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Explanation, Existence and Natural Properties in Mathematics – A Case Study: Desargues' Theorem

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1. Introduction

In practice, mathematicians have long distinguished proofs that *explain why* a given theorem holds from proofs that merely *establish that* it holds. For instance, in the *Port-Royal Logic* of 1662, Pierre Nicole and Antoine Arnauld characterized indirect proof (that is, proof of p by showing that $\sim p$ implies a contradiction) as "useful" but non-explanatory:

> such Demonstrations constrain us indeed to give our Consent, but no way clear our Understandings, which ought to be the principal End of Sciences: for our Understanding is not satisfied if it does not know not only that a thing is, but why it is? which cannot be obtain'd by a Demonstration reducing to Impossibility. (Nicole and Arnauld 1717, 422 (Part IV, chapter ix))

Nicole and Arnauld took explanation ("divining into the true reason of things" (1717, 427)) to be as important in mathematics as it is in natural science. More recently, the mathematician William Byers (2007, 337) has characterized a "good" proof as "one that brings out clearly the reason why the result is valid". Likewise, empirical researchers on mathematics education have recently argued that students who have proved and are convinced of a mathematical result often still want to know why the result is true (Mudaly and de Villiers 2000), that students assess alternative proofs for their "explanatory power" (Healy and Hoyles 2000, 399), and that students expect a "good" proof "to convey an insight into *why* the proposition is true" even though explanatory power "does not affect the *validity* of a proof" (Bell 1976, 24). However, none of this work investigates what it is that makes certain proofs but not others explanatory.

Recently, as Mancosu (2008) and Tappenden (2008a) note, philosophers have renewed their interest in mathematical explanation, which has received

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considerably less philosophical attention than scientific explanation.[2] With this renewed interest in mathematical explanation have come frequent calls for case studies that are attentive to the details of mathematical research. This paper presents such a case study and extracts from it some general morals regarding mathematical explanation, unification, coincidence, existence and natural properties.[3]

It is relatively easy to find suggestive toy examples of these ideas.[4] Take an ordinary calculator keyboard:

[2] Notable among the few earlier papers that investigated mathematical explanation are Steiner (1978a), Kitcher (1984, esp. 208–209, 227; 1989, esp. 423–426, 437), and Resnik and Kushner (1987). More recent work includes Hafner and Mancosu (2005), Tappenden (2005), Baker (2009) along with Lange (2010; 2014). I am concerned here with explanations of mathematical theorems rather than with explanations of natural phenomena where mathematical theorems do the heavy lifting; the latter has been explored recently by Colyvan (2001; 2002), Melia (2002), Baker (2005) and Lange (2013), and earlier by Steiner (1978b) and Nerlich (1979).

In a conversation, we might 'explain' why (or how) some mathematical proof works (either by giving its overall strategy or by making more explicit the transitions between steps), or we might 'explain' how to carry out some mathematical process. A textbook might 'explain' how to multiply matrices, for example, or a mathematics popularizer might 'explain' an obscure theorem by clarifying it or making it more accessible. However, none of these is the kind of 'mathematical explanation' with which I am concerned here, since none involves explaining why some result holds – just as Hempel (2001, 80) pointed out that an account of scientific explanation does not aim to account for what he does when he uses gestures to explain to a Yugoslav garage mechanic how his car has been misbehaving. By the same token, I shall not appeal to 'understanding', 'insight' or 'enlightenment' in order to capture mathematical explanation, just as these notions are too psychological and too imprecise to figure in an account of scientific explanation. It is also important to distinguish my project from the historical, sociological and psychological project of explaining why mathematicians hold various beliefs or how a given mathematician managed to make a certain discovery. Finally, questions asking for good reasons for some belief (e.g., 'Why do you think that this strategy for proving the theorem is going to work?', 'Why do you think that this mathematical claim is true?') are sometimes expressed as why questions, but these questions are not answered by mathematical explanations of the sort I am investigating.

[3] By focusing on proofs that mathematicians themselves recognize as explanatory, I do not mean to suggest that philosophers must unquestioningly accept the verdicts of mathematicians. But just as an explication of scientific explanation should do justice to scientific practice (without having to fit every judgment of explanatory value made by every scientist), so an explication of mathematical explanation should do justice to mathematical practice. Regarding the examples I shall discuss, I have found that working mathematicians' judgments of which proofs explain, and which do not, are widely shared and relatively easily appreciated by mathematicians and non-mathematicians alike. Accordingly, it is especially important that an account of mathematical explanation fit such cases.

[4] This example appears in Roy Sorensen's unpublished manuscript "Mathematical Coincidences". I discuss it further in Lange (2014), where I give a host of other cases, drawn from mathematical practice, where mathematicians have distinguished proofs that explain from proofs that do not. What I have to say in the present paper may well be deemed insufficient to fully justify my claim that this distinction plays an important role in mathematical practice and is worthy of philosophical investigation. For additional arguments and examples, readers may consult Lange (2010; 2014; forthcoming).

7	8	9
4	5	6
1	2	3

We can form a six-digit number by taking the three digits on any row, column, or main diagonal on the keyboard in forward and then in reverse order. For instance, the bottom row taken from left to right, and then right to left, yields 123321. There are 16 such numbers: 123321, 321123, 456654, 654456, 789987, 987789, 147741, 741147, 258852, 852258, 369963, 963369, 159951, 951159, 357753, and 753357. By checking each of these 'calculator numbers' separately, we can *prove that* all of the calculator numbers are divisible by 37, but we have not *explained why* this is so. As far as this brute-force proof tells us, there might be no explanation; it might be just a coincidence. However, there is another proof that, as Nummela (1987) says, shows it to be no coincidence; it explains why they are all divisible by 37:

> Consider any three integers a, $a+d$, and $a+2d$. Then $10^5a+10^4(a+d)+10^3(a+2d)$ $+10^2(a+2d)+10(a+d)+a=a(10^5+10^4+10^3+10^2+10+1)+d(10^4+2\text{x}10^3+2\text{x}10^2+$ $10)=111111a+12210d=1221(91a+10d)$. So the number is divisible by 37, since $1221=3 \text{ x } 11 \text{ x } 37$.

Crucial to this proof's explanatory power, I suspect, is the way it identifies a property *common* to each calculator number, *unifying* them rather than treating each number separately (in the manner of the brute-force proof). Every calculator number can be expressed as $10^5a+10^4(a+d)+10^3(a+2d)+10^2(a+2d)+10(a+d)+a$; the three digits on the calculator keypad that are used to form it are a, $a+d$, and $a+2d$. The proof that explains why they are all divisible by 37 treats all of the calculator numbers *uniformly*.

I will ultimately work out these thoughts in connection with an example in which mathematical explanation, existence and naturalness have played more important roles: Desargues' theorem, which I will introduce in section 2. There I will present three proofs of the theorem in Euclidean geometry, only one of which mathematicians regard as explaining why it is true. We will see that the explanatory power or impotence of these various proofs is not well accounted for either by Mark Steiner's (1978a) or by Philip Kitcher's (1984; 1989) accounts of explanation in mathematics. In section 3, I will argue that the details of this example suggest an account of why this proof alone is explanatory. This proof explains Desargues' theorem only because a certain feature of the theorem strikes us as remarkable.[5] In this context, what it *means* to ask for an *explanation* over and above a proof of

[5] There may be other proofs that also explain why Desargues' theorem holds – especially in contexts where different features of the theorem are salient. Desargues' theorem has many rich connections to other parts of mathematics.

Desargues' theorem is to ask for a proof that exploits some other feature of the given case that is similar to the remarkable feature of the theorem. Outside of such a context, there is no sense in which one proof is privileged over another as explanatory. As in the calculator-number example, the proof that explains Desargues' theorem in Euclidean geometry reveals the theorem to be no mathematical coincidence.

However, mathematicians say that Desargues' theorem naturally belongs to projective rather than to Euclidean geometry. In section 4, I show how an explanation of Desargues' theorem in projective geometry unifies what Euclidean geometry portrays as a motley collection of special cases. Euclidean geometry is *mistaken* in portraying as coincidental certain results about Euclidean points, lines and planes that in fact have a common, unified explanation. Our study of Desargues' theorem will suggest that projective geometry's talk of 'points at infinity' is not a mere *façon de parler*; rather, features of those points explain facts about Euclidean points, lines and planes. Points at infinity exist in *Euclidean* geometry by virtue of their playing such an explanatory role.[6]

This common, unified explanation of Desargues' theorem in projective geometry strongly suggests that a proof's explanatory power is independent of its purity. However, this explanation presupposes that various properties (such as the property of being a point, whether a Euclidean point or a 'point at infinity') are *natural* rather than disjunctive. Here we seem to be caught in a vicious circle: the proof's explanatory power (indeed, even the 'Why?' question demanding an explanation of the theorem) presupposes that certain properties are natural, but presumably, they are natural purely in virtue of their role in such explanations. In section 5, I argue that the naturalness of these properties and the explanatory power of these proofs arise together; neither is prior to the other. The case of Desargues' theorem also illustrates how mathematicians discover that the properties in a given family are natural by finding them in many, diverse proofs that (mathematicians

[6] Admittedly, this is a radical view. Perhaps the idea that Euclidean geometry is in this respect mistaken about Euclidean points, lines and planes – and that projective geometry uncovers the truth about the explanation of certain facts about Euclidean entities – is part of what Cassirer is driving at in passages such as the following, which concern "new elements" in mathematics such as points at infinity: "For it is not enough that the new elements should prove equally justified with the old, in the sense that the two can enter into a connection that is free from contradiction – it is not enough that the new should take their place beside the old and assert themselves in this juxtaposition. This merely formal combinability would not in itself provide a guarantee for a true inner conjunction, for a homogeneous logical structure of mathematics. Such a structure is secured only if we show that the new elements are not simply adjoined to the old ones as elements of a different kind and origin, but the new are a systematically necessary unfolding of the old. And this requires that we demonstrate a primary logical kinship between the two. Then the new elements will bring nothing to the old, other than what was implicit in their original meaning. If this is so, we may expect that the new elements, instead of fundamentally changing this meaning and replacing it by another, will first bring it to its full development and clarification. And when we survey the history of the ideal elements in mathematics, this expectation is never disappointed." (Cassirer 1957, 392)

recognize) would be explanatory, if those properties were natural. In section 6, I draw conclusions about explanations' mathematical importance.

Mathematical explanation, existence and naturalness constitute tremendously exciting and generally underexplored topics. But I will generally avoid pursuing them beyond the limits of this particular case study. I will stick to elaborating the lessons that (I argue) are suggested by some mathematical work on Desargues' theorem. Although this focus will not allow me to argue fully for my proposals, I hope that by delving fairly deeply into one example of mathematical explanation, I can do some justice to the roles that mathematical explanation, existence and natural properties have actually played in one sliver of mathematics.

2. *Three proofs – but only one explanation – of Desargues' theorem in two-dimensional Euclidean geometry*

Here is Desargues' theorem in two-dimensional Euclidean geometry:

> If two triangles are so situated that the three lines joining their corresponding vertices all meet at a single point, then the points of intersection of the two triangles' corresponding sides – if those intersection points exist – all lie on one line.

This is easier to understand with a figure (see Figure 1).

Triangles ABC and A′B′C′ lie on the same Euclidean plane and their corresponding vertices (point A corresponding to point A′, B to B′, and C to C′) are connected by lines that all meet at a single point (O). The two triangles are said to be 'in perspective from O'. Desargues' theorem concerns pairs of corresponding sides of the two triangles, where side CA corresponds to C′A′, for example. Line CA may intersect line C′A′ (remembering that each of these lines extends infinitely beyond the segment forming a side of one of the two triangles in

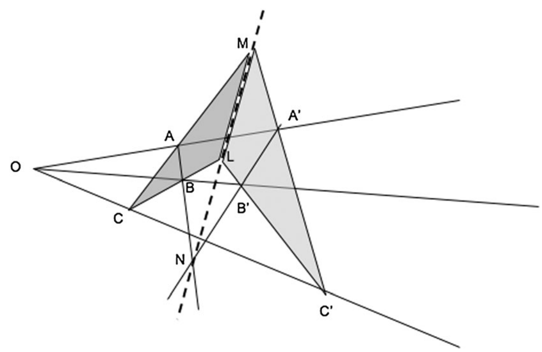

Figure 1. Desargues' theorem in two-dimensional Euclidean geometry.

perspective); unless CA and C′A′ are parallel, they will intersect somewhere on the plane. In the figure, M is their point of intersection. Likewise, N lies at the intersection of AB and A′B′, and L lies at the intersection of CB and C′B′. The theorem says that these three points of intersection, if they exist, are collinear. (In Figure 1, they all lie on the dashed line.)

