

The Spatial Model of Politics

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November 21, 2007

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Foreword

The last half century has witnessed the confluence of several intellectual tributaries in formal political theory. The earliest of these, dating from the second decade of the twentieth century and crowned by a magnum opus in 1944, was cooperative game theory. Von Neumann and Morgenstern's *The Theory of Games and Economic Behavior* (Princeton, 1944) is arguably the most significant single achievement in social science theory in the last century. And Luce and Raiffa's *Games and Decisions* (Wiley, 1957) served as a seminal textbook, popularizing many of the tools developed there. Commencing not too long after was social choice theory, which may be dated with the appearance of Kenneth Arrow's *Social Choice and Individual Values* (Wiley, 1951), though it was preceded in the 1940s by several of his own papers as well as those by Duncan Black (and several centuries earlier by the work of Condorcet and Borda). Spatial models, the third tributary, also is associated with work early in the twentieth century, principally that of Harold Hotelling. But its real relevance for political science emerged in the 1950s with Anthony Downs's *An Economic Theory of Democracy* (Harper and Row, 1957) and Duncan Black's *Theory of Committees and Elections* (Cambridge, 1958). A fourth tributary is non-cooperative game theory, developed as an offshoot of the earlier work by von Neumann and Morgenstern in seminal papers by John Nash, but reached its zenith in the late 1970s as it took over field after field in economics (for overviews, see especially James Friedman, *Oligopoly and the Theory of Games* (North-Holland, 1977), Drew Fudenberg and Jean Tirole, *Game Theory* (MIT, 1991), and David Kreps, *A Course in Microeconomic Theory* (Princeton, 1990)). A fifth tributary, another offshoot of cooperative game theory, was coalition theory, developed most prominently in political science by William Riker, *The Theory of Political Coalitions* (Yale, 1962). Needless to say, there were second-order intellectual descendants, but these are the principal branches.

For quite some time each of these lived an independent existence,

their fruits harvested by separate communities of scholars with only the occasional cross-over (as in William Riker and Peter Ordeshook, *An Introduction to Positive Political Theory* (Prentice-Hall, 1972)). One of the leading innovators, drawing from (and making important contributions to) all of these traditions, is Norman Schofield. The present volume is something of a grand synthesis. Its substantive focus is elections with its electoral deals, party activists, and voters on the one hand, and parliaments with their coalition-building and government-forming maneuverings on the other. In terms of tools, the arguments found in this volume draw heavily on social choice theory, the spatial model, and cooperative game theory. Indeed, the first five chapters constitute just about all one needs to know of social choice theory.

But the core (pun intended) of this fine volume is found in four very rich applied chapters, constituting a profound synthesis of elections and parliaments – of voters and activists choosing political agents, agents in turn choosing governments, and governments governing. Along the way we learn about voting rules, electoral systems, the ecology of government coalitions, precipitating events, and quite a lot about the empirical condition of electorates, parliaments, and legislatures in the advanced industrial democracies of the West. (The intensive treatment of an incredibly complex coalitional situation found in Israel over the last two decades is highly instructive.)

I want to single out two especially imaginative treatments found in the later chapters of this volume – imaginative both theoretically and empirically. First, Schofield provides one of the most elaborated theoretically grounded typologies of coalitional arrangements in parliamentary democracies (in a literature rich in typologies based mainly, even exclusively, on empirical patterns alone) which, in turn, provides insights into the frequent absence of centripetal forces in multiparty regimes. Second, Schofield elaborates an analysis of electoral activists that goes farther, and is founded on a more firm theoretical basis, than anything that presently exists in the literature. Taking a highly original turn, Schofield applies the logic of Duverger to interest groups, suggesting how the electoral rule (plurality vs. proportional representation) will affect activist coalition building. Application to the building of electoral coalitions in Argentina and the United States is quite provocative.

Each of these will be of separate interest to research communities. To-

gether, however, they provide the underpinnings for a net assessment of the effects of the centripetal pull of the voting electorate and the centrifugal impact of activists. The reader may have to burn a lot of intellectual energy to get to these points, but getting there not only is half the fun but also makes strikingly evident that Schofield's large theoretical buildup is not merely an occasion for play in the theory sandbox.

Norman Schofield has the soul of a mathematician and the heart of a political scientist. He has, over a long career but especially in the present volume, combined these two impulses elegantly. In one sense this book is a finished product. In another it is but the beginning of a conversation.

Kenneth A. Shepsle
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September 2007

Preface

William H. Riker's book, *The Theory of Political Coalitions* (1962) attempted to answer the question why plurality rule in the United States electoral system seemed to be the cause of both minimal winning coalitions and the two-party system. His later book, *Positive Political Theory* (1973) with Peter Ordeshook, presented the theory on two-party elections, suggesting that parties would be compelled to compete over the electoral center. In fact, there is no empirical evidence for such convergence either in the U.S. polity or in those European polities whose electoral systems are based on proportional representation.

This volume is, in a sense, the fourth part of a tetralogy aimed at providing a theory why political convergence does not occur.

The first part of the tetralogy was the book, *Multiparty Government* (1990) with Michael Laver. The key idea of this book was to use the social choice notion of the *core* to explain the occurrence of minority governments in multiparty polities based on proportional representation. Social choice theory suggests that a large and centrally located party could dominate politics, and possibly create a minority government. This possibility seemed to provide a reason why minimal winning coalition governments were not the norm in such multiparty systems.

Although the idea of the core seemed very attractive, it could only be used to account for government formation in very specific situations in the post-election period when party positions and seat strengths were determined. There was no theory to account for the nature of coalition government in the absence of a core. Secondly, no theory was provided of the pre-election choice of party positions and the electoral mechanism by which seats were allocated to the parties.

The second volume on *Multiparty Democracy* (Schofield and Sened, 2006) presented a formal theory of elections, together with empirical analyses of elections in Britain, Israel, Italy, the Netherlands and the United States. The key idea of that book was that elections involve judgements by the electorate of the quality, or *valence* of the candidates,

or party leaders. In elections with multiple parties, the theory suggests that low valence parties will be compelled to vacate the electoral center in order to garnish votes. Moreover, it is possible that parties can use the resources provided by activists to enhance the electoral perception of the party leaders.

The third volume, *Architects of Political Change* (Schofield, 2006a), addressed a number of the historical questions raised by Riker in his *Liberalism against Populism* (1982) and *The Art of Manipulation* (1986), using as a conceptual basis the formal electoral model presented in *Multiparty Democracy*. The key idea was that if elections depend on electoral judgements, then the bases of these judgements may be transformed at the time of crucial *social quandaries*. The book considered a number of events in U.S. political history, including the decision to declare independence in 1776, the choice over the ratification of the Constitution in 1787, the elections of Jefferson in 1800 and of Lincoln in 1860, and finally the passing of civil rights legislation in 1964.

The current volume is intended as an exposition first of social choice theory, leading to the idea of the *core* and the *heart*. The notion of the heart is intended to apply to a committee, or legislature, when positions and strengths are known. It is a domain in the policy space, bounded by the compromise sets of various winning coalitions. The technical theory of social choice is presented in Chapters 2 to 5 of this book, and the application of the theory to multiparty coalitions is given in Chapter 6. To account for the positioning and power of parties, Chapter 7 presents a general model of elections, using the idea of activist support. It is suggested that the heterogeneity of activist groups in polities based on proportional representation accounts for the great variety of political configurations in these polities. Chapter 8 applies this theory to show that it can lead to a significant transformation of the winning party position, particularly in a polity whose electoral system is based on plurality rule.

Finally, the theory is applied in Chapter 9 to recent elections in the United States. It is argued that party positions are based on both economic and social dimensions, and that the changing power of activist coalitions in this two-dimensional domain has brought about a certain confusion of policy goals within the two-party structure of the United States.

Chapter 10 draws together the various threads of the book, by quoting

from Madison's dual theory of the Republic. On the one hand, Madison saw the President as a natural way to prevent mutability or disorder in the legislature, while the ability of Congress to veto presidential risk-taking could prevent autocracy. At the same time, the election of the president would, in the extended republic, enhance the "probability of a fit choice." Madison's argument is interpreted in the light of the model of elections presented in the previous chapters.

A number of chapters of this book use some figures and tables from previous work. Cambridge University Press kindly gave permission to use material from Schofield (2006a), Schofield and Sened (2006) and Miller and Schofield (2008). I am grateful to Blackwell for permission to use material from Schofield (1993) and Schofield and Miller (2007), to Sage for permission to use material from Schofield (1995), to Springer to use material from Schofield (1996) and Schofield (2006b) and to North Holland to use material from Schofield and Cataife (2007).

I received very helpful comments on the versions of the last four chapters of the book, presented at various conferences and seminars: the World Public Choice Meeting, Amsterdam, the ISNIE conference, Boulder, the conference on European Governance, Emory University, conferences on political economy in Cancún and Guanajuato, Mexico, the Meeting of the Society for the Advancement of Economic Theory, Vigo, Spain, the Conference on Modernization of the State and the Economy, Moscow, and other conferences at the University of Virginia, Charlottesville, at SUNY, Binghamton, and at the University of Hamburg. Versions of some of this work were presented at the Department of Economics, Concordia University, Montreal, at the Center for Mathematical and Statistical Modeling, Wilfrid Laurier University, Waterloo, at the Center for Mathematical Modeling, University of California at Irvine, at the Higher School of Economics, St. Petersburg, at George Mason University, and at the California Institute of Technology.

The original versions of most of the chapters were typed by Cherie Moore, and many of the diagrams were drawn by Ugur Ozdemir. I am grateful to Cheryl Eavey, Joseph Godfrey, Eric Linhart, Evan Schnidman and Suumu Shikano for permission to make use of their work. I thank my coauthors, Guido Cataife, Gary Miller and Ugur Ozdemir for their collaboration. Ken Benoit and Michael Laver graciously gave permission for me to use their estimates of party position for many of the European

polities (Benoit and Laver, 2006).

I appreciate the support of the NSF (under grants SES 0241732 and 0715929), and of Washington University. The Weidenbaum Center at Washington University provided support during a visit at the International Center for Economic Research in Turin. I thank Enrico Colombatto and Alessandra Calosso for the hospitality I enjoyed at ICER. A year spent at Humboldt University, Berlin, under the auspices of the Fulbright Foundation, as distinguished professor of American Studies during 2002–2003, gave me the opportunity to formulate an earlier version of the formal electoral model. Finally, my thanks to Terry Clague and Robert Langham, the editors at Routledge, for their willingness to wait for a number of years while the manuscript was in preparation.

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4 November 2007

Chapter 1

Introduction

1.1 Representative Democracy

A fundamental question that may be asked about a political, economic or social system is whether it is responsive to the wishes or opinions of the members of the society and, if so, whether it can aggregate the conflicting notions of these individuals in a way which is somehow rational. More particularly, is it the case, for the kind of configuration of preferences that one might expect, that the underlying decision process gives rise to a set of outcomes which is natural and stable, and more importantly, “small” with respect to the set of all possible outcomes? If so, then it may be possible to develop a theoretical or “causal” account of the relationship between the nature of the decision process, along with the pattern of preferences, and the behavior of the social and political system. For example, microeconomic theory is concerned with the analysis of a method of preference aggregation through the market. Under certain conditions this results in a particular distribution of prices for commodities and labor, and thus income. The motivation for this endeavor is to match the ability of some disciplines in natural science to develop causal models, tying initial conditions of the physical system to a small set of predicted outcomes. The theory of democracy is to a large extent based on the assumption that the initial conditions of the political system are causally related to the essential properties of the system. That is to say it is assumed that the interaction of cross-cutting interest groups in a democracy leads to an “equilibrium” outcome that is natural in the sense of balancing the divergent interests of the members of

the society. One aspect of course of this theoretical assumption is that it provides a method of legitimating the consequences of political decision making.

The present work directs attention to those conditions under which this assumption may be regarded as reasonable. For the purposes of analysis it is assumed that individuals may be represented in a formal fashion by preferences which are “rational” in some sense. The political system in turn is represented by a social choice mechanism, such as, for example, a voting rule. The purpose is to determine whether such a formal political system is likely to exhibit an equilibrium. It turns out that a stable social equilibrium in a pure (or *direct*) democracy is a rare phenomenon. This seems to suggest that if the political system is in fact in equilibrium, then it is due to the nature of the method of representation.

As Madison argued in *Federalist X*,

[I]t may be concluded that a pure democracy . . . can admit of no cure for the mischiefs of faction Hence it is that such democracies have ever been spectacles of turbulence and contention; have ever been found incompatible with personal security. . . and have in general been as short in their lives, as they have been violent in their deaths. . . .

A republic, by which I mean a government in which the scheme of representation takes place, opens a different prospect . . .

[I]f the proportion of fit characters be not less in the large than in the small republic, the former will present a greater option, and consequently a greater probability of a fit choice.¹

Social choice is a theory of direct or pure democracy which seeks to understand the connection between individual preferences, institutional rules and outcomes. The theory suggests that Madison’s intuition was largely correct. In any direct democracy, if there is no great concentration of power, in the form of an oligarchy or dictator, then decision making can be incoherent. Madison, in *Federalist LXII*, commented on the “mischievous effects of mutable government”:

It will be of little avail to the people that the laws are made by men of their own choice, if laws be so . . . incoherent that they

¹James Madison, *Federalist X* (1787) in Rakove (1999: 164).

cannot be understood; if they be repealed or revised before they are promulgated, or undergo such incessant changes that no man who knows what the law is today can guess what it will be tomorrow²

The opposite of chaos is equilibrium, or rationality, what Madison called “stability in government” in *Federalist XXXVII*::

Stability in government, is essential to national character, and to . . . that repose and confidence in the minds of the people . . . An irregular and mutable legislation is not more an evil in itself, than it is odious to the people[.]³

This volume may be regarded as a contribution to the development of Madison’s intuition. Chapters 2 to 5 present a self-contained exposition of social choice theory on the possibility of aggregating individual preferences into a social preference in a direct democracy. Chapter 6 considers legislative bargaining in polities based on proportional representation, while Chapters 7 to 9 present a theory of elections—the selection of the representatives in an *indirect* democracy, what Madison called a *republic*. A brief concluding Chapter 10 attempts to relate social choice theory, the models of election and Madison’s theory of the Republic in terms of the aggregation of preferences and beliefs.

In a sense, Chapter 6 combines elements of social choice theory with the theory of elections that is to follow in the later chapters. It applies the theoretical notions developed in the early chapters to examine bargaining in a legislature. In particular, it assumes that the representatives have policy preferences, induced from the preferences of their constituents and the activists for the various parties. The party leaders must bargain among themselves in order to form a governing coalition, in the situation most common under proportional representation, that the election has led to a number of parties, none of which commands a majority.

Social choice theory suggests that there are two fundamentally different bargaining situations in such a multiparty legislature. The first is where there is a large, centrally located party in the policy space. Such a party is located at what is known as the “core.” No combination of other parties can agree to overturn the position of this “core” party. Conse-

²James Madison, *Federalist LXII* (1787) in Rakove (1999: 343).

³James Madison, *Federalist XXXVII* (1787) in Rakove (1999: 196).

quently the “core” party can, if it so chooses, form a minority government, one without a majority of the seats in the legislature. This property of the core provides an explanation for what has appeared to be a puzzle. The data set collected by Laver and Schofield (1990) dealing with coalition governments in 12 European countries in the period 1945–1987 shows that about one-third of the governments were minority. About one-third were minimal winning, with just enough seats for a majority, and the remaining third were surplus, with parties included in the coalition unnecessary for the majority. In the absence of a core, the spatial theory suggests that bargaining between the parties will focus on a domain in the policy space known as the “heart.” In the simplest case where it is assumed that parties have “Euclidean” preferences determined by policy distance, the “heart” will be a domain bounded by the compromise sets of various minimal winning coalitions. These minimal winning coalitions are natural candidates for coalition government. Indeed, in some cases a bounding minimal winning coalition may costlessly include a surplus party. This notion of the heart suggests that in the absence of a core, one or other of these minimal winning or surplus coalitions will form.

Chapter 6 illustrates the difference between a core and the heart by considering recent elections in Israel in the period 1988 to 2006 and in the Netherlands in 1977, 1981 and 2006. In Israel, the core party was Labor, under Rabin in 1992, and a new party, “Kadima,” founded by Ariel Sharon in 2005, but under the leadership of Ehud Olmert. After the elections of 1988 and 2003 the bargaining domain of the heart was bounded by various coalitions, involving the larger parties, Labor and Likud, and smaller parties like Shas.

These examples raise another theoretical problem: if party leaders are aware that by adopting a centrist position they can create minority, dominant government, then why are parties located so far from the electoral center? Chapter 6 illustrates the great variety of political configurations in Europe: bipolar political systems, such as the Netherlands and Finland; left unipolar systems such as Denmark, Sweden and Norway; center unipolar systems such as Belgium, Luxembourg and Ireland; right unipolar such as Iceland. Italy is unique in that it had a dominant center party, the Christian Democrats until 1994, after which the political system was totally transformed by the elimination of the core.

Models of elections also suggest that the electoral center will be an at-

tractor for political parties, since parties will calculate that they will gain most votes at the center.⁴ Chapter 7 presents an electoral model where this centripetal tendency will only occur under specific conditions. The model is based on the idea of *valence*, derived from voters' judgements about characteristics of the candidates, or party leaders. These valences or judgements are first assumed to be independent of the policy choice of the party. The theory shows that parties will converge to the electoral center only if the valence differences between the parties are small, relative to the other parameters of the model.

The empirical analysis considers elections in Israel in 1996, in Turkey in 1999 and 2002, in the Netherlands in 1977–1981 and in Britain in 1997. The results show that the estimated parameters of the model did not satisfy the necessary condition for convergence in Israel. The theory thus gives an explanation for the dispersion of political parties in Israel and Turkey along a principal electoral axis.

However, the condition sufficient for convergence of the parties was satisfied in the British election of 1997, and in the Dutch elections of 1977–1981. Because there was no evidence of convergence in these elections, the conflict between theory and evidence suggests that the stochastic electoral model be modified to provide a better explanation of party policy choice. The chapter goes on to consider a more general valence model based on activist support for the parties⁵. This activist valence model presupposes that party activists donate time and other resources to their party. Such resources allow a party to present itself more effectively to the electorate, thus increasing its valence. The main theorem of this chapter indicates how parties might balance the centrifugal tendency associated with activist support, and the centripetal tendency generated by the attraction of the electoral center.

One aspect of this theory is that it implies that party leaders will act as though they have policy preferences, since they must accommodate the demands of political activists to maintain support for future elections. A further feature is that party positions will be sensitive to the nature

⁴An extensive literature has developed in an attempt to explain why parties do not converge to the electoral center. See, for example, Adams (1999a,b, 2001); Adams and Merrill (1999a, 2005); Adams, Merrill and Grofman (2005); Merrill and Grofman (1999); Merrill and Adams (2001); Macdonald and Rabinowitz (1998).

⁵See Aldrich (1983a,b, 1995); Aldrich and McGinnis (1989).

of electoral judgements and to the willingness of activists to support the party. As these shift with time, then so will the positions of the parties. The theory thus gives an explanation of one of the features that comes from the discussion in Chapter 6: the general configuration of parties in each of the countries shifts slowly with time. In particular, under proportional representation, there is no strong impulse for parties to cohere into blocks. As a consequence, activist groups may come into existence relatively easily, and induce the creation of parties, leading to political fragmentation.

Chapters 8 and 9 apply this activist electoral model to examine elections under plurality rule. Chapter 8 considers presidential elections in Argentina in 1989 and 1995. In 1989, a populist leader on the left, Carlos Menem, was able to use a new dimension of policy (defined in terms of the financial structure of the economy) to gain new middle-class activist supporters, and win the election of 1995. Chapter 9 considers recent elections in the United States, and argues that there has been a slow realignment of the principal dimensions of political competition. Since the presidential contest between Johnson and Goldwater in 1964, the party positions have rotated (in a clockwise direction) in a space created by economic and social axes. In recent elections, the increasing importance of the social dimension, characterized by attitudes associated with civil and personal rights, have made policy making for political candidates very confusing. Aspects of policy making, such as stem cell research and immigration, are discussed at length to give some background to the nature of current politics in the United States.

It is worth summarizing the results from the formal model and the empirical analyses presented in this volume.

1. The results on the formal spatial model, presented in Chapters 2 to 5, indicate that the occurrence of a core, or unbeaten alternative, is very unlikely in a direct democracy using majority rule, when the dimension of the policy is at least two. However, a social choice concept known as the heart, a generalization of the core, will exist, and converges to the core when the core is non-empty. A legislative body, made up of democratically elected representatives, can be modeled in social choice terms. Because party strengths will be disparate, a large, centrally located party may be located at a core position. Such a party, in a situation with no majority party, may be able to form a

minority government.

2. A more typical situation is one with no core party. In such a case, the legislative heart can give an indication of the nature of bargaining between parties as they attempt to form a winning coalition government. This theory of legislative behavior takes as given the position and strengths of the parties. Because a centrally located party may dominate coalitional bargaining, and because such a party should be able to garner a large share of the vote, there would appear to be a strong centripetal tendency in all electoral systems.
3. Estimates of party positions suggest that parties adopt quite heterogeneous positions. This suggests that there is a countervailing or centrifugal force that affects all parties. While core parties can be observed in some of the Scandinavian polities, in Israel, and in Italy, the dominance of such central parties can be destroyed particularly if there is a tendency to political fragmentation.
4. It is very unlikely that the heterogeneous positions of the parties can be accounted for in terms of a stochastic model of elections based simply on *exogenous* valence alone. This stochastic model is reasonably competent to model elections in Israel, but does not account for party divergence in the Netherlands for example. Other work on Italy (Giannetti and Sened, 2004; Schofield and Sened, 2006) came to the same conclusion. This suggests that party location can be better modeled as a balancing act between the centripetal electoral pull, and the centrifugal pull of activist groups.
5. Under *proportional electoral methods*, there need be no strong tendency forcing activist groups to coalesce, in order to concentrate their influence. If activist groups respond to this impulse, then activist fragmentation will result in party fragmentation. Illustrations of party positions in Chapter 6 show that parties tend to be scattered throughout the policy space. Activist groups, linked to small parties, may aspire to political office. This is indicated by the observation that the bargaining domain in the legislature (the heart) often includes small parties. In some countries (such as Italy), a centrist core party can dominate the political landscape. To maintain dominance, such a party requires a high valence leader who can also maintain a flow of resources from a centrist activist group. By definition, however, an

activist group will tend to be located at a policy extreme. Thus a core party may need the support of an activist group that is not concerned about policy per se, but about monetary rewards. Thus there may be a link between core dominance and corruption. This suggests the underlying reason for the collapse of core dominance in Italy.

