathbf{A}|$, $m \in |\mathbf{B}|$, p is an embedding from the classical structure $\mathfrak{A}_k$ at node k to the classical structure $\mathfrak{B}_m$ at node m, and q is an embedding from $\mathfrak{B}_m$ to $\mathfrak{A}_k$. $\mathcal{B}$ must also satisfy the following conditions.*

1. *If $\langle k, p, q, m\rangle \in \mathcal{B}$, then for every $f \in F(\mathfrak{A}, k, k')$ and $\mathcal{L}$-sentence φ, $(\mathfrak{A}, k') \Vdash {}^f\varphi$ if and only if $(\mathfrak{A}, k') \Vdash {}^{fqp}\varphi$.*

2. *If $\langle k, p, q, m\rangle \in \mathcal{B}$, then for every $g \in F(\mathfrak{B}, m, m')$ and $\mathcal{L}$-sentence φ, $(\mathfrak{B}, m') \Vdash {}^g\varphi$ if and only if $(\mathfrak{B}, m') \Vdash {}^{gpq}\varphi$.*

3. *If $\langle k, p, q, m\rangle \in \mathcal{B}$, then for every $f \in F(\mathfrak{A}, k, k')$ and finite $A_0 \subseteq A_k$, there is $\langle k', p', q', m'\rangle \in \mathcal{B}$ such that there exists $g \in F(\mathcal{B}, m, m')$ such that $g(p \restriction A_0) = p'(f \restriction A_0)$.*

4. *If $\langle k, p, q, m\rangle \in \mathcal{B}$, then for every $g \in F(\mathfrak{B}, m, m')$ and finite $B_0 \subseteq B_m$, there is $\langle k', p', q', m'\rangle \in \mathcal{B}$ such that there exists $f \in F(\mathcal{A}, k, k')$ such that $f(q \restriction B_0) = q'(g \restriction B_0)$.*

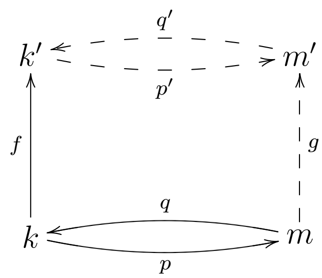

Figure 3: A graph bisimulation diagram

We can represent Criterion 3 (and 4) diagrammatically in Figure 3, as we did for Zig and Zag above. Recall, however, that this diagram only commutes for the specific choices of a morphism f and a finite set A_0.

Definition 3.9. *A graph bisimulation $\mathcal{B}$ from Kripke model $\mathfrak{A}$ to Kripke model $\mathfrak{B}$ is a* **full graph bisimulation** *if for every $k \in |\mathbf{A}|$, there is an $m \in |\mathbf{B}|$ and embeddings p and q such that $\langle k, p, q, m \rangle \in \mathcal{B}$, and for every $m \in |\mathbf{B}|$, there is a $k \in |\mathbf{A}|$ and embeddings p and q such that $\langle k, p, q, m \rangle \in \mathcal{B}$.*

Our definition departs from the standard one in two important ways. First, we relate the corresponding classical structures of the two Kripke models via the embeddings p and q instead of an isomorphism i and its inverse. Consequently, we need to add Criteria 1 and 2 to maintain the desired relationship between forcing in $\mathfrak{A}$ and forcing in $\mathfrak{B}$ (Lemma 3.10). The second main departure is that Criteria 3 and 4 demand the composition of morphisms be equal only on a predetermined finite set, whereas Zig and Zag require the compositions to be equal everywhere. In many model theoretic proofs, we only require these properties to hold for the parameters of a given sentence, so Criteria 3 and 4 suffice. This property is what allowed us to build a JRS Kripke model with only countably many morphisms in Section 1.3; see Corollary 3.13.

The reader familiar with bisimulations might also notice that our notation has hidden a third difference between graph bisimulation and the usual notion. Most definitions of bisimulation are written in the paradigm of Kripke models on partial orders. The Criterion Zig is often stated as something closer to "If $\langle k, i, j, m\rangle \in \mathcal{B}$ and $k \leq k'$, then ...", whereas in our paradigm there may be several morphisms from $\mathfrak{A}_k$ to $\mathfrak{A}_{k'}$. Replacing "$k \leq k'$" with "there is $f \in F(\mathfrak{A}, k, k')$" provides a definition that works in either paradigm.

Lemma 3.10. *Let $\mathcal{B}$ be a graph bisimulation between $\mathfrak{A}$ and $\mathfrak{B}$. Then, for every $\langle k, p, q, m\rangle \in \mathcal{B}$ and for every $\mathcal{L}(A_k)$-sentence φ, $(\mathfrak{A}, k) \Vdash \varphi$ if and only if $(\mathfrak{B}, m) \Vdash {}^p\varphi$.*

Proof. Fix an $\mathcal{L}(A_k)$-sentence φ such that $(\mathfrak{A}, k) \Vdash \varphi$ and $\langle k, p, q, m\rangle \in \mathcal{B}$. We proceed by induction on the complexity of sentences simultaneously for all quadruples and all nodes to show that if $(\mathfrak{A}, k) \Vdash \varphi$, then $(\mathfrak{B}, m) \Vdash {}^p\varphi$.

If φ is an atom, then $(\mathfrak{B}, m) \Vdash {}^p\varphi$ since p is an embedding. Conversely, if $(\mathfrak{B}, m) \Vdash {}^p\varphi$, then $(\mathfrak{A}, k) \Vdash {}^{qp}\varphi$ and therefore $(\mathfrak{A}, k) \Vdash \varphi$ by Criterion 1 of graph bisimilarity. The cases of conjunction, disjunction and existential quantifier are straightforward.

Suppose $\varphi \equiv \psi \to \theta$. Fix $g \in F(\mathfrak{B}, m, m')$ such that $(\mathfrak{B}, m') \Vdash {}^{gp}\psi$. Then by Criterion 4 of graph bisimilarity, there is an $f \in F(\mathfrak{A}, k, k')$, and embeddings p' and q' such that $\langle k', p', q', m'\rangle \in \mathcal{B}$ and fq and $q'g$ agree on the parameters of φ. Now, by Criterion 2, $(\mathfrak{B}, m') \Vdash {}^{gp}\psi$ if and only if $(\mathfrak{B}, m') \Vdash {}^{p'q'gp}\psi$ which, by the inductive hypothesis, is true if and only if $(\mathfrak{A}, k') \Vdash {}^{q'gp}\psi$. Since $(q'g)p = (fq)p$ on the parameters of φ, we have $(\mathfrak{A}, k') \Vdash {}^f\psi$ by Criterion 1. Since $(\mathfrak{A}, k) \Vdash \varphi$, $(\mathfrak{A}, k') \Vdash {}^f\theta$. By Criteria 1 and 4 again, $(\mathfrak{A}, k') \Vdash {}^{q'gp}\theta$. By the inductive hypothesis, $(\mathfrak{B}, m') \Vdash {}^{p'q'gp}\theta$. Criterion 2 implies that for any sentence α, $(\mathfrak{B}, m') \Vdash \alpha$ if and only if $(\mathfrak{B}, m') \Vdash {}^{p'q'}\alpha$. So we have

$(\mathfrak{B}, m') \Vdash {}^{gp}\theta$. Therefore, $(\mathfrak{B}, m) \Vdash {}^{p}\varphi$.

Suppose $\varphi \equiv \forall x \psi$. Fix $g \in F(\mathfrak{B}, m, m')$ and $b \in B_{m'}$. We must show that $(\mathfrak{B}, m') \Vdash {}^{gp}\psi(b)$. By Criterion 4 of graph bisimilarity there is an $f \in F(\mathfrak{A}, k, k')$, and embeddings p' and q' such that $\langle k', p', q', m' \rangle \in \mathcal{B}$ and fq and $q'g$ agree on the parameters of ${}^{p}\varphi$. Since $(\mathfrak{A}, k) \Vdash \varphi$, we have $(\mathfrak{A}, k') \Vdash {}^{f}\varphi$. By Criterion 1, we have $(\mathfrak{A}, k') \Vdash {}^{fqp}\varphi$, and so $(\mathfrak{A}, k') \Vdash {}^{q'gp}\varphi$. Since ${}^{q'}b \in A_{k'}$, $(\mathfrak{A}, k') \Vdash {}^{q'gp}\psi({}^{q'}b)$. By the inductive hypothesis, $(\mathfrak{B}, m') \Vdash {}^{p'q'gp}\psi({}^{p'q'}b)$. By Criterion 2, $(\mathfrak{B}, m') \Vdash {}^{gp}\psi(b)$. Therefore, $(\mathfrak{B}, m) \Vdash {}^{p}\varphi$.

As graph bisimulation is clearly a symmetric relation, a symmetric argument shows that if $(\mathfrak{B}, m) \Vdash {}^{p}\varphi$, then $(\mathfrak{A}, k) \Vdash {}^{qp}\varphi$, and therefore $(\mathfrak{A}, k) \Vdash \varphi$. □

Note that the proof of Lemma 3.10 would still work if p and q were simply morphisms. However, the lemma shows that any morphisms p and q must actually be embeddings, so changing the definition of graph bisimilarity to require that p and q be morphisms would be no more general than our definition. In [19], Połacik defines a notion of bounded bisimulation wherein the maps p and q are finite isomorphisms between $\mathfrak{A}_k$ and $\mathfrak{B}_m$. That approach is appealing, applies to Kripke models with uncountable classical node structures, and is somewhat similar to the finiteness conditions we impose in Criteria 3 and 4 in Definition 3.8. In more general settings, Połacik's definition seems overly technical for our purposes here (his notion of bisimilarity is indexed by a triple of natural numbers corresponding to three pairs of Zig/Zag conditions), though much of that technical machinery is not needed in our particular situation[1]. Our goal is to be

[1]That is, if $\mathfrak{A}_k$ and $\mathfrak{B}_m$ are countable JRS models, π is a partial isomorphism such that π and π^{-1} satisfy Criteria 3 and 4 in Definition 3.8, and $\pi : k \sim_{0,0,0} m$, then by a back and forth argument using the JRS axioms, $\pi : k \sim_{\infty} m$, thereby rendering the triple indexing unnecessary.

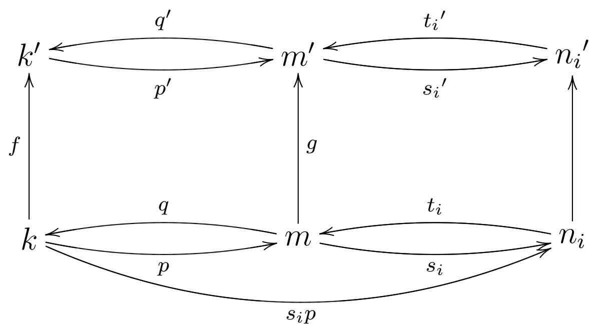

Figure 4: Full graph bisimilarity is transitive

able to justify the argument made in Section 1.3 that JRS Kripke models can be chosen to have countably many morphisms (see Corollary 3.13); with this in mind, we choose a notion of bisimilarity more general than Visser's, but not as broad as Połacik's.

Lemma 3.11. *Full graph bisimilarity is an equivalence relation.*

Proof. Full graph bisimilarity is clearly reflexive and symmetric. Given a full graph bisimulation $\mathcal{B}$ between $\mathfrak{A}$ and $\mathfrak{B}$, and a full graph bisimulation $\mathcal{B}'$ between $\mathfrak{B}$ and $\mathfrak{C}$, construct the full graph bisimulation $\mathcal{B}''$ between $\mathfrak{A}$ and $\mathfrak{C}$ as follows. Given $\langle k, p, q, m\rangle \in \mathcal{B}$, for every $\langle m, s_i, t_i, n_i\rangle \in \mathcal{B}'$, add $\langle k, (s_i p), (q t_i), n_i\rangle$ to $\mathcal{B}''$. Given $\langle m, s, t, n\rangle \in \mathcal{B}'$, for every $\langle k_j, p_j, q_j, m\rangle \in \mathcal{B}$, add $\langle k_j, (s p_j), (q_j t), m\rangle$ to $\mathcal{B}''$. Some of these morphisms are represented in Figure 4.