There are various ways of proving Desargues' theorem. For example, Girard Desargues (who first discovered the theorem in the early 1600s) used Menelaus' theorem (discovered by Menelaus of Alexandria, c. 100 AD), which says:

> Consider triangle RST, and let R_1, S_1, and T_1 be points on lines ST, TR, and RS, respectively. Then R_1, S_1, and T_1 are collinear iff $(RT_1/ST_1)(SR_1/TR_1)(TS_1/RS_1) = 1$.

Here is Desargues' proof of his theorem, first published in 1648 (Field and Gray 1987, 161–164).

> Consider triangle OBC (from Figure 1): L lies on BC, B′ lies on OB, and C′ lies on OC. From the collinearity of L, B′, and C′, Menelaus' theorem in the left-to-right direction entails that
>
> $$(CL/BL)(BB'/OB')(OC'/CC') = 1.$$
>
> Likewise, from triangle OAB and line NA′B′, Menelaus' theorem entails that
>
> $$(BN/AN)(AA'/OA')(OB'/BB') = 1.$$
>
> Similarly, from triangle OAC and line MA′C′, Menelaus' theorem entails that
>
> $$(AM/CM)(CC'/OC')(OA'/AA') = 1.$$
>
> By multiplying all of the left sides together and all of the right sides together, we find
>
> $$(CL/BL)(BB'/OB')(OC'/CC')(BN/AN)(AA'/OA')(OB'/BB')$$
> $$(AM/CM)(CC'/OC')(OA'/AA') = 1.$$
>
> Three fortuitous cancellations (e.g., BB′/OB′ with OB′/BB′) produce
>
> $$(CL/BL)(BN/AN)(AM/CM) = 1.$$
>
> By the right-to-left direction of Menelaus' theorem applied to triangle ABC, it follows that L, N and M are collinear.

This argument, though successful at proving Desargues' theorem, is typically characterized by mathematicians as failing to *explain why* it is true. For example, Zvezdelina Stankova says, “A serious drawback of this solution is that it doesn’t give us a clue *really why* Desargues’ Theorem works” (Stankova 2004, 175). What is missing from the proof, depriving it of explanatory power? A general answer to this question would reveal to us what it is (at least in a certain class of cases) to give a mathematical explanation.

One clue to this proof’s shortcomings is that mathematicians typically describe this proof (and others like it) “as ingenious exercises in Euclidean geometry” (Gray 2007, 28), where ‘ingenious’ here means ‘clever’ (in the pejorative sense of ‘merely clever’). What is clever about this proof is the way that the three equations ‘magically’ cancel one another’s inconvenient terms. This cancellation appears out of nowhere, and the theorem arises from it. This proof thus makes it seem like an accident of algebra, as it were, that everything cancels out so nicely, leaving us with just the terms needed for Menelaus’ theorem to yield the collinearity of L, M and N.

Of course, nothing in mathematics is genuinely accidental; the result is mathematically necessary. Nevertheless, I think that many of us, after working through the above proof, are inclined to suspect that there is some *reason why* everything works out (and *has* to work out) so neatly in the end – that is, a reason why all of the terms with primes ultimately disappear from the calculation. This reason eludes the above proof and must somehow explain why Desargues’ theorem holds. Similarly, after checking each ‘calculator number’ individually and finding them all to be divisible by 37, we might well suspect that there is a reason why they are all alike in this respect, this suspicion motivating us to search for this reason. (Of course, our evidence does not guarantee that such a reason exists, and in some examples, it does not; see the discussion of ‘mathematical coincidences’ in Lange (2010; 2014).)

To try to understand why this proof fails to explain, we can compare it to another proof of Desargues’ theorem. This proof introduces a third dimension above and below the Euclidean plane on which the two triangles lie in perspective. One way to picture this third dimension is to imagine pulling line OCC′ below the plane of the paper. Then the two shaded regions in Figure 1 slice up through the paper’s plane at AB and A′B′, respectively, and from there rise above the plane of the paper to meet along a peak at line LM. The dotted line is then envisioned as slanting from N in the paper’s plane up through L above the paper’s plane, ultimately rising further to M. (Picture the dotted line as like the line along which the two sides of a pitched roof meet.) Suitably positioned light sources below the paper’s plane would project the shadows of triangles CAB and C′A′B′ onto the corresponding triangles lying on the plane.

For any arrangement of coplanar triangles in perspective from O, there is such an arrangement (indeed, there are many such arrangements) of corresponding triangles jutting into the third dimension. To construct one (see Figure 2, which

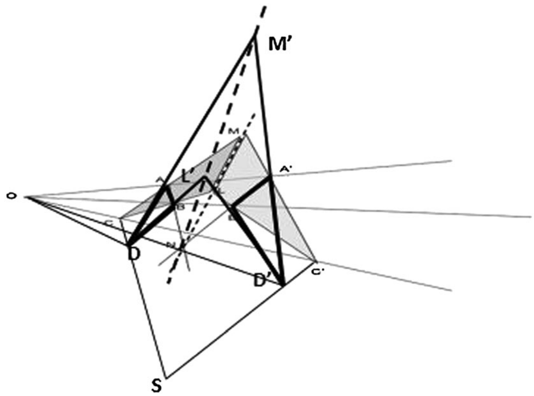

Figure 2. A three-dimensional proof of Desargues' theorem.

includes Figure 1 on a plane seen edge-on), select any point S outside the plane of the two triangles in perspective from O. (S in Figure 2 is drawn below that plane.) Draw lines SC and SC′. Choose any line from O that intersects SC; let D be their point of intersection. Likewise, choose any line from O that intersects SC′; let D′ be their point of intersection. Now triangle BAD (outlined in bold in Figure 2), extending below the original plane, projects onto triangle BAC lying on that plane, and likewise B′A′D′ projects onto B′A′C′. (Figure 2 is obviously busy; Figure 3 shows how two classic textbooks attempt to represent this proof.) Any such projection preserves collinearity, so to show that L, M and N are collinear, it suffices to show that the corresponding points in the three-dimensional figure are collinear. This is easily done. Let's return to Figure 1, now thinking of the two shaded triangles as jutting into the third dimension above and below the plane of the page. Each of them lies on its own plane slanting through the plane of the page. These two planes meet, and any two planes that intersect meet at exactly one line. Since L, M and N lie on this line, they are collinear. In other words (switching to Figure 2, where the two shaded triangles from Figure 1 are lying on the original plane): M′ (floating above the original plane, at the intersection of DA and D′A′) is common to the plane containing triangle DAB and the plane containing triangle D′A′B′ (since DA lies on the former plane, D′A′ lies on the latter, and any point on one of these lines is on that line's plane). Likewise, L′ (at the intersection of DB and D′B′) is common to the two triangles' planes, and the same for N (on the original plane – where BA and B′A′ intersect). Since the same pair of planes is involved in all three cases, and since any two planes that intersect meet at exactly one line, the three points (L′, M′ and N) are collinear – and so are their projections onto the original plane (L, M and once again N).

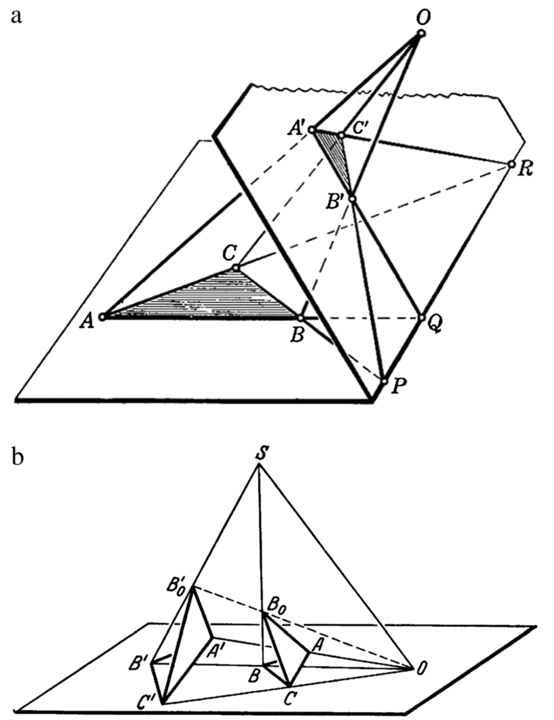

Figure 3. Two illustrations of the proof exiting to the third dimension – from (Courant et al. 1996, 171) and (Hilbert and Cohn-Vóssen 1952, 122).

This proof of Desargues' theorem is generally recognized by mathematicians as explaining why Desargues' theorem holds. It holds in Euclidean geometry because the two triangles in perspective from O are projections of triangles jutting into the third dimension, and since the planes of those triangles must meet at a line, their projections must, too. Of course, we can appreciate this proof's explanatory power (especially by contrasting it with the proof using Menelaus' theorem) without seeing *precisely* what *makes* this proof explanatory. For instance, Jeremy Gray says:

> How do we feel about this proof? We've changed the subject, of course, from two dimensions to three. We need to convince ourselves that any two-dimensional figure can be drawn in three dimensions. That's easy enough if the triangles don't cross, but what if they do? Still, this ability to see the figure and see the truth of the theorem is a very powerful guide to understanding it. It conveys what a lengthy calculation may not always manage, a sense of the inevitability of the result. (Gray 2007, 29)

The proof using Menelaus' theorem employs just such a "lengthy calculation", where the three crucial cancellations seem fortuitous – coincidental rather than "inevitable". Yet Gray's attempt to contrast these proofs obviously remains unsatisfactory since, as I just mentioned, everything here (including these cancellations) is inevitable in being mathematically necessary.

In the next section, I will return to Gray's remark.[7] But for now, let's try to improve our grip on the contrast between the two preceding proofs of Desargues' theorem by looking briefly at another route to proving it – namely, by using coordinate geometry. The textbook technique (e.g., McLeod and Baart 1998, 149–150) is to use 'homogeneous coordinates'.[8] That is (briefly), three coordinates (x,y,z) rather than the usual two are used to represent each point on a plane figure; the third coordinate supplies some redundancy so that (x,y,z) and (nx,ny,nz) are the same point. Then for two arbitrary distinct points H (x_1,y_1,z_1) and J (x_2,y_2,z_2), line HJ consists of all points (x,y,z) such that for some real numbers a and b that are not both zero

$$x = ax_1 + bx_2,\ y = ay_1 + by_2,\ z = az_1 + bz_2$$

and so the equation for the line HJ is

$$(y_1z_2 - y_2z_1)\,x + (z_1x_2 - z_2x_1)\,y + (x_1y_2 - x_2y_1)\,z = 0.$$

In Figure 1, we can let

A be (1,0,0)
B be (0,1,0)
C be (0,0,1)
O be (α,β,γ)

By the above, line OA consists of all points (x,y,z) such that for some real numbers a and b that are not both zero,

$$x = a\alpha + b1,\ y = a\beta + b0,\ z = a\gamma + b0.$$

If $a=0$, then $y=z=0$ and so any point A′ on line OA is (b,0,0), which (by the redundancy in representing points) is the same point as A. Since A and A′ in the

[7] Sawyer (1955, 148–149) and Stankova (2004, 175–178) contrast the proof exiting to the third dimension with Euclid-style proofs. I think it is fair to read both authors as seeing the former proof as possessing explanatory power absent from the latter. (See the passage from Stankova quoted above and notes 9 and 12.)