6. Under *plurality rule*, small parties face the possibility of extinction. Unlike the situation in a polity based on proportional rule, an activist group linked to a small party in a plurality polity has little expectation of influencing government policy. Thus activist groups face increasing returns to size. The activist model of elections presented in chapters 8 and 9 suggests that when there are two dimensions of policy, then there will tend to be at most four principal activist groups. The nature of the electoral contest generally forces these four principal activist groups to coalesce into at most two, as in the United States and the United Kingdom.
7. In the United States, plurality rule induces the two-party system, through this effect on activist groups. Although the two-party configuration may be in equilibrium at any time, the tension within the activist coalitions induces a slow rotation, and thus political realignment. Presidential candidates must balance the centripetal electoral effect against the centrifugal valence effect. It is plausible that, in general, the relative electoral effect is stronger under plurality than under proportional rule. On occasion in the United States, the conflict within activist groups is so pronounced that the two-party system breaks down. Such a collapse of the activist cohesion may herald a major realignment, induced by the creation of a new policy dimension, such as civil rights.

The fundamental theory presented in this book in Chapters 2 to 5 concerns the application of social choice theory to modeling political choice in direct and representative democracy. This theory is quite technical, and to provide a guide, the following section provides a brief overview of the theory.

1.2 The Theory of Social Choice

Each individual i in a society $N = \{1, \dots, n\}$ is characterized by a “ra-

tional" preference relation p_i . The society is represented by a profile of preference relations, $p = (p_1, \dots, p_n)$, one for each individual. Let the set of possible alternatives be $W = \{x, y, \dots\}$. If person i prefers x to y then write $(x, y) \in p_i$, or more commonly $x p_i y$. The social mechanism or *preference function*, σ , translates any profile p into a preference relation $\sigma(p)$. The point of the theory is to examine conditions on σ which are sufficient to ensure that whatever "rationality properties" are held by the individual preferences, then these same properties are held by $\sigma(p)$. Arrow's Impossibility Theorem (1951) essentially showed that if the rationality property under consideration is that preference be a weak order then σ must be dictatorial. To see what this means, let R_i be the weak preference for i induced from p_i . That is to say $x R_i y$ if and only if it is not the case that $y p_i x$. Then p_i is called a *weak order* if and only if R_i is *transitive*, i.e., if $x R_i y$ and $y R_i z$ for some x, y, z in W , then $x R_i z$. Arrow's theorem effectively demonstrated that if $\sigma(p)$ is a weak order whenever every individual has a weak order preference then there must be some dictatorial individual i , say, who is characterized by the ability to enforce every social choice.

It was noted some time afterwards that the result was not true if the conditions of the theorem were weakened. For example, the requirement that $\sigma(p)$ be a weak order means that "social indifference" must be transitive. If it is only required that strict social preference be transitive, then there can indeed be a non-dictatorial social preference mechanism with this weaker rationality property (Sen, 1970). To see this, suppose σ is defined by the *strong Pareto rule*: $x \sigma(p) y$ if and only if there is no individual who prefers y to x but there is some individual who prefers x to y . It is evident that σ is non-dictatorial. Moreover if each p_i is transitive then so is $\sigma(p)$. However, $\sigma(p)$ cannot be a weak order. To illustrate this, suppose that the society consists of two individuals $\{1, 2\}$ who have preferences

1	2
x	y
z	x
y	z

This means $x p_1 z p_1 y$ etc. Since $\{1, 2\}$ disagree on the choice between x and y and also on the choice between y and z both x, y and y, z must be

socially indifferent. But then if $\sigma(p)$ is to be a weak order, it must be the case that x and z are indifferent. However, $\{1, 2\}$ agree that x is superior to z , and by the definition of the strong Pareto rule, x must be chosen over z . This of course contradicts transitivity of social indifference.

A second criticism due to Fishburn (1970) was that the theorem was not valid in the case that the society was infinite. Indeed since democracy often involves the aggregation of preferences of many millions of voters the conclusion could be drawn that the theorem was more or less irrelevant.

However, three papers by Gibbard (1969), Hanssen (1976) and Kirman and Sondermann (1972) showed that the result on the existence of a dictator was quite robust. The first three sections of Chapter 2 essentially parallel the proof by Kirman and Sondermann. The key notion here is that of a *decisive coalition*: a coalition M is decisive for a social choice function, σ , if and only if $x p_i y$ for all i belonging to M for the profile p implies $x \sigma(p) y$. Let \mathbb{D}_σ represent the set of decisive coalitions defined by σ . Suppose now that there is some coalition, perhaps the whole society N , which is decisive. If σ preserves transitivity (i.e., $\sigma(p)$ is transitive) then the intersection of any two decisive coalitions must itself be decisive. The intersection of all decisive coalitions must then be decisive: this smallest decisive coalition is called an *oligarchy*. The oligarchy may indeed consist of more than one individual. If it comprises the whole society then the rule is none other than the Pareto rule. However, in this case every individual has a veto. A standard objection to such a rule is that the set of chosen alternatives may be very large, so that the rule is effectively indeterminate. Suppose the further requirement is imposed that $\sigma(p)$ always be a weak order. In this case it can be shown that for any coalition, M , either M itself or its complement $N \setminus M$ must be decisive. Take any decisive coalition A , and consider a proper subset B say of A . If B is not decisive then $N \setminus B$ is, and so $A \cap (N \setminus B) = A \setminus B$ is decisive. In other words every decisive coalition contains a strictly smaller decisive coalition. Clearly, if the society is finite then some individual is the smallest decisive coalition, and consequently is a dictator. Even in the case when N is infinite, there will be a smallest “invisible” dictator. It turns out, therefore, that reasonable and relatively weak rationality properties on σ impose certain restrictions on the class \mathbb{D}_σ of decisive coalitions. These restrictions on \mathbb{D}_σ do not seem to be similar to the characteristics that po-

litical systems display. As a consequence these first attempts by Sen and Fishburn and others to avoid the Arrow Impossibility Theorem appear to have little force.

A second avenue of escape is to weaken the requirement that $\sigma(p)$ always be transitive. For example a more appropriate mechanism might be to make a choice from W of all those *unbeaten* alternatives. Then an alternative x is chosen if and only if there is no other alternative y such that $y\sigma(p)x$. The set of unbeaten alternatives is also called the *core* for $\sigma(p)$, and is defined by

$$\text{Core}(\sigma, p) = \{x \in W : y\sigma(p)x \text{ for no } y \in W\}.$$

In the case that W is finite the existence of a core is essentially equivalent to the requirement that $\sigma(p)$ be *acyclic* (Sen, 1970). Here a preference, p , is called acyclic if and only if whenever there is a chain of preferences

$$x_0 p x_1 p x_2 p \cdots p x_r$$

then it is not the case that $x_r p x_0$.

However, acyclicity of σ also imposes a restriction on \mathbb{D}_σ . Define the *collegium* $\kappa(\mathbb{D}_\sigma)$ for the family \mathbb{D}_σ of decisive coalitions of σ to be the intersection (possibly empty) of all the decisive coalitions. If the collegium is empty then it is always possible to construct a “rational” profile p such that $\sigma(p)$ is cyclic (Brown, 1973). Therefore, a necessary condition for σ to be acyclic is that σ exhibit a non-empty collegium. We say σ is *collegial* in this case. Obviously, if the collegium is large then the rule is indeterminate, whereas if the collegium is small the rule is almost dictatorial.

A third possibility is that the preferences of the members of the society are restricted in some way, so that a natural social choice function, such as majority rule, will be “well behaved.” For example, suppose that the set of alternatives is a closed subset of a single dimensional “left–right” continuum. Suppose further that each individual i has *convex* preference on W , with a most preferred point (or bliss point) x_i , say.⁶ Then a well-known result by Black (1958) asserts that the core for majority rule is

⁶Convexity of the preference p just means that for any y the set $\{x : xpy\}$ is convex. A natural preference to use is Euclidean preference defined by xp_iy if and only if

$\|x - x_i\| < \|y - x_i\|$, for some bliss point, x_i , in W , and norm $\|\cdot\|$ on W . Clearly Euclidean preference is convex.

the median most preferred point. On the other hand, if preferences are not convex, then as Kramer and Klevorick (1974) demonstrated, the social preference relation $\sigma(p)$ can be cyclic, and thus have an empty core. However, it was also shown that there would be a *local core* in the one-dimensional case. Here a point is in the local core, $LCore(\sigma, p)$, if there is some neighborhood of the point which contains no socially preferred points.

The idea of preference restrictions sufficient to guarantee the existence of a majority rule core was developed further in a series of papers by Sen (1966), Inada (1969) and Sen and Pattanaik (1969). However, it became clear, at least in the case when W had a geometric form, that these preference restrictions were essentially only applicable when W was one-dimensional.

To see this suppose that there exists a set of three alternatives $X = \{x, y, z\}$ in W , and three individuals $\{1, 2, 3\}$ in N whose preferences on X are:

1	2	3
x	y	z
y	z	x
z	x	y

The existence of such a *Condorcet cycle* is in contradiction to all the preference restrictions. If a profile p on W , containing such a Condorcet cycle, can be found then there is no guarantee that $\sigma(p)$ will be acyclic or exhibit a non-empty core. Kramer (1973a) demonstrated that if W were two-dimensional then it was always possible to construct convex preferences on W such that p contained a Condorcet cycle. Kramer's result, while casting doubt on the likely existence of the core, did not, however, prove that it was certain to be empty. On the other hand, an earlier result by Plott (1967) did show that when the W was a subset of Euclidean space, and preference convex and smooth, then, for a point to be the majority rule core, the individual bliss points had to be symmetrically distributed about the core. These Plott symmetry conditions are sufficient for existence of a core when n is either odd or even, but are necessary when n is odd. The "fragility" of these conditions suggested that a majority rule core was unlikely in some sense in high enough dimension (McKelvey and Wendell, 1976). It turns out that these symmetry condi-

tions are indeed fragile in the sense of being “non-generic” or atypical.

An article by Tullock (1967) at about this time argued that even though a majority rule core would be unlikely to exist in two dimensions, nonetheless it would be the case that cycles, if they occurred, would be constrained to a central domain in the Pareto set (i.e., within the set of points unbeaten under the Pareto rule).

By 1973, therefore, it was clear that there were difficulties over the likely existence of a majority rule core in a geometric setting. However, it was not evident how existence depended on the number of dimensions. The results by McKelvey and Schofield (1987) and Saari (1997) discussed in Chapter 5 indicate how the behavior of a general social choice rule is dependent on the dimensionality of the space of alternatives.

1.2.1 Restrictions on the Set of Alternatives

One possible way of indirectly restricting preferences is to assume that the set of alternatives, W , is of finite cardinality, r , say. As Brown (1973) showed, when the social preference function σ is not collegial then it is always possible to construct an acyclic profile such that $\sigma(p)$ is in fact cyclic. However, as Ferejohn and Grether (1974) proved, to be able to construct such a profile it is necessary that W have a sufficient cardinality. These results are easier to present in the case of a *voting rule* σ . Such a rule, σ , is determined completely by its decisive coalitions, \mathbb{D}_σ . That is to say:

$x\sigma(p)y$ if and only if xp_iy for every $i \in M$, for some $M \in \mathbb{D}_\sigma$.

An example of a voting rule is a q -rule, written σ_q , and the decisive coalitions for σ_q are defined to be

$$\mathbb{D}_q = \{M \subset N : |M| \geq q\}.$$

Clearly if $q < n$ then \mathbb{D}_q has an empty collegium. Ferejohn and Grether (1974) showed that if

$$q > \left(\frac{r-1}{r}\right)n \text{ where } |W| = r$$

then no acyclic profile, p , could be constructed so that $\sigma(p)$ was acyclic.

Conversely, if

$$q \leq \left(\frac{r-1}{r} \right) n$$

then such a profile could certainly be constructed. Another way of expressing this is that a q -rule σ is acyclic for all acyclic profiles if and only if

$$|W| < \frac{n}{n-q}.$$

Note that we assume that $q < n$.

Nakamura (1979) later proved that this result could be generalized to the case of an arbitrary social preference function. The result depends on the notion of a *Nakamura number* $v(\sigma)$ for σ . Given a non-collegial family \mathbb{D} of coalitions, a member M of \mathbb{D} is *minimal decisive* if and only if M belongs to \mathbb{D} , but for no member i of M does $M \setminus \{i\}$ belong to \mathbb{D} . If \mathbb{D}' is a subfamily of \mathbb{D} consisting of minimal decisive coalitions, and moreover \mathbb{D}' has an empty collegium then call \mathbb{D}' a *Nakamura subfamily* of \mathbb{D} . Now consider the collection of all Nakamura subfamilies of \mathbb{D} . Since N is finite these subfamilies can be ranked by their cardinality. Define $v(\mathbb{D})$ to be the cardinality of the smallest Nakamura subfamily, and call $v(\mathbb{D})$ the *Nakamura number* of \mathbb{D} . Any Nakamura subfamily \mathbb{D}' , with cardinality $|\mathbb{D}'| = v(\mathbb{D})$, is called a *minimal non-collegial subfamily*. When σ is a social preference function with decisive family \mathbb{D}_σ define the Nakamura number $v(\sigma)$ of σ to be equal to $v(\mathbb{D}_\sigma)$. More formally

$$v(\sigma) = \min\{|\mathbb{D}'| : \mathbb{D}' \subset \mathbb{D} \text{ and } \kappa(\mathbb{D}') = \Phi\}.$$

In the case that σ is collegial then define

$$v(\sigma) = v(\mathbb{D}_\sigma) = \infty \text{ (infinity)}.$$

Nakamura showed that for any voting rule, σ , if W is finite, with $|W| < v(\sigma)$, then $\sigma(p)$ must be acyclic whenever p is an acyclic profile. On the other hand, if σ is a social preference function and $|W| \geq v(\sigma)$ then it is always possible to construct an acyclic profile on W such that $\sigma(p)$ is cyclic. Thus the cardinality restriction on W which is necessary and sufficient for σ to be acyclic is that $|W| < v(\sigma)$. To relate this to Ferejohn–Grether’s result for a q -rule, define $v(n, q)$ to be the largest integer such that

$$v(n, q) < \frac{q}{n-q}.$$

It is an easy matter to show that when σ_q is a q -rule then

$$v(\sigma_q) = 2 + v(n, q).$$

The Ferejohn–Grether restriction $|W| < \frac{n}{n-q}$ may also be written

$$|W| < 1 + \frac{q}{n-q}$$

which is the same as

$$|W| < v(\sigma_q).$$

Thus Nakamura's result is a generalization of the earlier result on q -rules.

The interest in this analysis is that Greenberg (1979) showed that a core would exist for a q -rule as long as preferences were convex and the choice space, W , was of restricted dimension. More precisely suppose that W is a compact,⁷ convex subset of Euclidean space of dimension w , and suppose each individual preference is continuous⁸ and convex. If $q > (\frac{w}{w+1})n$ then the core of $\sigma(p)$ must be non-empty, and if $q \leq (\frac{w}{w+1})n$ then a convex profile can be constructed such that the core is empty. From a result by Walker (1977) the second result also implies, for the constructed profile p , that $\sigma(p)$ is cyclic. Rewriting Greenberg's inequality it can be seen that the necessary and sufficient dimensionality condition (given convexity and compactness) for the existence of a core and the non-existence of cycles for a q -rule, σ_q , is that $\dim(W) \leq v(n, q)$ where $\dim(W) = w$ is the dimension of W .

Since

$$v(\sigma_q) = 2 + v(n, q).$$

where $v(\sigma_q)$ is the Nakamura number of the q -rule, this suggests that for an arbitrary non-collegial voting rule σ there is a *stability dimension*, namely $v^*(\sigma) = v(\sigma) - 2$, such that $\dim(W) \leq v^*(\sigma)$ is a necessary and sufficient condition for the existence of a core and the non-existence of

⁷Compactness just means the set is closed and bounded.

⁸The continuity of the preference, p , that is required is that for each $x \in W$, the set $p^{-1}\{y \in W : xpy\}$ is open in the topology on W .

cycles. Chapters 3, 4 and 5 of this volume prove this result and present a number of further applications.

An important procedure in this proof is the construction of a *representation* ϕ for an arbitrary social preference function.

Let $\mathbb{D} = \{M_1, \dots, M_v\}$ be a minimal non-collegial subfamily for σ . Note that \mathbb{D} has empty collegium and cardinality $v(\sigma) = v$. Then σ can be represented by a $(v - 1)$ dimensional simplex Δ in \mathbb{R}^{v-1} . Moreover, each of the v faces of this simplex can be identified with one of the v coalitions in \mathbb{D} . Each proper subfamily $\mathbb{D}_t = \{\dots, M_{t-1}, M_{t+1}, \dots\}$ has a non-empty collegium, $\kappa(\mathbb{D}_t)$, and each of these can be identified with one of the vertices of Δ . To each $i \in \kappa(\mathbb{D}_t)$ we can assign a preference p_i , for $i = \{1, \dots, v\}$ on a set $x = \{x_1, x_2, \dots, x_v\}$ giving a *permutation profile*

$$\begin{array}{ccccccc} \kappa(\mathbb{D}_1) & \kappa(\mathbb{D}_2) & \dots & \kappa(\mathbb{D}_v) & & & \\ x_1 & x_2 & & x_v & & & \\ x_2 & x_3 & & x_1 & & & \\ \cdot & \cdot & & \cdot & \dots & & \\ \cdot & \cdot & & \cdot & & & \\ \cdot & \cdot & & \cdot & & & \\ x_v & x_1 & \dots & x_{v-1} & & & \end{array}$$

From this construction it follows that

$$x_1 \sigma(p) x_2 \cdots \sigma(p) x_v \sigma(p) x_1.$$

Thus whenever W has cardinality at least v , then it is possible to construct a profile p such that $\sigma(p)$ has a permutation cycle of this kind. This representation theorem is used in Chapter 4 to prove Nakamura's result and to extend Greenberg's Theorem to the case of an arbitrary rule.

The principal technique underlying Greenberg's Theorem is an important result due to Fan (1961). Suppose that W is a compact convex subset of \mathbb{R}^w , and suppose P is a correspondence from W into itself which is convex and continuous.⁹ Then there exists an "equilibrium" point x in W such that $P(x)$ is empty. In the case under question if each individual preference, p_i , is continuous, then so is the preference correspondence P associated with $\sigma(p)$. Moreover, if W is a subset of Euclidean space with

⁹Again, continuity of the preference correspondence, P , means that for each $x \in W$, the set $P^{-1}(x) = \{y \in W : x \in P(y)\}$ is open in the topology on W .

dimension no greater than $v(\sigma) - 2$, then using Caratheodory's Theorem it can be shown that P is also convex. Then by Fan's Theorem, P must have an equilibrium in W . Such an equilibrium is identical to the core, $Core(\sigma, p)$.

On the other hand, suppose that $\dim(W) = v(\sigma) - 1$. Using the representation theorem, the simplex Δ representing σ can be embedded in W . Let $Y = \{y_1, \dots, y_v\}$ be the set of vertices of Δ . As above, let $\{\kappa(\mathbb{D}_t) : t = 1, \dots, v\}$ be the various collegia. Each player $i \in \kappa(\mathbb{D}_t)$, is associated with the vertex y_t and is assigned a "Euclidean" preference of the form $x p_i z$ if and only if $\|x - y_i\| < \|z - y_i\|$. In a manner similar to the situation with W finite, it is then possible to show, with the profile p so constructed, that for every point z in W there exists x in W such that $x \sigma(p) z$. Thus the core for $\sigma(p)$ is empty and $\sigma(p)$ must be cyclic. In the case that W is compact and convex, and preference is continuous and convex, then a necessary and sufficient condition for the existence of the core, and non-existence of cycles is that $\dim(W) \leq v^*(\sigma)$, where $v^*(\sigma) = v(\sigma) - 2$ is called the *stability dimension*. This result was independently obtained by Schofield (1984a,b) and Strnad (1985).

This result on the Nakamura number is extended by showing that even with non-convex preference, a "critical" core called $\Theta(\sigma, p)$, which contains the *local core*, $LCore(\sigma, p)$, will exist as long as $\dim(W) \leq v^*(\sigma)$. It is an easy matter to show that for majority rule $v^*(\sigma) \geq 1$, and so this gives an analog of the Kramer–Klevorick (1974) Theorem.

Chapter 5 examines in more detail the case when $\dim(W) \geq v^*(\sigma) + 1$. The purpose here is essentially to extend Kramer's (1973a) result from the three-person case to that of an arbitrary voting rule. Given a profile p on a topological space W , say a point x in W belongs to the *local cycle set* $LC(\sigma, p)$ for $\sigma(p)$ if and only if, in every neighborhood V of x , there exists a $\sigma(p)$ cycle. In Theorem 5.1.2 it is essentially shown that the local cycle set contains the interior of the simplex Δ associated with the Euclidean preference profile constructed above. In effect $\dim(W) \geq v^*(\sigma) + 1$ is a sufficient condition not only for the non-existence of the local core $LO(\sigma, p)$ but also for the non-emptiness of the local cycle set $LC(\sigma, p)$.

This result has an important bearing on *manipulation* of a choice function, C , derived from a social preference function σ . Consider a choice $C(W, p)$ from W which is compatible with σ in some sense. The choice

is manipulable if the members of some coalition may lie about their preferences (and so change p to p') so that $C(W, p')$ is preferred by them to $C(W, p)$. Maskin (1999) used the term “implementable in Nash equilibria of the underlying game form” for what I call “non-manipulation.” Maskin has shown that if the choice is to be non-manipulable then it must be *monotonic*. Monotonicity is the condition that whenever $\text{not}(yp_i x)$ implies $\text{not}(yp'x)$ for all $y \neq x$ and all i , then $x \in C(W, p)$ implies $x \in C(W, p')$. For an arbitrary choice function, C , define $v(C)$ to be the same as the Nakamura number, $v(\sigma)$, of the underlying social preference function σ . The existence of local σ -cycles whenever $\dim(W) \geq v(C) - 1$ implies that C cannot be monotonic. This suggests that a non-collegial voting procedure cannot be implemented by an appropriate game form.