We sketch the proof that $\mathcal{B}''$ is a full graph bisimulation. By Lemma 3.10 together with the symmetry of full graph bisimilarity, $(\mathfrak{A}, k) \Vdash \varphi$ if and only if $(\mathfrak{B}, m) \Vdash {}^{p}\varphi$ if and only if $(\mathfrak{C}, n_i) \Vdash {}^{s_i p}\varphi$ if and only if $(\mathfrak{B}, m) \Vdash {}^{t_i s_i p}\varphi$ if and only if $(\mathfrak{A}, k) \Vdash {}^{q t_i s_i p}\varphi$, so $\mathcal{B}''$ meets Criterion 1 of Definition 3.8. To show that $\mathcal{B}''$ meets Criterion 3, fix $\langle k, p, q, m\rangle \in \mathcal{B}$, $f \in F(\mathfrak{A}, k, k')$, and finite $A_0 \subseteq A_k$. Fix $\langle k', p', q', m'\rangle \in \mathcal{B}'$ and $g \in F(\mathfrak{B}, m, m')$ as in Criterion 3 applied to $\mathcal{B}$. Now use Criterion 3 in $\mathcal{B}'$ for the finite set $f(A_0) \subseteq B_m$ to get

$\langle m, s, t, n \rangle \in \mathcal{B}'$ and compose arrows in the obvious way. $\square$

The following is a corollary of Lemma 3.10.

Corollary 3.12. *If Kripke models $\mathfrak{A}$ and $\mathfrak{B}$ are fully graph bisimilar, then* $\mathrm{Th}(\mathfrak{A}) \restriction \mathcal{L} = \mathrm{Th}(\mathfrak{B}) \restriction \mathcal{L}$.

Let us now return to considering Kripke models with a single classical node structure (although we do not yet assume that structure is a JRS model). While perhaps not as technical as some other choices of bisimulation, the notion of graph bisimulation may not be immediately transparent. It may be easier to think about graph bisimulation and Lemma 3.10 in less generality, as in the next corollary.

Let $\mathfrak{B}$ be a locally-$\mathfrak{A}$, single-node Kripke model. Choose $F(\mathfrak{B})$ such that for every endomorphism f of $\mathfrak{A}$ and for every finite subset A_0 of A, there is a $g \in F(\mathfrak{B})$ that agrees with f on A_0. Let $\mathfrak{C}$ be the locally-$\mathfrak{A}$, single-node Kripke model where $F(\mathfrak{C})$ consists of all endomorphisms of $\mathfrak{A}$. Then by Lemma 3.10 and Corollary 3.12, we have that $\mathrm{Th}(\mathfrak{B}) = \mathrm{Th}(\mathfrak{C})$. This generalizes the argument made in Section 1.3. We can more generally express this relationship as follows.

Corollary 3.13. *Let $\mathfrak{B}$ and $\mathfrak{C}$ be locally-$\mathfrak{A}$, single-node Kripke models. If the set of finite subgraphs of morphisms in $F(\mathfrak{B})$ is the same as the set of finite subgraphs of morphisms in $F(\mathfrak{C})$, then* $\mathrm{Th}(\mathfrak{B}) = \mathrm{Th}(\mathfrak{C})$.

We also have an easy generalization of Theorem 1.16.

Theorem 3.14. *Let $\mathfrak{B}$ be any single-node Kripke model. Then* $\mathrm{Th}(\mathfrak{B}) \restriction \mathcal{L}$ *is complete.*

Proof. Let φ be an $\mathcal{L}$-sentence. If $\mathfrak{B} \Vdash \varphi$, then we are done. Otherwise, $\mathfrak{B} \nVdash \varphi$. As φ is an $\mathcal{L}$-sentence, for every $f \in F(\mathfrak{B})$, ${}^f\varphi$ is the same syntactic object as φ. So, for every $f \in F(\mathfrak{B})$, $\mathfrak{B} \nVdash {}^f\varphi$. Therefore $\mathfrak{B} \Vdash \neg\varphi$. □

Finally, we return to considering single-node Kripke models whose classical node structure is a JRS model.

Definition 3.15. 1. *For $\mathfrak{A}$ a classical* JRS *model, let* M *be the monoid of all endomorphisms of* $\mathfrak{A}$.

2. *For* $\{\text{id}\} \subseteq K \subseteq \text{M}$, *let* $\mathfrak{A}_K$ *be the locally-*$\mathfrak{A}$*, single-node Kripke model where* $F(\mathfrak{A}_K) = K$.

With this notation, $\mathfrak{A}_\text{M}$ is still the Kripke model defined in Section 1.3, and $\mathfrak{A}_{\{\text{id}\}} = \mathfrak{A}_\text{JRS} = \mathfrak{A}$.

A corollary to Lemma 3.10 now answers the question of what monoids of morphisms preserve the intuitionistic theory Γ_M.

Corollary 3.16. *Let K be a monoid of endomorphisms of the classical* JRS *model* $\mathfrak{A}$. *If the set of finite subgraphs of morphisms in K equals the set of finite subgraphs of morphisms in* M, *then* $\text{Th}(\mathfrak{A}_K) \restriction \mathcal{L} = \Gamma_\text{M}$.

3.3 Preserving Quantifier Elimination

Corollary 3.16 gives us a way to change the monoid of morphisms of a single-node Kripke model without changing the intuitionistic theory of the Kripke model. However, as we have seen in Section 1.3, if M is non-trivial, then changing the monoid of morphisms does effect the theory of the model: $\Gamma_{\mathrm{M}} = \mathrm{Th}(\mathfrak{A}_{\mathrm{M}}) \neq \mathrm{Th}(\mathfrak{A}_{\{\mathrm{id}\}}) = \Gamma_{\mathrm{JRS}}$. However, both of these theories admit quantifier elimination. In this section, we investigate for what monoids K $\mathrm{Th}(\mathfrak{A}_K) \restriction \mathcal{L}$ admits quantifier elimination.

First, a note about classical homogeneity. A standard definition of a countable **homogeneous** model (from Marker [17], for example) is that $\mathfrak{A}$ is homogeneous if for every finite set $B \subset A$, every partial elementary embedding $f : B \to A$, and every $a \in A$, there is a partial elementary embedding $f^* : B \cup \{a\} \to A$ such that $f^* \supseteq f$. Other references ([14] and [15], for example) use the term **ultrahomogeneous** to describe what we have called Fraïssé homogeneous (Definition 1.2). In general, homogeneity is a weaker property than Fraïssé homogeneity. However, in our case the two notions coincide: finite substructures correspond exactly to complete $\mathcal{A}\mathrm{t}^{\pm}$-types, so the proof to Theorem 1.3 shows that classical JRS models are both homogeneous and ultrahomogeneous.

Recall that for monoids K such that $K \subseteq \mathrm{M}$ where M is the monoid of all endomorphisms of the classical JRS structure $\mathfrak{A}$, Kripke model $\mathfrak{A}_K$ is the Kripke model with single node structure $\mathfrak{A}$ with $F(\mathfrak{A}_K) = K$ and $\Gamma_K = \mathrm{Th}(\mathfrak{A}_K) \restriction \mathcal{L}$. In this section, we look to find monoids K (besides $\{\mathrm{id}\}$ and M) such that Γ_K admits quantifier elimination. To this end, we essentially retrace the steps we took in Section 1.4 to show that Γ_{M} admits quantifier elimination. Some of the proofs from that section used properties of the Fraïssé homogeneity of $\mathfrak{A}$, so we will consider monoids K that preserve properties of this homogeneity.

Definition 3.17. *We say that a monoid K such that $K \subseteq \mathrm{M}$ satisfies the* **Fraïssé condition** *if it meets the following criterion.*

$$\text{If } \mathrm{tp}_{\mathbf{a}} = \mathrm{tp}_{\mathbf{b}}, \text{ then for every } f \in K, \text{ there is a } g \in K \text{ such that } \mathrm{tp}_{f\mathbf{a}} = \mathrm{tp}_{g\mathbf{b}}.$$

Classically, if two tuples in a homogeneous structure satisfy the same types, then the tuples satisfy all of the same $\mathcal{L}$-formulas (see [17, Theorem 4.2.11] for example). Our Fraïssé condition says that if two tuples satisfy the same $\mathcal{A}\mathrm{t}^{\pm}$-types then in some sense they have the same futures in the Kripke model. Lemma 3.18 shows the Fraïssé condition is enough to prove an intuitionistic version of the classical characterization of homogeneity for single-node Kripke models.

Lemma 3.18. *For all monoids K satisfying the Fraïssé condition, for all tuples* $\mathbf{a}$ *and* $\mathbf{b}$*, and for all $\mathcal{L}(\mathbf{x})$-formulas φ, if $\mathrm{tp}_{\mathbf{a}} = \mathrm{tp}_{\mathbf{b}}$ then $\mathfrak{A}_K \Vdash \varphi(\mathbf{a})$ if and only if $\mathfrak{A}_K \Vdash \varphi(\mathbf{b})$.*

Proof. By symmetry, it suffices to show only the left to right direction of the bi-implication. We proceed by induction on the complexity of φ simultaneously for all $\mathbf{a}$ and $\mathbf{b}$ of the same length. If φ is an atom, then $\mathfrak{A}_K \Vdash \varphi(\mathbf{a})$ if and only if $\varphi(\mathbf{x}) \in \mathrm{tp}_{\mathbf{a}}(\mathbf{x})$ if and only if $\varphi(\mathbf{x}) \in \mathrm{tp}_{\mathbf{b}}(\mathbf{x})$ if and only if $\mathfrak{A}_K \Vdash \varphi(\mathbf{b})$. The inductive steps for $\wedge$ and $\vee$ are straightforward.

Let $\varphi(\mathbf{x}) \equiv \exists y \psi(\mathbf{x}y)$. Suppose $\mathfrak{A}_K \Vdash \varphi(\mathbf{a})$. Then there is $c \in A$ such that $\mathfrak{A}_K \Vdash \psi(\mathbf{a}c)$. Let $u = \mathrm{tp}_{\mathbf{a}c}$. Since $d(u) = \mathrm{tp}_{\mathbf{a}} = \mathrm{tp}_{\mathbf{b}}$, we have $\mathfrak{A} \models \pi_{d(u)}(\mathbf{b})$. By the JRS axiom δ_u, there is $d \in A$ such that $\mathfrak{A} \models \pi_u(\mathbf{b}d)$. So $\mathrm{tp}_{\mathbf{a}c} = \mathrm{tp}_{\mathbf{b}d}$. By the inductive hypothesis, $\mathfrak{A}_K \Vdash \psi(\mathbf{b}d)$, and therefore $\mathfrak{A}_K \Vdash \varphi(\mathbf{b})$.

Let $\varphi \equiv \psi \to \theta$. Suppose $\mathfrak{A}_K \Vdash \varphi(\mathbf{a})$. Fix $g \in K$ such that $\mathfrak{A}_K \Vdash \psi({}^g\mathbf{b})$. Since K satisfies the Fraïssé condition, we get $f \in K$ such that $\mathrm{tp}_{{}^f\mathbf{a}} = \mathrm{tp}_{{}^g\mathbf{b}}$. By the inductive hypothesis, $\mathfrak{A}_K \Vdash \psi({}^f\mathbf{a})$, so $\mathfrak{A}_K \Vdash \theta({}^f\mathbf{a})$. Again by the inductive hypothesis, $\mathfrak{A}_K \Vdash \theta({}^g\mathbf{b})$, and therefore $\mathfrak{A}_K \Vdash \varphi(\mathbf{b})$.