[8] Homogeneous coordinates are standardly used because textbook authors are anticipating the fact that (as we will see) the theorem's 'natural setting' is projective rather than Euclidean geometry, and homogeneous coordinates assign finite coordinates to the points at infinity figuring in projective geometry. (My thanks to Jamie Tappenden for discussion of this point and others in the vicinity and for calling my attention to the passages I cite in note 10.)

configuration with which Desargues' theorem is concerned (Figure 1) are distinct points, $a \neq 0$ and A' is $(a\alpha+b, a\beta, a\gamma)$, which (by the redundancy) is the same point as $(\alpha+d, \beta, \gamma)$ for some real number d. By analogous reasoning, B' is $(\alpha, \beta+e, \gamma)$ and C' is $(\alpha, \beta, \gamma+f)$ for some real numbers e and f. The above equation for line HJ yields

$$\text{For line BC: } (1\cdot 1 - 0\cdot 0)x + (0\cdot 0 - 1\cdot 0)y + (0\cdot 0 - 0\cdot 1)z = 0, \text{ i.e., } x = 0$$

$$\text{For line } B'C'\text{: } [(\beta+e)(\gamma+f) - \beta\gamma]x + [\gamma\alpha - (\gamma+f)\alpha]y + [\alpha\beta - \alpha(\beta+e)]z = 0,$$

$$\text{i.e., } [ef + \beta f + e\gamma]x - f\alpha y - \alpha ez = 0.$$

These two lines meet at $x = 0$ and

$$-f\alpha y - \alpha ez = 0, \text{ i.e., } \alpha(fy + ez) = 0.$$

If $\alpha = 0$, then BC and $B'C'$ are the same line, but they are distinct in the configuration with which Desargues' theorem is concerned (Figure 1). So the intersection L of BC and $B'C'$ is where $x = 0$ and $fy + ez = 0$, i.e., the point $(0, e, -f)$. By analogous reasoning, lines CA and $C'A'$ meet at $M = (-d, 0, f)$ and lines AB and $A'B'$ meet at $N = (d, -e, 0)$. Again using the above equation for line HJ, we find

$$\text{For line NM: } (-ef - 0\cdot 0)x + [0(-d) - fd]y + [d\cdot 0 - (-d)(-e)]z = 0, \text{ i.e.,}$$

$$efx + fdy + dez = 0.$$

$$\text{For line NL: } [(-e)(-f) - e\cdot 0]x + [0\cdot 0 - (-f)d]y + [de - 0(-e)]z = 0, \text{ i.e.,}$$

$$efx + fdy + dez = 0.$$

So NM and NL are the same line; L, M and N are collinear.

This proof is widely regarded as failing to explain why Desargues' theorem holds. For example, in contrasting the proof exiting to the third dimension with a proof using homogeneous coordinates, the mathematicians Robin McLeod and Louisa Baart (1998, 125) say that "synthetic proofs [such as the former] tend to give more insight than algebraic ones [such as the latter]".[9] The coordinate-geometry proof seems to depend on another 'algebraic miracle' at the end, where everything fortuitously turns out so nicely.

[9] I read McLeod and Baart as using "insight" here to refer to explanatory power, but admittedly, one might try to argue that they have something else in mind.

The coordinate-geometry proof is a perfect example of what mathematicians call a 'brute-force' approach. That is, it simply calculates everything directly, plugging in everything we know and then grinding out the result.[10] Mathematicians generally agree on whether or not a proof is aptly characterized as 'brute force', just as they do on whether or not a proof is explanatory. I suggest that no 'brute-force' proof is explanatory. A brute-force approach is not selective; it sets aside no features of the problem as irrelevant. Rather, it just "ploughs ahead" like a "bulldozer" (Atiyah 1988, 215), plugging everything in and calculating everything out. In contrast, an explanation must be selective; it must identify a particular feature of the set up as responsible for (and other features as failing to account for) the result being explained. The proof that proceeds by exiting to the third dimension identifies the key feature as the fact that the two coplanar triangles in perspective are the projections of triangles on different planes (which perforce intersect in a line).[11]

The proof using coordinate geometry proceeds directly from the essential features of the set up: that the two triangles are in perspective is encoded directly into the coordinates of the various points, and the rest is mere algebra. One might well have supposed that to explain why a given geometric theorem holds, it suffices to deduce the theorem directly from the 'natures' or 'essences' of the elements in the figure, just as the proof using coordinate geometry does (with the assistance of the 1:1 correspondence between real numbers and points on a line). This idea is the core of Mark Steiner's (1978a) account of the difference between mathematical proofs that explain and proofs that fail to explain what they prove. According to Steiner, a proof that all S_1's are P_1 (e.g., that all triangles in perspective have a given property) explains why the theorem holds if and only if it reveals how the theorem depends on

[10] Actually, matters are somewhat more complicated. Although the coordinate-geometry proof that I gave is plug-and-chug brute force, there are more elegant analytic arguments. (See, for instance, Lord 2013, 34–36 and Borceux 2014, 215–217.) Dieudonné (1985, 9) writes that "by the use of homogeneous coordinates accompanied by a harmonious choice of indexing notation" we can maintain "a symmetry and a clarity in the calculations so that they closely follow the geometric argument". Perhaps, then, a coordinate-geometry proof, despite its metric character, can allow a sufficiently proficient mathematician to see through the calculations to the non-metric, explanatory proof. (For a similar point, see note 28 of Lange (2014).) In that event, a coordinate-geometry proof would possess explanatory power for the same reason (to be elaborated in section 3) as the proof exiting to the third dimension.

[11] In the following passage, Ruelle (2007, 18) seems to be expressing the idea that a plug-and-chug coordinate-geometry proof is brute force and therefore is seen by at least some mathematicians as not explanatory. He describes one method of proof as follows: "Use brute force. In fact, for problems of elementary geometry one can always (as we shall see) introduce coordinates, write equations for the lines that occur, and reduce the problem to checking some algebra. This method is due to Descartes. It is effective but cumbersome. It is often long and inelegant and some mathematicians will say that it teaches you nothing: you don't get a real understanding of the problem you have solved". Mathematicians commonly say that a brute-force solution supplies "little understanding" and fails to show "what's going on" (e.g., Levi 2009, 29–30).

S_1's 'characterizing property'– that is, on the property essential to being S_1 that is just sufficient to distinguish S_1's from other entities S_2, S_3,… in the same 'family' (for example, to distinguish triangles in perspective from other kinds of triangles or from other kinds of polygons in perspective). Nevertheless, the proof using coordinate geometry is not respected as explaining why Desargues' theorem holds. Indeed, whereas the proof from coordinate geometry fails to explain despite merely unpacking the definitions of the elements in the theorem's set up, the explanatory proof invokes considerations exogenous to the defining features of those elements. It introduces a third dimension that is not mentioned by the two-dimensional theorem. Here we have a potentially important and perhaps surprising lesson about mathematical explanation (though, of course, this single case study may not suffice to fully demonstrate the point).

The great practical advantage of a brute-force proof (such as the coordinate-geometry proof of Desargues' theorem) is that the same plug-and-chug approach can be used to prove an enormously wide range of theorems (see note 11). We could, for instance, use the strategy of unpacking the set up in terms of coordinate geometry, and then working through the algebra, to prove not only Desargues' theorem, but also (say) that that the midpoints of any quadrilateral are the vertices of a parallelogram. A proof instantiating this argument scheme operates by brute force. Philip Kitcher (1984, esp. 208–209, 227; 1989, esp. 423–426, 437) proposes that arguments (whether in mathematics or science) are explanatory precisely when they instantiate argument schemes in the optimal collection ("the explanatory store") – optimal in that arguments instantiating these schemes manage to cover the most facts with the fewest different argument schemes placing the most stringent constraints upon arguments. An argument instantiating an argument scheme excluded from the explanatory store fails to explain. Not every brute-force proof instantiates the same scheme; there isn't a unique 'brute-force' scheme. But a given brute-force proof instantiates a very widely applicable scheme – a particular 'plug and chug' approach that can be used to prove a great many theorems. Any of the brute-force proof-schemes is likely to belong to Kitcher's "explanatory store". Nevertheless, proofs instantiating a brute-force scheme tend to lack explanatory power. The coordinate-geometry proof of Desargues' theorem is unilluminating because it begins by expressing the entire set up in terms of coordinate geometry and never goes on to privilege certain features of triangles in perspective (such as their being projections of triangles on different planes) as responsible for the theorem. The meat-grinder of a brute-force approach is not capable of drawing these distinctions.

But *why* is Desargues' theorem in two-dimensional Euclidean geometry *explained* by the fact that two coplanar triangles in perspective are projections of two triangles in perspective on different planes jutting into the third dimension? Why is the proof that exits to the third dimension explanatory – despite introducing auxiliary lines and, for that matter, an entire dimension extraneous to a statement of Desargues' theorem in two-dimensional Euclidean geometry?

3. Why Desargues' theorem in two-dimensional Euclidean geometry is explained by an exit to the third dimension

Desargues' theorem strikes us as remarkable because it identifies something common to the three points L, M and N – namely, that they lie on the same line. (Of course, any *two* points are collinear, but here we have *three* points on the same line.) This commonality impresses us; it prompts us to ask why the theorem holds. In view of the salience of this feature of Desargues' theorem, an explanation of the theorem must (I suggest) reveal something *else* given as common to these three points from which their collinearity follows. Now each of the three points is specified in the theorem's set up as the intersection of two lines that form corresponding sides of the two triangles in perspective. (For instance, L is where CB and C′B′ intersect.) So to reveal a feature common to each of the three points is to reveal a feature common to each of the three pairs of lines (CB and C′B′; BA and B′A′; AC and A′C′) joining corresponding vertices.

Of course, each *is* a pair of lines that form corresponding sides of two triangles in perspective; that is obviously a feature common to each of these pairs of lines. But this feature is just the set up of the theorem. To explain why the theorem holds, a proof must pick out from this set up some particular feature from which the result follows. As we have seen, a brute-force proof fails to do that. It fails to isolate any particular feature of the triangles in perspective as responsible for the theorem. Our task now is to understand the explanatory power of the feature isolated by the proof exiting to the third dimension, namely, that the two triangles in perspective are projections of triangles on different planes.

As I said, the striking feature of Desargues' theorem is that it reveals a property common to each of the three pairs of lines that form corresponding sides of two triangles in perspective. Hence (I have just suggested), in order for a proof to explain why Desargues' theorem holds, it must trace the result to some other property common to each of these pairs of lines. If a proof fails to trace the salient feature shared by these three pairs of lines to some other feature they share, then the proof treats it as *coincidental* that the three pairs of lines share the feature identified by Desargues' theorem. By contrast, a mathematical explanation of Desargues' theorem would show it to be no coincidence. Thus, Gray was on to something: there is a sense in which an explanation of Desargues' theorem conveys the result's 'inevitability'. In contrast, a proof that first deduces the equation of NM and then separately deduces the equation of NL, 'miraculously' finding them to be the same, portrays this fact as coincidental – though nevertheless mathematically necessary. (I will shortly return to this idea.)

In the proof that exits to the third dimension, each of the three pairs of lines forming corresponding sides of the triangles in perspective is the projection onto the original plane of a pair of lines, one line joining a pair of points on one plane jutting into the third dimension, and the other line joining another pair of points on another such plane. Crucially, *it is the same two planes for all three pairs*. By this proof, points L, M and N have a line in common because the three pairs of lines that give rise to them have two planes in common. That the proof traces the salient commonality to another commonality is, I suggest, the source of its explanatory power.

This proof explains Desargues' theorem only because a certain feature of the theorem strikes us as remarkable: its identification of a property common to each of the three points at which lines forming corresponding sides of the triangles intersect. (This commonality strikes us forcibly as soon as we understand the theorem, but it seems even more remarkable in light of the 'magical' way it emerges from the proof using Menelaus' theorem and the brute-force proof using coordinate geometry.) In this context, what it *means* to ask for an *explanation* over and above a proof of Desargues' theorem is to ask for a proof that exploits some other feature common to the three points (or, equivalently, to the three pairs of lines from which they arise). Outside such a context, there is no sense in which one proof is privileged over another as explanatory.