For a general voting rule σ , if $\dim(W) \leq v(\sigma) - 1$ then $LC(\sigma, p)$ may be non-empty, but it will be contained within the Pareto set. Since $v(\sigma) - 1 = 2$ for majority rule in general, this supports Tullock’s (1967) argument that voting cycles are not very important in two dimensions. However, in the case $\dim(W) = v(\sigma)$ then for the Euclidean preference profile, p , constructed above, the local cycle set $LC(\sigma, p)$ is open dense and path connected. This means essentially that there is a profile p on W such that the set $LC(\sigma, p)$ has the following property: for almost any two points x, y in W , there exists a voting trajectory between x and y which is contained in $LC(\sigma, p)$ such that successive manipulations by various coalitions can force the choice from x to y . Thus, as the dimension of W increases from the stability dimension $v^*(\sigma)$ to $v^*(\sigma) + 2$, the existence of the core can no longer be guaranteed, and instead cycles, and indeed open dense cycles can be created.

1.2.2 Structural Stability of the Core

Although the σ -core cannot be guaranteed in dimension $v^*(\sigma) + 1$ or more, nonetheless it is possible for a core to exist in a “structurally stable” fashion. We now assume that each preference p_i can be represented by a smooth utility function $u_i: W \rightarrow \mathbb{R}$. As before, this means simply that

$$xp_i y \text{ if and only if } u_i(x) > u_i(y).$$

A smooth profile for the society N is a differentiable function

$$u = (u_1, \dots, u_n): W \rightarrow \mathbb{R}^n.$$

We assume in the following analysis that W is compact, and let $U(W)^N$ be the space of all such profiles endowed with the Whitney C^1 -topology (Golubitsky and Guillemin, 1973; Hirsch, 1976). Essentially two profiles u^1 and u^2 are close in this topology if all values and the first derivatives are close.

Restricting attention to smooth utility profiles whose associated preferences are convex gives the space $U_{con}(W)^N$. We say that the core $Core(\sigma, u)$ for a rule σ is *structurally stable* (in $U_{con}(W)^N$) if $Core(\sigma, u)$ is non-empty and there exists a neighborhood V of u in $U_{con}(W)^N$ such that $Core(\sigma, u')$ is non-empty for all u' in V . To illustrate, if $Core(\sigma, u)$ is non-empty but not structurally stable then an arbitrary small perturbation of u , to a different but still convex smooth preference profile, u' , is sufficient to destroy the core by rendering $Core(\sigma, u')$ empty.

By the previous result if $\dim(W) \leq v^*(\sigma)$ then $Core(\sigma, u)$ is non-empty for every smooth, convex profile, and thus this dimension constraint is sufficient for $Core(\sigma, u)$ to be structurally stable.

It had earlier been shown by Rubinstein (1979) that the set of continuous profiles such that the majority rule core is non-empty is in fact a nowhere dense set in a particular topology on profiles, independently of the dimension. However, the perturbation involved deformations induced by creating non-convexities in the preferred sets. Thus the construction did not deal with the question of structural stability in the topological space $U_{con}(W)^N$.

Chapter 5 continues with the result by McKelvey and Schofield (1987) and Saari (1997) which indicates that, for any q -rule, σ_q , there is an *instability dimension*, $w(\sigma_q)$. If $\dim(W) \geq w(\sigma_q)$ and W has no boundary then the σ_q -core will be empty for a dense set of profiles in $U_{con}(W)^N$. This immediately implies that the core cannot be structurally stable, so any sufficiently small perturbation in $U_{con}(W)^N$ will destroy the core. The same result holds if W has a non-empty boundary but $\dim(W) \geq w(\sigma_q) + 1$. Theorem 5.1.1 shows that if a point belongs to the core of a voting game, defined by a set, \mathbb{D} , of decisive coalitions, then the direction gradients must satisfy certain generalized symmetry conditions on the utility gradients of the voters at that point. This theorem is an exten-

sion of an earlier result by Plott (1967) for majority rule.

The easiest case to examine is where the core, $Core(\sigma, u)$, is characterized by the property that exactly one individual has a bliss point at the core. We denote this by $BCore(\sigma, u)$. The Thom Transversality Theorem can then be used to show that $BCore(\sigma_q, u)$ is generically empty (in the space $U_{con}(W)^N$), whenever the dimension exceeds $2q - n + 1$. This suggests that the instability dimension satisfies $w(\sigma_q) = 2q - n + 1$.

Saari (1997) extended this result in two directions, by showing that if $\dim(W) \leq 2q - n$ then $BCore(\sigma_q, u)$ could be structurally stable. Moreover, he was able to compute the instability dimension for the case of a non-bliss core, when no individual has a bliss point at the core.

For example, with majority rule the instability dimension is two or three depending on whether n is odd or even. For n odd, neither bliss nor non-bliss cores can be structurally stable in two or more dimensions, since the Plott (1967) symmetry conditions cannot be generically satisfied. On the other hand, when $(n, q) = (4, 3)$, the Nakamura number is four, and hence a core will exist in two dimensions. Indeed, both bliss cores and non-bliss cores can occur in a structurally stable fashion. However, in three dimensions the cycle set is contained in, but fills the Pareto set. For all majority rules with $n \geq 6$, and even, a structurally stable bliss-core can occur in two dimensions. However, when n even, in three dimensions the core cannot be structurally stable and the cycle set need not be constrained to the Pareto set (in contradiction to Tullock's hypothesis).

For general weighted voting games, defined by a non-collegial family, \mathbb{D} , the core symmetry condition can be satisfied in a structurally stable fashion. This provides a technique for examining when a core exists in the legislatures discussed in Chapter 6.

Since the core may be empty, the notion of the heart is presented as an alternative solution idea. The heart can be interpreted in terms of a local uncovering relation, and can be shown to be non-empty under fairly weak conditions. This idea is illustrated by considering various voting rules in low dimensions. The last section of Chapter 5 presents the experimental results obtained by Fiorina and Plott (1978), McKelvey, Ordeshook and Winer (1978), Laing and Olmstead (1978) and Eavey (1996) to indicate the nature of the heart in two dimensions.

Chapter 2

Social Choice

2.1 Preference Relations

Social choice is concerned with a fundamental question in political or economic theory: is there some process or rule for decision making which can give consistent social choices from individual preferences?

In this framework denoted by W is a *universal set of alternatives*. Members of W will be written x, y etc. The *society* is denoted by N , and the individuals in the society are called $1, \dots, i, \dots, j, \dots, n$. The values of an individual i are represented by a preference relation p_i on the set W . Thus $x p_i y$ is taken to mean that individual i prefers alternative x to alternative y . It is also assumed that each p_i is strict, in the way to be described below. The rest of this section considers the abstract properties of a preference relation p on W .

Definition 2.1.1. A strict preference relation p on W is

- (i) *Irreflexive*: for no $x \in W$ does $x p x$;
- (ii) *Asymmetric*: for any $x, y, \in W$; $x p y \Rightarrow \text{not}(y p x)$.

The strict preference relations are regarded as fundamental primitives in the discussion. No attempt is made to determine how individuals arrive at their preferences, nor is the problem considered how preferences might change with time. A preference relation p may be *represented* by a utility function.

Definition 2.1.2. A preference relation p is *representable* by a utility function

$$u: W \rightarrow \mathbb{R} \text{ for any } x, y \in W; xpy \Leftrightarrow u(x) > u(y).$$

Alternatively, p is representable by u whenever

$$\{x: u(x) > u(y)\} = \{x: xpy\} \text{ for any } y \in W.$$

If both $u^1, u^2: W \rightarrow \mathbb{R}$ represent p then write $u^1 \sim u^2$. The equivalence class of real valued functions which represents a given p is called an *ordinal utility function* for p , and may be written u_p . If p can be represented by a continuous (or smooth) utility function, then we may call p continuous (or smooth).

By some abuse of notation we shall write:

$$u_p(x) > u_p(y)$$

to mean that for any $u: W \rightarrow \mathbb{R}$ which represents p it is the case that $u(x) > u(y)$. We also write $u_p(x) = u_p(y)$ when for any u representing p , it is the case that $u(x) = u(y)$.

From the primitive strict preference relation p define two new relations known as *indifference* and *weak preference*. These satisfy various properties.

Definition 2.1.3. A relation q on W is:

- (i) *symmetric* iff $xqy \Rightarrow yqx$ for any $x, y \in W$.
- (ii) *reflexive* iff xqx for all $x \in W$.
- (iii) *connected* iff xqy or yqx , for any $x, y \in W$.
- (iv) *weakly connected* iff $x \neq y \Rightarrow xqy$ or yqx for $x, y \in W$.

Definition 2.1.4. For a strict preference relation p , define the symmetric component $I(p)$ called *indifference* by:

$$xI(p)y \text{ iff not}(xpy) \text{ and not}(ypx).$$

Define the reflexive component $R(p)$, called *weak preference* by

$$xR(p)y \text{ iff } xpy \text{ or } xI(p)y.$$

Note that since p is assumed irreflexive, then $I(p)$ must be reflexive. From the definition $I(p)$ must also be symmetric, although $R(p)$ need not

be. From the definitions:

$$xpy \text{ or } xI(p)y \text{ or } ypx,$$

so that $xR(p)y \Leftrightarrow \text{not}(ypx)$. Furthermore either $xR(p)y$ or $yR(p)x$ must be true for any $x, y \in W$, so that $R(p)$ is *connected*. In terms of an ordinal utility function for p , it is the case that for any x, y in W :

- (i) $xI(p)y$ iff $u_p(x) = u_p(y)$
- (ii) $xR(p)y$ iff $u_p(x) \geq u_p(y)$.

If p is representable by u , then from the natural orderings on the real line, \mathbb{R} , it follows that p must satisfy certain consistency properties.

If

$$u(x) > u(y)$$

and

$$u(y) > u(z)$$

it follows that

$$u(x) > u(z).$$

Thus it must be the case that

$$xpy \text{ and } ypz \Rightarrow xpz.$$

This property of a preference relation is known as *transitivity* and may be seen as a desirable property for preference even when p itself is not representable by a utility function. The three consistency properties for preference that we shall use are the following.

Definition 2.1.5. A relation q on W satisfies

- (i) *Negative transitivity* iff $\text{not}(xqy)$ and $\text{not}(yqz) \Rightarrow \text{not}(xqz)$.
- (ii) *Transitivity* iff xqy and $yqz \Rightarrow xqz$.
- (iii) *Acyclicity* iff for any finite sequence x_1, \dots, x_r in W it is the case that if x_jqx_{j+1} for $j = 1, \dots, r - 1$ then $\text{not}(x_rqx_1)$. If q fails acyclicity then it is called *cyclic*.

The class of strict preference relations on W will be written $B(W)$. If $p \in B(W)$ and is moreover negatively transitive then it is called a *weak order*. The class of weak orders on W is written $O(W)$. In the same way if p is a transitive strict preference relation then it is called a *strict partial*

order, and the class of these is written $T(W)$. Finally the class of *acyclic* strict preference relations on W is written $A(W)$.

If $p \in O(W)$ then it follows from the definition that $R(p)$ is transitive. Indeed $I(p)$ will also be transitive.

Lemma 2.1.1. If $p \in O(W)$ then $I(p)$ is transitive.

Proof. Suppose $xI(p)y, xI(p)z$ but not $(xI(p)z)$. Because of not $(xI(p)z)$ suppose xpz . By asymmetry of p , not (zpx) so $xR(p)z$. By symmetry of $I(p)$, $yI(p)x$ and $zI(p)y$, and thus $yR(p)x$ and $zR(p)y$. But $xR(p)z$ and $zR(p)y$ and $yR(p)x$ contradicts the transitivity of $R(p)$. Hence not (xpz) . In the same way not (zpx) , and so $xI(p)z$, with the result that $I(p)$ must be transitive. ■

Lemma 2.1.2. If $p \in O(W)$ then

$$xR(p)y \text{ and } ypz \Rightarrow xpz.$$

Proof. Suppose $xR(p)y, ypz$ and not (xpz) .

But

$$\text{not}(xpz) \Leftrightarrow zR(p)x.$$

By transitivity of $R(p)$, $zR(p)y$.

By definition not (ypz) which contradicts ypz by asymmetry. ■

Lemma 2.1.3. $O(W) \subset T(W) \subset A(W)$.

Proof.

- (i) Suppose $p \in O(W)$ but xpy, ypz yet not (xpz) , for some x, y, z . By p asymmetry, not (ypx) and not (zpy) . Since $p \in O(W)$, not (xpz) and not $(zpy) \Rightarrow$ not (xpy) . But not (ypx) . So $xI(p)y$. But this violates xpy . By contradiction, $p \in T(W)$.
- (ii) Suppose $x_j p x_{j+1}$ for $j = 1, \dots, r-1$. If $p \in T(W)$, then $x_1 p x_r$. By asymmetry, not $(x_r p x_1)$, so p is acyclic. ■

2.2 Social Preference Functions

Let the society be $N = \{1, \dots, i, \dots, n\}$. A *profile* for N on W is an

assignment to each individual i in N of a strict preference relation p_i on W . Such an n -tuple (p_1, \dots, p_n) will be written p . A subset $M \subset N$ is called a *coalition*. The restriction of p to M will be written

$$p/M = (\dots p_i \dots : i \in M).$$

If p is a profile for N on W , write $x p_N y$ iff $x p_i y$ for all $x p_i y$ for all $i \in N$. In the same way for M a coalition in N write $x p_M y$ whenever $x p_i y$ for all $i \in M$.

Write $B(W)^N$ for the class of profiles on N . When there is no possibility of misunderstanding we shall simply write B^N for $B(W)^N$.

On occasion the analysis concerns profiles each of whose component individual preferences are assumed to belong to some subset $F(W)$ of $B(W)$; for example $F(W)$ might be taken to be $O(W)$, $T(W)$ or $A(W)$. In this case write $F(W)^N$, or F^N , for the class of such profiles.

Let X be the class of all subsets of W . A member $V \in X$ will be called a *feasible set*.

Suppose that $p \in B(W)^N$ is a profile for N on W . For some $x, y \in W$ write “ $p_i(x, y)$ ” for the preference expressed by i on the alternatives x, y under the profile p . Thus “ $p_i(x, y)$ ” will give either $x p_i y$ or $x I(p_i) y$ or $y p_i x$.

If $f, g \in B(W)^N$ are two profiles on W , and $V \in X$, use $f/M = g/M$ on V to mean that for any $x, y \in V$, any $i \in M$,

$$f_i(x, y) = g_i(x, y).$$

In more abbreviated form write $f/M^V = g/M^V$. Implicitly this implies consideration of a restriction operator

$$\begin{matrix} V \\ M \end{matrix} : B(W)^N \rightarrow B(V)^M : f \rightarrow f/M^V,$$

where $B(V)^M$ means naturally enough the set of profiles for M on V .

A *social preference function* is a method of aggregating preference information, and *only* preference information, on a feasible set in order to construct a social preference relation.

Definition 2.2.1. A *method of preference aggregation* (MPA), σ , assigns to any feasible set V , and profile p for N on W a strict social preference relation $\sigma(V, p) \in B(V)$. Such a method is written as $\sigma: X \times B^N \rightarrow B$.

As before write $\sigma(V, p)(x, y)$ for $x, y \in V$ to mean “the social preference relation declared by $\sigma(V, p)$ between x and y .” If $f, g \in B^N$, write

$$\sigma(V, f) = \sigma(V, g)$$

whenever $\sigma(V, f)(x, y) = \sigma(V, g)(x, y)$ for any $x, y \in V$.

Definition 2.2.2. A method of preference aggregation $\sigma: X \times B^N \rightarrow B$ is said to satisfy the *weak axiom of independence of infeasible alternatives* (II) iff

$$f^V = g^V \Rightarrow \sigma(V, f) = \sigma(V, g).$$

Such a method is called a *social preference function* (SF). Note that an SF, σ , is functionally dependent on the feasible set V . Thus there need be no specific relationship between $\sigma(V_1, f)$ and $\sigma(V_2, f)$ for $V_2 \subset V_1$ say.

However, suppose $\sigma(V_1, f)$ is the preference relation induced by σ from f on V_1 . Let $V_2 \subset V_1$, and let $\sigma(V_1, f)/V_2$ be the preference relation induced by $\sigma(V_1, f)$ on V_2 from the definition

$$[\sigma(V_1, f)/V_2](x, y) = [\sigma(V_1, f)(x, y)]$$

whenever $x, y \in V_2$.

A binary preference function is one which is consistent with this restriction operator.

Definition 2.2.3. A social preference function σ is said to satisfy the *strong axiom of independence of infeasible alternatives* (II*) iff for $f \in B(V_1)^N$, $g \in B(V_2)^N$, and $f^V = g^V$ for $V = V_1 \cap V_2$ non-empty, then

$$\sigma(V_1, f)/V = \sigma(V_2, g)/V.$$

For $\sigma(V_1, f)$ to be meaningful when σ is an SF, we only require that f be a profile defined on V_1 . This indicates that II* is an extension property. For suppose f, g are defined on V_1, V_2 respectively, and agree on V . Then it is possible to find a profile p defined on $V_1 \cup V_2$, which agrees with f on V_1 and with g on V_2 . Furthermore if σ is an SF which satisfies II*, then

$$\begin{aligned} \sigma(V_1 \cup V_2, p)/V_1 &= \sigma(V_1, f) \\ \sigma(V_1 \cup V_2, p)/V_2 &= \sigma(V_2, g) \\ \sigma(V_1 \cup V_2, p)/V_1 \cap V_2 &= \sigma(V_1, f)/V = \sigma(V_2, g)/V. \end{aligned}$$

Consequently if $V_2 \subset V_1$ and f is defined on V_1 , let f^{V_2} be the restriction of f to V_2 . Then for a BF,

$$\sigma(V_2, f^{V_2}) = \sigma(V_1, f)/V_2.$$

The attraction of this axiom is clear. It implies that one can piece together the observed social preferences on various feasible sets to obtain a universal social preference on W .

Moreover in the definition we need only consider V to be a pair of alternatives and construct the social preference $\sigma(f)$ from the pairwise comparisons. It is for this reason that a SF satisfying II^* is called a *binary social preference function* (BF). We may regard a BF as a function $\sigma: B^N \rightarrow B$.

To illustrate the differences between an MPA, an SF and a BF, consider the following adaptation of an example due to Plott (1976).

Example 2.2.1. Three individuals i, j, k seek to choose a candidate for a job from a short list $V = \{x, y, z, w\}$. For purposes of illustration take the *universal* set to be $W = V \cup \{m\} \cup \{j\} \cup \{s\}$, where M, J, S stand for Madison, Jefferson and J.S. Mill respectively. The preferences (f) of the individuals are:

i	j	k	Borda count
m	y	z	$z : 16$
J	z	w	$y : 15$
x	w	x	$x : 14$
y	x	y	$m : 13$
z	m	m	$w : 12$
s	j	j	$j : 10$
w	s	s	$s : 4$

- (i) The Borda count is used on W : that is the most preferred candidate if each individual scores 7 and the least preferred 1. On W , z wins with 16, and y is second. Assume social preference on V is induced by restriction from W . With the profile f , we obtain $z\sigma y\sigma x\sigma w$. Now change i 's preferences to the following: x is preferred to y to M to J to z to S to w . With this new profile, g , the induced preference on V is $y\sigma zI(\sigma)x\sigma w$, where $zI(\sigma)x$ means z and x are

socially indifferent. This decision rule is an MPA, because although $\sigma(V, f) \neq \sigma(V, g)$, f and g are not identical on W . Although the method uses restriction as required for a BF, it satisfies neither II nor II*. This can be seen since

$$f^V = g^V$$

yet

$$\sigma(W, f)/V \neq \sigma(W, g)/V.$$

- (ii) Alternatively suppose that on each subset V' of W the Borda count is recomputed. Thus on V , an individual's best alternative scores 4 and the worst 1. The scores for (z, y, x, w) are now $(9, 8, 7, 6)$, so $z\sigma y\sigma x\sigma w$. Clearly σ satisfies II and is an SF, since by definition, if f and g agree on V , so must the scores on V .

However, this method is not a BF, since this social preference cannot be induced by restriction from $\sigma(W, g)$.

More importantly, consider the restriction of the method to binary choice. For example on $\{x, y\}$, x scores 5 and y only 4 so $x\sigma y$. Indeed under this binary majority rule, $z\sigma w\sigma x\sigma y$ yet $y\sigma z$, a *cyclic* preference. Thus the social preference on V cannot be constructed simply by pairwise comparisons.

One method of social decision that is frequently recommended is to assign to each individual i a utility function u_i , representing p_i , and to define the social utility function by

$$u_\sigma(x) = \sum_{i \in N} \lambda_i u_i(x), \quad \text{with all } \lambda_i \geq 0.$$

Social preference can be obtained from u_σ in the obvious way by

$$x\sigma(p)y \Leftrightarrow u_\sigma(x) > u_\sigma(y).$$

See for example Harsanyi (1976), Rawls (1971) and Sen (1973). Unfortunately in the ordinal framework, each u_i is only defined up to an equivalence relation, and in this setting the above expression has no meaning, and so $\sigma(p)$ is not well defined. Such a procedure in general cannot be used then to define a social preference function. However, if each feasible set is finite, then as the Borda count example shows we may define

$u_i(x) = (v - r_i)$ where $|V| = v$ and r_i is the rank that x has in i 's preference schedule. Although this gives a well-defined SF, $\sigma(V, p)$, it nonetheless results in a certain inconsistency, since $\sigma(V_1, p_1)$ and $\sigma(V_2, p_2)$ may not agree on the intersection $V_1 \cap V_2$, even though p_1 and p_2 do.

Although a BF avoids this difficulty, other inconsistencies are introduced by the strong independence axiom.