Let $\varphi(\mathbf{x}) \equiv \forall y \psi(\mathbf{x}y)$. Suppose $\mathfrak{A}_K \Vdash \varphi(\mathbf{a})$. Fix $g \in K$ and $d \in A$. We wish to show that $\mathfrak{A}_K \Vdash \psi({}^g(\mathbf{b})d)$. Since K satisfies the Fraïssé condition, there is $f \in K$

such that $\mathrm{tp}_{^g\mathbf{b}} = \mathrm{tp}_{^f\mathbf{a}}$. Let $u = \mathrm{tp}_{^g(\mathbf{b})d}$. By the JRS axiom δ_u, there is $c \in A$ such that $\mathrm{tp}_{^f(\mathbf{a})c} = \mathrm{tp}_{^g(\mathbf{b})d}$. By supposition, $\mathfrak{A}_K \Vdash \psi(^f(\mathbf{a})c)$. By the inductive hypothesis, $\mathfrak{A}_K \Vdash \psi(^g(\mathbf{b})d)$ and therefore $\mathfrak{A}_K \Vdash \varphi(\mathbf{b})$. □

Working with a monoid K that satisfies the Fraïssé condition, we may now retrace our steps through Section 1.4. Many of the notions first defined in that section need to be redefined in order to incorporate their dependence on K.

Definition 3.19. *Let $\mathfrak{C}_K(\mathbf{x})$ be the following Kripke model.*

1. *As nodes for the underlying category $\mathbf{C}_K(\mathbf{x})$ we take all complete $\mathcal{A}\mathrm{t}^{\pm}(\mathbf{x})$-types t that are consistent with the classical* JRS *theory* Γ*.*

(So $|\mathbf{C}_K(\mathbf{x})| = |\mathbf{C}(\mathbf{x})|$.)

2. *Turn $\mathbf{C}_K(\mathbf{x})$ into a poset category as follows. Given a pair of nodes t and u, we set $t \leq u$ exactly when there are $\mathbf{a} \in A$ and $f \in K$ such that $t = \mathrm{tp}_{\mathbf{a}}$ and $u = \mathrm{tp}_{^f(\mathbf{a})}$.*

(So $\mathbf{C}_K(\mathbf{x})$ is a subcategory of the category $\mathbf{C}(\mathbf{x})$ from Definition 1.20.)

Definition 3.20. *For each quantifier-free φ, let $[\![\varphi(\overline{\mathbf{x}})]\!] = \{t \in |\mathbf{C}_K(\mathbf{x})| : t \Vdash \varphi(\overline{\mathbf{x}}(t))\}$. (Compare to Definition 1.22.)*

Definition 3.21. 1. *Given $t \in |\mathbf{C}_K(\mathbf{x})|$, define $\hat{t} = \{u \in |\mathbf{C}_K(\mathbf{x})| \,:\, t \leq u\}$.*

2. *Let $\check{t}$ be the set $\{u \in |\mathbf{C}_K(\mathbf{x})| \,:\, u \not\leq t\}$.*

(Compare to Definition 1.25.)

Definition 3.22. 1. *Let $\bigwedge\{\pi_u^+ \to \sigma_u^- \,:\, u \in |\mathbf{C}_K(\mathbf{x})|, t^+ \subseteq u^+ \text{ and } t \not\leq u\}$ be the formula ρ_t^-.*

2. Let $\pi_t^+ \wedge \rho_t^-$ *be the formula* ρ_t^+*.*

(Compare with Definition 1.27.)

The analog of Lemma 1.21 requires an appeal to the Fraïssé condition; this is done via the use of Lemma 3.18.

Lemma 3.23. *Let* $\varphi(\mathbf{x})$ *be a quantifier-free* $\mathcal{L}$*-formula, let* $\mathbf{a} \in A$*, and let* K *satisfy the Fraïssé condition. Then* $\mathfrak{A}_K \Vdash \varphi(\mathbf{a})$ *if and only if* $\mathrm{tp}_{\mathbf{a}} \Vdash \varphi(\overline{\mathbf{x}}(\mathrm{tp}_{\mathbf{a}}))$*.*

Proof. We complete the proof by induction on the complexity of φ for all elements $\mathbf{a}$ simultaneously. The case for atoms and the induction steps for $\wedge$ and $\vee$ are easy. Let φ equal $\psi \to \theta$.

Suppose $\mathfrak{A}_K \Vdash \psi(\mathbf{a}) \to \theta(\mathbf{a})$. Let $\mathrm{tp}_{\mathbf{a}} \leq u$ such that $u \Vdash \psi(\overline{\mathbf{x}}(u))$. It suffices to show that $u \Vdash \theta(\overline{\mathbf{x}}(u))$. By Definition 3.19.2, there is $f \in K$ and $\mathbf{b} \in A$ such that $\mathrm{tp}_{\mathbf{a}} = \mathrm{tp}_{\mathbf{b}}$ and $u = \mathrm{tp}_{f(\mathbf{b})}$. By Lemma 3.18, $\mathfrak{A}_K \Vdash \psi(\mathbf{b}) \to \theta(\mathbf{b})$, and therefore $\mathfrak{A}_K \Vdash \psi({}^f\mathbf{b}) \to \theta({}^f\mathbf{b})$. Since $u \Vdash \psi(\overline{\mathbf{x}}(u))$, the inductive hypothesis tells us that $\mathfrak{A}_K \Vdash \psi({}^f\mathbf{b})$, so $\mathfrak{A}_K \Vdash \theta({}^f\mathbf{b})$. Again by the inductive hypothesis, $u \Vdash \theta(\overline{\mathbf{x}}(u))$. Therefore, $\mathrm{tp}_{\mathbf{a}} \Vdash \varphi(\overline{\mathbf{x}}(\mathrm{tp}_{\mathbf{a}}))$.

Conversely, suppose $\mathrm{tp}_{\mathbf{a}} \Vdash \psi(\overline{\mathbf{x}}(\mathrm{tp}_{\mathbf{a}})) \to \theta(\overline{\mathbf{x}}(\mathrm{tp}_{\mathbf{a}}))$. Let $f \in K$ be such that $\mathfrak{A}_K \Vdash \psi({}^f\mathbf{a})$. It suffices to show $\mathfrak{A}_K \Vdash \theta({}^f\mathbf{a})$. By the inductive hypothesis, we have $\mathrm{tp}_{f(\mathbf{a})} \Vdash \psi(\overline{\mathbf{x}}(\mathrm{tp}_{{}^f\mathbf{a}}))$. By Definition 3.19.2, $\mathrm{tp}_{\mathbf{a}} \leq \mathrm{tp}_{{}^f\mathbf{a}}$ so, by supposition, we have $\mathrm{tp}_{{}^f\mathbf{a}} \Vdash \theta(\overline{\mathbf{x}}(\mathrm{tp}_{{}^f\mathbf{a}}))$. Again by the inductive hypothesis, $\mathfrak{A}_K \Vdash \theta({}^f\mathbf{a})$. □

The proofs of the following results (Lemmas 3.24 to 3.27) are trivial modifications of the corresponding proofs in Section 1.4 (those of Lemmas 1.24 to 1.30). As such, we state the new results, now dependent on the monoid K, and omit the proofs.

Lemma 3.24. *For a monoid K satisfying the Fraïssé condition, there are for each* $\mathbf{x}$ *only finitely many quantifier-free formulas with all free variables from* $\mathbf{x}$*, modulo provable equivalence over* Γ_K*. For all quantifier-free formulas* $\varphi(\mathbf{x})$ *and* $\psi(\mathbf{x})$ *we have* $\Gamma_K \vdash \forall\mathbf{x}(\varphi \to \psi)$ *exactly when* $[\![\varphi(\overline{\mathbf{x}})]\!] \subseteq [\![\psi(\overline{\mathbf{x}})]\!]$.

Lemma 3.25. *Let K be a monoid satisfying the Fraïssé condition, and let* $t \in |\mathbf{C}_K(\mathbf{x})|$*. Then* $\check{t} = [\![\pi_t^+(\overline{\mathbf{x}}) \to \sigma_t^-(\overline{\mathbf{x}})]\!]$.

Lemma 3.26. *Let K be a monoid satisfying the Fraïssé condition, and let* $t \in |\mathbf{C}_K(\mathbf{x})|$*. Then* $\hat{t} = [\![\rho_t^+(\overline{\mathbf{x}})]\!]$*. So all open subsets of* $\mathbf{C}_K(\mathbf{x})$ *are definable.*

Lemma 3.27. *In* $\mathbf{C_K}(\mathbf{x})$*, each open subset equals a finite union of prime open subsets. A nonempty open subset is prime if and only if it is of the form* $\hat{t}$*, for some* $t \in |\mathbf{C}_K(\mathbf{x})|$.

Corollary 3.28. *If K satisfies the Fraïssé condition, then over* Γ_K*, every quantifier-free formula* φ *is equivalent to both the formula* $\bigvee\{\rho_t^+ \ : \ t \in [\![\varphi]\!]\}$ *and the formula* $\bigwedge\{\pi_t^+ \to \sigma_t^- \ : \ t \notin [\![\varphi]\!]\}$.

Proof. The first equivalence follows from Corollary 1.31, while the second follows from Theorem 2.14. □

The sentence in the following lemma in some sense axiomatizes the Fraïssé homogeneity of the classical JRS structure; see the discussion of axiom Ha4 in Section 2.1. Not unexpectedly, the proof of the next lemma relies on K satisfying the Fraïssé condition (via Lemma 3.18).

Lemma 3.29. *If K satisfies the Fraïssé condition, then for all $\mathcal{L}$ formulas* $\varphi(\mathbf{x}x_n)$*, and for all* $t \in \mathbf{C}_K(\mathbf{x}x_n)$*,* Γ_K *includes the sentence:*

$$\forall \mathbf{x} x_n(\varphi \wedge \rho_t^+ \rightarrow (\sigma_t^- \vee \forall x_n(\rho_t^+ \rightarrow \varphi))).$$

Proof. Fix φ, $t \in \mathbf{C}_K(\mathbf{x}x_n)$ and $\mathbf{a}, b \in A$ and suppose $\mathfrak{A}_K \Vdash \varphi(\mathbf{a}b) \wedge \rho_t^+(\mathbf{a}b)$. If $\mathfrak{A}_K \Vdash \sigma_t^-(\mathbf{a}b)$ then we are done, so suppose not. Then $t = \mathrm{tp}_{\mathbf{a}b}$. We need to show that for arbitrary $c \in A_{\mathrm{JRS}}$ and $f \in K$, if $\mathfrak{A}_K \Vdash \rho_t^+({}^f(\mathbf{a})c)$ then $\mathfrak{A}_K \Vdash \varphi({}^f(\mathbf{a})c)$. Fix such an element c and such a morphism $f \in K$. Since $\mathfrak{A}_K \Vdash \rho_t^+({}^f(\mathbf{a})c)$, we have $\mathrm{tp}_{{}^f(\mathbf{a})c} \in \hat{t}$ by Lemma 3.26. So $\mathrm{tp}_{\mathbf{a}b} \leq \mathrm{tp}_{{}^f(\mathbf{a})c}$. By the definition of $\leq$, this means that there is a tuple $\mathbf{d}e$ and morphism $g \in K$ such that $\mathrm{tp}_{\mathbf{a}b} = \mathrm{tp}_{\mathbf{d}e}$ and ${}^g(\mathbf{d}e) = {}^f(\mathbf{a})c$. Since $\mathfrak{A}_K \Vdash \varphi(\mathbf{a}b)$, Lemma 3.18 gives us that $\mathfrak{A}_K \Vdash \varphi(\mathbf{d}e)$, and so $\mathfrak{A}_K \Vdash \varphi({}^g(\mathbf{d}e))$. Therefore $\mathfrak{A}_K \Vdash \varphi({}^f(\mathbf{a})c)$. □

We are now ready to prove our main result, quantifier elimination. The proof to Theorem 3.30 below is identical to the proof of Theorem 1.33, with the exception that the latter cites Lemmas 3.23 to 3.29 instead of Lemmas 1.21 to 1.32.