The calculator-number result has a different striking feature: that it identifies a property that is common to every one of the 16 'calculator numbers'. The proof we saw earlier that explains this theorem derives it by exploiting another feature that all of the calculator numbers have in common: that each can be expressed in the form $10^5a+10^4(a+d)+10^3(a+2d)+10^2(a+2d)+10(a+d)+a$. (Of course, other numbers besides the 'calculator numbers', such as 630036, also possess this feature – and hence they, too, are divisible by 37.) Another mathematical result may exhibit a remarkable symmetry, for example (see Lange 2014). In that event, a mathematical proof that traces the result to a similar symmetry in the problem would count as explaining why the result holds. I (Lange 2014) have proposed that what it means to ask for a mathematical proof that explains why some result holds is to ask for a proof that exploits a certain kind of feature in the 'given' – the same kind of feature that captured our attention in the result. There is no distinction between proofs that explain and proofs that merely prove except in a context where some feature of the result being proved is salient. That feature's salience in that context may make some proof(s) explanatory there.

This is the general conception of mathematical explanation that seems to me to be suggested by the explanation of Desargues' theorem. To elaborate and defend this conception further would require us to look at a variety of other mathematical explanations, diverting us from our focus on Desargues' theorem. (I pursue this project and elaborate this account of mathematical explanation in Lange (2014; forthcoming).)

Fortunately, I will not need to argue for this general conception of mathematical explanation in order to extract further lessons about explanation, existence and natural properties in mathematics from our case study of the explanation of Desargues' theorem.

Desargues' theorem in *two*-dimensional Euclidean geometry is explained by a proof that exits to the third dimension. This proof proceeds by first proving Desargues' theorem in *three*-dimensional Euclidean geometry – that is, as a theorem concerning two triangles that are *not* coplanar (but are in perspective from O). This explanation of Desargues' two-dimensional theorem is prompted by that theorem's having revealed a feature common to the three pairs of lines forming corresponding sides of two coplanar triangles in perspective (namely, that each pair's point of intersection is on the same line). Accordingly, I have suggested, the explanation works by tracing Desargues' theorem to another feature common to these three pairs of lines – but requiring three dimensions: that if the two triangles are *pulled out* of their original plane onto intersecting planes (as in Figure 2), then each pair of lines forming corresponding sides involves the same two planes – namely, the planes of the two triangles. These planes must meet along a line. The proof thus explains not only why Desargues' *two*-dimensional theorem holds, but also why Desargues' *three*-dimensional theorem holds.

We can now better appreciate why it is helpful to introduce the third dimension: because it supplies a second way to pick out the line LMN. If we stick to two dimensions, then the only way to pick out this line is as the line that runs through a given pair of points. (Through any two points, there is exactly one line.) But with the third dimension, we can also identify the line as where two given planes intersect. Those two planes, in turn, are picked out as those containing one or the other of the two triangles in perspective, once those triangles have been pulled out of their original plane. (Any plane is individuated by three non-collinear points on it.) The two planes, then, *unite* the three pairs of lines forming corresponding sides, since the same two planes are common to each pair. Without the third dimension, there is no such unity and so (considering the salient feature of Desargues' result) no explanation.

Thus, the third dimension is not actually artificial to Desargues' theorem in two-dimensional Euclidean geometry. Rather, because the third dimension provides an alternate means of picking out a line, it supplies the resources for specifying another feature common to points L, M and N. A proof that proceeds entirely in two dimensions can compare lines NM and NL only by computing their equations (whether by coordinate geometry, as we have seen, or by using vectors or in some other way) and concluding that they are identical. More broadly, a proof confined to two dimensions must use *metrical considerations* (such as the ratios in Menelaus' theorem). As we have seen, these considerations end up depriving the proof of explanatory power. In contrast, when the third dimension is introduced, Desargues' theorem regarding non-coplanar triangles in perspective follows entirely from axioms of incidence (that two points determine a line, three noncollinear points determine a plane, two intersecting planes determine a line, two intersecting lines determine a point, a line lies entirely in a given plane if two points on that line do), without any appeal to metrical considerations. The three-dimensional theorem's projection onto a plane yields the two-dimensional theorem. By appealing only to the axioms of incidence, the proof that exits to the third dimension avoids the 'algebraic coincidences' on which metrical proofs depend and explains why Desargues' theorem holds in two-dimensional Euclidean geometry.[12]

I have just now again invoked the notion of a mathematical 'coincidence'. Like mathematical explanation, mathematical coincidence is a puzzling feature of mathematical practice. How can a mathematical necessity nevertheless be coincidental? How, for instance, can a cancellation in the derivation of Desargues' theorem from Menelaus' theorem really be 'fortuitous'?

Some mathematical necessities are indeed commonly termed 'coincidental' – for example, as Davis (1981, 312) says, that 9 is both the thirteenth digit of the decimal representation of π (=3.14 159 265 358 979 3…) and the thirteenth digit of the decimal representation of e (=2.71 828 182 845 904 5…). To appreciate that the necessity of these mathematical facts (and hence of their conjunction) is compatible with their conjunction being coincidental, consider that some natural necessities are standardly characterized as coincidences. For instance, nineteenth-century chemists believed it naturally necessary that all noncyclic alkane hydrocarbons differ in molecular weight by integral multiples of 14 'atomic mass units' (amu) – since they differ by multiples of one carbon atom (12 amu) and two

[12] Without exiting to three dimensions, the axioms of incidence do not suffice to prove Desargues' theorem in two dimensions. There are non-Euclidean geometries where Desargues' theorem in two dimensions fails, but the axioms of incidence regarding two dimensions hold. See Baker (1922, 120) and Arana and Mancosu (2012). The latter presents a passage where "the eminent algebraist" Marshall Hall (1943) says that the explanatory contribution made by exiting to three dimensions is to supply another way of picking out line LMN – namely, as the intersection of two planes. Hall writes that "the kernel of the proof" exiting to three dimensions "lies in the identification of [LM, LN, and MN] with [the two planes' line of intersection] and hence with each other" (1943, 233). He says that these "forced identifications" explain "why" Desargues' theorem holds in three-dimensional projective geometry but not in two-dimensional projective geometry. (I shall say more about projective geometry in the following sections.) Hall (Ib, 232) elaborates: "One way of answering this fundamental question is the following: In a space configuration of three or more dimensions the identification of a line, constructed as the intersection of two planes, with a line, constructed as the union of two points forces the identification of further constructed elements and the establishment of non-trivial configurations such as the Desargues configuration, while in the plane a line may be constructed only as a union of two points, a point as the intersection of two lines and there are no forced identifications".

hydrogen atoms (1 amu each).[13] These chemists also believed it naturally necessary that the atomic weight of nitrogen is 14 amu. But they termed it 'coincidental' (albeit naturally necessary) that all noncyclic alkanes differ in molecular weight by integral multiples of the atomic weight of nitrogen. (See, for instance, van Spronsen 1969, 73–74.)

It is easy to see roughly what makes this fact coincidental: alkane hydrocarbons contain no nitrogen. In other words, that 14 amu is nitrogen's atomic weight does not explain why alkane hydrocarbons differ by integral multiples of 14 amu, nor does any explanation run in the other direction, nor is there a common explanation. Though each of these two facts is naturally necessary, the two are not united in explanation, and that is the sense in which it just happens to be the case that the same quantity, 14 amu, figures in both of them. In somewhat the same way, the components of a mathematical coincidence, though mathematically necessary, have no common explanation in that there is no single, unified proof that explains both. Roughly speaking, any proof of both is nothing but a cobbling together of separate proofs of each. The steps needed to prove one do not suffice for proving the other. For instance, there is no common explanation of the values of π and e.

Likewise, it is just a coincidence that these two Diophantine equations (that is, equations where the variables can take only integer values)

$$2x^2\left(x^2-1\right) = 3\left(y^2-1\right)$$

and

$$x(x-1)/2 = 2^n-1$$

have exactly the same five positive solutions (namely, $x = 1$, 2, 3, 6, and 91) (Guy 1988, 704). The two equations have nothing to do with each other. Of course, we could take separate procedures for solving the two equations and cobble them together into one proof. But the steps of the procedure for solving one equation could always be omitted without impeding the proof from solving the other equation.

In this light, let's reconsider the proof of Desargues' theorem using coordinate geometry. It shows that lines NM and NL are the same by ascertaining NM's equation, separately ascertaining NL's equation, and then finding these equations to be identical. This proof is similar to a proof regarding the above two Diophantine

[13] My use of "atomic mass units" here is anachronistic; nineteenth-century chemists would have denoted the same unit differently. Today, moreover, we realize that an element's atomic weight reflects the abundances that its various isotopes happen to have on earth and so is not a natural necessity. But nineteenth-century chemists believed atomic weights (like other characteristic features of the chemical elements) to be fixed by natural law. Despite being scientifically out of date, this example nicely illustrates how science leaves room for plenty of naturally necessary coincidences.

equations that proceeds by solving one equation, separately solving the other, and finally noting that the two sets of solutions are exactly the same. Likewise, in the derivation of Desargues' theorem from Menelaus' theorem, the cancellations (e.g., of BB′/OB′ by OB′/BB′) are genuinely fortuitous in that the appearance of a given term in one application of Menelaus' theorem, and of its reciprocal in a separate application of Menelaus' theorem, is a mathematical coincidence. They arise independently. (The same applies to the brute-force proof of the calculator-number theorem, where each of the 16 calculator numbers is proved separately to be divisible by 37.)

The proof of Desargues' two-dimensional theorem that exits to the third dimension reveals the collinearity of L, M and N to be no mathematical coincidence. Below we will see another example of a mathematical explanation of Desargues' theorem that reveals its target to be no coincidence. This will lead to a better understanding of mathematical coincidence.[14]

4. Desargues' theorem in projective geometry: unification and existence in mathematics

Here again is Desargues' theorem in two-dimensional Euclidean geometry:

> If two triangles are so situated that the three lines joining their corresponding vertices all meet at a single point, then the points of intersection of the two triangles' corresponding sides – if those intersection points exist – all lie on one line.

The qualification "if those intersection points exist" is included to acknowledge that there need not be three points of intersection (one for each pair of corresponding sides); if two corresponding sides are parallel, then in Euclidean geometry, they do not intersect. It may even be that *each* pair of corresponding sides consists of parallel lines, so there are *no* points of intersection at all (see Figure 4). Hence, in Euclidean geometry, we cannot refer to "the points of intersection" without

[14] Tappenden (personal communication) suggests that part of Gray's point in the passage I quoted in section 2 is that it is the ability to visualize the proof that is responsible for giving us a sense of the result's inevitability. I interpret Gray's point in terms of my view that 'understanding' the theorem (i.e., knowing its explanation, that is, why it is true) comes from recognizing that all three cases involve the same pair of planes (and how that leads to their all involving the same line), which seeing the figure greatly assists us in appreciating – but this does not mean that the proof's explanatory power derives primarily from its visualizability per se. More important is *what* is visualized: the common pair of planes and how it produces the common line. The figure is a 'powerful guide' to understanding the theorem because it helps us to appreciate how the common line comes from the common pair of planes.

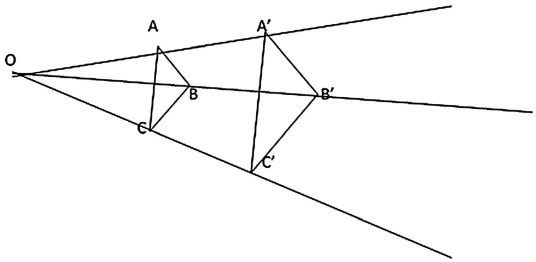

Figure 4. One arrangement of triangles in perspective.

giving the qualification "if they exist" (as given, for instance, by Whitehead 1971 [1907], 16; Stankova 2004, 173). Of course, if there are only two points of intersection, then they are necessarily collinear (since a line runs through any two points), and if there are none or exactly one, then collinearity among the points of intersection that exist is trivially achieved.[15] These are 'special cases' of Desargues' theorem; the above proofs of Desargues' two-dimensional theorem in Euclidean geometry cover only the non-trivial case: where there are three intersection points.

However, Desargues' theorem is usually understood as a theorem of *projective* rather than Euclidean geometry. In projective geometry, any two coplanar lines meet. Parallel lines meet at a point infinitely far away in the lines' direction (that is, at a single point, which is infinitely far away in *either* direction, one and the same point being reachable 'either way'). This 'point at infinity' is not located on the Euclidean plane (since for any two Euclidean points, there is some finite distance between them). All of the members of a set of mutually parallel lines have the same single point at infinity in common, and for each different orientation that coplanar parallel lines can take, there is a distinct point at infinity. All (and only) the points at infinity arising from lines on a given plane lie on a given 'line at infinity'. A 'projective plane' thus consists of a 'finite plane' (a.k.a. a Euclidean plane) plus a line at infinity.