2.3 Arrowian Impossibility Theorems

This section considers the question of the existence of a binary social preference function, $\sigma: F^N \rightarrow F$, where F is some subset of B . In this notation $\sigma: F^N \rightarrow F$ means the following:

Let V be any feasible set in W , and $F(V)^N$ the set of profiles, defined on V , each of whose component preferences belong to F . The domain of σ is the union of $F(V)^N$ across all V in W . That is for each $f \in F(V)^N$, we write $\sigma(f)$ for the binary social preference on V , and require that $\sigma(f) \in F(V)$.

Definition 2.3.1. A BF $\sigma: B^N \rightarrow B$ satisfies

(i) The *weak Pareto property* (P) iff for any $p \in B^N$, any $x, y \in W$,

$$xp_Ny \Rightarrow x\sigma(p)y.$$

(ii) *Non-dictatorship* (ND) iff there is no $i \in N$ such that for all x, y in W ,

$$xp_iy \Rightarrow x\sigma(p)y.$$

A BF σ which satisfies (P) and (ND) and maps $O^N \rightarrow O$ is called a *binary welfare function* (BWF).

Arrow's Impossibility Theorem 2.3.1. For N finite, there is no BWF.

This theorem was originally obtained by Arrow (1951). To prove it we introduce the notion of a *decisive coalition*.

Definition 2.3.2. Let M be a coalition, and σ a BF.

(i) Define M to be *decisive under σ for x against y* iff for all $p \in B^N$

$$xp_My \Rightarrow x\sigma(p)y.$$

- (ii) Define M to be *decisive under σ* iff for all $x, y \in W$, M is decisive for x against y .
- (iii) Let $\mathbb{D}_\sigma(x, y)$ be the family of decisive coalitions under σ for x against y , and \mathbb{D}_σ be the family of decisive coalitions under σ .

To prove the theorem we first introduce the idea of an ultrafilter.

Definition 2.3.3. A family of coalitions can satisfy the following properties.

- (F1) *monotonicity*: $A \subset B$ and $A \in \mathbb{D} \Rightarrow B \in \mathbb{D}$;
- (F2) *identity*: $N \in \mathbb{D}$ and $\Phi \in \mathbb{D}$ (where Φ is the empty set);
- (F3) *closed intersection*: $A, B \in \mathbb{D} \Rightarrow A \cap B \in \mathbb{D}$;
- (F4) *negation*: for any $A \subset N$, either $A \in \mathbb{D}$ or $N \setminus A \in \mathbb{D}$.

A family \mathbb{D} of subsets of N which satisfies (F1), (F2) and (F3) is called a *filter*. A filter \mathbb{D}_1 is said to be *finer* than a filter \mathbb{D}_2 if each member of \mathbb{D}_2 belongs to \mathbb{D}_1 . \mathbb{D}_1 is *strictly finer* than \mathbb{D}_2 iff \mathbb{D}_1 is finer than \mathbb{D}_2 and there exists $A \in \mathbb{D}_1$ with $A \notin \mathbb{D}_2$. A filter which has no strictly finer filter is called an *ultrafilter*. A filter is called *free* or *fixed* depending on whether the intersection of all its members is empty or non-empty. In the case that N is finite then by (F2) and (F3) any filter, and thus any ultrafilter, is fixed.

Lemma 2.3.2. (Kirman and Sondermann, 1972). If $\sigma: O^N \rightarrow O$ is a BF and satisfies the weak Pareto property (P), then the family of decisive coalitions, \mathbb{D}_σ , satisfies (F1), F(2), F(3) and F(4).

We shall prove this lemma below. Arrow's theorem follows from Lemma 2.3.2 since \mathbb{D}_σ will be an ultrafilter which defines a unique dictator. This can be shown by the following three lemmas.

Lemma 2.3.3. Let \mathbb{D} be a family of subsets of N , which satisfies (F1), (F2), (F3) and (F4). Then if $A \in \mathbb{D}$, there is some proper subset B of A which belongs to \mathbb{D} .

Proof. Let B be a proper subset of A with $B \notin \mathbb{D}$. By (F4), $N \setminus B \in \mathbb{D}$. But then by (F3),

$$A \cap (N \setminus B) = A \setminus B \in \mathbb{D}.$$

Hence if $B \subset A$, either $B \in \mathbb{D}$ or $A \setminus B \in \mathbb{D}$. ■

Lemma 2.3.4. If \mathbb{D} satisfies (F1), (F2), (F3) and (F4) then it is an ultrafilter.

Proof. Suppose \mathbb{D}_1 is a filter which is strictly finer than \mathbb{D} . Then there is some $A, B \in \mathbb{D}_1$, with $A \in \mathbb{D}$ but $B \notin \mathbb{D}$. By the previous lemma, either $A \setminus B$ or $A \cap B$ must belong to \mathbb{D} . Suppose $A \setminus B \in \mathbb{D}$. Then $A \setminus B \in \mathbb{D}_1$. But since \mathbb{D}_1 is a filter $(A \setminus B) \cap B = \Phi$ must belong to \mathbb{D} , which contradicts (F2). Hence $A \cap B$ belongs to \mathbb{D} . But by (F1), $B \in \mathbb{D}$. Hence \mathbb{D} is an ultrafilter. ■

Lemma 2.3.5. If N is finite and \mathbb{D} is an ultrafilter with $\mathbb{D} = \{A_j\}$ then

$$\bigcap A_j = \{i\},$$

where $\{i\}$ is decisive and consists of a single member of N .

Proof. Consider any $A_j \in \mathbb{D}$, and let $i \in A_j$. By (F4) either $\{i\} \in \mathbb{D}$ or $A_j - \{i\} \in \mathbb{D}$. If $\{i\} \in \mathbb{D}$, then $A_j - \{i\} \in \mathbb{D}$. Repeat the process a finite number of times to obtain a singleton $\{i\}$, say, belonging to \mathbb{D} . ■

Proof of Theorem 2.3.1. For N finite, by the previous four lemmas, the family of σ -decisive coalitions forms an ultrafilter. The intersection of all decisive coalitions is a single individual i , say. Since this intersection is finite, $\{i\} \in \mathbb{D}_\sigma$. Thus i is a *dictator*. Consequently any BF $\sigma: O^N \rightarrow O$ which satisfies (P) must be dictatorial. Hence there is no BWF. ■

Note that when N is infinite there can exist a BWF σ (Fishburn, 1970). However, its family of decisive coalitions still forms an *ultrafilter*. See Schmitz (1977) and Armstrong (1980) for further discussion on the existence of a BWF when N is an infinite society.

The rest of this section will prove Lemma 2.3.2. The following definitions are required.

Definition 2.3.4. Let σ be a BF, M a coalition, p a profile, $x, y \in W$.

- (i) M is *almost decisive for x against y with respect to p* iff $x p_M y$, $y p_{N-M} x$ and $x \sigma(p) y$.

- (ii) M is *almost decisive* for x against y iff for all $p \in B^N$, $x p_M y$ and $p_{N-M} \Rightarrow x \sigma(p) y$. Write $\mathbb{D}_\sigma^0(x, y)$ for the family of coalitions almost decisive for x against y .
- (iii) M is *almost decisive* if it is almost decisive for x against y for all $x, y \in W$. Write \mathbb{D}_σ^0 for this family.

As before let $\mathbb{D}_\sigma(x, y)$ be the family of coalitions decisive under σ for x against y , and \mathbb{D}_σ be the family of decisive coalitions (see Definition 2.3.2). Note that

$$\begin{array}{ccc} \mathbb{D}_\sigma & \subset & \mathbb{D}_\sigma(x, y) \\ \cap & & \cap \\ \mathbb{D}_\sigma^0 & \subset & \mathbb{D}_\sigma^0(x, y) \end{array}$$

since being *decisive* is a stronger property than being *almost decisive*.

Lemma 2.3.6. If $\sigma: B^N \rightarrow B$ is a BF, and M is almost decisive for x against y with respect to some f , then $M \in \mathbb{D}_\sigma^0(x, y)$.

Proof. Suppose there is some f such that $x f_M y, y f_{N-M} x$ and $x \sigma(f) y$. Let g be any profile in B^N which agrees with f on $\{x, y\}$. Since $x \sigma(f) y$ by the strong independence axiom, Π^* , we obtain $x \sigma(g) y$.

So $M \in \mathbb{D}_\sigma^0(x, y)$. ■

Lemma 2.3.7. (Sen, 1970). Suppose $\sigma: T^N \rightarrow T$ is a BF. Then it satisfies

$$\mathbb{D}_\sigma = \mathbb{D}_\sigma(x, y)$$

for any x, y .

Proof.

(i) We seek first to show that

$$\mathbb{D}_\sigma^0(x, y) \subset \mathbb{D}_\sigma(x, z)$$

for any $z \neq x$ or y . Let $M \in \mathbb{D}_\sigma^0(x, y)$. We need to show that

$$x f_M z \Rightarrow x \sigma(f) z$$

for any $f \in T^N$. Let $g \in T^N$, and suppose $x g_M y g_M z$ and $y g_{N-M} x, y g_{N-M} z$. Thus $x g_M z$, by transitivity of $g_i, i \in M$. Since $M \in \mathbb{D}_\sigma^0(x, y)$, $x \sigma(g) y$. By the Pareto property, $y \sigma(g) z$. By transitivity

$x\sigma(g)z$. Let xf_Mz , and choose g such that $f = g$ on $\{x, z\}$. Since $x\sigma(g)z$, by Π^* , $x\sigma(f)z$, so $M \in \mathbb{D}_\sigma(x, z)$.

(ii) Now we show that $\mathbb{D}_\sigma^0(x, y) \subset \mathbb{D}_\sigma(z, y)$ for $z \neq x$ or y . Let $M \in \mathbb{D}_\sigma^0(x, y)$. In the same way, let $h \in T^N$ with zh_Mxh_My , $zh_{N-M}xy$ and $h_{N-M}x$. Then zh_My by transitivity of $h_i, i \in M$. Since $M \in \mathbb{D}_\sigma^0(x, y)$, $x\sigma(h)y$. By the Pareto property, $z\sigma(h)x$. By transitivity $z\sigma(h)y$. Suppose $f \in T^N$, with zf_My and construct $f = h$ on $\{z, y\}$. By Π^* , $z\sigma(f)y$, so $M \in \mathbb{D}_\sigma(z, y)$.

(iii) By reiteration of (i) and (ii), $\mathbb{D}_\sigma^0(x, y) \subset \mathbb{D}_\sigma(u, v)$ for any $u, v \in W$. But since $\mathbb{D}_\sigma^0(x, y) \subset \mathbb{D}_\sigma(x, y)$, this shows that $\mathbb{D}_\sigma^0(x, y) \subset \mathbb{D}_\sigma$. However, by definition $\mathbb{D}_\sigma \subset \mathbb{D}_\sigma^0(x, y)$, so

$$\mathbb{D}_\sigma = \mathbb{D}_\sigma^0 = \mathbb{D}_\sigma(x, y) = \mathbb{D}_\sigma^0(x, y), \text{ for any } x, y. \quad \blacksquare$$

Lemma 2.3.8. (Hanssen, 1976). If $\sigma: T^N \rightarrow T$ is a BF and satisfies (P) then \mathbb{D}_σ is a *filter*.

Proof.

(F1) Suppose $A \in \mathbb{D}_\sigma$ and $A \subset B$. Now

$$\begin{aligned} xf_By &\Rightarrow xf_Ay \Rightarrow x\sigma(f)y, \\ \text{so } A \in \mathbb{D}_\sigma(x, y) &\Rightarrow B \in \mathbb{D}_\sigma(x, y) \end{aligned}$$

and by the previous lemma, $B \in \mathbb{D}_\sigma$.

(F2) By (P), $N \in \mathbb{D}_\sigma$. Suppose that $\Phi \in \mathbb{D}_\sigma$. But this would imply, for some p , xp_Ny and $y\sigma(p)x$ which contradicts (P).

(F3) Suppose that $A, B \in \mathbb{D}_\sigma$. Let

$$\begin{aligned} V_1 &= A \cap B, \\ V_2 &= A \cap (N \setminus B), \\ V_3 &= (N \setminus A) \cap B, \\ V_4 &= N \setminus (A \cup B). \end{aligned}$$

Define p on $\{x, y, z\}$, in the following way. To each individual in group V_i , for $i = 1, \dots, 4$, assign the preference p_i in the following

fashion:

$$zp_1xp_1y$$

$$xp_2yp_2z$$

$$yp_3zp_2x$$

$$yp_4xp_4z.$$

Since

$$A = V_1 \cup V_2 \in \mathbb{D}_\sigma, \text{ we obtain } x\sigma(p)y,$$

$$B = V_1 \cup V_3 \in \mathbb{D}_\sigma, \text{ we obtain } z\sigma(p)y.$$

By transitivity, $z\sigma(p)y$. Now $zp_{V_1}y$, $yp_{N-V_1}z$ and $z\sigma(p)y$. By Lemma 2.3.6, $V_1 \in \mathbb{D}_\sigma^0(z, y) = \mathbb{D}_\sigma$. Thus $A \cap B \in \mathbb{D}_\sigma$. ■

This lemma demonstrates that if $\sigma: T^N \rightarrow T$ is a BF which satisfies (P) then \mathbb{D}_σ is a filter. However, if $p \in O^N$ then $p \in T^N$, and if $\sigma(p) \in O$ then $\sigma(p) \in T$, by Lemma 2.1.3. Hence to complete the proof of Lemma 2.3.2 only the following lemma needs to be shown.

Lemma 2.3.9. If $\sigma: O^N \rightarrow O$ is a BF and satisfies (P) then \mathbb{D}_σ satisfies (F4).

Proof. Suppose $M \in \mathbb{D}_\sigma$. We seek to show that $N \setminus M \in \mathbb{D}_\sigma$. If for any f , there exist $x, y \in W$ such that yf_Mx and $y\sigma(f)x$, then M would belong to $\mathbb{D}_\sigma(x, y)$ and so be decisive. Thus for any f , there exist $x, y \in W$, with yf_Mx and not $(y\sigma(f)x)$ i.e., $xR(\sigma(f))y$. Now consider $g \in O^N$, with $g = f$ on $\{x, y\}$ and $xg_{N-M}z$, $yg_{N-M}z$ and yg_Mz . By Π^* , $xR(\sigma(g))y$. By (P), $y\sigma(g)z$. Since $\sigma(g) \in O$ it is negatively transitive, and by Lemma 2.1.2, $x\sigma(g)z$. Thus $N \setminus M \in \mathbb{D}_\sigma(x, z)$ and so $N \setminus M \in \mathbb{D}_\sigma$. ■

2.4 Power and Rationality

Arrow's theorem showed that there is no binary social preference function which maps weak orders to weak orders and satisfies the Pareto and non-dictatorship requirements when N is finite. Although there may exist a BWF when N is infinite, nonetheless "power" is concentrated in the sense that there is an "invisible dictator." It can be argued that the require-

ment of negative transitivity is too strong, since this property requires that indifference be transitive. Individual indifference may well display intransitivities, because of just perceptible differences, and so may social indifference. To illustrate the problem with transitivity of indifference, consider the binary social preference function, called the *weak Pareto rule* written σ_n and defined by:

$$x\sigma_n(p)y \text{ iff } xp_Ny.$$

In this case $\{N\} = \mathbb{D}_{\sigma_n}$. This rule is a BF, satisfies (P) by definition, and is non-dictatorial. However, suppose the preferences are

$$zp_Mxp_My$$

$$yp_{N-M}zp_{N-M}x$$

for some proper subgroup M in N . Since there is not unanimous agreement, this implies $xI(\sigma_n)yI(\sigma_n)z$. If negative transitivity is required, then it must be the case that $xI(\sigma_n)z$. Yet zp_Nx , so $z\sigma(p)x$. Such an example suggests that the Impossibility Theorem is due to the excessive rationality requirement. For this reason Sen (1970) suggested weakening the rationality requirement.

Definition 2.4.1. A BF $\sigma: O^N \rightarrow T$ which satisfies (P) and (ND) is called a *binary decision function* (BDF).

Lemma 2.4.1. There exists a BDF.

To show this, say a BF σ satisfies the *strong Pareto property* (P*) iff, for any $p \in B^N$, yp_ix for no $i \in N$, and xp_jy for some $j \in N \Rightarrow x\sigma(p)y$. Note that the strong Pareto property (P*) implies the weak Pareto property (P).

Now define a BF $\bar{\sigma}_n$, called the *strong Pareto rule*, by:

$$x\bar{\sigma}_n(p)y \text{ iff } yp_ix \text{ for no } i \in N \text{ and } xp_jy \text{ for some } j \in N.$$

$\bar{\sigma}_n$ may be called the *extension* of σ_n , since it is clear that

$$x\sigma_n(p)y \Rightarrow x\bar{\sigma}_n(p)y.$$

Obviously $\bar{\sigma}_n$ satisfies (P*) and thus (P). However, just as σ_n violates transitive indifference, so does $\bar{\sigma}_n$. On the other hand $\bar{\sigma}_n$ satisfies transitive strict preference.

Lemma 2.4.2. (Sen, 1970). $\bar{\sigma}_n$ is a BDF.

Proof. Suppose $x\bar{\sigma}_n(p)y$ and $y\bar{\sigma}_n(p)z$. Now

$$x\bar{\sigma}_n(p)y \Leftrightarrow xR(p_i)y \text{ for all } i \in N \text{ and } xp_jy \text{ for some } j \in N.$$

Similarly for $\{y, z\}$. By transitivity of $R(p_i)$, we obtain $xR(p_i)z$ for all $i \in N$. By Lemma 2.1.2, xp_jz for some $j \in N$. Hence $x\bar{\sigma}_n(p)z$. ■

While this seems to refute the relevance of the impossibility theorem, note that the only decisive coalition for $\bar{\sigma}_n$ is $\{N\}$. Indeed the strong Pareto rule is somewhat indeterminate, since any individual can effectively veto a decision. Any attempt to make the rule more “determinate” runs into the following problem.

Definition 2.4.2. An *oligarchy* θ_σ for a BF σ is a minimally decisive coalition which belongs to every decisive coalition.

Lemma 2.4.3. (Gibbard, 1969). If N is finite, then any BF $\sigma: O^N \rightarrow T$ which satisfies P has an oligarchy.

Proof. Restrict σ to $\sigma: T^N \rightarrow T$. By Lemma 2.3.8, since σ satisfies (P), its decisive coalitions form a filter. Let $\theta = \cap A_j$, where the intersection runs over all $A_j \in \mathbb{D}_\sigma$. Since N is finite, this intersection is finite, and so $\theta \in \mathbb{D}_\sigma$. Obviously $\theta - \{i\} \in \mathbb{D}_\sigma$ for any $i \in \theta$. Consequently θ is a minimally decisive coalition or oligarchy. ■

The following lemma shows that members of an oligarchy can block social decisions that they oppose.

Lemma 2.4.4. (Schwartz, 1986). If $\sigma: O^N \rightarrow T$ and $p \in O^N$, and θ is the oligarchy for σ , then define

$$\begin{aligned} \theta_x(p) &= \{i \in \theta: xp_iy\} \\ \theta_y(p) &= \{j \in \theta: yp_jx\}. \end{aligned}$$

Then

- (i) $\theta_x(p) \neq \Phi$ and $\theta = \theta_x(p) \cup \theta_y(p) \Rightarrow \text{not } (y\sigma(p)x)$
- (ii) $\theta_x(p) \neq \Phi, \theta_y(p) \neq \Phi$ and $\theta = \theta_x \cup \theta_y \Rightarrow xI(\sigma(p))y$.

Individuals in the oligarchy may thus block decisions in the sense implied by this lemma. From these results it is clear that BDF must concentrate power within some group in the society. If the oligarchy is large, as for $\bar{\sigma}_n$, then we may infer that decision-making costs would be high. If the oligarchy is small, then one would be inclined to reject the rule on normative grounds. Consider for a moment an economy where trades are permitted between actors. With unrestricted exchange any particular coalition M is presumably decisive for *certain* advantageous trades. If we require the resulting social preference to be a BDF, then by Lemma 2.3.7, this coalition M has to be (globally) decisive. Consequently there *must* be some oligarchy. In a free exchange economy there is however no oligarchy, and so the social preference relation must violate either the fundamental independence axiom II*, or the rationality condition. This would seem to be a major contradiction between social choice theory and economic equilibrium theory.

Lemma 2.4.3 suggests that the rationality condition be weakened even further to acyclicity. It will be shown below that acyclicity of a BF is sufficient to define a well-behaved choice procedure.

Definition 2.4.3. A BF $\sigma: A^N \rightarrow A$ which satisfied (P) is called a *binary acyclic preference function* (BAF).

Definition 2.4.4. Let $\mathbb{D} = M_1, \dots, M_r$ be a family of subsets of N . \mathbb{D} is called a *prefilter* iff \mathbb{D} satisfies (F1), (F2) and *non-empty intersection* (F0), namely that there is a non-empty *collegium*, $\kappa(\mathbb{D}) = M_1 \cap M_2 \cdots \cap M_r$.

If σ is a BF, \mathbb{D}_σ is its family of decisive coalitions, and $\kappa_\sigma = \kappa(\mathbb{D}_\sigma)$ is non-empty, then σ is said to be *collegial*. Otherwise σ is said to be *non-collegial*.

Theorem 2.4.5. (Brown, 1973). If $\sigma: A^N \rightarrow A$ is a BF which satisfies (P) then σ is collegial and \mathbb{D}_σ is a prefilter.

Proof. (F1) and (F2) follow as in Lemma 2.3.8. To prove (F0), suppose there exists $\{M_j\}_{j=1}^r$ where each $M_j \in \mathbb{D}_\sigma$, yet this family has empty intersection. Let

$$V = \{a_1, \dots, a_r\}$$

be a collection of distinct alternatives.