Theorem 3.30. *If K satisfies the Fraïssé condition, then the theory Γ_K admits quantifier elimination.*

Now let us re-examine Figure 1 at the end of Section 3.1. By Theorem 3.14, each theory in the table is complete. Below, we show that each monoid appearing in the table satisfies the Fraïssé condition. Fix $\mathbf{a}$ and $\mathbf{b}$ in A such that $\mathrm{tp}_{\mathbf{a}} = \mathrm{tp}_{\mathbf{b}}$.

- $F(\mathfrak{C}_1) = \mathrm{Aut}(\mathfrak{A})$ satisfies the Fraïssé condition. For every $f \in \mathrm{Aut}(\mathfrak{A})$, $\mathrm{tp}_{\mathbf{b}} = \mathrm{tp}_{\mathbf{a}} = \mathrm{tp}_{{}^f\mathbf{a}}$. Since the JRS model $\mathfrak{A}$ is homogeneous, there is an automorphism g taking $\mathbf{b}$ to ${}^f\mathbf{a}$. (Also notice that $\{\mathrm{id}\}$ trivially satisfies the Fraïssé condition.)

- $F(\mathfrak{C}_4) = \mathrm{M}$, the collection of all endomorphisms of $\mathfrak{A}$, satisfies the Fraïssé condition. Since $\mathfrak{A}$ is homogeneous, there is an automorphism $h \in \mathrm{M}$ such that ${}^h\mathbf{b} = \mathbf{a}$. For $f \in \mathrm{M}$, we can choose g to be fh to satisfy the Fraïssé condition.

- $F(\mathfrak{C}_2)$ satisfies the Fraïssé condition. As each $f \in F(\mathfrak{C}_2)$ is also an element of M, consider the morphism g given by the Fraïssé condition in M. We have $\mathrm{tp}_{{}^f\mathbf{a}} = \mathrm{tp}_{{}^g\mathbf{b}}$, so we need only determine if $g \in F(\mathfrak{C}_2)$. We are given that $R(\mathbf{x}) \in \mathrm{tp}_{\mathbf{b}}$ if and only if $R(\mathbf{x}) \in \mathrm{tp}_{\mathbf{a}}$. Since $f \in F(\mathfrak{C}_2)$, this occurs if and only if $R(\mathbf{x}) \in \mathrm{tp}_{{}^f\mathbf{a}}$ which, by the Fraïssé condition in M, occurs if and only if $R(\mathbf{x}) \in \mathrm{tp}_{{}^g\mathbf{b}}$. Therefore, $g \in F(\mathfrak{C}_2)$.

- The argument to show that $F(\mathfrak{C}_3)$ satisfies the Fraïssé condition is similar to the one for $F(\mathfrak{C}_2)$.

So each monoid of morphisms mentioned in Figure 1 satisfies the Fraïssé condition. Therefore Theorem 3.30 applies, and we are justified in concluding that we have found four different, single-node Kripke models with the same underlying classical structure that have complete, non-comparable intuitionistic theories.

As a corollary to Theorem 3.30 we obtain the following.

Theorem 3.31. *Let $\varphi(\mathbf{x})$ be a formula. If K satisfies the Fraïssé condition, then over Γ_K, φ is equivalent to a disjunction of formulas ρ_t^+ with $t \in |\mathbf{C}_K(\mathbf{x})|$, as well as to a conjunction of formulas of the form $\pi_t^+ \to \sigma_t^-$.*

Lastly, we again point out make another small observation. Recall Definition 1.36, and compare the following to Theorem 1.38.

Theorem 3.32. *Let $\varphi(\mathbf{x})$ be a formula. If K satisfies the Fraïssé condition, then over Γ_K, φ is equivalent to a quantifier-free universal formula.*

Proof. The result follows from Theorem 3.31 since each formula ρ_t^+ and each formula $\pi_t^+ \to \sigma_t^-$ is a universal formula. $\square$

Chapter 4

JRS **Kripke Models with Multiple Nodes**

In this chapter, we broaden our definition of classical JRS theories. In so doing, we gain the ability to create JRS Kripke models with multiple (different) node structures. In addition to generalizing many of the results of previous chapters, this broadening also gives us a semantic way to investigate the relationship between a given classical JRS theory Γ and its corresponding intuitionistic theory Γ_{M}. Further, this change anticipates future work. A crucial step towards proving that Γ_{M} (or Γ_K where K satisfies the Fraïssé condition) admits quantifier elimination is the fact that every open set in $\mathbf{C}(\mathbf{x})$ (or $\mathbf{C}_K(\mathbf{x})$) is a finite union of definable open sets (see Lemmas 1.30 and 3.27). The $\mathbf{C}(\mathbf{x})$ and $\mathbf{C}_K(\mathbf{x})$ topologies in some sense degenerately meet that requirement, as they are both finite topologies. Allowing for JRS Kripke models with multiple nodes is a first step on the path towards constructing Kripke models where the topologies on the associated auxiliary Kripke models are infinite but satisfy some compactness property. We hope to generalize our techniques and find a still broader class of intuitionistic theories that admit quantifier elimination.

4.1 Nullary Predicates and JRS Theories

To further our investigation of different kinds of intuitionistic JRS theories, we broaden our notion of a classical JRS theory. We still consider a language $\mathcal{L}$ that has finitely many predicates $\{R_i\}_{i<r}$, but we now allow predicates to be nullary.

The inclusion of nullary predicates in our language warrants a brief discussion of the nature of morphisms. A morphism f from a classical structure $\mathfrak{A}$ to a classical structure $\mathfrak{B}$ is usually thought of as a map from A to B such that for every atomic $\mathcal{L}(A)$-sentence δ, if $\mathfrak{A} \models \delta$ then $\mathfrak{B} \models {}^f\delta$. However, in the presence of nullary predicates, determining the behavior of a morphism by its action on the elements of the domain is no longer sufficient to deduce its behavior on atoms in the target domain. One way to sidestep this difficulty is to demand that a morphism f from $\mathfrak{A}$ to $\mathfrak{B}$ be a triple $(\mathfrak{A}, \mathfrak{B}, f')$ where f' is a morphism in the usual sense. The distinction is minor, however, and we will continue to identify morphisms as functions from A to B while additionally specifying their behavior on any nullary predicates in $\mathcal{L}$.

All of our notation from Section 1.2.1 remains the same. However, we make a slight modification to the definition of a JRS theory.

Definition 4.1. *Given a universal theory* Π*, we define the* JRS **extension** Γ *of* Π *as the theory axiomatizable by* Π *and all* JRS *sentences* δ_u *for which* $\Pi \nvdash_c \forall \mathbf{x} \neg \pi_u$*.*

Theorem 4.2. *Let* Π *be a consistent universal theory. Then the* JRS *extension* Γ *of* Π *is consistent if and only if the collection of models of the form* $\mathfrak{A}_t$*, where* t *is a complete* $\mathcal{A}\mathrm{t}^{\pm}(\mathbf{x})$*-type such that* $\Pi \nvdash_c \forall \mathbf{x} \neg \pi_t$*, has the amalgamation property. If* Γ *is consistent, then* $\Gamma_\forall = \Pi$*.*

Proof. First, suppose Γ is consistent. Consider finite models $\mathfrak{A}_t$, $\mathfrak{A}_u$, and $\mathfrak{A}_v$ of $\Gamma_\forall$ and suppose that $\mathfrak{A}_t$ embeds in both $\mathfrak{A}_u$ and $\mathfrak{A}_v$. Without loss of generality, we may assume that u and v are complete $\mathcal{A}\mathrm{t}^{\pm}(\mathbf{x}x_n)$-types and that t is a complete $\mathcal{A}\mathrm{t}^{\pm}(\mathbf{x})$-type. There is a model $\mathfrak{A}$ of Γ such that for some $\mathbf{a} \in A$, $\mathfrak{A}$ satisfies $\pi_t(\mathbf{a})$, δ_u and δ_v. So we have $\mathfrak{A} \models \exists x \pi_u(\mathbf{a}x) \wedge \exists x \pi_v(\mathbf{a}x)$. Fix $\mathbf{a}$, b and c such that $\mathfrak{A} \models \pi_u(\mathbf{a}b) \wedge \pi_v(\mathbf{a}c)$. Let $w = \mathrm{tp}_{\mathbf{a}bc}$. Then $\mathfrak{A}_w$ is the amalgam of $\mathfrak{A}_u$ and $\mathfrak{A}_v$ over $\mathfrak{A}_t$.

Conversely, suppose that the collection of models of Π of the form $\mathfrak{A}_t$ has the amalgamation property. We sketch a construction of a model $\mathfrak{A}$ of Γ as the limit of an ω-chain of models of the form $\mathfrak{A}_t$. Suppose we have a model $\mathfrak{A}_t$ of size n. For each complete $\mathcal{A}\mathrm{t}^{\pm}(\mathbf{x}x_n)$-type u consistent with Π and for all $\mathbf{a} \in A_t$ such that $\mathfrak{A}_t \models \pi_{d(u)}(\mathbf{a})$ there is an amalgam $\mathfrak{A}_{(u,\mathbf{a})}$ of $\mathfrak{A}_t$ and $\mathfrak{A}_u$ over $\mathfrak{A}_{d(u)}$. As the next model in the ω-chain, take the amalgam of all $\mathfrak{A}_{(u,\mathbf{a})}$ over $\mathfrak{A}_t$. So Γ is consistent.

For the last claim, it suffices to show that every finite model of Π embeds into a model of Γ. Let u be a complete $\mathcal{A}\mathrm{t}^{\pm}$-type consistent with Π. Then $\mathfrak{A}_u$ is a model of Π and by the previous paragraph, it embeds into a model of Γ. □

Definition 4.3. 1. *We say* Γ *is a* JRS **theory** *if there is a consistent universal theory* Π *such that* Γ *is the* JRS *extension of* Π.

2. *A model of a classical* JRS *theory is called a* JRS **model**.

In previous chapters, we defined JRS theories as follows (Definition 1.1.12).

> A (consistent) classical theory Γ over $\mathcal{L}$ is called a JRS theory if for all $\mathbf{x}x_n$ and complete $\mathcal{A}\mathrm{t}^{\pm}(\mathbf{x}x_n)$-types u that are consistent with Γ, we have $\delta_u \in \Gamma$.

The following theorem shows that Definition 1.1.12 is a special case of the new Definition 4.3.1.

Theorem 4.4. *Let* Γ *be a* JRS *theory such that* $\Gamma \vdash_c R$ *or* $\Gamma \vdash_c \neg R$ *for each nullary predicate* R. *Then, up to isomorphism,* Γ *has exactly one model of size* $\leq \omega$. *Additionally, this model is Fraïssé homogeneous.*

Proof. The proof uses the axioms δ_u to complete a standard back and forth construction to extend finite isomorphisms to automorphisms. □

The assumption that each nullary predicate is decidable reduces us to the original Definition 1.1.12. For a JRS theory Γ, each model of $\Gamma_\forall$ with empty domain extends to a unique (up to isomorphism) model of Γ of size $\leq \omega$. If $\mathcal{L}$ has r nullary predicates (other than $\top$ and $\bot$), there are up to isomorphism at most 2^r such models.

For the next theorem, recall Definition 1.4 and Theorem 1.5.