One way to think about a point at infinity is as the point where endless railroad tracks on a Euclidean plane, viewed in perspective, are seen to meet (Figure 5). Admittedly, this way of thinking merely heightens our inclination to pose the question: 'Of course, we can choose to speak in terms of "points (and lines) at infinity", if we wish, but do they really exist? After all, railroad tracks never really meet, no matter how far they run. Do parallel lines really meet, or do they only "meet" inside inverted commas – that is, in the projective-geometry sense?'. But as soon as we ask this question, we begin to worry that it is ill-posed. After

[15] But there cannot be exactly one – as I will show in a moment.

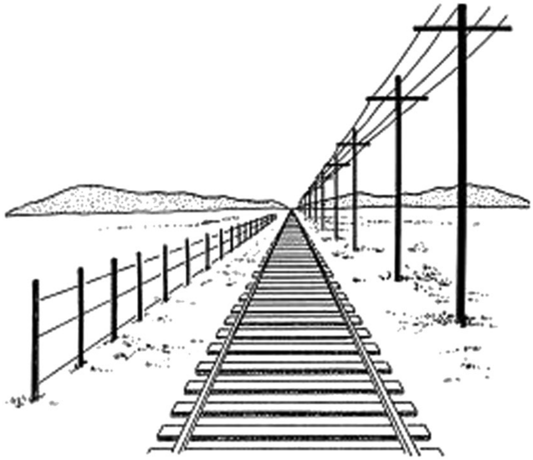

Figure 5. Railroad tracks viewed in perspective.

all, mathematics is not concerned with whether Euclidean points (never mind points at infinity) exist *physically*, and clearly (a familiar thought runs) they exist *mathematically* – in Euclidean geometry (as when a Euclidean geometer says 'There exists a point at which the diagonals of a square meet'). In the same internal, purely mathematical sense, then, 'points at infinity' exist in projective geometry.[16] When doing mathematics, we can choose to study Euclidean geometry or to study projective geometry without any fear that our selection might be erroneous, since both are true; the former accurately describes Euclidean planes and the latter accurately describes projective planes. Considering the straightforward way that we just introduced the notion of 'points at infinity', we might well doubt whether it could possibly make any difference (other than to our convenience) whether we choose to say 'Two coplanar parallel lines meet at a point at infinity' or 'Two coplanar parallel lines never meet', since anything expressed in one way can be translated into the other.

This last thought suggests that it is merely for the sake of simplicity or convenience that Desargues' theorem is generally expressed in terms of projective geometry rather than Euclidean geometry. Yet mathematicians generally say that

[16] Here is a typical statement of this familiar view: 'In the case of ordinary geometrical elements our intuition makes us feel at ease as far as their "existence" is concerned. But all we really need in geometry, considered as a mathematical system, is the validity of certain rules by means of which we can operate with these concepts, as in joining points, finding the intersection of lines, etc. … the mathematical existence of "points at infinity" will be assured as soon as we have stated in a clear and consistent manner the mathematical *properties* of these new entities, i.e., their relations to "ordinary" points and to each other' (Courant et al. 1996, 181).

projective geometry is where Desargues' theorem really resides. This does not sound like a mere matter of convenience. But then what does it mean?

Since any two coplanar lines intersect in projective geometry, Desargues' theorem in two-dimensional projective geometry does not need the qualification "if those intersection points exist". It says simply:

> If two triangles are so situated that the three lines joining their corresponding vertices all meet at a single point, then the three points of intersection of the two triangles' corresponding sides are collinear.

Let's see how Desargues' theorem in two-dimensional projective geometry applies to the case depicted in Figure 4, where each of the three pairs of corresponding sides consists of a pair of parallel lines, so none of the intersection points L, M or N exists on the Euclidean plane (so trivially in Euclidean geometry all of the points of intersection lie on one line, as required by Desargues' theorem in Euclidean geometry). The lines in each pair do intersect on the projective plane: at infinity. Since the three points of intersection are all on the same line at infinity, they are collinear, as demanded by Desargues' theorem in projective geometry.[17] In projective geometry, this is not a 'special case' of Desargues' theorem, requiring separate treatment. Rather, it is proved automatically by the proof that exits to the third dimension and explains why Desargues' theorem holds in projective geometry. Whether the intersection points are at infinity or on the Euclidean plane, there are three intersection points, and so the case is covered by the proof in projective geometry. The proof that exits to the third dimension proves Desargues' theorem in two-dimensional projective geometry without having to treat any cases separately; they get proved automatically along with the rest, not as special cases.

Now suppose that two of the three pairs of corresponding sides are pairs of parallel lines. As we will now see, it follows that the third pair must also consist of parallel lines, so none of L, M or N exists on the Euclidean plane. Here is a proof in Euclidean geometry (using the labels in Figure 4):

> Suppose lines BC and B′C′ are parallel and that lines AB and A′B′ are parallel. Show that lines AC and A′C′ are parallel:
> Consider triangles OBC and OB′C′. They have the same angle O. Since BC and B′C′ are parallel lines cut by the transversal OB′, angles B and B′ are corresponding angles, and so are equal. So two angles of triangle OBC are equal to two angles of triangle OB ′C′. Therefore, the triangles must be similar. For analogous reasons, triangles OAB and OA′B′ are similar. Similar triangles have all of their corresponding sides in the

[17] The theorem in projective geometry also allows O to be at infinity (so OA′, OB′ and OC′ are parallel) or some of the vertices (A, B, C, A′, B′, C′) of the two triangles to lie at infinity, whereas Euclidean geometry does not.

> same ratio. From the similarity of OAB and OA′B′, OA/OA′ = OB/OB′, and from the similarity of OBC and OB′C′, OB/OB′ = OC/OC′. Hence, OA/OA′ = OC/OC′. Hence, two sides of triangle OAC are in the same ratio as two sides of triangle OA′C′, and the included angle O is the same. Hence, OAC and OA′C′ are similar, so their corresponding angles A and A′ must be equal. That is, lines AC and A′C′ make equal angles with transversal OA′, so AC and A′C′ must be parallel.

As we saw earlier, Desargues' theorem in two-dimensional Euclidean geometry does *not* entail the result we just proved, i.e., that it is impossible for exactly one of the three pairs of corresponding sides to have a point of intersection on the Euclidean plane. As far as that theorem is concerned, it could be that exactly one of the three pairs of lines has an intersection point; in that case, it is trivial that all of the intersection points are collinear. This is a 'special case' of Desargues' theorem in Euclidean geometry, since (whether there is one intersection point or none) the theorem then holds trivially. By contrast, Desargues' theorem in two-dimensional *projective* geometry does entail this result. Suppose two of the three pairs of corresponding sides are pairs of parallel lines. Then in projective geometry, the lines in each of those pairs intersect at infinity and the unique line through those two intersection points is the line at infinity associated with the given plane. By Desargues' theorem in projective geometry, all three points of intersection must be collinear. The only way for the third point of intersection to lie on the same line as the other two intersection points – namely, on the line at infinity – is for the third point also to lie at infinity, and hence for the third pair of corresponding sides to consist of two parallel lines.

Thus, Desargues' theorem in projective geometry has an implication regarding the Euclidean plane that fails to follow at all from Desargues' theorem in Euclidean geometry. Furthermore, this additional implication is not arbitrarily tacked on to the rest of the theorem, requiring a separate proof from the rest. Rather, in proving Desargues' theorem in projective geometry by exiting to the third dimension, one does not treat this case separately. Cases where one or more of the points of intersection are at infinity are not 'special cases' of Desargues' theorem in projective geometry. All of this stands in stark contrast to Euclidean geometry: a proof of Desargues' theorem in Euclidean geometry does not thereby prove the further result we just discovered. Rather, a Euclidean proof of that result (which I just gave) is entirely separate from any proof of Desargues' theorem in Euclidean geometry.

We could, of course, supplement Desargues' theorem in Euclidean geometry so that the strengthened theorem does imply that it is impossible for exactly one of the three pairs of corresponding sides to have a point of intersection on the Euclidean plane. Here is the strengthened theorem:

> If two triangles are so situated that the three lines joining their corresponding vertices all meet at a single point, then the three points of intersection of the two triangles'

> corresponding sides – if those intersection points exist – all lie on one line, and if two pairs of corresponding sides are such that the two sides in each pair fail to intersect, then the two sides in the third pair also fail to intersect.

However, in Euclidean geometry, this strengthened theorem is merely two theorems cobbled together in that they have no common proof. Of course, we could cobble together explanations of each into a single proof. But it would be no common proof since some of the steps needed to prove one could then be weakened or omitted from the proof without compromising the derivation of the other – just like solutions to the two Diophantine equations in the previous section. In terms of the notion of a mathematical 'coincidence' that we encountered there, the strengthened theorem constitutes a mathematical coincidence according to Euclidean geometry. But Euclidean geometry is mistaken in so depicting it. Projective geometry reveals the theorem to be in fact no coincidence; the two components of the strengthened Euclidean theorem have a common proof in an explanation of Desargues' theorem in projective geometry.

It is not merely the case that Euclidean geometry and projective geometry *differ* in whether or not they treat the strengthened Desargues' theorem as a coincidence. Rather, Euclidean geometry is *mistaken*; projective geometry reveals that these various 'special cases' are actually no coincidence – in the same way as the complex numbers reveal certain results involving real numbers alone to be no coincidence (though they appear to be coincidental when considered in the context of the real numbers alone). For instance (as I discussed in Lange 2010, 329–332), it is no coincidence that the two Taylor series

$$1 / \left(1 - x^2\right) = 1 + x^2 + x^4 + x^6 + \ldots$$
$$1 / \left(1 + x^2\right) = 1 - x^2 + x^4 - x^6 + \ldots$$

are alike in that, for real x, each converges when $|x| < 1$ but diverges when $|x| > 1$. The reason why these functions both exhibit this convergence behavior is that both of them (considered as functions of complex numbers) are undefined for some z on the unit circle of the complex plane and because all of these cases can be treated together in a proof of the 'radius of convergence' theorem:

> For any power series $\sum a_n z^n$ (from $n = 0$ to ∞), either it converges for all complex numbers z, or it converges only for $z = 0$, or there is a number $R > 0$ such that it converges if $|z| < R$ and diverges if $|z| > R$.

Just as considerations from the real numbers alone mischaracterize as coincidental the common convergence behavior of the two Taylor series, so considerations from Euclidean geometry alone fail to recognize that the various components of the strengthened Desargues' theorem are no coincidence.

Desargues' theorem nicely illustrates the fact that a proof's explanatory power is distinct from its 'purity' in the rough sense of making use of no concepts foreign to the concepts in the theorem being proved (or in their definitions). Points at infinity do not figure in Desargues' theorem or its strengthened version above, so an appeal to them in proving that theorem is a violation of purity. Despite its foreign elements, the projective proof is explanatory. Explanatory power is just one of the many respects in which one proof can be better or worse than another proof of the same theorem; purity is another ideal, and beauty, brevity and accessibility are still others. Explanatory power is distinct from these other virtues.

There are further respects in which Desargues' theorem in projective geometry unifies what Euclidean geometry treats as special cases, thereby revealing various results depicted as coincidental by Euclidean geometry to be no coincidence at all. Consider the case where exactly two of the three pairs of corresponding sides are pairs of lines that intersect on the Euclidean plane. As we have seen, a proof of Desargues' theorem in Euclidean geometry must treat this as a special case; the proofs in section 2 all concern only the case where there exist three points of intersection. (For example, we cannot make all of the requisite fortuitous cancellations with only two applications of Menelaus' theorem.) To treat it as a special case is no problem, of course, since when there are exactly two points of intersection, their collinearity is trivial. In projective geometry, by contrast, points at infinity are just points like any other. The case of two intersection points on the finite plane, but one at infinity, requires no special treatment; an explanatory proof does not proceed by cases at all.