For each pair $\{a_j, a_{j+1}\}$, $j = 1, \dots, r - 1$, let p^j be a profile defined on $\{a_j, a_{j+1}\}$ such that $a_j p_i^j a_{j+1}$ for all $i \in A_j$. Thus $a_j \sigma(p^j) a_{j+1}$. In the same way let p^r be defined on $\{a_r, a_1\}$ such that $a_r p_i^r a_1$ for all $i \in A_r$. Thus $a_r \sigma(p^r) a_1$. Now extend $\{p^1, \dots, p^r\}$ to a profile p on V , in such a way that each p_i is acyclic.¹⁰ By the extension property of Π^* ,

$$a_j \sigma(p^j) a_{j+1} \Leftrightarrow a_j \sigma(p) a_{j+1} \text{ etc.}$$

Hence

$$a_1 \sigma(p) a_2 \sigma(p) a_3 \cdots a_r \sigma(p) a_1.$$

This gives an acyclic profile p such that $\sigma(p)$ is cyclic. By contradiction the family $\{M_j\}_{j=1}^r$ must have non-empty intersection. ■

Even acyclicity requires some concentration in power, though the existence of a collegium is of course much less unattractive than the existence of an oligarchy or dictator.

The next section turns to the question of the existence of choice procedures associated with binary preference functions, and relates consistency properties of these procedures to rationality properties of the preference functions.

2.5 Choice Functions

Instead of seeking a preference function σ that satisfies certain rationality conditions, one may seek a procedure which “selects” from a set V some subset of V , in a way which is determined by the profile.

Definition 2.5.1. A *choice function* C is a mapping $C: X \times B^N \rightarrow X$ with the property that $\Phi \neq C(V, p) \subset V$ for any $V \in X$. Note that the notational convention that is used *only* requires that the profile p be defined on V . If f is defined on V_1 , g is defined on V_2 and $f^V = g^V$ for $V = V_1 \cap V_2 \neq \Phi$, then it must be the case that $C(V, f^V) = C(V, g^V)$. Thus by definition a choice function satisfies the analogue of the weak independence axiom (II). Note that there has as yet been no requirement that C satisfy the analogue of the strong independence axiom.

¹⁰A later result, Lemma 3.2.6, shows that this can indeed be done, as long as r is of sufficient cardinality.

Definition 2.5.2. A choice function $C: X \times B^N \rightarrow X$ satisfies the *weak axiom of revealed preference* (WARP) iff whenever $V \subset V'$, and p is defined on V' , with $V \cap C(V', p) \neq \Phi$, then

$$V \cap C(V', p) = C(V, p^V),$$

where p^V is the restriction of p to V . Note the analogue with (II*). If we write $C(V', p)/V$ for $V \cap C(V', p)$ when this is non-empty, then WARP requires that

$$C(V, p^V) = C(V', p)/V.$$

Definition 2.5.3.

(i) A choice function $C: X \times B^N \rightarrow X$ is said to be *rationalized* by an SF $\sigma: X \times B^N \rightarrow B$ iff for any $V \in X$ and any $p \in B^N$,

$$C(V, p) = \{x: y\sigma(V, p)x \text{ for no } y \in V\}.$$

(ii) A choice function $C: X \times B^N \rightarrow X$ is said to be *rationalized* by a BF $\sigma: B^N \rightarrow B$ iff for any $p \in B^N$, and any $x, y \in W, x \neq y$,

$$C(\{x, y\}, p) = \{x\} \Leftrightarrow x\sigma(p)y.$$

(iii) A choice function $C: X \times B^N \rightarrow X$ is said to satisfy the *binary choice axiom* (BICH) iff there is a BF $\sigma: B^N \rightarrow B$ such that for any $V \in X$, any $p \in B^N$,

$$C(V, p) = \{x \in V: y\sigma(p)x \text{ for no } y \in V\}.$$

Say C satisfies BICH w.r.t. σ in this case.

(iv) Given a choice function $C: X \times B^N \rightarrow X$ define the induced BF $\sigma_C: B^N \rightarrow B$ by

$$C(\{x, y\}, p) = \{x\} \Leftrightarrow x\sigma_C(p)y.$$

(v) Given a BF $\sigma: B^N \rightarrow B$ define the *choice procedure*

$$C_\sigma: X \times B^N \rightarrow X$$

by

$$C_\sigma(V, p) = \{x \in V: y\sigma(p)x \text{ for no } y \in V\}.$$

Note that $C_\sigma(V, p)$ may be empty for some V, p .

Lemma 2.5.1. If C satisfies BICH w.r.t. σ then σ rationalizes C .

Proof.

(i) $C(\{x, y\}, p) = \{x\} \Rightarrow \text{not}(y\sigma(p)x)$. If $yI(\sigma(p))x$ then $\text{not}(x\sigma(p)y)$, so $y \in C(\{x, y\}, p)$.

Hence $C(\{x, y\}, p) = \{x\} \Rightarrow x\sigma(p)y$.

(ii) $x\sigma(p)y \Rightarrow \text{not}(y\sigma(p)x)$. Hence $C(\{x, y\}, p) = \{x\}$. ■

Another way of putting this lemma is that if $C = C_\sigma$ is a choice function then $\sigma = \sigma_C$.

In the following we delete reference to p when there is no ambiguity, and simply regard C as a mapping from X to itself.

Example 2.5.1.

(i) Suppose C is defined on the pair sets of $W = \{x, y, z\}$ by

$$\begin{aligned} C(\{x, y\}) &= \{x\}, \\ C(\{y, z\}) &= \{y\} \end{aligned}$$

and

$$C(\{x, z\}) = \{z\}.$$

If C satisfies BICH w.r.t. σ , then it is necessary that $x\sigma y\sigma z\sigma x$, so

$C(\{x, y, z\}) = \Phi$. Hence C cannot satisfy BICH.

(ii) Suppose

$$\begin{aligned} C(\{x, y\}) &= \{x\} \\ C(\{y, z\}) &= \{y, z\}. \end{aligned}$$

If

$$C(\{x, z\}) = \{x, z\},$$

then $x\sigma yI(\sigma)z$ and $xI(\sigma)z$, so

$$C(\{x, y, z\}) = \{x, z\}.$$

While C satisfies BICH w.r.t. σ , σ does not give a weak order, although σ may give a strict partial order.

Theorem 2.5.1. (Sen, 1970). Let the universal set, W , be of finite cardinality.

- (i) If a choice function C satisfies BICH w.r.t. σ , then $\sigma = \sigma_C$ is a BAF.
- (ii) If σ is a BAF then C_σ , restricted to $X \times A^N$, is a choice function.

Proof.

- (i) By Lemma 2.5.1, if C satisfies BICH w.r.t. σ then σ rationalizes C , and so by definition the induced BF, σ_C , is identical to σ . We seek to show that σ is a BAF, or that $\sigma: A^N \rightarrow A$. Suppose on the contrary that σ is not a BAF. Since σ is a BF and W is of finite cardinality, this assumption is equivalent to the existence of a finite subset $V = \{a_1, \dots, a_r\}$ of W , a profile $p \in A(V)^N$, and a cycle

$$a_1\sigma(p)a_2\sigma(p)\cdots a_r\sigma(p)a_1.$$

Let $a_r \equiv a_0$. Then for each $a_j \in V$ it is the case that $a_{j-1}\sigma(p)\sigma_j$. Since C satisfies BICH with respect to σ , it is evident that $C(V, p) = \Phi$. By contradiction, σ is a BAF.

- (ii) We seek now to show that for any finite set, V , if $A(V)^N$ and $p \in A(V)^N$ and $\sigma(p) \in A(V)$ then $C_\sigma(V, p) \neq \Phi$. First, let $I(\sigma(p))$ and $R(\sigma)$ represent the indifference and weak preference relations defined by $\sigma(p)$. Suppose that $V = \{x_1, \dots, x_r\}$. If

$$x_1I(\sigma(p))x_2 \dots x_{r-1}I(\sigma(p))x_r$$

then $C_\sigma(V, p) = V$. So suppose that for some $a_1, a_2 \in V$ it is the case that $a_2\sigma(p)a_1$. If $a_2 \in C_\sigma(V, p)$ then there exists a_3 , say, such $a_3\sigma(p)a_2$. If $a_1\sigma(p)a_3$, then by acyclicity, not $(a_2\sigma(p)a_1)$. Since $\sigma(p)$ is a strict preference relation, this is a contradiction. Hence not $(a_1\sigma(p)a_3)$, and so $a_3 \in C_\sigma(\{a_1, a_2, a_3\}, p)$. By induction,

$$C_\sigma(V', p) \neq \Phi \Rightarrow C_\sigma(V'', p) \neq \Phi$$

whenever $|V'| + 1 = |V''|$ and $V' \subset V'' \subset W$. Thus $C(V, p) \neq \Phi$ for any finite subset V of W . ■

Lemma 2.5.3. (Schwartz, 1976). A choice function C satisfies BICH iff

for any V_1, V_2 ,

$$C(V_1) \cap C(V_2) = C(V_1 \cup V_2) \cap V_1 \cap V_2.$$

Consider for the moment $V_1 \subset V_2$. By the above

$$C(V_1 \cup V_2) \cap V_1 \cap V_2 = C(V_2) \cap V_1 = C(V_1) \cap C(V_2) \subseteq C(V_1).$$

Brown (1973) had shown this earlier. Since this is part of the WARP condition, WARP must imply BICH.

Lemma 2.5.4. (Arrow, 1959). A choice function, C , satisfies WARP iff C satisfies BICH and σ_C is a BF $\sigma_C: O^N \rightarrow O$.

Even though WARP is an attractive property of a choice function, it requires that σ_C satisfy the strong rationality condition sufficient to induce a dictator. Consider now the properties of a choice function when $\sigma_C: T^N \rightarrow T$.

Definition 2.5.4. The choice function $C: X \rightarrow X$ satisfies

(i) *Independence of path* (IIP) iff

$$C(\cup_{j=1}^r C(V_j)) = C(V)$$

whenever

$$V = \cup_{j=1}^r V_j.$$

(ii) *Exclusion* (EX) iff

$$V_1 \subset V \setminus C(V) \Rightarrow C(V \setminus V_1) \subseteq C(V).$$

Example 2.5.2. To illustrate (EX), consider Example 2.2.1 above and let C be the procedure which selects from V the top-most ranked alternative under the Borda count on V . Thus suppose $V = \{x, y, z, w\}$ and consider the profile

i	j	k
x	y	z
y	z	w
z	w	x
w	x	y

On V , the Borda count for $\{z, y, x, w\}$ is $\{9, 8, 7, 6\}$. Thus $C(V) = \{z\}$. Let $V_1 = \{w\}$, and observe that $V_1 \subset V \setminus C(V)$. Now perform the Borda count on $V \setminus V_1 = \{x, y, z\}$. However, $C(\{x, y, z\}) = \{x, y, z\} \not\subset \{z\}$, so EX is violated. The exclusion axiom is sometimes confused with the independence of infeasible alternatives for choice functions. Schwartz (1976) and Plott (1970, 1973) have examined the nature of the conditions (EX) and IIP.

Lemma 2.5.5. (Schwartz, 1976). A choice function C satisfies (EX) iff C satisfies BICH and σ_C is a BF : $T^N \rightarrow T$.

Lemma 2.5.6. (Plott, 1970, 1973).

- (i) If a choice function C satisfies II P then the BF $\sigma_C: T^N \rightarrow T$ and $C \subset C_{\sigma_C}$.
- (ii) If $\sigma: T^N \rightarrow T$ is a BF, then C_σ satisfies IIP.

Note that if a choice function C satisfies II P then $x \in C(V)$ implies there is no y s.t. $y \sigma_C x$. Suppose if the following property on C is satisfied: [for all $x, y, \in V, C\{x, y\} = \{x, y\} \Rightarrow C(V) = V$]. Then if C satisfies II P it is the case that $C = C_{\sigma_C}$. Since non-oligarchic binary preference functions cannot map $T^N \rightarrow T$, Ferejohn and Grether (1977) have proposed weakening II P in the following way.

Definition 2.5.5. A $C: X \rightarrow X$ satisfies *weak path independence* (*IIP) iff

$$C(\cup_{j=1}^r C(V_j)) \subset C(V)$$

whenever

$$V = \cup_{j=1}^r V_j.$$

Lemma 2.5.7. (Ferejohn and Grether, 1977). Let C be a choice function $C: X \times B^N \rightarrow X$ which satisfies *IIP. If V is a $\sigma_C(p)$ cycle, then

- (i) $C(V, p) = V$.
- (ii) Moreover, if for any $x \in W \setminus V$ there is some set $Y \subset W$ such that

$$C(Y \cup \{x\}, p) \subset V,$$

then

$$V \subset C(W, p).$$

Example 2.5.3. If majority rule is used with the profile given in Example 2.5.2, then there is a cycle

$$z\sigma(p)w\sigma(p)x\sigma(p)y\sigma(p)z.$$

So any choice function C which satisfies *II P has to choose $C(V, p) = V = \{x, y, z, w\}$. However, zp_Nw , so the choice function can choose alternatives which are beaten under the weak Pareto rule (i.e., are not Pareto optimal).

If one seeks a choice function which satisfies the strong consistency properties of WARP or EX, then choices must be made by binary comparisons (BICH), and consequently the Arrowian Impossibility Theorems are relevant. If one seeks only IIP, then $C \subset C_{\sigma_C}$, and again binary comparisons must be made, so the Impossibility Theorems are once more relevant. The attraction of *IIP is that it permits choice to be done by division. Suppose a decision problem, V , is divided into components V_j , choice made from V_j , and then choice made from these. Then the resultant decision must be compatible with whatever choice would have been made from V . *II P would seem to be a minimal consistency property of a choice procedure. Unfortunately it requires the selection of cycles, no matter how large these are.

The next chapter examines the occurrence of cycles for arbitrary voting rules. Since cycles, and particular non-Paretian cycles, will occur under such rules, there is a contradiction between implementability (or path independence) and Pareto optimally for general voting processes.

Chapter 3

Voting Rules

3.1 Simple Binary Preference Functions

The previous chapter showed that for a binary preference function σ to satisfy certain rationality postulates it is *necessary* that the family of decisive coalitions obey various filter properties. A natural question is whether the previous restrictions on power, imposed by the filter properties, are *sufficient* to ensure rationality. In general, however, this is not the case. To see this, for a given class of coalitions define a new BF as follows.

Definition 3.1.1. Let N be a fixed set of individuals, and \mathbb{D} a family of subsets of N . Define the BF $\sigma^{\mathbb{D}}: B^N \rightarrow B$ by:

$$x\sigma^{\mathbb{D}}(p)y \Leftrightarrow \{i \in N : xp_iy\} \in \mathbb{D}, \text{ whenever } x, y \in W.$$

For a given BF $\sigma: B^N \rightarrow B$, \mathbb{D}_σ is defined to be its family of decisive coalitions. Consequently there are two transformations:

$$\sigma \rightarrow \mathbb{D}_\sigma \text{ and } \mathbb{D}_\sigma \rightarrow \sigma^{\mathbb{D}_\sigma}.$$

In terms of these transformations, the previous results may be written:

Lemma 3.1.1. If σ is a BF which satisfies (P) and

- (i) $\sigma: O^N \rightarrow O$ then \mathbb{D}_σ is an ultrafilter;
- (ii) $\sigma: T^N \rightarrow T$ then \mathbb{D}_σ is a filter;
- (iii) $\sigma: A^N \rightarrow A$ then \mathbb{D}_σ is a prefilter.

Lemma 3.1.2. (Ferejohn, 1977). If \mathbb{D} is

- (i) an ultrafilter then $\sigma^{\mathbb{D}}: O^N \rightarrow O$ is a BF and satisfies (P);
- (ii) a filter then $\sigma^{\mathbb{D}}: T^N \rightarrow T$ is a BF and satisfies (P);
- (iii) a prefilter then $\sigma^{\mathbb{D}}: A^N \rightarrow A$ is a BF and satisfies (P).

However, even though \mathbb{D}_σ satisfies one of the filter properties, σ need not satisfy the appropriate rationality property. The problem is that the transformation

$$\sigma \rightarrow \sigma^{\mathbb{D}_\sigma}$$

is “structure forgetting.” It is easy to see that for any $x, y \in W, p \in B^N$,

$$x\sigma^{\mathbb{D}_\sigma}(p)y \Rightarrow x\sigma(p)y.$$

Thus $\sigma^{\mathbb{D}_\sigma} \subset \sigma$. For this reason it may be the case that, for some x, y, p , we obtain $x\sigma(p)y$ but also not $(x\sigma^{\mathbb{D}_\sigma}(p)y)$. To see this consider the following example due to Ferejohn and Fishburn (1979).

Example 3.1.1. Let $N = \{1, 2\}, W = \{x, y, z\}$ and T be the cyclic relation

$$xTy, yTz, zTx.$$

Define

$$x\sigma(p)y \Leftrightarrow xp_1y \text{ or } [xI(p_1)y \text{ and } xTy].$$

It follows from this definition that

$$\mathbb{D}_\sigma = \{\{1\}, \{1, 2\}\} \text{ is an ultrafilter.}$$

Obviously $x\sigma^{\mathbb{D}_\sigma}(p)y \Leftrightarrow xp_1y$. Hence $\sigma^{\mathbb{D}_\sigma}: O^N \rightarrow O$ is dictatorial. On the other hand, if p is a profile under which $\{1\}$ is indifferent on $\{x, y, z\}$ then $x\sigma(p)y\sigma(p)z\sigma(p)x$, so $\sigma(p)$ is cyclic.

Definition 3.1.2. If σ_1, σ_2 are two binary preference functions on W , and for all $p \in B^N$

$$x\sigma_1(p)y \Rightarrow x\sigma_2(p)y \text{ for any } x, y \in W,$$

then say that σ_2 is *finer* than σ_1 , and write $\sigma_1 \subseteq \sigma_2$. If in addition $x\sigma_2(p)y$ yet not $(x\sigma_1(p)y)$ for some x, y , then say σ_2 is *strictly finer* than σ_1 , and write $\sigma_1 \subset \sigma_2$.

If σ_2 is strictly finer than σ_1 then σ_1 may satisfy certain rationality

properties, such as acyclicity, although σ_2 need not. On the other hand, if σ_1 fails a rationality property, such as acyclicity, then so will σ_2 .

From the above discussion, $\sigma^{\mathbb{D}\sigma} \subseteq \sigma$. Indeed, in Example 3.1.1, σ is strictly finer than $\sigma^{\mathbb{D}\sigma}$, and is cyclic, even though $\sigma^{\mathbb{D}\sigma}$ is acyclic. To induce rationality conditions on σ from properties of $\mathbb{D}\sigma$, we can require $\sigma = \sigma^{\mathbb{D}\sigma}$ by assuming certain additional properties on σ .

Definition 3.1.3. Let p, q be any profiles in B^N and x, y alternatives in W . A BF σ is

(i) *decisive* iff $\{i \in N : xp_iy\} = \{i \in N : xq_iy\}$ implies that $x\sigma(p)y \Rightarrow x\sigma(q)y$;

(ii) *neutral* iff

$$\{i : xp_iy\} = \{i : aq_ib\}$$

and

$$\{j : yp_jx\} = \{j : bq_ja\}$$

implies that $\sigma(p)(x, y) = \sigma(q)(a, b)$;

(iii) *monotonic* iff

$$\{i : xp_iy\} \subset \{i : aq_ib\}$$

and

$$\{j : yp_jx\} \supset \{j : bq_ja\}$$

implies that $[x\sigma(p)y \Rightarrow a\sigma(q)b]$;

(iv) *anonymous* iff

$$\sigma(p) = \sigma(s(p)),$$

where $s : N \rightarrow N$ is any permutation of N , and

$$s(p) = (p_{s(1)}, p_{s(2)}, \dots, p_{s(n)});$$

(v) *simple* iff $\sigma = \sigma^{\mathbb{D}\sigma}$.

It readily follows that a simple rule is characterized by its decisive coalitions.

Lemma 3.1.3. A BF σ is *simple* iff σ is *decisive*, *neutral* and *monotonic*.

To distinguish between neutrality and decisiveness consider the following example, adapted from Ferejohn and Fishburn (1979).

Example 3.1.2.

(i) Let $\sigma = \sigma_n \cup \sigma'$ on $W = \{a, b, c\}$, where, as before,

$$x\sigma_n(p)y \Leftrightarrow xp_Ny,$$

and for the fixed pair $\{a, b\}$,

$$a\sigma'(p)b \text{ iff } ap_1b \text{ and } aI(p_i)b, \forall i \neq 1.$$

It is clear that $\mathbb{D}_\sigma = \{N\}$. However, σ is *not* decisive. To see this construct two profiles p, q such that:

$$aq_1b \text{ yet } bq_i a \text{ for } i \neq 1$$

$$ap_1b \text{ and } aI(p_i)b \text{ for } i \neq 1$$

$$\text{with } p = q \text{ on } \{a, b\}.$$

Although $\{i: aq_i b\} = \{i: ap_i b\}$ it is the case that $a\sigma(p)b$ yet not $[a\sigma(q)b]$. Hence σ is neither decisive nor neutral.

(ii) Let $\sigma = \sigma_n \cup \sigma'$ where for any $x, y \in W$ $x\sigma'(p)y$ iff xp_1y and $xI(p_i), \forall i \neq 1$.

As above, σ is not decisive, but it is neutral.

(iii) Let $\sigma = \sigma_n \cup \sigma'$ where σ' is the decisive BF defined by

$$\mathbb{D}_{\sigma'}(a, b) = \{1\}, \mathbb{D}_{\sigma'}(b, c) = \{2\}, \mathbb{D}_{\sigma'}(c, a) = \{3\}.$$

While σ is decisive, it is not neutral.

A simple BF is called a simple voting rule. To illustrate various kinds of simple voting rules consider the following.

Definition 3.1.4.

- (i) A voting rule, σ , is called a *simple weighted majority rule* iff:
- (a) each individual i in N is assigned a real valued integer *weight* $s(i) \geq 0$;
 - (b) each coalition M is assigned the weight

$$s(M) = \sum_{i \in M} s(i);$$

- (c) q is a real valued integer with $\frac{s(N)}{2} < q \leq s(N)$ such that $M \in$

\mathbb{D}_σ iff $s(M) \geq q$;

(d) $\sigma = \sigma^{\mathbb{D}_\sigma}$.

(ii) A simple weighted majority rule is written $\sigma_{q(s)}$, where

$$q(s) = [q: s(1), \dots, s(i), \dots, s(n)].$$

If $s(i) = 1$ for each $i \in N$, and $q > \frac{n}{2}$, then the voting rule is called the *simple q -majority rule*, or q -rule, and denoted σ_q .