Theorem 4.5. *Let* Γ *be a* JRS *theory, and let* $\exists x_n \varphi(\mathbf{x}x_n)$ *be a primitive formula. Then* $\Gamma \vdash_c \exists x_n \varphi \leftrightarrow \bigvee_{s \in S} \pi_{d(s)}$, *where*

$$S = \{s \;:\; s \textit{ is a complete } \mathcal{A}\mathrm{t}^{\pm}(\mathbf{x}x_n)\textit{-type consistent with } \Gamma \textit{ and } \Gamma \vdash_c \pi_s \to \varphi\}.$$

In particular, JRS *theories admit quantifier elimination.*

Proof. The formula $\exists x_n \varphi$ is equivalent to $\bigvee_{s \in S} \exists x_n \pi_s$, where an empty disjunction is identified with $\bot$. Apply the JRS sentences of Γ: $\exists x_n \varphi$ is equivalent to $\bigvee_{s \in S} \pi_{d(s)}$. □

As in Section 3.1, we can build a new JRS theory from other JRS theories. Compare to Definition 3.1.

Definition 4.6. *Let* Γ_1 *and* Γ_2 *be* JRS *theories over a language* $\mathcal{L}$. *Let* $\mathcal{L}' = \mathcal{L} \cup \{R\}$ *where* R *is a new nullary predicate. Then let* $\Gamma_1 \rtimes_R \Gamma_2$ *be the* $\mathcal{L}'$ *theory* $\{R \to \gamma_1 \;:\; \gamma_1 \in \Gamma_1\} \cup \{\neg R \to \gamma_2 \;:\; \gamma_2 \in \Gamma_2\}$.

Compare the following result with Lemma 3.2.

Lemma 4.7. $\Gamma_1 \rtimes_R \Gamma_2$ *is a* JRS *theory.*

Proof. Let $\Gamma = \Gamma_1 \rtimes_R \Gamma_2$, and let u be a complete $\mathcal{A}t^{\pm}(\mathbf{x})$-type over $\mathcal{L}'$ consistent with Γ. Suppose $u = t \cup \{R\}$ where t is a complete $\mathcal{A}t^{\pm}(\mathbf{x})$-type over $\mathcal{L}$ (the case where $u = t \cup \{\neg R\}$ is similar). Then t is consistent with Γ_1, so $\Gamma_1 \vdash_c \delta_t$. So $\Gamma \vdash_c R \to \delta_t$. But $\vdash_c \delta_u \leftrightarrow (R \to \delta_t)$, so $\Gamma \vdash_c \delta_u$. □

Corollary 4.8. *Let* Γ *be a* JRS *theory over language* $\mathcal{L}$*, and let* $\mathcal{L}' = \mathcal{L} \cup \{R\}$ *where* R *is a new nullary predicate. Then* Γ *is a* JRS *theory over* $\mathcal{L}'$*.*

Proof. By Lemma 4.7, $\Gamma \rtimes_R \Gamma$ is axiomatizable by Γ. □

Note, however, that $\Gamma_1 \rtimes_R \Gamma_2$ is not complete; the theory does not prove R nor $\neg R$.

4.2 JRS Kripke Models with Multiple Nodes

We construct an intuitionistic version of our generalized classical JRS theories as follows.

Definition 4.9.

1. *Given a language* $\mathcal{L}$*, let* $\mathcal{L}^{\downarrow} \subseteq \mathcal{L}$ *be the sublanguage of* $\mathcal{L}$ *whose only nullary predicates are* $\top$ *and* $\bot$*. We call* $\mathcal{L}^{\downarrow}$ *the* **core language***.*

2. *Given a classical* JRS *model* $\mathfrak{A}$ *in the language* $\mathcal{L}$*, let the* **core** JRS **model of** $\mathfrak{A}$*,* $\mathfrak{A}^{\downarrow}$*, be the reduct of* $\mathfrak{A}$ *to* $\mathcal{L}^{\downarrow}$*. Let* $\Gamma = \mathrm{Th}_c(\mathfrak{A})$*, and let* $\Gamma^{\downarrow} = \mathrm{Th}_c(\mathfrak{A}^{\downarrow})$*.*

Note that $\mathfrak{A}^{\downarrow}$ is again a JRS model, by Theorem 4.2 and the fact that amalgamation for the finite models of $\Gamma_{\forall}$ implies amalgamation for the finite models of $(\Gamma^{\downarrow})_{\forall}$.

Definition 4.10. *Consider a* JRS *model* $\mathfrak{A}$ *in a language* $\mathcal{L}$*, where* $\mathcal{L}$ *has* r *many nullary predicates other than* $\top$ *and* $\bot$*.*

1. *Let* $\mathcal{R} = \{R_0, ..., R_{r-1}\}$*, the collection of nullary predicates other than* $\top$ *and* $\bot$*.*

(So up to isomorphism, there are 2^r *many countable* JRS *models that have* $\mathfrak{A}^{\downarrow}$ *as its core* JRS *model.)*

2. *Each* $k \subseteq \mathcal{R}$ *is an* $\mathcal{A}\mathrm{t}^{\pm}(\emptyset)$*-type; as such, the sentence* π_k^+ *is defined as in Definition 1.1.5.*

3. *The sentence* σ_k^- *is defined as the disjunction of all* $R_i \in \mathcal{R}$ *such that* $R_i \notin k$*.*

4. *The sentence* π_k *is defined as* $\pi_k^+ \wedge \neg\sigma_k^-$*.*

5. *Define the classical* JRS *model* ${}^k\mathfrak{A}$ *to be the* $\mathcal{L}$*-structure with* $\mathfrak{A}^{\downarrow}$ *as its core* JRS *model such that* ${}^k\mathfrak{A} \models \pi_k$*.*

In Chapter 3, we built a single node JRS Kripke model $\mathfrak{A}_{\mathrm{M}}$ by starting with a classical JRS model and including all of its endomorphisms. We then investigated Kripke submodels of $\mathfrak{A}_{\mathrm{M}}$ by changing the monoid of morphisms. We generalize that technique here.

Definition 4.11. *Given a* JRS *model* $\mathfrak{A}$ *in the language* $\mathcal{L}$*, we define the Kripke model* $\mathfrak{R}$ *as follows.*

1. *The domain* $|\mathbf{R}|$ *of the underlying category* $\mathbf{R}$ *is* $\mathcal{P}(\mathcal{R})$*, the power set of* $\mathcal{R}$*.*

2. *To each node* $k \in |\mathbf{R}|$*, assign the classical* JRS *structure* ${}^k\mathfrak{A}$ *whose core* JRS *model is* $\mathfrak{A}^{\downarrow}$*.*

3. *For every* k *and* $m \in |\mathbf{R}|$*, let* $F(\mathfrak{R}, k, m)$ *be the collection of all morphisms from* ${}^k\mathfrak{A}$ *to* ${}^m\mathfrak{A}$*.*

By an argument similar to Corollary 3.13, we could instead choose to include a smaller (possibly countable) collection of morphisms and still get a Kripke model with the same intuitionistic theory as $\mathfrak{R}$. We choose not to do so as to keep the results of this section more straightforward.

As in our work in Chapter 3, we broaden our investigation by allowing changes to the monoids (and nodes) of $\mathfrak{R}$.

Definition 4.12. *For any Kripke submodel $\mathfrak{D}$ of $\mathfrak{R}$, and for a tuple $\mathbf{a} \in A$, define the* $\mathcal{A}\mathrm{t}^{\pm}(\mathbf{x})$**-type of a in $\mathfrak{D}$ at node** k*, written* $\mathrm{tp}^k_{\mathbf{a}}$*, as the collection of $\mathcal{L}$-atoms and negated $\mathcal{L}$-atoms such that $\delta(\mathbf{x}) \in \mathrm{tp}^k_{\mathbf{a}}$ if and only if ${}^k\mathfrak{A} \models \delta(\mathbf{a})$ where $\delta \in \mathcal{A}\mathrm{t}^{\pm}(\mathbf{x})$.*

Definition 4.13. 1. *We say that the Kripke model $\mathfrak{D}$ is a* **Fraïssé submodel** *of $\mathfrak{R}$ if $\mathfrak{D}$ is a Kripke submodel of $\mathfrak{R}$ and for every k and $m \in |\mathbf{D}|$, $F(\mathfrak{D}, k, m)$ satisfies the Fraïssé condition, that is, if $\mathrm{tp}^k_{\mathbf{a}} = \mathrm{tp}^k_{\mathbf{b}}$, then for every $f \in F(\mathfrak{D}, k, m)$, there is a $g \in F(\mathfrak{D}, k, m)$ such that $\mathrm{tp}^m_{f\mathbf{a}} = \mathrm{tp}^m_{g\mathbf{b}}$.*

2. *Write Γ_D for the (intuitionistic) theory of $\mathfrak{D}$.*

The following two lemmas are straightforward observations, but could play an important role in future generalizations.

Lemma 4.14. *For a Fraïssé submodel $\mathfrak{D}$ of $\mathfrak{R}$, and for k and m nodes of $|\mathbf{D}|$, $(\mathfrak{D}, k) \Vdash \pi^+_m$ if and only if $m \subseteq k$.*

While perhaps the nodes of a Fraïssé submodel $\mathfrak{D}$ are not definable in the strictest sense, the next Lemma shows we can still identify nodes using forcing.

Lemma 4.15. *For a Fraïssé submodel $\mathfrak{D}$ of $\mathfrak{R}$, and for k and m nodes of $|\mathbf{D}|$, the following are equivalent.*

1. $k = m$

2. $(\mathfrak{D}, k) \Vdash \pi_m^+$ *and for all* $n \in |\mathbf{D}|$ *such that* $m \subsetneq n$, $(\mathfrak{D}, k) \nVdash \pi_n^+$

Proof. First suppose 1. Then $(\mathfrak{D}, k) \Vdash \pi_k^+$ by Lemma 4.14. Fix $n \in |\mathbf{D}|$ such that $k \subsetneq n$. Then $(\mathfrak{D}, k) \nVdash \pi_n^+$. Conversely, suppose 2. Lemma 4.14 gives us both that $m \subseteq k$ and that for all $n \supsetneq m$, $n \not\subseteq k$. Therefore, $k = m$. □

Definition 4.16. *For a tuple* $\mathbf{x}$ *and a Fraïssé submodel* $\mathfrak{D}$*, construct auxiliary Kripke model* $\mathfrak{C}_D(\mathbf{x})$ *as follows.*

1. *Working in the language* $\mathcal{L}$*, let the underlying category* $|\mathbf{C}_D(\mathbf{x})|$ *be the collection of all complete* $\mathcal{A}\mathrm{t}^{\pm}(\mathbf{x})$*-types* t *that are (classically) consistent with* $\Gamma^{\downarrow}$*.*

(Note that if u *is a complete* $\mathcal{A}\mathrm{t}^{\pm}(\mathbf{x}) \cap \mathcal{L}^{\downarrow}(\mathbf{x})$*-type consistent with* $\Gamma^{\downarrow}$*, then for every* $k \in |\mathbf{D}|$, $u \cup \{\pi_k\}$ *generates a complete* $\mathcal{A}\mathrm{t}^{\pm}(\mathbf{x})$*-type (in the language* $\mathcal{L}$*) consistent with* $\Gamma^{\downarrow}$*.)*

2. *We turn* $\mathbf{C}_D(\mathbf{x})$ *into a poset category as follows. Given a pair of nodes* t *and* u*, we set* $t \leq u$ *exactly when there is* $\mathbf{a} \in A$ *and a morphism* $f \in F(\mathfrak{D}, k, m)$ *such that* $t = \mathrm{tp}_{\mathbf{a}}^k$ *and* $u = \mathrm{tp}_{f(\mathbf{a})}^m$*.*

(As t *and* u *are complete* $\mathcal{A}\mathrm{t}^{\pm}(\mathbf{x})$*-types in the language* $\mathcal{L}$*, this relation is well-defined by Lemma 4.15. As before, note that* $t \leq u$ *implies* $t^+ \subseteq u^+$*.)*

3. *To each node* t *we associate a finite classical* $\mathcal{L}(\mathrm{D})$*-structure* $\mathfrak{A}_t$*, as defined in Section 1.4.*

Note that for each $k \in |\mathbf{D}|$, the subcategory of $\mathbf{C}_D(\mathbf{x})$ consisting of all nodes u such that $u \vdash_c \pi_k$ is isomorphic to the underlying category $\mathbf{C}_K(\mathbf{x})$ of the auxiliary Kripke model $\mathfrak{C}_K(\mathbf{x})$ associated to $\mathfrak{A}^{\downarrow}$ where $K = F(\mathfrak{D}, k)$; see Section 1.4. (In fact, $\mathfrak{C}_K(\mathbf{x})$ is essentially identical to the Kripke submodel formed by restricting $\mathfrak{C}_D(\mathbf{x})$ to the above subcategory.) For each complete $(\mathcal{A}\mathrm{t}^{\pm} \cap \mathcal{L}^{\downarrow})(\mathbf{x})$-type t, the subcategory of $\mathbf{C}_D(\mathbf{x})$ consisting of all nodes u such that $u \vdash_c t$ is isomorphic to the category $\mathbf{D}$.