Furthermore, if exactly two of the intersection points lie on the Euclidean plane (i.e., exactly one of the three pairs of corresponding sides consists of two parallel lines), then the line through those two intersection points is parallel to the two parallel sides. (See Figure 6, where line NL is parallel to lines AC and A′C′.) Once again, this result in Euclidean geometry does not follow from Desargues' theorem in Euclidean geometry. It can be proved in Euclidean geometry, but this proof is

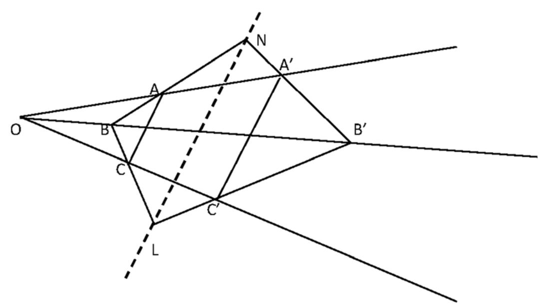

Figure 6. A special case of Desargues' theorem.

separate from a proof of Desargues' theorem. Once again, we could strengthen Desargues' theorem in Euclidean geometry by conjoining it with this result. But the strengthened theorem would have no common, unified explanation in Euclidean geometry. Its various cases would have to be proved separately. Accordingly, mathematicians commonly refer to the strengthened theorem in Euclidean geometry as a mere collection of special cases. (See, for example, Jones 1986, 556; Silvester 2001, 251; Gray 2007, 29.) In contrast, this result follows from Desargues' theorem in projective geometry by negligible additional steps and without having to be treated separately as concerning a special case: the intersection point M at infinity is collinear with the two intersection points L and N on the finite plane,[18] as demanded by Desargues' theorem in projective geometry, only if M lies at the intersection of line LN and the line at infinity, so lines LN, AC, and A′C′ meet at a point at infinity and therefore must be parallel. Thus, an explanation of Desargues' theorem in projective geometry shows it to be no coincidence that this result holds together with Desargues' theorem in Euclidean geometry.

The phenomenon I have just described is not a peculiarity of Desargues' theorem. Projective geometry characteristically unifies what Euclidean geometry treats as separate theorems and special cases. This unification has long been recognized as among projective geometry's great achievements (Chasles 1837, 75–76, 87; Dieudonné 1985, 8; Lord 2013, 11). As Descartes wrote to Desargues on 19 June 1639:

> Pour votre façon de considérer les lignes parallèles, comme si elles s'assemblaient à un but à distance infinie, afin de les comprendre sous le même genre que celles qui tendent à un point, elle est fort bonne … .[19] (Descartes 1639, 555).

Indeed, the search for such unified explanations – the suspicion that Euclidean geometry proves but fails to explain certain geometrical facts, that it incorrectly characterizes them as coincidences – was one of the original motivations for developing projective geometry. Consider Jean-Victor Poncelet, whose 1822 work made the first contribution to projective geometry after Desargues:

> The lesson Poncelet set about drawing … was that there should be a better way of reasoning geometrically, one that did not pursue the argument down a maze of bifurcating cases: one when there are four points, another when there are two, a third when there are none, a fourth when two points coincide; one when this segment is less than that one, another when it is greater … (Gray 2007, 46)

[18] Using the labels from Figure 6 – though, of course, M does not appear anywhere on that figure! M is the intersection of parallel lines AC and A′C′.

[19] As regards your way of considering parallel lines as if they met at a point infinitely distant, in order to include them in the same genus as those that go toward a point – it's very good …

We have here a nice example of the role that concepts such as mathematical explanation, unification and coincidence play in mathematical practice.

This role is not merely heuristic and pragmatic. Projective geometry is not merely a convenient way of proving theorems in Euclidean geometry – shortening the proofs, making them more efficient, allowing more to be proved at once. Rather, I have suggested that Euclidean geometry is *mistaken* in portraying certain results as coincidental and certain theorems as mere collections of special cases. Projective geometry reveals facts about Euclidean points, lines and planes that escape Euclidean geometry – not because they are too difficult to prove in Euclidean geometry, but because Euclidean geometry gets them wrong. These facts concern whether certain results in Euclidean geometry have a common, unified explanation. As Dieudonné (1985, 9) puts it: "the projective view exposes properties that appear accidental" to be otherwise.

That Euclidean geometry can (in this respect) be *incorrect* regarding Euclidean points, lines and planes runs contrary to the familiar thought I mentioned near the start of this section: that when doing mathematics, we can choose to study Euclidean geometry or to study projective geometry without any fear that our selection might be erroneous, since both are true; the former accurately describes Euclidean planes and the latter accurately describes projective planes. Although the theorems of Euclidean geometry are true of Euclidean points, lines and planes, Euclidean geometry taken more broadly (to include the proofs – and hence the explanations – that may be given of these theorems) may nevertheless mischaracterize those objects.

Earlier I also mentioned the familiar thought that to ask whether points at infinity *really* exist is to make a fundamental mistake. Trivially (this familiar thought runs), they exist in projective geometry but do not exist in Euclidean geometry. However, our case study of Desargues' theorem suggests that points at infinity are not a mere *façon de parler*. Rather, their features genuinely explain facts about Euclidean points, lines, and planes. In other words, points and lines at infinity exist in *Euclidean* geometry; mathematicians discovered that they do (via 'inference to the best mathematical explanation') by working in projective geometry.

My argument can be put roughly as follows:

P1: Certain facts about points at infinity explain certain facts about Euclidean points, lines and planes. (I have argued for this claim in the preceding sections.)

P2: What explains a fact about some entities must be on an ontological par with those entities. (Roughly: only facts about what exists can explain facts about what exists.)

C: Points at infinity exist in Euclidean geometry.

In arguing for P1, I have argued against the view that certain facts about Euclidean points, lines and planes have an explanation *in projective geometry* but have no explanation *in Euclidean geometry*. Rather, they have an explanation, period; whether they have an explanation is not relative to some mathematical field. Of course, their explanation requires the resources of projective geometry, but it does not follow that their explanation is relative to the geometry in question – any more than whether a given empirical fact has an explanation is relative to the scientific theory in question. For example, certain facts about the observed motions of the planets in Earth's night sky have an explanation, which is supplied by the Copernican theory of the heavens. The Ptolemaic theory says that these astronomical facts have no explanation. It is the case, of course, that these astronomical facts have an explanation according to the Copernican theory and no explanation according to the Ptolemaic theory. But what the two theories say is not all there is to the matter; whether these astronomical facts do indeed possess an explanation is not relative to the theory in question. Rather, it turns out that the astronomical facts have an explanation, as the Copernican theory correctly says; the Ptolemaic theory is mistaken in portraying them as brute. Likewise, Euclidean geometry is mistaken in some of what it says about the explanation of certain facts about Euclidean points, lines and planes. Perhaps what it is for points at infinity to exist in Euclidean geometry is for them to play an explanatory role there.

I conclude that it is not merely for the sake of simplicity or convenience that Desargues' theorem is generally expressed in terms of projective geometry rather than Euclidean geometry. As mathematicians say, projective geometry is where Desargues' theorem naturally belongs.[20]

[20] After completing this paper, I found that Wilson (1992) also argues that the points introduced by projective geometry are not mere conveniences, prettifying Euclidean theorems, nor is it the case that "any self consistent domain is equally worthy of mathematical investigation" (152). Rather, those theorems "cry out for the extended, projective setting" (153). Although Wilson says that nineteenth-century mathematicians compared appeal to ideal points to "physical explanations by appeal to unseen molecular structures" (151), Wilson does not offer an account of a theorem's 'proper setting', much less unpack it in terms of mathematical explanation. His concerns lie elsewhere. Wilson, in turn, notes that Manders (1987; 1989) has also argued that points at infinity "unify concepts, in a technical sense which covers widely cited advantages of simplification and clarity" (1989, 554). But Manders elaborates the conceptual unification and "more systematic understanding" (Ib., 561) provided by these posits not fundamentally in terms of mathematical explanation, but rather through model-theoretic notions such as "closure" and "completeness" (as when the addition of complex numbers allows all quadratic equations in one unknown to have solutions) that need not bring any increased explanatory power by my lights. Nevertheless, I agree with Manders that "we can have explicit epistemological grounds for commitment to those domains of entities by which certain prior domains of inquiry are made more understandable" (Ib., 562).

5. Desargues' theorem in projective geometry: explanation and natural properties in mathematics

I have suggested that in projective geometry, the proof exiting to the third dimension explains Desargues' theorem only because it exploits a feature common to the three points at which pairs of corresponding sides of the two triangles intersect. This feature gives the proof explanatory power, on my view, only because a certain feature of Desargues' theorem strikes us as remarkable: its identification of a property common to each of the three intersection points – or (equivalently) each of the three pairs of corresponding sides of the two triangles. The salience of this feature makes meaningful the why question demanding the theorem's explanation over and above its proof. But the content of Desargues' theorem in projective geometry strikes us in this way (as identifying a property common to each of the three intersection points) only if *we already recognize points at infinity as just like other points* – as all "le même genre" (as Descartes said above). Otherwise, Desargues' theorem in projective geometry consists of Desargues' theorem in Euclidean geometry plus various other theorems (two of which I gave in the previous section). So understood, the theorem is a motley collection of results. It does not identify a property common to each of the three points of intersection. Therefore, only in projective geometry does it make sense even to ask *why* these results (the various components of Desargues' theorem in projective geometry) all hold.

I have just presumed that a genuine resemblance among the three intersection points is distinguished from a difference among them – even though we could paper over that difference by using the same specious term to describe them all. I am thus invoking a familiar philosophical distinction between what David Armstrong (1978, 38–41) and David Lewis (1999, 10–13) call "natural" (i.e., 'sparse') properties – that is, respects in which things may genuinely resemble each other – on the one hand, and mere shadows of predicates (i.e., 'abundant' properties), on the other hand. Consider some of the properties figuring in projective geometry. For example, take the property of being a point, which is instantiated by both points on the finite plane and points at infinity. Or take the property of being a line, which is instantiated by both Euclidean lines and lines at infinity. These are genuine properties according to projective geometry. In fact, according to projective geometry, they mark off natural kinds. In projective geometry "all lines are 'created' equal, regardless of whether they are usual lines or the 'lines at infinity'" (Stankova 2004, 176; cf. Courant et al. 1996, 181). Explanatory proofs in projective geometry treat all lines in the same way; lines at infinity are not 'special cases'. For instance, in the proof explaining Desargues' theorem in projective geometry, we saw that the case where points L, M and N all lie on the line at infinity does not require special treatment. Rather, that case is treated together with the others as all constituting instances of collinearity.

But any of these properties that is natural, according to projective geometry, is instead a gerrymandered, artificial, wildly disjunctive, unnatural 'property' (all of these pejorative terms being roughly synonymous) according to Euclidean geometry – akin to Nelson Goodman's (1983) famous example of being "grue" (of being green and observed before the year 3000 [to update Goodman's example] or blue and unobserved before the year 3000). For instance, the projective property of being collinear, as applying to the three intersection points according to Desargues' theorem in projective geometry, is understood in Euclidean geometry as the property of being collinear, if the three intersection points are on the Euclidean plane, *or* of the third intersection point's 'existing at infinity' (i.e., the two corresponding sides being parallel), if the other two intersection points 'exist at infinity', *or*… (with each disjunct corresponding to a separate theorem in Euclidean geometry).