(iii) *Simple majority rule*, written as σ_m , is the q -rule defined in the following way:

if $n = 2k + 1$ is odd, then $q = k + 1 = m$;

if $n = 2k$ is even, then $q = k + 1 = m$.

(iv) In the case $q = n$, this gives the *weak Pareto rule* σ_n .

Note that σ_q is *anonymous* as well as *simple*. We shall often refer to a simple weighted majority rule as a $q(s)$ -rule. Two further properties of a voting rule σ are as follows.

Definition 3.1.5. A voting rule σ is

(i) *proper* iff for any $A, B \in \mathbb{D}_\sigma$, $A \cap B \neq \Phi$;

(ii) *strong* iff $A \in \mathbb{D}_\sigma$ then $N \setminus A \in \mathbb{D}_\sigma$.

For example consider a $q(w)$ -simple weighted majority rule, σ . Because $q > \frac{s(N)}{2}$ then $M \in \mathbb{D}_\sigma$ implies that

$$s(N \setminus A) = s(N) \setminus w(A) < \frac{s(N)}{2}.$$

Hence, if $B \subset N \setminus A$ then $B \in \mathbb{D}_\sigma$. Thus σ must be proper. On the other hand, suppose σ is simple majority rule with $|N| = 2k$, an even integer. Then if $|A| = k$, $A \in \mathbb{D}_\sigma$ but $|N \setminus A| = k$ and $N \setminus A \notin \mathbb{D}_\sigma$. Thus σ is not strong. However, if $|N| = 2k + 1$, an odd integer, and $|A| = k$ then $A \notin \mathbb{D}_\sigma$ but $|N \setminus A| = k + 1$ and so $N \setminus A \in \mathbb{D}_\sigma$. Thus σ is strong. Another interpretation of these terms is as follows. If $A \notin \mathbb{D}_\sigma$ then A is said to be *losing*. On the other hand if A is such that $N \setminus A \notin \mathbb{D}_\sigma$ then call A

blocking. If σ is strong, then no losing coalition is blocking, and if σ is proper then every winning coalition is blocking.

Given a $q(s)$ -rule, σ , it is possible to define a new rule $\bar{\sigma}$, called the *extension* of σ , such that $\bar{\sigma}$ is finer than σ .

Definition 3.1.6.

(i) For a $q(w)$ rule, σ , define its extension $\bar{\sigma}$ by

$$x\bar{\sigma}(p)y \Leftrightarrow s(M_{xy}) \geq q \left(\frac{s(M_{xy}) + s(M_{yx})}{s(N)} \right)$$

where

$$M_{xy} = \{i: xp_iy\}$$

and

$$M_{yx} = \{j: yp_jx\}.$$

Write $\bar{\sigma}_q$ for the extension of the simple q -majority rule, σ_q . Then

$$x\bar{\sigma}_q(p)y \Leftrightarrow |M_{xy}| \geq q \left(\frac{|M_{xy}| + |M_{yx}|}{|N|} \right).$$

(ii) The *weak Pareto rule*, σ_n is defined by

$$x\bar{\sigma}_n(p)y \Leftrightarrow |M_{xy}| = n.$$

(iii) The *strong Pareto rule*, $\bar{\sigma}_n$ is defined analogously by

$$x\bar{\sigma}_n(p)y \Leftrightarrow |M_{xy}| \geq |M_{xy}| + |M_{yx}|.$$

That is to say

$$|M_{yx}| = \Phi.$$

(iv) *Plurality rule*, written σ_{plur} , is defined by:

$$x\sigma_{plur}(p)y \Leftrightarrow |M_{xy}| > |M_{yx}|.$$

Lemma 3.1.4. The simple q -majority rules and their extensions are *nested*:

i.e., for any $q, n/2 < q \leq n$,

$$\begin{array}{ccc} \sigma_n & \subset & \bar{\sigma}_n \\ \cap & & \cap \\ \sigma_q & \subset & \bar{\sigma}_q \\ \cap & & \cap \\ \sigma_m & \subset & \bar{\sigma}_m \end{array}$$

where as before $\sigma_1 \subset \sigma_2$ iff $x\sigma_1(p)y \Rightarrow x\sigma_2(p)y$
wherever $x, y \in W, p \in B^N$.

Notice that it is possible that $x\sigma_{plur}(p)y$ yet not $[x\sigma_m(p)y]$. For example, if $n = 4$, and $|M_{xy}| = 2, |M_{yx}| = 1$, we obtain $x\sigma_{plur}(p)y$. However,

$$|M_{xy}| < \left\lceil \frac{3}{4} \right\rceil [|M_{xy}| + |M_{yx}|]$$

so not $[x\bar{\sigma}_m(p)y]$.

From Brown's result (Theorem 2.4.5) for a voting rule, σ , to be *acyclic* it is necessary that there be a collegium κ . Indeed, for a collegial voting rule, each member i of the collegium has the veto power:

$$xp_iy \Rightarrow \text{not } (y\sigma(p)x).$$

As we have seen $\bar{\sigma}_n$ maps O^N to T , and σ_n maps $T^N \rightarrow T$.

However, any anonymous q -rule σ , with $q < n$, is non-collegial, and so it is possible to find a profile p such that $\sigma(p)$ is *cyclic*.

The next section shows that such a profile must be defined on a feasible set containing a sufficiently large number of alternatives.

3.2 Acyclic Voting Rules on Restricted Sets of Alternatives

In this section we shall show that when the cardinality of the set of alternatives is suitably restricted, then a voting rule will be acyclic.

Let $B(r)^N$ be the class of profiles, each defined on a feasible set of *at most* r alternatives, and let $F(r)^N$ be the natural restriction to a subclass defined by $F \subset B$. Thus $A(r)^N$ is the set of *acyclic* profiles defined on feasible sets of cardinality at most r .

Lemma 3.2.1. (Ferejohn and Grether, 1974). Let $q^r = \left(\frac{r-1}{r}\right)n$, for a given $|N| = n$.

- (i) A q -rule maps $A(r)^N \rightarrow A(r)$ iff $q > q^r$.
- (ii) The extension of a q -rule maps $O(r)^N \rightarrow A(r)$ iff $q > q^r$.

Comment 3.2.1. Note that the inequality $q > \left(\frac{r-1}{r}\right)n$ can be written $rq > rn - n$ or $r < \frac{n}{n-q}$ if $q \neq n$. In the case $q \neq n$, if we define the integer $v(n, q) = \left\lfloor \frac{q}{n-q} \right\rfloor$ to be the greatest integer which is strictly less than $\frac{q}{n-q}$. Then the inequality $q > q^r$ can be written $r \leq v(n, q) + 1$.

Lemma 3.2.1 can be extended to cover the case of a general non-collegial voting rule where the restriction on the size of the alternative set involves not $v(n, q)$ but the *Nakamura* number of the rule.

Definition 3.2.1.

- (i) Let \mathbb{D} be a family of subsets of N . If the collegium, $\kappa(\mathbb{D})$, is non-empty then \mathbb{D} is called *collegial* and the *Nakamura number* $v(\mathbb{D})$ is defined to be ∞ .
- (ii) A member M of \mathbb{D} is *minimal decisive* if and only if M belongs to \mathbb{D} , but for no member i of M does $M \setminus \{i\}$ belong to \mathbb{D} .
- (iii) If the collegium $\kappa(\mathbb{D})$ is empty then \mathbb{D} is called *non-collegial*. If \mathbb{D}' is a subfamily of \mathbb{D} consisting of minimal decisive coalitions, with $\kappa(\mathbb{D}') = \Phi$ then call \mathbb{D}' a *Nakamura subfamily* of \mathbb{D} .
- (iv) Consider the collection of all Nakamura subfamilies of \mathbb{D} . Since N is finite these subfamilies can be ranked by their cardinality. Define the *Nakamura number*, $v(\mathbb{D})$, by

$$v(\mathbb{D}) = \min\{|\mathbb{D}'| : \mathbb{D}' \subset \mathbb{D} \text{ and } \kappa(\mathbb{D}') = \Phi\}$$

A *minimal non-collegial subfamily* is a Nakamura subfamily, \mathbb{D}_{\min} , such that $|\mathbb{D}_{\min}| = v(\mathbb{D})$.

- (v) If σ is a BF with \mathbb{D}_σ its family of decisive coalitions, then define the *Nakamura number*, $v(\sigma)$, to be $v(\mathbb{D}_\sigma)$, and say σ is *collegial* or *non-collegial* depending on whether \mathbb{D}_σ is collegial or not.

Example 3.2.1. As an example, consider the $q(w)$ -rule given by

$$q(w) = [q: w_1, w_2, w_3, w_4] = [6: 5, 3, 2, 1].$$

We may take $\mathbb{D}_{\min} = \{\{1, 4\}, \{1, 3\}, \{2, 3, 4\}\}$ so $v(\sigma_{q(w)}) = 3$.

For a non-collegial q -rule, σ_q , we can relate $v(\sigma_q)$ to $v(n, q)$.

Lemma 3.2.2.

(i) For any non-collegial voting rule σ , with a society of size n ,

$$v(\sigma) \leq n.$$

(ii) For a non-collegial q -rule σ_q ,

$$\begin{aligned} v(\sigma_q) &= 2 + v(n, q), \\ \text{so that } v(\sigma_q) &< 2 + q. \end{aligned}$$

(iii) For any *proper* voting rule σ ,

$$v(\sigma) \geq 3.$$

(iv) For simple majority rule σ_m , $v(\sigma_m) = 3$ except when $(n, q) = (4, 3)$ in which case $v(\sigma_3) = 4$.

Proof.

(i) Consider any Nakamura subfamily, \mathbb{D}' of $\mathbb{D}_{\sigma'}$ where $|\mathbb{D}'| = h$ and each coalition M_i in \mathbb{D}' is of size $m_i \leq n - 1$. Then $|M_i \cap M_j| \leq n - 2$, for any $M_i, M_j \in \mathbb{D}'$. Clearly $|\kappa(\mathbb{D}')| \leq n - h$. In particular, for the *minimal non-collegial subfamily*, \mathbb{D}_{\min} , $h = v(\sigma)$ and $0 = |\kappa(\mathbb{D}_{\min})| \leq n - v(\sigma)$. Thus $v(\sigma) \leq n$.

(ii) For a q -rule, σ_q , let \mathbb{D}_q be its family of decisive coalitions, and let \mathbb{D}_{\min} be a *minimal non-collegial subfamily*. If $M_1, M_2 \in \mathbb{D}_{\min}$ then $|M_1 \cap M_2| \geq 2q - n$. By induction if $|\mathbb{D}'| = h$ for any $\mathbb{D}' \subset \mathbb{D}_{\min}$ then $|\kappa(\mathbb{D}')| \geq hq - (h - 1)n$. Thus $h < \frac{n}{n - q} \Rightarrow |\kappa(\mathbb{D}')| > 0$. Now

$$v(n, q) < \frac{q}{n - q} \text{ so } 1 + v(n, q) < \frac{n}{n - q}.$$

Hence $|\mathbb{D}'| \leq 1 + v(n, q) \Rightarrow |\kappa(\mathbb{D}')| > 0$. Therefore $v(\sigma_q) > 1 + v(n, q)$. On the other hand, there exists \mathbb{D}' such that $|\kappa(\mathbb{D}')| = hq -$

$(h-1)n$. Consequently $h \geq \frac{n}{n-q} \Rightarrow |\kappa(\mathbb{D}')| = 0$. But $1 + v(n, q) < \frac{n}{n-q} \leq 2 + v(n, q)$. Thus $h \geq 2 + v(n, q) \Rightarrow |\kappa(\mathbb{D}')| = 0$. Hence $v(\sigma_q) = 2 + v(n, q)$. If $q = n$, then σ_q is collegial. When $q \leq n-1$, clearly

$$v(\sigma_q) = 2 + v(n, q) < 2 + \frac{q}{n-q} \leq 2 + q.$$

Hence $v(\sigma_q) < 2 + q$.

- (iii) By definition σ is *proper* when $M_1 \cap M_2 \neq \Phi$ for any $M_1 \cap M_2 \in \mathbb{D}_\sigma$. Clearly $\kappa(\mathbb{D}') \neq \Phi$ when $|\mathbb{D}'| = 2$ for any $\mathbb{D}' \subset \mathbb{D}_\sigma$ and so $v(\sigma) \geq 3$.
- (iv) Majority rule is a q -rule with $q = k+1$ when $n = 2k$ or $n = 2k+1$. In this case $\frac{q}{n-q} = \frac{k+1}{k} = 1 + 1/k$ for n odd or $\frac{k+1}{k-1} = \frac{2}{(k-1)}$ for n even. For n odd ≥ 3 , $k \geq 2$ and so $v(n, q) = 1$. For n even ≥ 6 , $k \geq 3$ and so $1 < \frac{q}{n-q} \leq 2$. Thus $v(n, q) = 1$ and $v(\sigma_m) = 3$. Hence $v(\sigma_m) = 3$ except for the case $(n, q) = (4, 3)$. In this case, $k = 2$, so $\frac{q}{n-q} = 3$ and $v(4, 3) = 2$ and $v(\sigma_3) = 4$. ■

Comment 3.2.2. To illustrate the Nakamura number, note that if σ is proper, strong, and has two distinct decisive coalitions then $v(\sigma) = 3$. To see this suppose M_1, M_2 are minimal decisive. Since σ is proper $A = M_1 \cap M_2 \neq \Phi$, must also be losing. But then $N \setminus A \in \mathbb{D}$ and so the collegium of $\{M, M', N \setminus A\}$ is empty. Thus $v(\sigma) = 3$.

By Lemma 3.2.1, a q -rule maps $A(r)^N \rightarrow A(r)$ iff $r \leq v(n, q) + 1$. By Lemma 3.2.2, this cardinality restriction may be written as $r \leq v(\sigma_q) - 1$. The following Nakamura Theorem gives an extension of the Ferejohn–Grether lemma.

Theorem 3.2.3. (Nakamura, 1978). Let σ be a simple voting rule, with Nakamura number $v(\sigma)$. Then $\sigma(p)$ is acyclic for all $p \in A(r)^N$ iff $r \leq v(\sigma) - 1$.

Before proving this theorem it is useful to define the following sets.

Definition 3.2.2. Let σ be a BF, with decisive coalitions \mathbb{D}_σ and let p be a profile on W .

(i) For a coalition $M \subset N$, define the *Pareto set* for M (at p) to be

$$Pareto(W, M, p) = \{x \in W : \nexists y \in W \text{ s.t. } y p_i x \forall i \in M\}.$$

If $M = N$ then this set is simply called the *Pareto set*.

(ii) The core of $\sigma(p)$ is

$$Core(\sigma, W, N, p) = \{x \in W : \nexists y \in W \text{ s.t. } y \sigma(p) x\}.$$

Thus

$$Core(\sigma, W, N, p) \subset \bigcap [Pareto(W, M, p)],$$

where the intersection is taken over all $M \in \mathbb{D}_\sigma$. If σ is a voting rule, then this inclusion is an equality.

(iii) An alternative $x \in W$ belongs to the *cycle set*, $Cycle(\sigma, W, N, p)$, of $\sigma(p)$ in W iff there exists a $\sigma(p)$ -cycle

$$x \sigma(p) x_2 \sigma(p) \dots \sigma(p) x_r \sigma(p) x.$$

If there is no fear of ambiguity write $Core(\sigma, p)$ and $Cycle(\sigma, p)$ for the core and cycle set respectively. Note that by Theorem 2.5.1, if $\sigma(p)$ is acyclic on a *finite* alternative set W , so that $Cycle(\sigma, W, N, p)$ is empty, then the $Core(\sigma, W, N, p)$ is non-empty. Of course the choice and cycle sets may both be non-empty. We are now in a position to prove the sufficiency part of Nakamura's Theorem.

Lemma 3.2.4. Let σ be a non-collegial, simple voting rule with Nakamura number $v(\sigma)$ on the set W . If $p \in A(W)^N$ and $Cycle(\sigma, W, N, p) \neq \Phi$ then $|W| \geq v(\sigma)$.

Proof. Since $Cycle(\sigma, W, N, p) \neq \Phi$ there exists a set

$$Z = \{x_1, \dots, x_r\} \subset W$$

and a $\sigma(p)$ -cycle (of length r) on Z :

$$x_1 \sigma(p) x_2 \dots x_r \sigma(p) x_1.$$

Write $x_r \equiv x_0$. For each $j = 1, \dots, r$, let M_j be the decisive coalition such that $x_{j-1} p_i x_j$ for all $i \in M_j$. Without loss of generality we may suppose that all M_1, \dots, M_r are distinct and minimally decisive and $|W| \geq r$.

Let $\mathbb{D}' = \{M_1, \dots, M_r\}$ and suppose that $\kappa(\mathbb{D}') \neq \Phi$. Then there exists $i \in \kappa(\mathbb{D}')$ such that

$$x_1 p_i x_2 \cdots x_r p_i x_1.$$

But by assumption, $p_i \in A(W)$. By contradiction, $\kappa(\mathbb{D}') = \Phi$, and so, by definition of $v(\sigma)$, $|\mathbb{D}'| \geq v(\sigma)$. But then $r \geq v(\sigma)$ and so $|W| \geq v(\sigma)$. ■

This proves the sufficiency of the cardinality restriction of Theorem 3.2.3, since if $r \leq v(\sigma) - 1$, then there can be no $\sigma(p)$ -cycle for $p \in A(r)^N$. We now prove necessity, by showing that if $r \geq v(\sigma)$ then there exists a profile $p \in A(r)^N$ such that $\sigma(p)$ is cyclic.

To prove this we introduce the notion of a σ -complex by an example. First we define the convex hull of set.

Definition 3.2.3.

(i) If $x, y, \in \mathbb{R}^w$ then $\text{Con}[\{x, y\}]$ is the *convex combination* of $\{x, y\}$, and is the set defined by

$$\text{Con}[\{x, y\}] = \{z \in \mathbb{R}^w : z = \alpha x + (1 - \alpha) y \text{ where } \alpha \in [0, 1]\}.$$

(ii) The convex hull $\text{Con}[Y]$ of a set $Y = \{y_1, \dots, y_v\}$ is defined by

$$\text{Con}[Y] = \{z \in \mathbb{R}^w : z = \sum_{y_j \in Y} \alpha_j y_j, \text{ where } \sum \alpha_j = 1, \text{ all } \alpha_j \geq 0\}.$$

Example 3.2.2. Consider the voting rule, σ , with six players $\{1, \dots, 6\}$ whose minimal decisive coalitions are $\mathbb{D}_{\min} = \{M_1, M_2, M_3, M_4\}$ where $M_1 = \{2, 3, 4\}$, $M_2 = \{1, 3, 4\}$, $M_3 = \{1, 2, 4, 5\}$, $M_4 = \{1, 2, 3, 5\}$. Clearly $v(\sigma) = 4$.

We represent σ in the following way. Since $\kappa(\mathbb{D}_{\min} \setminus \{M_j\}) = \{j\}$, for $j = 1, \dots, 4$, we let $Y = \{y_1, y_2, y_3, y_4\}$ be the set of vertices, and let each y_j represent one of the players $\{1, \dots, 4\}$. Let Δ be the convex hull of Y in \mathbb{R}^3 . We define a representation ϕ by $\phi(\{j\}) = y_j$ and $\phi(M_j) = \text{Con}[Y \setminus \{y_j\}]$ for $j = 1, \dots, 4$. Thus $\phi(M_j)$ is the face opposite y_j . Now player 5 belongs to both M_3 and M_4 , but not to M_1 or M_2 , and so we place y_5 at the center of the intersection of the faces corresponding to M_3 and M_4 . Finally, since player 6 belongs to no minimally decisive coalition let $\phi(\{6\}) = \{y_6\}$, an isolated vertex. Thus the complex Δ_σ consists of the four faces of Δ together with $\{y_6\}$. See Figure 3.1.

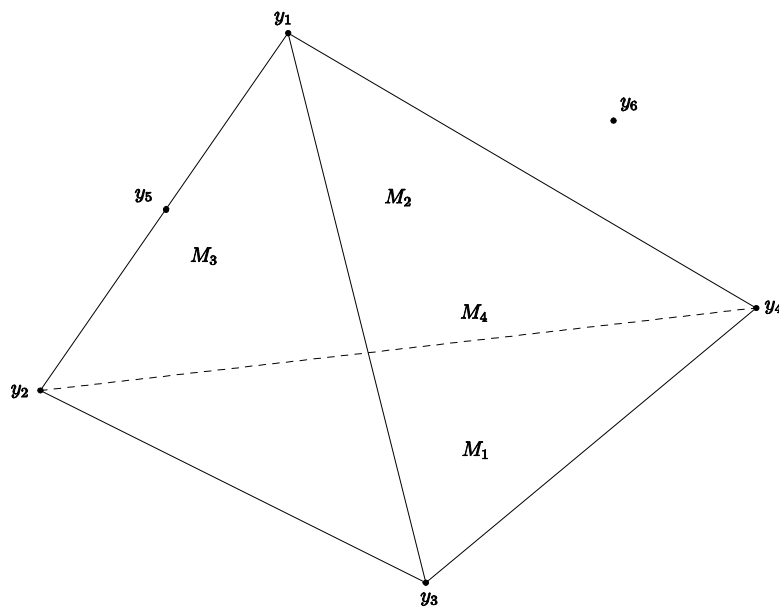


Figure 3.1: A voting complex

A representation, ϕ , of σ allows us to construct a profile p , on a set of cardinality $v(\sigma)$ such that $\sigma(p)$ is cyclic. Note that the simplex is situated in dimension $v(\sigma) - 1$. We use this later to construct cycles in dimension $v(\sigma) - 1$.

We now define the notion of a complex.