The proof of the following result is nearly identical to the proof of Lemma 3.18.

Lemma 4.17. *For a Fraïssé submodel $\mathfrak{D}$ of $\mathfrak{R}$, and for all $k \in |\mathbf{D}|$, for all tuples $\mathbf{a}$ and $\mathbf{b}$, and for all $\mathcal{L}(\mathbf{x})$-formulas φ, if $\mathrm{tp}^k_{\mathbf{a}} = \mathrm{tp}^k_{\mathbf{b}}$ then $(\mathfrak{D}, k) \Vdash \varphi(\mathbf{a})$ if and only if $(\mathfrak{D}, k) \Vdash \varphi(\mathbf{b})$.*

The proof of the next lemma is similar to the proof of Lemmas 3.23 and 1.21.

Lemma 4.18. *For a Fraïssé submodel $\mathfrak{D}$ of $\mathfrak{R}$, let $\varphi(\mathbf{x})$ be a quantifier-free $\mathcal{L}$ formula, $\mathbf{a} \in A$, $k \in |\mathbf{D}|$, and let $t = \mathrm{tp}^k_{\mathbf{a}}$. Then $t \Vdash \varphi(\overline{\mathbf{x}}(t))$ if and only if $(\mathfrak{D}, k) \Vdash \varphi(\mathbf{a})$.*

Proof. We proceed by induction on the complexity of φ for all elements $\mathbf{a}$ and all nodes k simultaneously. By construction of $\mathfrak{C}_D$, the case for atoms is clear. The induction steps for disjunction and conjunction are also straightforward. Let $\varphi \equiv \psi \to \theta$.

Suppose $(\mathfrak{D}, k) \Vdash \psi(\mathbf{a}) \to \theta(\mathbf{a})$. Let $t \leq u$ such that $u \Vdash \psi(\overline{\mathbf{x}}(u))$. It suffices to show that $u \Vdash \theta(\overline{\mathbf{x}}(u))$. By Definition 4.16.2, there is $m \in |\mathbf{D}|$, $f \in F(\mathfrak{D}, k, m)$ and $\mathbf{b} \in A$ such that $t = \mathrm{tp}^k_{\mathbf{b}}$ and $u = \mathrm{tp}^m_{f(\mathbf{b})}$. By Lemma 4.17, $(\mathfrak{D}, k) \Vdash \psi(\mathbf{b}) \to \theta(\mathbf{b})$, and therefore $(\mathfrak{D}, m) \Vdash \psi({}^f\mathbf{b}) \to \theta({}^f\mathbf{b})$. Since $u \Vdash \psi(\overline{\mathbf{x}}(u))$, the inductive hypothesis tells us that $(\mathfrak{D}, m) \Vdash \psi({}^f\mathbf{b})$, so $(\mathfrak{D}, m) \Vdash \theta({}^f\mathbf{b})$. Again by the inductive hypothesis, $u \Vdash \theta(\overline{\mathbf{x}}(u))$. Therefore, $t \Vdash \varphi(\overline{\mathbf{x}}(t))$.

Conversely, suppose $t \Vdash \psi(\overline{\mathbf{x}}(t)) \to \theta(\overline{\mathbf{x}}(t))$. Let $m \in |\mathbf{D}|$ and $f \in F(\mathfrak{D}, k, m)$ be such that $(\mathfrak{D}, m) \Vdash \psi({}^f\mathbf{a})$. It suffices to show $(\mathfrak{D}, m) \Vdash \theta({}^f\mathbf{a})$. By the inductive hypothesis, $\mathrm{tp}^m_{f(\mathbf{a})} \Vdash \psi(\overline{\mathbf{x}}(\mathrm{tp}^m_{f\mathbf{a}}))$. By Definition 4.16.2, $t \leq \mathrm{tp}^m_{f\mathbf{a}}$ so, by supposition, $\mathrm{tp}^m_{f\mathbf{a}} \Vdash \theta(\overline{\mathbf{x}}(\mathrm{tp}^m_{f\mathbf{a}}))$. Again by the inductive hypothesis, $(\mathfrak{D}, m) \Vdash \theta({}^f\mathbf{a})$, and therefore $(\mathfrak{D}, k) \Vdash \varphi(\mathbf{a})$. □

As in Definitions 1.22 and 3.20, for a given Fraïssé submodel $\mathfrak{D}$, the sets $[\![\varphi]\!]$ form a Heyting algebra of definable sets in the poset topology on $\mathbf{C}_D(\mathbf{x})$. Define the sets $\hat{t}$ and $\check{t}$, the formula ρ_t^+, and prime open sets as before (Definitions 3.21, 3.22, and 1.29, respectively). As Lemma 4.18 above, many of the following lemmas will be similar to those found in Sections 1.4 or 3.3. As in Section 3.3, we will omit those proofs that are trivial modifications of previous proofs.

Lemma 4.19. *For a Fraïssé submodel $\mathfrak{D}$ of $\mathfrak{R}$, let φ be an $\mathcal{L}(\mathbf{x})$-formula, let $\mathbf{a} \in A$, and let $t = \mathrm{tp}^k_{\mathbf{a}}$, for $k \in |\mathbf{D}|$. Then $(\mathfrak{D}, k) \Vdash \varphi(\mathbf{a})$ if and only if $t \in [\![\varphi]\!]$.*

Lemma 4.20. *For a Fraïssé submodel $\mathfrak{D}$ of $\mathfrak{R}$, and for all quantifier-free formulas $\varphi(\mathbf{x})$ and $\psi(\mathbf{x})$ we have $\Gamma_D \vdash \forall\mathbf{x}(\varphi \to \psi)$ exactly when $[\![\varphi(\overline{\mathbf{x}})]\!] \subseteq [\![\psi(\overline{\mathbf{x}})]\!]$. Modulo provable equivalence over Γ_D, there are for each $\mathbf{x}$ only finitely many quantifier-free formulas with all free variables from $\mathbf{x}$.*

Lemma 4.21. *For a Fraïssé submodel $\mathfrak{D}$ of $\mathfrak{R}$, let $t \in |\mathbf{C}_D(\mathbf{x})|$. Then $\check{t} = [\![\pi_t^+ \to \sigma_t^-]\!]$.*

Proof. Suppose $s \in |\mathbf{C}_D(\mathbf{x})|$ such that $s \not\leq t$. We must show $s \in [\![\pi_t^+ \to \sigma_t^-]\!]$. Fix $u \in |\mathbf{C}_D(\mathbf{x})|$ such that $s \leq u$. Then there are $\mathbf{a} \in A$, k and $m \in |\mathbf{D}|$, and $f \in F(\mathfrak{D}, k, m)$ such that $s = \mathrm{tp}^k_{\mathbf{a}}$, $u = \mathrm{tp}^m_{f\mathbf{a}}$, and $(\mathfrak{D}, m) \Vdash \pi_t^+({}^f\mathbf{a})$. Since $s \not\leq t$, $t \neq u$, so there is an atomic formula δ such that $(\neg\delta) \in t$ and $(\mathfrak{D}, m) \Vdash \delta({}^f\mathbf{a})$. So $(\mathfrak{D}, m) \Vdash \sigma_t^-({}^f\mathbf{a})$. Therefore, $(\mathfrak{D}, k) \Vdash \pi_t^+(\mathbf{a}) \to \sigma_t^-(\mathbf{a})$. By Lemma 4.19, $s \in [\![\pi_t^+ \to \sigma_t^-]\!]$.

Now suppose $s \leq t$. We must show $s \notin [\![\pi_t^+ \to \sigma_t^-]\!]$. There are $\mathbf{a} \in A$, $k \in |\mathbf{D}|$, and a morphism $f \in F(\mathfrak{D}, k, m)$ such that $s = \mathrm{tp}_{\mathbf{a}}^k$ and $t = \mathrm{tp}_{^f\mathbf{a}}^m$. Then $(\mathfrak{D}, m) \Vdash \pi_t^+(^f\mathbf{a})$ and $(\mathfrak{D}, m) \nVdash \sigma_t^-(^f\mathbf{a})$. So $(\mathfrak{D}, k) \nVdash \pi_t^+(\mathbf{a}) \to \sigma_t^-(\mathbf{a})$. By Lemma 4.19, $s \notin [\![\pi_t^+ \to \sigma_t^-]\!]$. $\square$

Lemma 4.22. *For a Fraïssé submodel $\mathfrak{D}$ of $\mathfrak{R}$, let $t \in |\mathbf{C}_D(\mathbf{x})|$. Then $\hat{t} = [\![\rho_t^+]\!]$. So all open subsets of $|\mathbf{C}_D(\mathbf{x})|$ are definable.*

Proof. To show $\hat{t} \subseteq [\![\rho_t^+]\!]$, it suffices to show $t \in [\![\rho_t^+]\!]$. Obviously, $t \in [\![\pi_t^+]\!]$. Fix u such that $t^+ \subseteq u^+$ and $t \not\leq u$. By Lemma 4.21, $t \in [\![\pi_u^+ \to \sigma_u^-]\!]$. Thus, $t \in [\![\rho_t^+]\!]$.