On my account, the proof exiting to the third dimension possesses the power to explain why Desargues' theorem in projective geometry holds partly by virtue of the proof's exploiting natural properties. Otherwise, the derivation would not constitute a common, unified proof of Desargues' theorem in projective geometry. Instead, its unity would be specious; it would be using disjunctive properties to cover (what Euclidean geometry portrays as) miscellaneous 'special cases'. After all, if we could help ourselves to disjunctive properties in our proofs, then we could always cobble together separate proofs into one, and a proof of one component of a coincidence would then contain no steps dispensable to proving the other component. The distinction I have tried to draw between common, unified proofs and the mere cobbling together of two unrelated proofs would vanish. Accordingly (since I have used this distinction to ground several others) no distinction would remain between the mathematical explanations we have seen and mere proofs, or between mathematical coincidences and joint results that are no coincidence. Whether the concept of a projective 'point' (covering both Euclidean points and points at infinity) picks out a natural class or is an artificial device for shortening proofs without genuinely unifying them makes a big difference to projective geometry's explanatory power.[21]

Thus, the why question regarding Desargues' theorem in projective geometry functions only in a context where the theorem is already appreciated as identifying

[21] In creating a specious unity, the concept of a projective 'point' would then be like the property *F* that applies to all and only things at worlds where a given deductive system of actual truths holds – thereby threatening to undermine Lewis's Best System Account of natural law by allowing a maximally simple formulation of a deductive system with maximal strength (Lewis 1999, 42). See also the bogus explanation created by disjunctive properties in the notorious footnote 33 in (Hempel 1965, 273).

something common to each of the three intersection points,[22] and the proof exiting to the third dimension succeeds in answering the why question only by virtue of exploiting another feature those points share. But here we seem to be caught in a vicious circle: the proof's explanatory power (indeed, the demand for an explanation) presupposes that certain properties are natural, but presumably, they are natural purely in virtue of their role in such explanations. What makes the points at infinity just more points is that they function in explanations no differently from points on the finite plane; explanatory proofs do not treat them as special cases. Two projective points behave in the same way for the same reasons, and this is what makes the projective points form a natural class.[23] A single term covering not only pairs of lines intersecting at Euclidean points, but also pairs of parallel lines (intersecting 'at infinity') could be stipulated within Euclidean geometry and used to present proofs more compactly. But only by discovering projective geometry's explanatory power do mathematicians discover that this term is not a mere *façon de parler*, but rather denotes a natural kind in mathematics.

[22] Thus, it is misleading to say simply that in Euclidean geometry, there is no explanation of the theorem that is 'Desargues' theorem in projective geometry', since this simple formulation suggests that the lack of any such explanation is felt in Euclidean geometry. Rather, in Euclidean geometry, we cannot even ask the why question that the explanation answers. Projective geometry is the natural habitat for Desargues' theorem in Euclidean geometry because only there does the theorem have a common, unifying explanation with various other Euclidean results (equivalent all together to Desargues' theorem in projective geometry), even though in Euclidean geometry, the lack of any such explanation is not felt; there it would not even make sense to ask for such an explanation.Here is a related point. I have understood a 'mathematical coincidence' to be two or more mathematical results with no common, unifying explanatory proof. In this sense, Desargues' theorem in projective geometry is portrayed as coincidental by Euclidean geometry because in Euclidean geometry, there is no such proof for Desargues' theorem in Euclidean geometry together with the additional content of the theorem in projective geometry. However, one might adopt a narrower understanding of 'mathematical coincidence' by also requiring, in order for two or more mathematical results to be a coincidence (or even to be no coincidence – the title of (Nummela 1987)), that the various mathematical results specify that various cases have something in common. On this reading, it is a mathematical coincidence that the two Diophantine equations given at the end of section 3 have the same positive solutions, but it is neither a mathematical coincidence nor no coincidence that there are five perfect solids and the two equations have the same solutions (even though the components of this result have no common, unifying explanatory proof). Euclidean geometry mistakenly depicts the components of Desargues' theorem in projective geometry as having no common, unifying explanatory proof – but does not mistakenly depict them as coincidental, on this narrower reading, since in Euclidean geometry, those results fail to show that various cases have something in common. Only if projective concepts denote natural properties do the results show that various cases have something in common. Therefore, on this narrower reading, only in projective geometry does it make sense even to ask whether this combination of results is coincidental or no coincidence.

[23] As indirect support for this idea, I argue in Lange (forthcoming, ch. 11) that roughly the same idea applies in science: many properties (such as having a given Reynolds number) become natural by virtue of their roles in certain scientific explanations.

How do mathematicians discover the explanatory power of some proof in projective geometry if they must already know that projective concepts denote natural properties, yet this knowledge, in turn, arises from their discovering projective geometry's explanatory power? Here we have the epistemic version of the ontological circularity I just mentioned. My suggestion is that (knowledge of) the naturalness of projective properties and (knowledge of) the explanatory power of proofs using those properties arise together; neither is prior to the other.

What makes a given proof in projective geometry explanatory is, in part, that it uses natural properties, and what makes those properties natural, in turn, is that they figure in other explanatory proofs. For each explanation, the existence of others secures the naturalness of the properties it uses. Of course, each of those others is beholden to others for the naturalness of the properties it uses. This holism does not involve vicious circularity because a given proof's status as an explanation *presupposes* the naturalness of the properties it exploits but does not also *make* those properties natural. An entire constellation of proofs that would be explanatory, were certain properties natural, is needed to make those properties natural. Insofar as those proofs are many and diverse (in that, e.g., one of these proofs is not contained in each one of the others, so we are not getting a multiplicity of proofs on the cheap), the properties qualify as natural and the proofs as explanatory.[24] This ontology is mirrored in epistemology. Mathematicians discover that the properties in a given family are natural by finding them in many, diverse proofs that (mathematicians recognize) would be explanatory, if those properties were natural.

A disjunctive property fails to figure in such a wide constellation of proofs. We could, of course, take two arbitrary theorems involving natural properties and make them into a single combined theorem by using disjunctive properties. We could likewise form a proof of the single combined theorem by taking a proof explaining why one of the original theorems holds and combining it with a proof explaining why the other holds – using further disjunctive properties to combine the first steps of the two proofs, the second steps, and so forth. The disjunctive

[24] By this "Insofar as …" formulation, I mean to recognize that a given mathematical property's naturalness and a given proof's explanatoriness can be matters of degree. (Lewis likewise regards naturalness as a matter of degree.) Also note that this view leaves room for a property to figure in an explanatory proof without its role there contributing at all toward making the property natural. The property must figure in the proof as a respect in which (if the property were natural) various things would be alike so as to enable the proof to explain by revealing how one (salient) similarity arises from another. Thus, the property cannot figure in the proof merely as the referent of a convenient notational device, for example. Note also that since an entire constellation of proofs is needed to render certain properties natural (and hence to make those proofs explanatory), the proof of Desargues' theorem that I have been examining is not enough by itself to make the property of being a projective point natural. (Nor, of course, have I aimed to give a full account of the mathematical significance of Desargues' theorem.)

properties figuring in the single combined theorem must appear in many other theorems: since the original two theorems involve natural properties, those properties figure in many other theorems as well, and so a given disjunctive property in the single combined theorem will appear in various combinations of these other theorems. However, although the disjunctive properties in the single combined theorem are guaranteed to appear in other theorems, no such guarantee applies to the other disjunctive properties in the steps of the combined proof of the single combined theorem. Rather, those properties are not apt to figure in proofs of any other theorems (apart from proofs of theorems logically related to the given theorem). The two original proofs have little in common; the disjunctive combinations created by the two proofs' combination are too idiosyncratic to be likely to arise in other proofs.

Now consider what happens when such seemingly disjunctive properties as 'Euclidean point or point at infinity' are used to combine various Euclidean theorems into Desargues' theorem in projective geometry.[25] We could take various proofs of these components individually (that exit to the third dimension) and combine them by using disjunctive properties. Lo and behold, the same few disjunctive properties (e.g., the properties in projective geometry of being a point, being a line, being a plane) arise in all of the combined steps – and exactly the same allegedly disjunctive properties appear in many other combinations of Euclidean theorems dealing with other allegedly special cases. Being collinear (whether the line is Euclidean plus a point at infinity or is a line at infinity), being coplanar, and other projective properties that would be ways of being alike, if these properties were natural, arise in many, otherwise diverse theorems and proofs that would be explanatory, if these properties were natural. Thus, the projective properties become natural. Epistemology mirrors this ontology: By discovering that the same few projective properties recur in all of these proofs, mathematicians discovered that they are natural and that these proofs explain those projective-geometry theorems.

In contrast, the unnatural properties needed to combine two arbitrary proofs are ad hoc. The combination of (e.g.) 'All triangles have interior angles adding to a straight angle' and 'All isosceles trapezoids have base angles that are congruent' involves the property of being a triangle or an isosceles trapezoid, for instance, which is guaranteed to figure in other theorems produced by similar combinations (since there are many other theorems concerning triangles and many others concerning isosceles trapezoids). But the combination of a step in a proof of the

[25] I say "*seemingly* disjunctive" because although the expression "being a Euclidean point *or* a point at infinity" appears to denote a genuinely disjunctive property, that appearance turns out to be deceptive; the property to which this expression refers was discovered to be mathematically natural. Whether a property is natural or disjunctive is not a matter of the syntax of the predicate picking it out.

triangle theorem with a step in a proof of the trapezoid theorem will involve properties such as being alternate interior angles where the transversal and one of the parallel lines are two sides of a triangle or being the foot of a perpendicular from a vertex to an isosceles trapezoid's base. There is no reason to expect this idiosyncratic property to recur in other such proofs – and if, remarkably, it does appear in some other proof, then the other disjunctive properties appearing there likely do not recur. Thus, it and those other properties are not natural and the proofs are not explanatory.

Among the few recent philosophical discussions of natural properties and kinds in mathematics are Tappenden (2008a; 2008b).[26] Both Tappenden and I take a property's naturalness as not determined by its contribution toward simplifying theorems or making proofs more efficient. Rather, Tappenden says, natural properties are "fruitful" and one kind of fruitfulness, perhaps more easily understood than other kinds, involves how a concept "contributes to addressing salient 'why?' questions" (2008a, 259). Though Tappenden offers no general account of mathematical explanation, I agree with him that explanatoriness and naturalness "interact in ways that make them hard to surgically separate" (2008a, 259). Discussing the function denoted by the Legendre symbol in number theory, he gives nice examples where unification is closely related to explanation.

Tappenden recognizes that unification and explanation are not achieved when gerrymandered properties are used. Therefore, Tappenden faces the task of specifying how the Legendre symbol creates genuine rather than spurious unification of what would otherwise constitute separate cases. Tappenden says that the unification results from the fact that the concept is "fruitful"; its use leads to lots of good mathematical theorems. But of course, unification is not supposed to be merely a heuristic matter and (as in the case of Desargues' theorem in projective geometry) many of these further theorems could be expressed (in more cumbersome ways) without the Legendre symbol. Moreover, if natural properties figure in lots of good mathematical results, then as I just pointed out, even an arbitrary disjunction of natural properties is guaranteed to figure in "grue"-some combinations of those results. (For example, Euclidean geometry might use the concept of 'points at infinity' to combine several theorems without purporting to unify them.) So an appeal to "fruitfulness" has got to go on to specify the particular kinds of fruitfulness that contribute to a mathematical property's naturalness. I agree with Tappenden that we should be guided here by the features that mathematicians themselves treat as significant in making a property natural (such as, he says, the way that the function expressed by the Legendre symbol turns out to be a

[26] Other discussions include Lakatos (1976) and Corfield (2003; 2005). I am grateful to Professor Tappenden for discussions of his views; of course, I am responsible for any remaining misunderstandings of them.

special case of a function central to the study of quadratic reciprocity, which itself turns out to connect to a wide variety of other mathematical domains). By getting a better grip on the kinds of fruitfulness that contribute to a mathematical property's naturalness, we hope to understand eventually how the fact that arbitrarily disjunctive properties figure in many mathematical results nevertheless fails to make those properties "fruitful" in a way that produces unification.

I have approached this issue by focusing on fruitfulness in connection with explanatory proofs. In particular, I have suggested that a given arbitrarily disjunctive property fails to belong to a family of properties any given member of which figures in proofs that would be explanatory, if the properties in the family were all natural, where these proofs are sufficiently numerous and diverse to make the given family member natural, if the proofs were all explanatory. But such service in explanatory proofs may well turn out to be only one of the possible contributors to a mathematical property's naturalness.

6. *Conclusion*

On the account I have sketched, certain mathematical properties are natural because they figure in proofs that (if these properties are natural) constitute common, unified explanations of various results that otherwise must describe a miscellany of special cases – where these proofs are of sufficient number and diversity to make these properties natural. Poincaré (1913, 375) nicely characterizes the unification achieved by mathematical proofs using natural properties and kinds, citing a host of examples including points at infinity:

> [M]athematics is the art of giving the same name to different things. … When the language has been well chosen, we are astonished to see that all the proofs made for a certain object apply immediately to many new objects; there is nothing to change, not even the words, since the names have become the same. A well-chosen word usually suffices to do away with the exceptions from which the rules stated in the old way suffer; this is why we have created negative quantities, imaginaries, points at infinity, and what not.