Definition 3.2.4. A complex Δ . Let Δ be the abstract simplex in \mathbb{R}^w of dimension $v - 1$, where $v - 1 \leq w$. The simplex Δ may be identified with the convex hull of a set of v district points, or *vertices*, $\{y_1, \dots, y_v\} = Y$. Opposite the vertex y_j is the *face* $F(j)$ where $F(j)$ is itself a simplex of dimension $(v - 2)$, and may be identified with the convex hull of the $(v - 1)$ vertices $\{y_1, \dots, y_{j-1}, y_{j+1}, \dots, y_v\}$. Say Δ is spanned by Y and write $\Delta(Y)$ to denote this. The *edge* of Δ is an intersection of faces. Let $V = \{1, \dots, v\}$. Then for any subset R of V , define the edge

$$F(R) = \bigcap_{j \in R} F(j).$$

Clearly $F(R)$ is spanned by $\{y_j : j \in V \setminus R\}$. It is a simplex of dimension $v - 1 - |R|$ opposite $\{y_j : j \in R\}$. In particular if $R = 1, \dots, j - 1, j + 1, \dots, v$ then $F(R) = \{y_j\}$. Finally if $R = V$ then $F(R) = \Phi$. If $\Delta(Y')$ is a simplex spanned by a subset of Y' of Y then the *barycenter* of $\Delta(Y')$ is the point

$$\theta(\Delta(Y')) = \frac{1}{|Y'|} \sum_{y_j \in Y'} y_j.$$

A complex Δ , of dimension $v - 1$, based on the vertices $Y = \{y_1, \dots, y_n\}$ is a family of simplices $\{\Delta(Y_k) : Y_k \subset Y\}$ where each simplex $\Delta(Y_k)$ in Δ has dimension at most $v - 1$, and the family is closed under intersection, so $\Delta(Y_j) \cap \Delta(Y_k) = \Delta(Y_j \cap Y_k)$.

Given a simplex $\Delta(Y)$, where $Y = \{y_1, \dots, y_v\}$, the natural complex Δ of dimension $(v - 2)$ on $\Delta(Y)$ is the family of faces of $\Delta(Y)$ together with all edges. If $\Delta(Y) \subset \mathbb{R}^w$ for $w \geq v - 1$, then the intersection of all faces of $\Delta(Y)$ will be empty.

Definition 3.2.5.

- (i) Let \mathbb{D} be a family of subsets of N , with Nakamura number, v . A *representation* $(\phi, \Delta, \mathbb{D})$ of \mathbb{D} is a complex Δ of dimension $(v - 1)$ in \mathbb{R}^w , for $w \geq v - 1$, spanned by $Y = \{y_1, \dots, y_n\}$ and a bijective

correspondence (or *morphism*)

$$\phi: (\mathbb{D}, \cap) \rightarrow (\Delta, \cap)$$

between the coalitions in \mathbb{D}_{\min} and the faces of Δ , which is natural with respect to intersection. That is to say for any subfamily \mathbb{D}' of \mathbb{D} ,

$$\phi(\kappa(\mathbb{D}')) = \bigcap_{M \in \mathbb{D}'} \phi(M).$$

Moreover, ϕ can be extended over N : if for some $i \in N$, $\mathbb{D}_i = \{M \in \mathbb{D} : i \in M\} \neq \Phi$ then $\phi(\{i\}) = \theta(\phi(\kappa(\mathbb{D}_i)))$, whereas if $\mathbb{D}_i = \Phi$ then $\phi(\{i\})$ is an isolated vertex in Δ .

- (ii) If there exists a representation $(\phi, \Delta, \mathbb{D})$ of \mathbb{D} then denote Δ by $\Delta(\mathbb{D})$.
- (iii) If σ is a BF with decisive coalitions \mathbb{D}_σ and $(\phi, \Delta, \mathbb{D}_\sigma)$ is a representation of \mathbb{D}_σ then write Δ as Δ_σ and say Δ_σ is the σ -*complex* which represents σ .

Schofield (1984a) has shown the following.

Theorem 3.2.4. Let \mathbb{D} be a family of subsets of N , with Nakamura number $v < \infty$. Let \mathbb{D}_{\min} be a minimal non-collegial subfamily of \mathbb{D} . Then there exists a simplex $\Delta(Y)$, in \mathbb{R}^{v-1} , spanned by $Y = \{y_1, \dots, y_v\}$ and a representation $\phi: (\mathbb{D}_{\min}, \cap) \rightarrow (\Delta, \cap)$ where Δ is the natural complex based on the faces of $\Delta(Y)$. Furthermore:

- (i) There exists a subset $V = \{1, \dots, v\}$ of N such that, for each $j \in V$, $\phi(\{j\}) = y_j$, a vertex of $\Delta(Y)$.
- (ii) After labelling appropriately, for each $M_j \in \mathbb{D}_{\min}$,

$$\phi(M_j) = F(j)$$

the face of $\Delta(Y)$ opposite y_j .

Proof. Each proper subfamily $\mathbb{D}_t = \{.., M_{t-1}, M_{t+1}, .. : t = 1, \dots, v\}$ of \mathbb{D}_{\min} has a non-empty collegium, $\kappa(\mathbb{D}_t)$, and each of these can be identified with a vertex, y_t of Δ . If $j \in \kappa(\mathbb{D}_t)$, then j is assigned the vertex y_t . Continue by induction: if $j \in \kappa(\mathbb{D}_t \cap \mathbb{D}_s) - \kappa(\mathbb{D}_t) - \kappa(\mathbb{D}_s)$ then j is assigned the barycenter of $[y_t, y_s]$. By this method we assign a vertex to each member of the set $N(\mathbb{D}_{\min})$ consisting of those individuals

who belong to at least one coalition in \mathbb{D}_{\min} . This assignment gives a representation $(\phi, \Delta, \mathbb{D}_{\min})$.

Corollary 3.2.1. Let σ be a voting rule with Nakamura number $v(\sigma)$. Then there exists a σ -complex Δ_σ , of dimension $v(\sigma) - 2$ in \mathbb{R}^w , for $w \geq v(\sigma) - 1$, which represents σ .

Proof. Let $v(\sigma) = v$. Let \mathbb{D}_{\min} be the minimal non-collegial subfamily of \mathbb{D}_σ and let $\phi: (\mathbb{D}_{\min}, \cap) \rightarrow (\Delta, \cap)$ be the representation constructed in Theorem 3.2.4. Extend ϕ to a representation $\phi: \mathbb{D}_\sigma \rightarrow \Delta(\mathbb{D}_\sigma)$ by adding new faces and vertices as required. Finally, the complex Δ_σ can be constructed so that, for any $\mathbb{D}' \subset \mathbb{D}$, then $\kappa(\mathbb{D}') = \Phi$ if and only if $\bigcap \phi(M_j) = \Phi$, where the intersection is taken over all $M_j \in \mathbb{D}'$. ■

We are now in a position to prove the necessity part of Nakamura's Theorem.

Corollary 3.2.2. Let σ be a voting rule, with Nakamura number $v(\sigma) = v$, on a finite alternative set W . If $|W| \geq v(\sigma)$ then there exists an acyclic profile p on W such that $\text{Cycle}(\sigma, W, N, p) \neq \Phi$ and $\text{Core}(\sigma, W, N, p) = \Phi$.

Proof. Construct a profile $p \in A(W)^N$ and a $\sigma(p)$ cycle on W as follows. Each proper subfamily

$$\mathbb{D}_t = \{M_1, \dots, M_{t-1}, M_{t+1}, \dots, M_v : t = 1, \dots, v\}$$

of \mathbb{D}_{\min} has a non-empty collegium, $\kappa(\mathbb{D}_t)$. By Corollary 3.2.1, each of these can be identified with a vertex, y_t of Δ . Without loss of generality we relabel so that $Y = \{y_1, \dots, y_t, \dots, y_v\} \subset W$. Let $V = \{1, \dots, v\}$. We assign preferences to the members of these collegia on the set Y as follows.

$$\begin{array}{cccc} p_1 : \kappa(\mathbb{D}_1) & p_2 : \kappa(\mathbb{D}_2) & \dots & p_v : \kappa(\mathbb{D}_v) \\ y_1 & y_2 & & y_v \\ y_2 & y_3 & & y_1 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ y_v & y_1 & \dots & y_{v-1} \end{array}$$

To any individual $j \in N(\mathbb{D}_{\min})$ who is assigned a position at the barycenter, $\theta(\Delta(Y'))$, for a subset $Y' = \{y_r; r \in R \subset V\}$, we let

$$p_j = \bigcap_{j \in R} p_r.$$

It follows from the construction that every member j of coalition M_t has a preference satisfying

$$\bigcap_{r \neq t} \{p_r \subseteq p_j\}.$$

The profile so constructed is called a σ -permutation profile.

It then follows that each $j \in M_t$ has the preference $y_{t+1}p_jy_t$, where we adopt the notational convention that $y_{v+1} = y_1$.

We thus obtain the cycle

$$y_1\sigma(p)y_v \cdots \sigma(p)y_2\sigma(p)y_1.$$

This profile can be extended over Y by assigning to an individual j not in $N(\mathbb{D}_{\min})$ the preference of complete indifference. Obviously

$$\text{Cycle}(\sigma, W, N, p) \neq \Phi \text{ and } \text{Core}(\sigma, W, N, p) = \Phi.$$

The argument obviously holds whenever $|W| > v(\sigma)$, again by assigning indifference to alternatives outside Y . ■

Note that Corollary 3.2.1 does not require that the voting rule be simple, since the construction holds for the simple rule $\sigma^{\mathbb{D}}$. In the same way, the corollary also holds for a BF, σ , by applying the construction to $\sigma^{\mathbb{D}}$.

Example 3.2.3. To illustrate the construction, consider the previous Example 3.2.2. The profile constructed according to the corollary is:

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ y_1 & y_2 & y_3 & y_4 \\ y_2 & y_3 & y_4 & y_1 \\ y_3 & y_4 & y_1 & y_2 \\ y_4 & y_1 & y_2 & y_3 \end{array}.$$

Because y_5 lies on the arc $[y_1, y_2]$ in the figure, we define $p_5 = p_1 \cap p_2$, so

$$y_2p_5y_3p_5y_4I_5y_1.$$

Individual 6 is assigned indifference.

$$y_1 I_6 y_2 I_6 y_3 I_6 y_4.$$

For this “permutation profile” we observe the following:

- (i) $M_1 = \{2, 3, 4\}$ and $y_4 p_i y_1$ for $i \in M_1$.
- (ii) $M_2 = \{1, 3, 4\}$ and $y_1 p_i y_2$ for $i \in M_2$.
- (iii) $M_3 = \{1, 2, 4, 5\}$ and $y_2 p_i y_3$ for $i \in M_3$.
- (iv) $M_4 = \{1, 2, 3, 5\}$ and $y_3 p_i y_4$ for $i \in M_4$.

Each of these coalitions belongs to \mathbb{D}_{\min} .

Thus we obtain the cycle

$$y_1 \sigma(p) y_2 \sigma(p) y_3 \sigma(p) y_4 \sigma(p) y_1.$$

Clearly $Cycle(\sigma, p) = \{y_1, y_2, y_3, y_4\} = Pareto(N, p)$ and $Core(\sigma, p) = \Phi$.

Lemma 3.2.4 and Corollary 3.2.2 together prove Nakamura’s Theorem. The demonstration in Lemma 3.2.6 of the existence of a σ -permutation preference profile on an alternative set of cardinality $v(\sigma)$ has a bearing on whether a choice mechanism can be manipulated. It is to this point that we now briefly turn.

3.3 Manipulation of Choice Functions

The existence of a permutation preference profile, of the kind constructed in the previous section, essentially means that a particular choice mechanism C can be *manipulated*.

The general idea is to suppose that the choice procedure is *implementable* in the sense that the outcomes selected by the choice procedure result from the individuals in the society selecting preference relations to submit to the choice procedure. These preference relations need not be “sincere” or truthful, but are in an appropriate sense optimal for the individuals in terms of their truthful preferences. An “implementable” choice procedure will then be *monotonic*. However, the existence of a σ -permutation profile means that any choice mechanism which is com-

patible with the voting rule, σ , cannot be monotonic and thus cannot be implementable. Full details can be found in Ferejohn, Grether and McKelvey (1982). Here we simply outline the proof that the existence of a σ -permutation profile means the choice mechanism is not monotonic.

Definition 3.3.1. Let $C: X \times B^N \rightarrow X$ be a choice function (where as before X is the set of all subsets of the universal set of alternatives):

(i) The choice function C is *monotonic* on W iff whenever $x \in W$ and $p, p' \in B(W)^N$ satisfy the property:

$$[x \in C(W, p) \text{ and } \forall i \in N, \forall y \in W \setminus \{x\}, xR(p_i)y \Rightarrow xR(p'_i)y]$$

then $x \in C(W, p')$.

(ii) The choice function, C , is *compatible* with a binary social preference function, σ , on W iff whenever $x \in W, p \in B(W)^N$ and $M \in \mathbb{D}_\sigma$ satisfy the property:

$$\text{for every } i \in M, \text{ there exists no } y_i \in W \text{ with } y_i p_i x$$

then $\{x\} = C(W, p)$.

(iii) If $p \in A(W)^N$ then a *manipulation* p' of p by a coalition M is a profile

$$p' = (p_1, \dots, p_n) \in A(W)^N$$

such that $p_i = p'_i$ for $i \in M$ and $p'_i \neq p_i$ for some $i \in M$.

(iv) If $x \in C(W, p)$, for $p \in A(W)^N$ then C is *manipulable by M at (x, p)* iff there exists a manipulation p' of p by M , with $x' = C(W, p')$ for some $x' \neq x$, where $x' p_i x$ for all $i \in M$.

We now show that if a choice function is compatible with a social preference function, σ , and there exists a σ -permutation preference profile p on W , for $|W| \geq v(\sigma)$, then p may be manipulated by some coalition in $\mathbb{D}_{\sigma'}$ in such a way that C cannot be monotonic.

Corollary 3.3.1. Let σ be a non-collegial BF, and C a σ -compatible choice function. If $|W| \geq v(\sigma)$ then C cannot be monotonic on W .

We can demonstrate this Theorem by using Example 3.2.3.

Example 3.3.1. Suppose C is a choice function compatible with σ , where

the decisive coalitions for σ are as given in the example. Let p be the permutation profile based on

1	2	3	4
y_1	y_2	y_3	y_4
y_2	y_3	y_4	y_1
y_3	y_4	y_1	y_2
y_4	y_1	y_2	y_3

with

$$y_2 p_5 y_3 p_5 y_4 I_5 y_1.$$

If $y_4 \in C(p, W)$ where $W = \{y_1, y_2, y_3, y_4, y_5\}$ consider $M_4 = \{1, 2, 3, 5\}$, and the manipulation p'

1	2	3	5
y_3	y_3	y_3	y_3
y_1	y_2	y_4	y_2
y_2	y_4	y_1	y_4, y_1
y_4	y_1	y_2	

Since $M_4 = \{1, 2, 3, 5\} \in \mathbb{D}$, then if C is σ -compatible, we obtain $\{y_3\} = C(W, p')$. But the preferences between y_3 and y_4 are identical in p and p' . Thus, if C were monotonic, we would obtain $y_4 \in C(p, W)$. This contradiction implies that C cannot be both monotonic and σ -compatible. In identical fashion, whichever alternative is selected by the choice function, one of the four decisive coalitions may manipulate p to its advantage.

Corollary 3.3.2. If $|W| \geq n$ and $|W \geq 3|$ then for no monotonic choice function C does there exist a non-collegial binary social preference function, σ , such that C is compatible with σ .

Proof. For any non-collegial voting rule, σ , it is the case that $v(\sigma) \leq n$. Thus if $W \geq n$, Corollary 3.3.1 applies to every choice function. ■

Ferejohn, Grether and McKelvey (1982) essentially obtained a version of Corollary 3.3.1 in the case that σ was a q -rule with $q = n - 1$. In this case they said that a choice function that was compatible with σ was *minimally democratic*. They then showed that a minimally democratic choice function could be neither monotonic nor implementable.

As we know from Lemma 3.2.2, $v(\sigma) = 3$ for majority rule other than when $n = 4$. Thus even with three alternatives, any choice function which is *majoritarian* is effectively manipulable. In a later chapter we shall show that Lemma 3.2.6 can be extended to show the existence of a permutation preference profile for a voting rule, σ , in dimension $v(\sigma) - 1$. “Spatial” voting rules will therefore be manipulable, in the sense described above, even in dimension $v(\sigma) - 1$. In particular, majoritarian rules will be manipulable in two dimensions.

3.4 Restrictions on the Preferences of Society

The results of the previous section show that non-manipulability, of a non-collegial voting rule cannot be guaranteed without some restriction on the size, r , of the set of alternatives. It is, however, possible that while majority rule, for example, need not be “rational” for general n and r , it is “rational” for “most” preference profiles. A number of authors have analyzed the probability of occurrence of voting cycles. Assume, for example, that each preference ranking on a set W with $|W| = r$ is equally likely. For given (n, r) it is possible to compute, for majority rule, the probability of

- (a) an unbeaten alternative;
- (b) a permutation preference profile, and thus a voting cycle, containing all r alternatives.

Niemi and Weisberg (1968) shows that for large n , the probability of (a) declined from about 0.923 when $r = 3$, to 0.188 when $r = 40$. By a simulation method Bell (1978) showed that the probability of total breakdown (b) increased from 0.084 (at $r = 3$) to 0.352 (at $r = 15$) to 0.801 (at $r = 60$).

Sen (1970) responded to the negative results of Niemi and Weisberg and others with the comment that the assumption of “equi-probable” preference orderings was somewhat untenable. The existence of classes in a society would surely restrict, in some complicated fashion, the variation in preferences. This assertion provides some motivation for studying restrictions on the domain of a binary social preference function, σ , which are sufficient to guarantee the rationality of the rule. These so-called *ex-*

clusion principles are sufficient to guarantee the transitivity of majority rule.

Definition 3.4.1.

- (i) A binary relation Q on W is a *linear order* iff it is asymmetric, transitive and weakly connected (viz. $x \neq y \Rightarrow xQy$ or yQx). Write $L(W)$ for the set of linear orders on W .
- (ii) A preference profile $p \in B(W)^N$ is *single peaked* iff there is a linear order Q on W such that for any x, y, z in W , if either $xQyQz$ or $zQyQx$, then for any $i \in N$ who is not indifferent on $\{x, y, z\}$ it is the case that
 - (a) $xR(p_i)z \Rightarrow yp_iz$;
 - (b) $zR(p_i)x \Rightarrow yp_ix$.

The class of single peaked preference profiles is written S^N , and the class of profiles whose component preferences are linear orders is written L^N .

Example 3.4.1. Suppose $N = \{1, 2, 3\}$ and the profile p on $\{x, y, z\}$ is given by

$$\begin{array}{ccc} 1 & 2 & 3 \\ x & & y \\ y & zy & z \\ z & x & x \end{array}$$

Then defining Q by $xQyQz$ it is easy to see the profile is single peaked.

Arrow (1951), Black (1958) and Fishburn (1973) have obtained the results given in Lemma 3.4.1.

Lemma 3.4.1.

- (i) If $p \in S^N \cap T^N$ then $\bar{\sigma}_m(p) \in T$.
- (ii) If $p \in S^N \cap L^N$ then $\bar{\sigma}_m(p) \in L$.
- (iii) If $p \in S^N \cap O^N$, n odd, and $xQyQz$ and there is no individual indifferent on $\{x, y, z\}$, then $\bar{\sigma}_m(p) \in O$.

Inada (1969), Sen and Pattanaik (1969) and Sen (1966) have extended the notion of single-peakedness by introducing the exclusion principles of *value restriction*, *extremal restriction* and *limited agreement*.

If the profile p satisfies value restriction or limited agreement and belongs to T^N , then $\bar{\sigma}_m(p)$ belongs to T , and if p satisfies extremal restriction and belongs to O^N , then $\bar{\sigma}_m(p)$ belongs to O .

However, all exclusion principles on a profile fail if there is a *Condorcet cycle* within the profile.

Definition 3.4.2.

- (i) There exists a *Condorcet Cycle* within a profile p on W iff there is a triple of individuals $N' = \{i, j, k\}$ and a triple of alternatives $V = \{x, y, z\}$ in W such that $p_{N'}^V$ is given by:

$$\begin{array}{ccc} i & j & k \\ x & y & z \\ y & z & x \\ z & x & y \end{array} .$$

- (ii) A profile is *Condorcet free* iff there is no Condorcet cycle $p_{N'}^V$, in p .

Schwartz (1986) has shown that if p_i is a linear order for each $i \in N$, so $p \in L^N$, and p is Condorcet free then $\sigma(p)$ is transitive, for σ belonging to a fairly general class of binary social preference functions.

It is obvious that if there is a Condorcet cycle in p , then p cannot be single-peaked, and indeed p must fail all the exclusion principles. On the other hand there may well be a Condorcet cycle in p , even though $\bar{\sigma}_m(p)$, for example, is transitive.

Chapter 4 will show, in two dimensions, that a Condorcet cycle typically exists. The analysis is developed to demonstrate that σ -voting cycles typically exist in \mathbb{R}^w as long as the dimension, w , is at least $v(\sigma) - 1$. This procedure clarifies the relationship between the behavior of a voting rule on a finite set of alternatives (the concern of this chapter) and the behavior of the rule on a policy space of particular dimension.

Chapter 4

The Core

In the previous chapter it was shown that a simple voting rule, σ , was acyclic on a finite set of alternatives, W , if and only if the cardinality $|W|$ of W satisfied $|W| \leq v(\sigma) - 1$. The same cardinality restriction was shown to be necessary and sufficient for the non-emptiness of the core. In this chapter an analogous result is obtained when the set of alternatives, W , is a compact, convex subset of \mathbb{R}^w .¹¹ With this assumption on the set of alternatives, $W \subseteq \mathbb{R}^w$, we shall show, for a simple voting rule, σ , that the core of $\sigma(p)$, for any convex and continuous preference profile p on W , will be non-empty if and only if the dimension of W is no greater than $v(\sigma) - 2$. We say that $v(\sigma) - 2 = v^*(\sigma)$ is the stability dimension. Thus, if $\dim(W) \geq v(\sigma) - 1$ then a profile can be constructed so that the core is empty and the cycle set non-empty. Indeed, above the stability dimension $v^*(\sigma)$, *local cycles* may occur, whereas below the stability dimension local cycles may not occur. In dimension $v^*(\sigma) + 1$, local cycles will be constrained to the Pareto set. In dimension above $v^*(\sigma) + 1$, these local cycles may extend beyond the Pareto set, suggesting a degree of chaos.

4.1 Existence of a Choice

We first show the sufficiency of the dimension restriction, by considering the preference correspondence associated with $\sigma(p)$.

