Polya Enumeration Theorem

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Cosets

- A **left** coset of H in G is gH where $g \in G$ (H is on the **right**).
- A **right** coset of H in G is Hg where $g \in G$ (H is on the **left**).

Theorem

If two left cosets of H in G intersect, then they coincide, and similarly for right cosets. Thus, G is a disjoint union of left cosets of H and also a disjoint union of right cosets of H.

Corollary(Lagrange's theorem) If G is a finite group and H is a subgroup of G, then the order of H divides the order of G. In particular, the order of every element of G divides the order of G.

Applications of Lagrange's Theorem

Theorem

For any integers $n \geq 0$ and $0 \leq m \leq n$, the number $\frac{n!}{m!(n-m)!}$ is an integer.

Theorem

For any positive integers a, b the ratios $\frac{(ab)!}{(a!)^b}$ and $\frac{(ab)!}{(a!)^b b!}$ are integers.

Theorem

For an integer $m > 1$ let $\varphi(m)$ be the number of invertible numbers modulo m. For $m \geq 3$ the number $\varphi(m)$ is even.

Polya's Enumeration Theorem

Theorem

Suppose that a finite group G acts on a finite set X. Then the number of colorings of X in n colors inequivalent under the action of G is

$$N(n) = \frac{1}{|G|} \sum_{g \in G} n^{c(g)}$$

where $c(g)$ is the number of cycles of g as a permutation of X.

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- There are 2 reflections with 2 cycles, and 2 reflections with 3 cycles, with contribute $2 \cdot 2^2 + 2 \cdot 2^3 = 24$.

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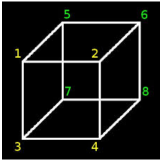
- For n colors, $N(n) = \frac{n^4+2n^3+3n^2+2n}{8}$. For example, $N(4) = 55$.

One more example



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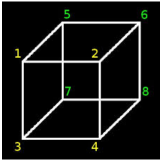

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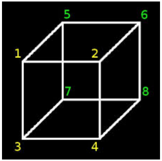

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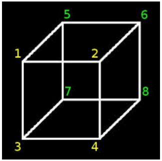

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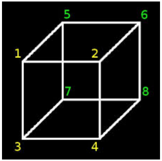

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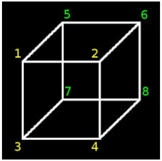

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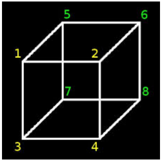

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- Summing up, $N(n) = \frac{n^6+3n^4+12n^3+8n^2}{24}$.

Cosets and orbits

Let G act on a set X, pick a point $x \in X$ and let Gx and G_x be its orbit and stabilizer.

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- Since $y = gx$ for some $g \in G$, then $G_y = gG_xg^{-1}$. Check that it indeed works!
- Note: the formula $g.h := ghg^{-1}$ actually defines an action of G on itself. This action is called **conjugation**. Moreover, for each g, $f_g : G \to G$ defined by $f_g(h) = ghg^{-1}$ is an isomorphism!

Proof of Polya's Theorem

Theorem

The number of colorings of X in n colors inequivalent under the action of G is

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- Let's instead count the pairs (g, C) with $C \in X_n$ a coloring and $g \in G_C \subset G$ an element of G preserving C.

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- The orbit GC of C has $|G|/|G_C|$ elements (**used Lemma 1**).
- Each element of GC will appear $|G_C|$ times in our counting (**used Lemma 2**).
- Thus each orbit of X_n will appear $|G_C| \cdot |G|/|G_C| = |G|$ many times in our counting. So to find $N(n)$ we will have to divide the result of

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- Summing over all $g \in G$ and dividing by $|G|$ gives the required formula.

Weighted Polya theorem

Let $c_m(g)$ denote the number of cycles of length m in $g \in G$ when permuting a finite set X.

Theorem (Weighted Polya theorem)

The number of colorings of X into n colors with exactly r_i occurrences of the i-th color is the coefficient of $t_1^{r_1} \dots t_n^{r_n}$ in the polynomial

$$P(t_1, \dots, t_n) = \frac{1}{|G|} \sum_{g \in G} \prod_{m \geq 1} (t_1^m + \dots + t_n^m)^{c_m(g)}$$

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- The previous formula is obtained by putting $t_1 = \dots = t_n = 1$.
- What is the number of necklaces with exactly 3 white and 3 black beads?