MATH 611 (Spring 2009) Homework #11 Due May 12th

(1) Prove that T is normal on H if and only if $\|T^*f\| = \|Tf\|$ for all $f \in H$.

(2) Suppose that T is invertible. Prove that $\sigma(T^{-1}) = \{\lambda^{-1} \ : \ \lambda \in \sigma(T)\}$.

(3) Suppose that T is normal. Prove that $\|T^n\| = \|T\|^n$ for all positive integers, n. [Hint: First show that $\|T^k\|^2 \leq \|T^{k+1}\| \, \|T^{k-1}\|$ and then use this result to prove that $\|T\|^n \leq \|T^n\|$ using an induction argument.]

(4) Let $\{\lambda_n\}$ be a sequence of non-zero real numbers that converges to zero. Prove that if H is an infinite-dimensional separable Hilbert space then there exists a self-adjoint (i.e. $T^* = T$), compact operator T on H whose set of non-zero eigenvalues is the set $\{\lambda_n\}$. [Hint: Use Hwk 9.5.]

(5) Let $T(x) = (x, y)z$ be the rank-one operator in Hwk 9.1 (and Hwk 8.5).
(a) Show that $T^2 = (z, y)T$.
(b) Let S be an operator so that $S^2 = \lambda S$. Show that, if $\lambda \neq 1$, then $I - S$ is invertible and calculate $(I - S)^{-1}$.
(c) Use (a), (b) to show that if H has dimension greater than 1, then $\sigma(T) = \{0, (z, y)\}$. [Hint: Apply (b) to $S = \frac{1}{\mu}T$.]

Additional Questions (Not to be handed in)

(6) Show that the Spectral Theorem for compact, normal operators on the Hilbert space $H = \mathbb{C}^n$ with the standard inner product is equivalent to the well known result that a normal matrix can be diagonalized a unitary matrix.

(7) Show that if some positive power of a compact self-adjoint operator T is zero, then T itself must be zero.

(8) Let T be a bounded linear operator on a Banach space. By considering the identity

$$(\lambda I - T)(\lambda I + T) = \lambda^2 I - T^2,$$

show that $\sigma(T^2) = \{\lambda^2 \ : \ \lambda \in \sigma(T)\}$. Deduce that for any $\lambda \in \sigma(T)$, we have $|\lambda| \leq \|T^2\|^{1/2}$.

(9) Let $T \in \mathcal{B}(H)$ be self-adjoint on a Hilbert space H.
(a) Prove that $\sigma(T) \subset [0, \infty)$ iff $(Tx, x) \geq 0$ for all $x \in H$. A self-adjoint operator that satisfies this condition is called *positive*.
(b) Prove that for any $T \in \mathcal{B}(H)$, the operator T^*T is self-adjoint and positive.

(c) Suppose that T is compact, self-adjoint and positive. Use the Spectral Theorem to construct a compact, self-adjoint, positive operator R so that $R^2 = T$.

(d) Let $T \in \mathcal{B}(H)$ be compact. Let λ_n be the non-zero eigenvalues of the positive operator T^*T and let $\{u_n\}$ be the corresponding orthonormal set of eigenvectors. Let $\mu_n = +\sqrt{\lambda_n}$. Show that there exists an orthonormal set $\{v_n\}$ in H so that $Tu_n = \mu_n v_n$, $T^* v_n = \mu_n u_n$ and

$$Tx \;=\; \sum \mu_n (x, u_n) v_n, \qquad x \in H. \tag{1}$$

The real numbers μ_n are called the singular values of T and (1) is the singular value decomposition of T. Prove that if T is self-adjoint then the non-zero singular values of T are the absolute values of the eigenvalues of T.

(10) Let $T \in \mathcal{B}(H)$ be self-adjoint. Prove that $r_\sigma(T) = \|T\|$ starting as follows. Let $\tau = \|T\|$. Show that

$$\|(\tau^2 - T^2)x\|^2 = \tau^4 \|x\|^2 - 2\tau^2 \|Tx\|^2 + \|T^2 x\|^2.$$

Choose x_n so that $\|x_n\| = 1$ and $\|Tx_n\| \to \|T\|$ and deduce that $\tau^2 \in \sigma(T^2)$.

(11) Let $T : \ell^1 \to \ell^1$ be defined by the infinite matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \\ 0 & 0 & \frac{1}{3} & 0 & 0 & \\ 0 & 0 & 0 & \frac{1}{4} & 0 & \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \end{pmatrix}$$

Recall that the norm of T is the supremum of the ℓ^1 norms of the column vectors in its matrix representation. Calculate that $\|T^n\| = \frac{1}{n!}$. Hence show that $\|T^n\|^{1/n} \to 0$ and $\sigma(T) = \{0\}$.

(12) Let ω be positive and not an integer multiple of π. Find the Green's function for the boundary value problem $f'' + \omega^2 f = g$, with $f'(0) = 0 = f'(1)$.

(13) Find the Green's function for $f'' = g$, $f(0) = 0$, $f(1) + f'(1) = 0$.

(14) Apply Sturm-Liouville theory to the system $f'' + \lambda f = 0$, $f(0) = 0 = f'(\pi)$ to show that for any $g \in L^2(0, \pi)$,

$$g(x) = \sum_{j=1}^{\infty} C_j \sin(j - \frac{1}{2})x$$

in $L^2(0, \pi)$ where

$$C_j = \frac{2}{\pi} \int_0^{\pi} g(x) \sin(j - \frac{1}{2})x \, dx$$