This is exactly what we found in the case of the proof explaining why Desargues' theorem holds in projective geometry; concerning Euclidean points and points at infinity, "the names have become the same". Poincaré regards these unifying proofs as explaining why various previous results had been so alike, his example here being groups and invariants:

> [These concepts] have made us see the essence of many mathematical reasonings; they have shown us in how many cases the old mathematicians considered groups without knowing it, and how, believing themselves far from one another, they

> suddenly found themselves near without knowing why. Today we should say that they had dealt with isomorphic groups. (1913, 375)

When we discover that the same natural property was instantiated in various, apparently disparate cases, we want to know why those cases turned out to be alike in so many ways. We may discover that it was no mathematical coincidence.

Our study of the explanatory contributions made by projective geometry allows us to appreciate one important role that explanation plays in mathematics. It has often been suggested (e.g., by Kitcher 2011) that mathematics consists of various interrelated 'games' of symbolic manipulation and that pure mathematicians have frequently extended their language and thereby begun to play new games that appeared to them to be worthwhile on purely mathematical grounds. A given game may be worth playing at least partly by virtue of its relations to other mathematical games that are independently worthwhile (and so mathematicians may take a given game to be worthwhile by virtue of its relation to other games that mathematicians already take to be worth playing). One of the features that can make a game mathematically worthwhile, I suggest, is its enabling mathematical explanations to be given (or demanded) that could never be given (or demanded) before. These new opportunities are especially worthwhile when the new explanations unify results that have already been arrived at (separately) in games that are already recognized as worthwhile. Projective geometry is a good example: part of what makes it worthwhile is that (as we have seen) it allows us to answer (and to ask) many why questions that could not be answered (or asked) in Euclidean geometry. It gives common, unified explanations in cases where Euclidean geometry gives none. Mathematical entities (such as points at infinity) are discovered when mathematical practices involving them are discovered to be worthwhile, and the explanations made possible by those practices often help to make them worthwhile. This is one of the most important roles played by explanations in mathematics.*

References

Arana, A. and Mancosu, P. 2012, "On the Relationship between Plane and Solid Geometry", *Review of Symbolic Logic*, **5**, pp. 294–353.

Armstrong, D. 1978, *Universals and Scientific Realism* (Volume 1), Cambridge: Cambridge University Press.

Atiyah, M. 1988, "How Research is Carried Out", in: M. Atiyah, ed., *Collected Works* (Volume 1), Oxford: Clarendon, pp. 211–216.

Baker, A. 2005, "Are There Genuine Mathematical Explanations of Physical Phenomena?", *Mind*, **114**, pp. 223–238.

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Baker, A. 2009, "Mathematical Accidents and the End of Explanation", in: O. Bueno and Ø. Linnebo, eds, *New Waves in the Philosophy of Mathematics*, Palgrave Macmillan: Basingstoke, pp. 137–159.

Baker, H. F. 1922, *Principles of Geometry* (Volume 1), Cambridge: Cambridge University Press.

Bell, A. W. 1976, "A Study of Pupils' Proof-Explanations in Mathematical Situations", *Educational Studies in Mathematics*, **7**, pp. 23–40.

Borceux, F. 2014, *An Algebraic Approach to Geometry*, Cham: Springer.

Byers, W. 2007, *How Mathematicians Think*, Princeton: Princeton University Press.

Cassirer, E. 1957, *The Philosophy of Symbolic Forms* (Volume 3), New Haven: Yale University Press.

Chasles, M. 1837, *Aperçu Historique sur l'Origine et le Developpement des Methods en Géométrie*, Paris: Gauthier-Villars.

Colyvan, M. 2001, *The Indispensability of Mathematics*, Oxford: Oxford University Press.

Colyvan, M. 2002, "Mathematics and Aesthetic Considerations in Science", *Mind*, **111**, pp. 69–78.

Corfield, D. 2003, *Towards a Philosophy of Real Mathematics*, Cambridge: Cambridge University Press.

Corfield, D. 2005, "Mathematical Kinds, or Being Kind to Mathematics", *Philosophica*, **74**, pp. 30–54.

Courant, R., Robbins, H. and Stewart, I. 1996, *What Is Mathematics?* (2nd edition), Oxford: Oxford University Press.

Davis, P. J. 1981, "Are There Coincidences in Mathematics?", *The American Mathematical Monthly*, **88**, pp. 311–320.

Descartes, R. 1639, Letter to Girard Desargues, *Œuvres Complètes de René Descartes*, Connaught Descartes Project, Past Masters database, online at: http://www.library.nlx.com/xtf/view?docId=descartes_fr/descartes_fr.00.xml;chunk.id=div.descartes.Correspondence.1619.pmpreface.pm.1;toc.depth=2;toc.id=div.descartes.Correspondence.1619.pmpreface.pm.1;hit.rank=0;brand=default (accessed 24 July 2009), Volume 1: Correspondence: 1619–1650, Correspondence 1639, II, pp. 554–556.

Dieudonné, J. 1985, *History of Algebraic Geometry*, Monterey, CA: Wadsworth.

Field, J. V. and Gray, J. J. 1987, *The Geometrical Work of Girard Desargues*, New York: Springer-Verlag.

Goodman, N. 1983, *Fact, Fiction, and Forecast* (4th edition), Cambridge: Harvard University Press.

Gray, J. 2007, *Worlds Out of Nothing*, London: Springer-Verlag.

Guy, R. K. 1988, "The Strong Law of Large Numbers", *The American Mathematical Monthly*, **95**, pp. 697–712.

Hafner, J. and Mancosu, P. 2005, "The Varieties of Mathematical Explanation", in: P. Mancosu, K.F. Jørgensen and S. A. Pedersen, eds, *Visualization, Explanation and Reasoning Styles in Mathematics*, Dordrecht: Springer, pp. 215–250.

Hall, M. 1943, "Projective Planes", *Transactions of the American Mathematical Society*, **54**, pp. 229–277.

Healy, L. and Hoyles, C. 2000, "A Study of Proof Conceptions in Algebra", *Journal for Research in Mathematics Education*, **31**, pp. 396–428.

Hempel, C. G. 1965, "Studies in the Logic of Explanation", in: C. G. Hempel, ed., *Aspects of Scientific Explanation*, New York: Free Press, pp. 245–290.

Hempel, C. G. 2001, "Explanation and Prediction by Covering Laws", in: J. Fetzer, ed., *The Philosophy of Carl G. Hempel*, New York: Oxford University Press, pp. 69–86.

Hilbert, D. and Cohn-Vóssen, S. 1952, *Geometry and the Imagination*, New York: Chelsea.

Jones, A. 1986, *Pappus of Alexandria, Book 7 of the* Collection, *Part 2*, New York: Springer-Verlag.

Kitcher, P. 1984, *The Nature of Mathematical Knowledge*, Oxford: Oxford University Press.

Kitcher, P. 1989, "Explanatory Unification and the Causal Structure of the World", in: P. Kitcher and W. C. Salmon, eds, *Scientific Explanation* (Minnesota Studies in the Philosophy of Science, Volume 13), Minneapolis: University of Minnesota Press, pp. 410–505.

Kitcher, P. 2011, "Epistemology Without History is Blind", *Erkenntnis*, **75**, pp. 505–524.

Lakatos, I. 1976, in: J. Worrall and E. Zahar, eds, *Proofs and Refutations*, Cambridge: Cambridge University Press.

Lange, M. 2010, "What are Mathematical Coincidences (and Why Does It Matter)?", *Mind*, **119**, pp. 307–340.

Lange, M. 2013, "What Makes a Scientific Explanation Distinctively Mathematical?", *British Journal for the Philosophy of Science*, **64**, pp. 485–511.
Lange, M. 2014, "Aspects of Mathematical Explanation: Symmetry, Unity, and Salience", *Philosophical Review*, **123**, pp. 485–531.
Lange, M. forthcoming, *Because Without Cause*, New York: Oxford University Press.
Levi, M. 2009, *The Mathematical Mechanic*, Princeton: Princeton University Press.
Lewis, D. 1999, "New Work for a Theory of Universals", in: D. Lewis, ed., *Papers in Metaphysics and Epistemology*, Cambridge: Cambridge University Press, pp. 8–55.
Lord, E. 2013, *Symmetry and Pattern in Projective Geometry*, London: Springer.
Mancosu, P. 2008, "Mathematical Explanation: Why It Matters", in: P. Mancosu, ed., *The Philosophy of Mathematical Practice*, Oxford: Oxford University Press, pp. 134–150.
Manders, K. 1987, *"Logic and Conceptual Relationships in Mathematics", in: Logic Colloquium '85* (Studies in Logic and the Foundations of Mathematics, Volume 122), Amsterdam: Elsevier, pp. 193–211.
Manders, K. 1989, "Domain Extension and the Philosophy of Mathematics", *Journal of Philosophy*, **86**, pp. 553–62.
McLeod, R. J. Y. and Baart, M. L. 1998, *Geometry and Interpolation of Curves and Surfaces*, Cambridge: Cambridge University Press.
Melia, J. 2002, "Response to Colyvan", *Mind*, **111**, pp. 75–79.
Mudaly, V. and de Villiers, M. 2000, "Learners' Needs for Conviction and Explanation Within the Context of Dynamic Geometry"", *Pythagoras*, **52**, August, pp. 20–23.
Nerlich, G. 1979, "What Can Geometry Explain?", *British Journal for the Philosophy of Science*, **30**, pp. 69–83.
Nicole, P. and Arnauld, A. 1717, *Logic; or the Art of Thinking*, London: William Taylor.
Nummela, E. 1987, "No Coincidence", *The Mathematical Gazette*, **71**, 456, pp. 147.
Poincaré, H. 1913, "The Future of Mathematics", in: H. Poincaré, ed., *Science and Method*, *The Foundations of Science*, trans. George Bruce Halsted, New York: Science Press, pp. 369–383.
Resnik, M. and Kushner, D. 1987, "Explanation, Independence and Realism in Mathematics", *British Journal for the Philosophy of Science*, **38**, pp. 141–158.
Ruelle, D. 2007, *The Mathematician's Brain*, Princeton: Princeton University Press.
Sawyer, W. W. 1955, *Prelude to Mathematics*, Harmondsworth: Penguin.
Silvester, J. R. 2001, *Geometry: Ancient and Modern*, Oxford: Oxford University Press.
Stankova, Z. 2004, "Geometric Puzzles and Constructions – Six Classical Geometry Theorems", in: D. F. Hayes and T. Shubin, (eds), *Mathematical Adventures for Students and Amateurs*, Washington, DC: Mathematical Association of America, pp. 169–184.
Steiner, M. 1978a, "Mathematical Explanation", *Philosophical Studies*, **34**, pp. 135–51.
Steiner, M. 1978b, "Mathematics, Explanation, and Scientific Knowledge", *Noûs*, **12**, pp. 17–28.
Tappenden, J. 2005, "Proof Style and Understanding in Mathematics I: Visualization, Unification, and Axiom Choice", in: P. Mancosu, K. F. Jørgensen and S. A. Pedersen, eds, *Visualization, Explanation, and Reasoning Styles in Mathematics*, Dordrecht: Springer, pp. 147–214.
Tappenden, J. 2008a, "Mathematical Concepts and Definitions", in: P. Mancosu, ed., *The Philosophy of Mathematical Practice*, Oxford: Oxford University Press, pp. 256–275.
Tappenden, J. 2008b, "Mathematical Concepts: Fruitfulness and Naturalness", in: P. Mancosu, ed., *The Philosophy of Mathematical Practice*, Oxford: Oxford University Press, pp. 276–301.
van Spronsen, J. W. 1969, *The Periodic System of Chemical Elements*, Amsterdam: Elsevier.
Whitehead, A. N. 1971 [1907], *The Axioms of Descriptive Geometry*, repr. New York: Hafner.
Wilson, M. 1992, "Frege: The Royal Road From Geometry", *Noûs*, **26**, pp. 149–180.