Conversely, suppose $v \in [\![\rho_t^+]\!]$. We must show $t \leq v$. Fix $\mathbf{a} \in A$ and $k \in |\mathbf{D}|$ such that $v = \mathrm{tp}_{\mathbf{a}}^k$. Then $(\mathfrak{D}, k) \Vdash \rho_t^+(\mathbf{a})$ by Lemma 4.19. So $(\mathfrak{D}, k) \Vdash \pi_t^+(\mathbf{a})$ and $t^+ \subseteq v^+$. Let u be such that $t^+ \subseteq u^+$ and $t \not\leq u$. Because $(\mathfrak{D}, k) \Vdash \rho_t^+(\mathbf{a})$, we have that $(\mathfrak{D}, k) \Vdash \pi_u^+(\mathbf{a}) \to \sigma_u^-(\mathbf{a})$. But $(\mathfrak{D}, k) \Vdash \pi_v^+(\mathbf{a})$ and $(\mathfrak{D}, k) \nVdash \sigma_v^-(\mathbf{a})$, so $u \neq v$. Thus, $t \leq v$. The second claim follows from the fact that all open sets are finite unions of sets of the form $\hat{t}$. $\square$

Lemma 4.23. *In $\mathbf{C_D}(\mathbf{x})$ where $\mathfrak{D}$ is a Fraïssé submodel, each open subset equals a finite union of prime open subsets. A nonempty open subset is prime if and only if it is of the form $\hat{t}$, for some $t \in |\mathbf{C}_D(\mathbf{x})|$.*

Corollary 4.24. *For a Fraïssé submodel $\mathfrak{D}$ of $\mathfrak{R}$, over Γ_D, every quantifier-free formula $\varphi(\mathbf{x})$ is equivalent to both the formula $\bigvee\{\rho_t^+ \ : \ t \in [\![\varphi]\!]\}$ and the formula $\bigwedge\{\pi_t^+ \to \sigma_t^- \ : \ t \notin [\![\varphi]\!]\}$.*

Lemma 4.25. *For a Fraïssé submodel $\mathfrak{D}$ of $\mathfrak{R}$, for all $\mathcal{L}$-formulas $\varphi(\mathbf{x}x_n)$ and for all $t \in |\mathbf{C}_D(\mathbf{x}x_n)|$, Γ_D includes the sentence*

$$\forall \mathbf{x} x_n(\varphi \wedge \rho_t^+ \rightarrow (\sigma_t^- \vee \forall x_n(\rho_t^+ \rightarrow \varphi))).$$

Proof. Fix an $\mathcal{L}(\mathbf{x}x_n)$-formula φ, $t \in |\mathbf{C}_D(\mathbf{x}x_n)|$, $\mathbf{a}b \in A$ and $k \in |\mathbf{D}|$. Suppose that $\ell \in |\mathbf{D}|$ and $f \in F(\mathfrak{D}, k, \ell)$ such that $(\mathfrak{D}, \ell) \Vdash \varphi({}^f(\mathbf{a}b)) \wedge \rho_t^+({}^f(\mathbf{a}b))$. If $(\mathfrak{D}, k) \Vdash \sigma_t^-({}^f(\mathbf{a}b))$, then we are done, so we may suppose $t = \mathrm{tp}^\ell_{f(\mathbf{a}b)}$. We must show

$$(\mathfrak{D}, k) \Vdash \forall x_n(\rho_t^+({}^f(\mathbf{a})x_n) \rightarrow \varphi({}^f(\mathbf{a})x_n)).$$

Fix $m \in |\mathbf{D}|$, $g \in F(\mathfrak{D}, \ell, m)$, and $c \in A$ such that $(\mathfrak{D}, m) \Vdash \rho_t^+({}^{gf}(\mathbf{a})c)$. We must show that $(\mathfrak{D}, m) \Vdash \varphi({}^{gf}(\mathbf{a})c)$. By Lemma 4.22, $\mathrm{tp}^m_{gf(\mathbf{a})c} \in \hat{t}$, so $\mathrm{tp}^\ell_{f(\mathbf{a}b)} \leq \mathrm{tp}^m_{gf(\mathbf{a})c}$. By Definition 4.16.2, there is $\mathbf{d}e \in A$ and $h \in F(\mathfrak{D}, \ell, m)$ such that $\mathrm{tp}_{\mathbf{d}e} = \mathrm{tp}_{f(\mathbf{a}b)}$ and ${}^h(\mathbf{d}e) = {}^{gf}(\mathbf{a})c$. Since $(\mathfrak{D}, \ell) \Vdash \varphi({}^f(\mathbf{a}b))$, Lemma 4.17 gives us $(\mathfrak{D}, \ell) \Vdash \varphi(\mathbf{d}e)$, and so $(\mathfrak{D}, m) \Vdash \varphi({}^h(\mathbf{d}e))$. Therefore, $(\mathfrak{D}, m) \Vdash \varphi({}^{gf}(\mathbf{a})c)$. □

Theorem 4.26. *For a Fraïssé submodel $\mathfrak{D}$ of $\mathfrak{R}$, Γ_D admits quantifier elimination.*

Proof. The proof is exactly the same as the proofs to Theorems 3.30 and 1.33, with one exception. In the base case where there are no free variables in φ, note that $|\mathbf{C}_D(\emptyset)| = |\mathbf{D}| \subseteq \mathcal{R}$. So $[\![\varphi]\!]$ is an open (in $\mathbf{D}$) subset of $\mathcal{R}$. By Lemma 4.14, $\Gamma_D \vdash \varphi \leftrightarrow \bigvee_{k \in [\![\varphi]\!]} \pi_k^+$, and therefore $\varphi \wedge \rho_t^+$ is equivalent to a quantifier-free formula. □

Once more, we point out that quantifier elimination gives us quantifier-free formulas of a simple form. Compare to Theorems 1.38 and 3.32.

Theorem 4.27. *Let $\varphi(\mathbf{x})$ be a formula. Over Γ_{D} where $\mathfrak{D}$ is a Fraïssé submodel of $\mathfrak{R}$, φ is equivalent to a quantifier-free universal formula.*

Proof. This easily follows from Corollary 4.24 since each ρ_t^+ (and each formula $\pi_t^+ \rightarrow \sigma_t^-$) is a universal formula. □

4.3 A Somewhat Universal Kripke Model

Classically, a model $\mathfrak{A} \models \Gamma$ is **universal** if for every $\mathfrak{B} \models \Gamma$ such that $|B| \leq |A|$, there is an elementary embedding of $\mathfrak{B}$ into $\mathfrak{A}$. Inspecting the restrictions we put on our choice for the Kripke model $\mathfrak{D}$ in Section 4.2 (Definition 4.13), there is an obvious candidate for a "largest" Kripke model, namely $\mathfrak{R}$. In some sense, $\mathfrak{R}$ (or more precisely, an "unraveled" version of $\mathfrak{R}$) is universal for certain Kripke models that have $\mathfrak{A}^{\downarrow}$ as the core JRS model at each node. We make this idea more precise in Theorem 4.32, but first a lemma.

Definition 4.28. *1. The* **type** *of a tuple* $\mathbf{a}$ *over a classical model* $\mathfrak{A}$ *is the set* $\{\varphi \,:\, \mathfrak{A} \models \varphi(\mathbf{a})\}$.

(Due to the finite relational language and quantifier elimination, over a JRS *model, every complete* $\mathcal{A}\mathrm{t}^{\pm}$*-type is a complete type.)*

2. *We call an* $\mathcal{A}\mathrm{t}^{\pm}(\mathbf{x})$*-type* t *an* n**-type** *when* $|\mathbf{x}| = n$.

3. *For a complete* n*-type* t *and a classical model* $\mathfrak{A}$*, we write* $t(A)$ *for the set* $\{\mathbf{a} \in A^n \,:\, t = \mathrm{tp}_{\mathbf{a}}\}$.

4. *For* $\mathbf{b} \in A^{n-1}$*, we write* $t(\mathbf{b}A)$ *for the set* $\{a \in A \,:\, t = \mathrm{tp}_{\mathbf{b}a}\}$.

Definition 4.29.

Definition 4.30. *Inductively construct a locally-*$\mathfrak{A}^{\downarrow}$ *Kripke model* $\mathfrak{R}^{\star}$*, the* **unraveling** *of* $\mathfrak{R}$*, on a rooted partial order* $\mathbf{R}^{\star}$ *as follows.*

1. *Let* ${}^{\emptyset}\mathfrak{A}$ *be the structure associated to the root node.*

2. *For every* $k \subseteq \mathcal{R}$ *and each* $f \in F(\mathfrak{R}, \emptyset, k)$*, we add a node directly above the root node of* $\mathbf{R}^\star$ *and associate to that node a copy of* ${}^k\mathfrak{A}$ *related to* ${}^\emptyset\mathfrak{A}$ *via that* f*.*

3. *For every node* ${}^m\mathfrak{A}$ *in the* n*th level of* $\mathbf{R}^\star$ *and for every* $g \in F(\mathfrak{R}, m, n)$ *we likewise associate a copy of* ${}^n\mathfrak{A}$ *above this node via* g*.*

4. *Write* λ *for the cardinality of the set of all endomorphisms of* $\mathfrak{A}$*. That is,* $\lambda = |F(\mathfrak{R})|$*.*

So we get the Kripke model $\mathfrak{R}^\star$ with an underlying rooted partial order $\mathbf{R}^\star$ that is λ-branching and countably high.

Lemma 4.31. $\mathfrak{R}$ *is fully graph bisimilar to* $\mathfrak{R}^\star$*.*

Proof. Let $\mathcal{B}$ be the collection of tuples of the form $\langle s, i_k, i_k, k\rangle$ where $s \in |\mathbf{R}^\star|$, $k \subseteq \mathcal{R}$, ${}^k\mathfrak{A}$ is the structure associated to node s in $\mathfrak{R}^\star$, and i_k is the identity map on this classical structure. We claim $\mathcal{B}$ is a full graph bisimulation from $\mathfrak{R}^\star$ to $\mathfrak{R}$. Criteria 1 and 2 of Definition 3.8 are trivially satisfied. Note that whenever ${}^m\mathfrak{A}$ is the structure associated to node $t \in |\mathbf{R}^\star|$, we have $\langle t, i_m, i_m, m\rangle \in \mathcal{B}$, where i_m is the identity map on ${}^m\mathfrak{A}$. Therefore, since every possible morphism is contained in each $F(\mathfrak{R}, k)$, Criterion 3 is satisfied. Additionally, since every morphism of the structure at node $s \in |\mathbf{R}^\star|$ is contained in $F(\mathfrak{R}^\star, s)$, $\mathcal{B}$ is a graph bisimulation. $\mathcal{B}$ is obviously a full graph bisimulation. □

Theorem 4.32. *Let* $\mathfrak{B}$ *be any Kripke model in the language* $\mathcal{L}$ *meeting the following criteria:*

1. $|\mathbf{B}| \leq \aleph_0$*,*

2. *each classical node structure of* $\mathfrak{B}$ *is countable, and*

3. $\mathfrak{B}$ *is locally-*$\Gamma^{\downarrow}$*.*

Then if λ *is infinite,* $\mathfrak{B}$ *is fully graph bisimilar to a Kripke submodel of* $\mathfrak{R}^{\star}$*.*

Proof. Note that by Lemma 3.10, for s and $t \in |\mathbf{R}^{\star}|$ and $k \subseteq \mathcal{R}$, if ${}^{k}\mathfrak{A}$ is the structure associated to both s and t, then $\mathrm{Th}(\mathfrak{R}^{\star}, s) = \mathrm{Th}(\mathfrak{R}^{\star}, t) = \mathrm{Th}(\mathfrak{R}, k)$. In fact, $F(\mathfrak{R}^{\star}, s) = F(\mathfrak{R}^{\star}, t) = F(\mathfrak{R}, k)$, that is, for every $k \subseteq \mathcal{R}$, every instance of ${}^{k}\mathfrak{A}$ in $\mathfrak{R}^{\star}$ "sees the same future". Also notice that since λ is infinite, the second level (and all subsequent levels) of the partial order $\mathbf{R}^{\star}$ has λ many nodes, and that for each $k \subseteq \mathcal{R}$, ${}^{k}\mathfrak{A}$ is the associated structure of λ many of those nodes.

Now consider a Kripke model $\mathfrak{B}$ meeting the hypotheses of the theorem. We will inductively build a graph bisimulation (indeed, a bisimulation) $\mathcal{B}'$ from (a Kripke submodel of) $\mathfrak{R}^{\star}$ to $\mathfrak{B}$. Index the nodes of $\mathbf{B}$ so that $|\mathbf{B}| = \{b_i\}_{i<N}$ where $N \leq \omega$. As $\mathfrak{B}$ is locally-$\Gamma^{\downarrow}$, each node structure is isomorphic to some ${}^{k}\mathfrak{A}$ for $k \subseteq \mathcal{R}$. For each b_i, choose node s_i in the second level of $\mathbf{R}^{\star}$ such that $i \neq j$ implies $s_i \neq s_j$ and h_i is an isomorphism from the associated node structure at s_i to the associated node structure at node b_i. For each $i \leq N$, include $\langle s_i, h_i, h_i^{-1}, b_i \rangle$ in $\mathcal{B}'$.

Now fix node s in the nth level of $\mathbf{R}^{\star}$ that is associated via $\mathcal{B}$' to $b \in |\mathbf{B}|$. Write $F(\mathfrak{B}, b) = \{f_\alpha\}_{\alpha<\kappa}$ where $\kappa = |F(\mathfrak{B}, b)|$, and for each $\alpha < \kappa$, write $f_\alpha \in F(\mathfrak{B}, b, b_\alpha)$. The structure at node b_α is isomorphic to ${}^{n}\mathfrak{A}$ for some $n \subseteq \mathcal{R}$ via isomorphism h_α. We wish to choose a node s_α in the $(n+1)$st level of $\mathbf{R}^{\star}$ to associate with b_α via $\mathcal{B}'$; however, there are many nodes above s in $\mathbf{R}^{\star}$ with associated node structure ${}^{n}\mathfrak{A}$. We choose s_α so that it satisfies Criterion 4 of Definition 3.8. That is, choose node s_α in the $(n+1)$st level of $\mathbf{R}^{\star}$ by following morphism $h_\alpha^{-1} f_\alpha h$ from node s; see Figure 5. For each $\alpha < \kappa$, include $\langle s_\alpha, h_\alpha, h_\alpha^{-1}, b_\alpha \rangle$ in $\mathcal{B}'$. Since $\kappa \leq \lambda$, there are enough copies of each ${}^{k}\mathfrak{A}$ associated

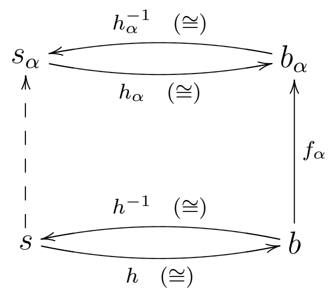

Figure 5: Choice of s_α in $\mathbf{R}^\star$

to nodes of the $(n+1)$st level in $\mathbf{R}^\star$ for us to choose each s_α at every stage such that $\alpha \neq \beta$ implies $s_\alpha \neq s_\beta$. This concludes the construction of $\mathcal{B}'$.

Again, Criteria 1 and 2 of Definition 3.8 are clearly satisfied by $\mathcal{B}'$. Criterion 4 is satisfied by our choices of s_α. We satisfy Criterion 3 by considering $\mathcal{B}'$ as a graph bisimulation from a Kripke submodel of $\mathfrak{R}^\star$ to $\mathfrak{B}$; namely, the Kripke submodel of $\mathfrak{R}^\star$ that only contains those morphisms explicitly used to choose the nodes of $|\mathbf{R}^\star|$ appearing in tuples in $\mathcal{B}'$ (the morphism $h_\alpha^{-1} f_\alpha h$ in the above diagram, for example) and their compositions. Note that if $f \in F(\mathfrak{B}, b_i, b_j)$, then the node b_j will correspond to a node s_j in the second level of $\mathfrak{R}^\star$, as well as a node $s_{i,\alpha}$ in the third level of $\mathfrak{R}^\star$ (above the node s_i corresponding to the node b_i). In this sense, for each node of $\mathfrak{B}$, $\mathcal{B}'$ determines one copy of that node and its entire future in $\mathfrak{R}^\star$ that is entirely disjoint from the (specified) images of the other nodes and their futures. □

Note that on the one hand, we should not call $\mathfrak{R}^\star$ a universal Kripke model for $\mathrm{Th}(\mathfrak{R})$. It is not true that all Kripke models of $\mathrm{Th}(\mathfrak{R})$ are locally $\Gamma^\downarrow$; see the examples in [20]. However, on the other hand, by Corollary 3.12, we could have chosen $\mathfrak{R}^\star$ so that for every $k \in |\mathbf{R}^\star|$, $|F(\mathfrak{R}^\star, k)| = \aleph_0$. Then we would still have $\mathfrak{B}$ "embed" in (be fully graph bisimilar to a Kripke submodel of) $\mathfrak{R}^\star$, even though $\mathbf{B}$ might be a bigger category than $\mathbf{R}^\star$.

Recall that we call a formula intuitionistically universal if it can be built from the atoms using the operations $\wedge$, $\vee$, $\to$ and $\forall$, with the restriction that no implications or universal quantifications occur in negative places (Definition 1.36). We call the collection of all universal formulas $\mathcal{U}_1$. See Section 1.5 and Fleischmann's [11] for more information.

Theorem 4.33. *Let Kripke model* $\mathfrak{B}$ *be as in Theorem 4.32. Then* $(\mathrm{Th}(\mathfrak{R}) \cap \mathcal{U}_1) \subseteq (\mathrm{Th}(\mathfrak{B}) \cap \mathcal{U}_1)$.

Proof. Theorems 4.32 and 1.37 give us that $(\mathrm{Th}(\mathfrak{R}^\star) \cap \mathcal{U}_1) \subseteq (\mathrm{Th}(\mathfrak{B}) \cap \mathcal{U}_1)$. By Lemma 4.31 and Corollary 3.12, $\mathrm{Th}(\mathfrak{R}) \restriction \mathcal{L} = \mathrm{Th}(\mathfrak{R}^\star) \restriction \mathcal{L}$, so the result follows. $\square$

4.4 Interaction of Classical and Intuitionistic JRS Theories

If we return to languages with no nullary predicates, then we have that both a classical JRS theory Γ and its corresponding intuitionistic theory Γ_K (for any monoid K satisfying the Fraïssé condition) are both complete theories (Theorem 3.14). As long as the classical JRS model has at least one non-embedding morphism, these two theories are incompatible. However, in some sense, both theories describe the same underlying structure. What do these related theories have in common, and how are they different? We now use the flexibility afforded us by the addition of nullary predicates to build a Kripke model that gives us a semantic way to partially answer these questions.

Definition 4.34. *Given a classical* JRS *model* $\mathfrak{A}$ *in a language* $\mathcal{L}^\downarrow$ *without nullary* *predicates and a monoid K of endomorphisms of $\mathfrak{A}$ satisfying the Fraïssé condition, construct the Kripke model $\mathfrak{B}$ as follows.*

1. *Add a single nullary predicate R to the language. That is, let $\mathcal{L} = \mathcal{L}^{\downarrow} \cup \{R\}$.*

2. *The partial order category $\mathbf{B}$ will have three nodes: $|\mathbf{B}| = \{r, k, m\}$. The partial order is given by $r \leq k$ and $r \leq m$ (that is, r is the root). The structure at node k is ${}^{R}\mathfrak{A}$, while the structure corresponding to nodes r and m is ${}^{\{\emptyset\}}\mathfrak{A}$. (Whenever possible, we will shorten $\{R\}$ to R.)*

3. *Set $F(\mathfrak{B}, r, r) = F(\mathfrak{B}, r, m) = F(\mathfrak{B}, m, m) = K$. Set $F(\mathfrak{B}, k, k) = \mathrm{id}_{\mathfrak{A}}$. Construct $F(\mathfrak{B}, r, k)$ by starting with the single morphism that is the identity on $\mathfrak{A}$ and takes $\neg R$ to R, and close under compositions with morphisms from K. (To be clear, $F(\mathfrak{B}, k, r) = F(\mathfrak{B}, m, r) = F(\mathfrak{B}, k, m) = F(\mathfrak{B}, m, k) = \emptyset$.)*

Lemma 4.35. $\mathrm{Th}(\mathfrak{B}, k) \cap \mathcal{L}^{\downarrow} = \Gamma$.

Proof. Since the only morphism from the node k is the identity, $\mathrm{Th}(\mathfrak{B}, k)$ is a classical theory by Theorem 1.44. Once we restrict to $\mathcal{L}^{\downarrow}$, the node structure at the node k is just $\mathfrak{A}$. $\square$

Corollary 4.36. $(\mathfrak{B}, k) \Vdash \Gamma$.

Lemma 4.37. *For every $\mathcal{L}^{\downarrow}$-formula φ, $\mathfrak{B} \Vdash \forall \mathbf{x}(R \to \varphi)$ if and only if $\mathfrak{A} \models \forall \mathbf{x} \varphi$.*

Proof. First, note that $\forall \mathbf{x}(R \to \varphi)$ is intuitionistically equivalent to $R \to \forall \mathbf{x} \varphi$. For the right to left direction, suppose $\mathfrak{A} \models \forall \mathbf{x} \varphi$. For any node $n \in |\mathbf{B}|$, we need only consider the morphisms $f \in F(\mathfrak{B}, n, k)$ (if they exist), as k is the only node forcing R. Note that

${}^f(\forall \mathbf{x}\varphi)$ is the same syntactic object as $\forall \mathbf{x}\varphi$. So by our supposition and Corollary 4.36, $(\mathfrak{B}, n) \Vdash R \to \forall \mathbf{x}\varphi$, so each node forces $\forall \mathbf{x}(R \to \varphi)$.

For the left to right direction, suppose $\mathfrak{B} \Vdash \forall \mathbf{x}(R \to \varphi)$. Then $(\mathfrak{B}, k) \Vdash R \to \forall \mathbf{x}\varphi$. Since $(\mathfrak{B}, k) \Vdash R$, we have $(\mathfrak{B}, k) \Vdash \forall \mathbf{x}\varphi$. Since Γ is a complete $\mathcal{L}^{\downarrow}$ theory, Corollary 4.36 tells us that $\Gamma \vdash \forall \mathbf{x}\varphi$. □

Lemma 4.38. $\mathrm{Th}(\mathfrak{B}, m) \cap \mathcal{L}^{\downarrow} = \Gamma_K$.

Proof. The full Kripke submodel of $\mathfrak{B}$ with root m reducted to $\mathcal{L}^{\downarrow}$ is essentially just $\mathfrak{A}_K$. □

Corollary 4.39. $(\mathfrak{B}, m) \Vdash \Gamma_K$.

Lemma 4.40. *For every $\mathcal{L}^{\downarrow}$-formula φ, $\mathfrak{B} \Vdash \forall \mathbf{x}(\neg R \to \varphi)$ if and only if $\mathfrak{A}_K \Vdash \forall \mathbf{x}\varphi$.*

Proof. Again, note that $\forall \mathbf{x}(\neg R \to \varphi)$ is intuitionistically equivalent to $\neg R \to \forall \mathbf{x}\varphi$. For the right to left direction, suppose $\mathfrak{A}_K \Vdash \forall \mathbf{x}\varphi$. For any node n, we need only consider morphisms sending n to m (if they exist), since m is the only node forcing $\neg R$. Again, ${}^f(\forall \mathbf{x}\varphi)$ is the same syntactic object as $\forall \mathbf{x}\varphi$. By our supposition and Corollary 4.39, $(\mathfrak{B}, n) \Vdash \forall \mathbf{x}(\neg R \to \varphi)$.

For the left to right direction, suppose $\mathfrak{B} \Vdash \forall \mathbf{x}(\neg R \to \varphi)$. Then $(\mathfrak{B}, m) \Vdash \forall \mathbf{x}(\neg R \to \varphi)$. Since $(\mathfrak{B}, m) \Vdash \neg R$, we have $(\mathfrak{B}, m) \Vdash \forall \mathbf{x}\varphi$. Since Γ_K is a complete $\mathcal{L}^{\downarrow}$ theory, Corollary 4.39 tells us that $\Gamma_K \vdash \forall \mathbf{x}\varphi$. □

Lemmas 4.37 and 4.40 immediately give us the following results.

Theorem 4.41. *The following relationships hold amongst* $\mathrm{Th}(\mathfrak{B})$, Γ *and* Γ_K.

1. $\mathrm{Th}(\mathfrak{B}) \cap \mathcal{L}^{\downarrow} \subseteq \Gamma \cap \Gamma_K$

2. $(\mathrm{Th}(\mathfrak{B}) \cup \{R\}) \cap \mathcal{L}^{\downarrow} = \Gamma$

3. $(\mathrm{Th}(\mathfrak{B}) \cup \{\neg R\}) \cap \mathcal{L}^{\downarrow} = \Gamma_K$

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