Let W be the set of alternatives and, as before, let X be the set of

¹¹In fact, the same result goes through when W is a subset of a topological vector space. The definitions of a *topological vector space*, and other notions such as *openness*, *compactness* and *continuity* are given in a brief Appendix to this chapter.

all subsets of W . If p is a preference relation on W , the *preference correspondence*, P , associated with p is the correspondence which associates with each point $x \in W$, the *preferred set* $P(x) = \{y \in W : ypx\}$. Write $P: W \rightarrow X$ or $P: W \rightrightarrows W$ to denote that the image of x under P is a set (possibly empty) in W . For any subset V of W , the restriction of P to V gives a correspondence $P_V: V \rightrightarrows V$, where for any $x \in V$, $P_V(x) = \{y \in V : ypx\}$. Define $P_V^{-1}: V \rightrightarrows V$ such that for each $x \in V$, $P_V^{-1}(x) = \{y \in V : ypy\}$. The sets $P_V(x)$, $P_V^{-1}(x)$ are sometimes called the *upper* and *lower* preference sets of P on V . When there is no ambiguity we delete the suffix V . The *choice* of P from W is the set

$$C(W, P) = \{x \in W : P(x) = \Phi\}.$$

The choice of P from a subset, V , of W is the set

$$C(V, P) = \{x \in V : P_V(x) = \Phi\}.$$

If the strict preference relation p is acyclic, then say the preference correspondence, P , is acyclic. In analogous fashion to the definition of Section 2.5 call C_P a choice function on W if $C_P(V) = C(V, P) \neq \Phi$ for every subset V of W . We now seek general conditions on W and P which are sufficient for C_P to be a choice function on W . Continuity properties of the preference correspondence are important and so we require the set of alternatives to be a topological space. For simplicity, we can just assume that W is a subset of \mathbb{R}^w , with the usual Euclidean topology (as defined in the Appendix to this chapter).

Definition 4.1.1. Let W, Y be two topological spaces. A correspondence $P: W \rightrightarrows Y$ is

- (i) *Lower hemi-continuous (lhc)* iff, for all $x \in W$, and any open set $U \subset Y$ such that $P(x) \cap U \neq \Phi$ there exists an open neighborhood V of x in W , such that $P(x') \cap U \neq \Phi$ for all $x' \in V$.
- (ii) *Upper hemi-continuous (uhc)* iff, for all $x \in W$ and any open set $U \subset Y$ such that $P(x) \subset U$, there exists an open neighborhood V of x in W such that $P(x') \subset U$ for all $x' \in V$.
- (iii) *Lower demi-continuous (ldc)* iff, for all $x \in Y$, the set $P^{-1}(x) = \{y \in W : x \in P(y)\}$ is open (or empty) in W .
- (iv) *Upper demi-continuous (udc)* iff, for all $x \in W$, the set $P(x)$ is open

(or empty) in Y

(v) *Continuous* iff P is both *ldc* and *udc*.

We shall use lower demi-continuity of a preference correspondence to prove existence of a choice. In some cases, however, it is possible to make use of lower hemi-continuity. For completeness we briefly show that the former continuity property is stronger than the latter.

Lemma 4.1.1. If a correspondence $P: W \rightrightarrows Y$ is *ldc* then it is *lhc*.

Proof. Suppose that $x \in W$ with $P(x) \neq \Phi$ and U is an open set in Y such that $P(x) \cap U \neq \Phi$. Then there exists $y \in U$ such that $y \in P(x)$. By definition $x \in P^{-1}(y)$. Since P is *ldc*, there exists a neighborhood V of x in W such that $V \subset P^{-1}(y)$. But then, for all $x' \in V$, $x' \in P^{-1}(y)$ or $y \in P(x')$. Since $y \in U$, $P(x') \cap U \neq \Phi$ for all $x' \in V$. Hence P is *lhc*. ■

We shall now show that if W is compact, and P is an acyclic and *ldc* preference correspondence $P: W \rightrightarrows W$, then $C(W, P) \neq \Phi$. First of all, say a preference correspondence $P: W \rightrightarrows W$ satisfies the *finite maximality property* (FMP) on W iff for every finite set V in W , there exists $x \in V$ such that $P(x) \cap V = \Phi$. Note that P is acyclic on W then P satisfies FMP. To see this, note that if P is acyclic on W then P is acyclic on any finite subset V of W , and so, by Theorem 2.5.1, $C(V, P) \neq \Phi$. But then there exists $x \in V$ such that $P(x) \cap V = \Phi$. Hence P satisfies FMP.

Lemma 4.1.2. (Walker, 1977). If W is a compact, topological space and P is an *ldc* preference correspondence that satisfies FMP on W , then $C(W, P) \neq \Phi$.

Proof. Suppose on the contrary that $C(W, P) = \Phi$. Then for every $x \in W$, there exists $y \in W$ such that $y \in P(x)$, and so $x \in P^{-1}(y)$. Thus $\{P^{-1}(y) : y \in W\}$ is an open cover for W . (Note that since P is *ldc*, each $P^{-1}(y)$ is open.) Moreover, W is compact and so there exists a finite subset, V , of W such that $\{P^{-1}(y) : y \in V\}$ is an open cover for W . But then for every $x \in W$ there exists $y \in V$ such that $x \in P^{-1}(y)$, and so $y \in P(x)$. Since $y \in V$, $y \in P(x) \cap V$. Hence $P(x) \cap V \neq \Phi$ for all $x \in W$ and thus for all $x \in V$. Thus P fails FMP. By contradiction $C(W, P) \neq \Phi$. ■

Corollary 4.1.1. If W is a compact topological space and P is an acyclic, ldc preference correspondence on W , then $C(W, P) \neq \Phi$.

Proof. If P is acyclic on W , then it satisfies FMP on W . By Lemma 4.1.2, $C(W, P) \neq \Phi$. ■

As Walker (1977) noted, when W is compact and P is ldc, then P is acyclic iff P satisfies FMP on W , and so either property can be used to show existence of a choice. A second method of proof to show that C_P is a choice function is to substitute a convexity property for P rather than acyclicity. First, remember that in Definition 3.3.3, we defined the convex combination of $\{x, y\}$ by

$$\text{Con}[\{x, y\}] = \{z \in \mathbb{R}^w : z = \alpha x + (1 - \alpha)y \text{ where } \alpha \in [0, 1]\}.$$

Definition 4.1.2.

- (i) The *convex hull* of W is the set, $\text{Con}[W]$, with $W \subseteq \text{Con}[W]$ defined by $\text{Con}[W] = \{z \in \text{Con}[x, y] \text{ for any } x, y \in W\}$.
- (ii) If $W \subseteq \mathbb{R}^w$ then W is *convex* iff $W = \text{Con}[W]$. (The empty set is also convex.)
- (iii) A subset $W \subseteq \mathbb{R}^w$ is *admissible* iff W is both compact and convex.
- (iv) A preference correspondence $P: W \rightarrow W$ on a convex set W is *convex* iff, for all $x \in W$, $P(x)$ is convex.
- (v) A preference correspondence $P: W \rightarrow W$ is *semi-convex* iff, for all $x \in W$, it is the case that $x \in \text{Con}(P(x))$.

Fan (1961) has shown that if W is admissible and P is ldc and semi-convex, then $C(W, P)$ is non-empty.

Theorem 4.1.1. (Fan, 1961). If W is admissible and $P: W \rightarrow W$ a preference correspondence on W which is ldc and semi-convex then $C(W, P) \neq \Phi$.

A proof of this theorem in the more general case that W is a compact, convex subset of Hausdorff topological vector space can be found in Schofield (2003a) using a lemma due to Knaster, Kuratowski and Mazurkiewicz (1929). There is a useful corollary to this theorem. Say a

preference correspondence on an admissible space W satisfies the *convex maximality property* (CMP) iff for any finite set V in W , there exists $x \in \text{Con}(V)$ such that $P(x) \cap \text{Con}(V) = \Phi$.

Corollary 4.1.2. Let W be admissible and $P: W \rightrightarrows W$ be ldc and semi-convex. Then P satisfies the convex maximality property.

The form of Theorem 4.1.1, originally proved by Fan, made the assumption that $P: W \rightrightarrows W$ was *irreflexive* (in the sense that $x \in P(x)$ for no $x \in W$) and *convex*. Together these two assumptions imply that P is semi-convex. Bergstrom (1975) extended Fan's original result to give the version presented above.

A different proof of Theorem 4.1.1 using a fixed-point argument has also been obtained by Yannelis and Prabhakar (1983). Note that Theorem 4.1.1 is valid without restriction on the dimension of W . Indeed, Aliprantis and Brown (1983) have used this theorem in an economic context with an infinite number of commodities to show existence of a price equilibrium. Bergstrom also showed that when W is finite dimensional then Theorem 4.1.1 is valid when the continuity property on P is weakened to lhc. Numerous applications of this theorem in the finite dimensional case to show existence of an equilibrium for an abstract economy with lhc preferences have been made by Shafer and Sonnenschein (1975) and Borglin and Keiding (1976). In the application that we shall make of this theorem we require the stronger property that the preference be ldc. Finally, note that Theorem 4.1.1 shows that C_P is a choice function, for every ldc and semi-convex preference correspondence $P: W \rightrightarrows W$, as long as the domain of C_P is restricted to admissible subsets of W .

4.2 Existence of the Core in Low Dimension

In the previous chapter we proved Nakamura's Theorem 3.2.3 for a simple, non-collegial voting rule, σ , in the case W was finite, and showed that this generalized the Ferejohn–Grether result for a q -rule, σ_q . Greenberg (1979) has extended the Ferejohn–Grether result to the case when W is admissible and individual preferences are continuous and convex, and shown that if $\dim(W) \leq v(n, q)$, then the q -rule, σ_q , has a non-empty

core.¹² We shall obtain a generalization of Greenberg's result, by showing that if σ is a general non-collegial voting rule with Nakamura number, $v(\sigma)$, then if $\dim(W) \leq v(\sigma) - 2$, and certain continuity, convexity and compactness properties are satisfied then σ has a core.

To make use of Theorem 4.1.1, we need to show that when individual preference correspondences are ldc then so is social preference. Suppose, therefore, that $p = (p_1, \dots, p_n)$ is a preference profile for society. Let σ be a voting rule, and \mathbb{D} be the family of decisive coalitions of σ .

Let $P = (P_1, \dots, P_n)$ be the family of preference correspondences defined by the profile $p = (p_1, \dots, p_n)$. Call P a *preference (correspondence) profile*. For any coalition $M \subset N$ define

$$P_M: W \rightarrow W \text{ by } P_M(x) = \bigcap_{i \in M} P_i(x).$$

For a family \mathbb{D} of subsets of N , define

$$P_{\mathbb{D}}: W \rightarrow W \text{ by } P_{\mathbb{D}}(x) = \bigcup_{M \in \mathbb{D}} P_M(x).$$

If σ is a BF, with \mathbb{D} its family of decisive coalitions then clearly

$$x \in P_{\mathbb{D}}(y) \Rightarrow x\sigma(p)y,$$

whereas when σ is a voting rule, then

$$x \in P_{\mathbb{D}}(y) \Leftrightarrow x\sigma(p)y.$$

In this latter case we sometimes write P_{σ} for the preference correspondence $P_{\mathbb{D}}$, where $\mathbb{D} = \mathbb{D}_{\sigma}$. We have defined the core of $\sigma(p)$ by

$$x \in \text{Core}(\sigma, W, N, p) \text{ iff } \nexists y \in W \text{ such that } y\sigma(p)x.$$

Thus, if σ is a BF with \mathbb{D} its family of decisive coalitions, then

$$\text{Core}(\sigma, W, N, p) \subseteq C(W, P_{\mathbb{D}}).$$

with equality in the case of a voting rule.

Theorem 4.2.1. (Schofield, 1984a; Strnad, 1985). Let W be admissible and let σ be a voting rule with Nakamura number $v(\sigma)$. If $\dim(W) \leq$

¹²Here $\dim(W)$ can be identified with the number of linear independent vectors that span W . Thus we can regard w as the smallest integer such that $W \subseteq \mathbb{R}^w$.

$v(\sigma) - 2$ and $p = (p_1, \dots, p_n)$ is a preference profile such that for each $i \in N$, the preference correspondence $P_i: W \rightarrow W$ is ldc and semi-convex. Then $\text{Core}(\sigma, W, N, p) \neq \Phi$.

Note that the theorem is valid in the case σ is collegial, with Nakamura number ∞ . We prove this theorem using the Fan Theorem and the following two lemmas. For convenience we say a profile $P = (P_1, \dots, P_n)$ satisfies a property, such as lower demi-continuity, iff each $P_i, i \in N$, satisfies the property.

Lemma 4.2.1. If W is a topological space and $P = (P_1, \dots, P_n)$ is an ldc preference profile then $P_{\mathbb{D}}: W \rightarrow W$ is ldc, for any family \mathbb{D} of coalitions in N .

Proof. We seek to show that $P_{\mathbb{D}}^{-1}(x)$ is open. Suppose that $y \in P_{\mathbb{D}}^{-1}(x)$. By definition $x \in P_{\mathbb{D}}(y)$ and so $x \in P_M(y)$ for some $M \in \mathbb{D}$. Thus $x \in P_i(y)$ for all $i \in M$. But $P_i^{-1}(x)$ is open for all $i \in N$, and so there exists an open set $U_i \subset W$ such that

$$y \in U_i \subset P_i^{-1}(x) \text{ for all } i \in M.$$

Let $U = \bigcap_{i \in M} U_i$. Then for all $z \in U$, it is the case that $z \in P_i^{-1}(x)$, $\forall i \in M$. Hence $x \in P_i(z) \forall i \in M$, or $x \in P_M(z)$, so $x \in P_{\mathbb{D}}(z)$. Thus $U \subset P_{\mathbb{D}}^{-1}(x)$ and is open, so $P_{\mathbb{D}}$ is ldc. ■

Comment 4.2.1. Note that if $P = (P_1, \dots, P_n)$ is a lhc profile on W then it is not necessarily the case that the preference correspondence $P_M: W \rightarrow W$ is lhc. For this reason we require the stronger continuity property of lower demi-continuity rather than lower hemi-continuity. An easy example in Yannelis and Prabhakar, 1983, shows that an lhc preference correspondence need not be ldc, although, as Lemma 4.1.1 showed, an ldc correspondence must be lhc.

Lemma 4.2.2. Let W be admissible and $P = (P_1, \dots, P_n)$ be a semi-convex profile. If \mathbb{D} is a family of subsets of N with Nakamura number $v(\mathbb{D}) = v$, and $\dim(W) \leq v - 2$, then $P_{\mathbb{D}}: W \rightarrow W$ is semi-convex.

Proof. Suppose, on the contrary, that for some $z \in W$, it is the case that $z \in \text{Con } P_{\mathbb{D}}(z)$. By Caratheodory's Theorem (Nikaido, 1968) there exists

$$x_1, \dots, x_{w+1} \in P_{\mathbb{D}}(z), \text{ where } w = \dim(W),$$

such that

$$z \in \text{Con}(\{x_1, \dots, x_{w+1}\}).$$

Let $V = \{1, \dots, w + 1\}$.

For each $j \in V$, $x_j \in P_{\mathbb{D}}(z)$ and so there exists $M_j \in \mathbb{D}$ such that $x_j \in P_{M_j}(x)$. Let $\mathbb{D}' = \{M_j : j \in V\}$. Observe that $\mathbb{D}' \subset \mathbb{D}$ and $|\mathbb{D}'| \leq w + 1 \leq v - 1$. By definition of the Nakamura number $\kappa(\mathbb{D}') \neq \Phi$. Thus there exists $i \in N$ such that $i \in M_j$ for all $M_j \in \mathbb{D}'$. Hence $x_j \in P_i(z)$ for all $j \in V$. But $z \in \text{Con}(\{x_j : j \in V\}) \subset \text{Con } P_i(z)$. Thus contradicts semi-convexity of P_i . Thus $z \in \text{Con } P_{\mathbb{D}}(z)$ for no $z \in W$. Hence $P_{\mathbb{D}}$ is semi-convex. ■

Lemma 4.2.3. If σ is a non-collegial voting rule, W is admissible with $\dim(W) \leq v(\sigma) - 2$, and $P = (P_1, \dots, P_n)$ is semi-convex, then

$$C(W, P_{\sigma}) \neq \Phi.$$

Proof. Since σ is a voting rule, $P_{\sigma} = P_{\mathbb{D}}$. Since $v(\sigma) = v(\mathbb{D})$ and $\dim(W) \leq v(\sigma) - 2$, by Lemma 4.2.2, $P_{\sigma} : W \rightarrow W$ is semi-convex. By Lemma 4.2.1 P_{σ} is also ldc. By Theorem 4.1.1, $C(W, P_{\sigma}) \neq \Phi$. ■

Note that the result also holds when σ is collegial. If $\kappa(\mathbb{D}_{\sigma}) \neq \Phi$, then $M \in \mathbb{D}_{\sigma}$ implies $\kappa(\mathbb{D}_{\sigma}) \subseteq M$ so

$$P_M(x) = \bigcap_{i \in M} P_i(x) \subseteq P_{\kappa(\mathbb{D}_{\sigma})}.$$

Thus

$$P_{\mathbb{D}_{\sigma}} : W \rightarrow W \text{ satisfies } P_{\mathbb{D}_{\sigma}}(x) \subseteq P_{\kappa(\mathbb{D}_{\sigma})}(x).$$

Just as in the proof of Lemma 4.2.3, $P_{\kappa(\mathbb{D}_{\sigma})}$ will be semi-convex, and so there is a choice $C(W, P_{\kappa(\mathbb{D}_{\sigma})})$. Clearly if

$$P_{\kappa(\mathbb{D}_{\sigma})}(x) = \Phi \text{ then } C(P_{\mathbb{D}_{\sigma}}(x)) = \Phi$$

so

$$C(W, P_{\kappa(\mathbb{D}_{\sigma})}) \subseteq C(W, P_{\mathbb{D}_{\sigma}}).$$

For a voting rule $C(W, P_{\sigma}) = \text{Core}(\sigma, W, N, p)$, and so Lemma 4.2.3 gives a proof of Theorem 4.2.1. As a further corollary, we obtain Greenberg's result. As a reminder, note that $v(n, q) = \left\lfloor \frac{q}{n-q} \right\rfloor$, for $q < n$, is the largest integer strictly less than the bracketed term, $\frac{q}{n-q}$.

Corollary 4.2.1. (Greenberg, 1979). If σ is a q -rule (with $n/2 < q < n$), and W is admissible with $\dim(W) \leq \left\lceil \frac{q}{n-q} \right\rceil$ and $P = (P_1, \dots, P_n)$ is an ldc and semi-convex preference profile then $C(W, P_\sigma) \neq \Phi$.

Proof. By Lemma 3.2.2, $v(\sigma) = v(n, q) + 2$ when σ is a q -rule. Thus $\dim(W) \leq v(\sigma) - 2$ iff $\dim(W) \leq \left\lceil \frac{q}{n-q} \right\rceil$. The result follows. ■

Clearly Theorem 4.2.1 is the analogue of the sufficiency part of Nakamura's Theorem 3.2.3, where the cardinality requirement, $|W| \leq v(\sigma) - 1$, and acyclicity of the profile are replaced by the dimensionality requirement $\dim(W) \leq v(\sigma) - 2$, together with lower demi-continuity and semi-convexity of preference. In parallel to the necessity part of Nakamura's Theorem, we shall now show that if $\dim(W) \geq v(\sigma) - 1$ then there exists a preference profile, p , satisfying the continuity and convexity properties such that the core is empty. We shall also show that σ -voting cycles may always be constructed whenever the dimension is at least $v(\sigma) - 1$.

A natural preference to use is *Euclidean preference* defined by $x p_i y$ if and only if $\|x - y_i\| < \|y - y_i\|$, for some bliss point, y_i , in W , and norm $\| - \|$ on W . Clearly Euclidean preference is convex. When preference is defined in this way, we say that the profile of correspondences $P = (P_1, \dots, P_n)$ as well as the profile of preference relations $p = (p_1, \dots, p_n)$ are Euclidean profiles. We now focus on constructing a Euclidean profile in the interior of W .

Lemma 4.2.4. Let $P = (P_1, \dots, P_n)$ be a Euclidean profile on W , with bliss points $\{y_i : i \in N\}$. Suppose that $\text{Con}(\{y_i : i \in N\})$, the convex hull of the bliss points, belongs to the interior of W . Then for any subset $M \subseteq N$, the choice set, $C(W, P_M)$, is given by $\text{Con}[\{y_i : i \in M\}]$.

We shall prove this lemma below. Assuming the lemma, we can now prove the necessity of the dimension condition.

Theorem 4.2.2. (Schofield, 1984b). Let σ be a non-collegial voting rule with Nakamura number $v(\sigma)$. Assume W is admissible with $\dim(W) \geq v(\sigma) - 1$. Then there exists a Euclidean preference profile

$$P = (P_1, \dots, P_n) \text{ such that } C(W, P_\sigma) = \Phi.$$

Proof. There exists a minimal non-collegial subfamily \mathbb{D}_{\min} of \mathbb{D}_σ such that $|\mathbb{D}_{\min}| = v = v(\sigma)$ and $\kappa(\mathbb{D}_{\min}) = \Phi$. As in Corollary 3.2.1, let

$(\phi, \Delta(Y), \mathbb{D}_{\min})$ be the representation of \mathbb{D}_{\min} based on the $(v - 1)$ -dimensional simplex spanned by $Y = \{y_1, \dots, y_v : y_i \in \text{Int}W\}$ and let $\Delta(\mathbb{D}_{\min})$ be the complex consisting of the faces of $\Delta(Y)$. By the corollary, the representation ϕ of \mathbb{D}_{\min} has the property, that, for any $M_j \in \mathbb{D}_{\min}, i \in M_j \Leftrightarrow \phi(\{i\}) \in F(j) = \phi(M_j)$, the face representing M_j .

Without loss of generality we may regard the barycenter of $\Delta(Y)$ as the origin $\{0\}$ in W . Let

$$N(\mathbb{D}_{\min}) = \{i : i \in M_j, \text{ for some } M_j \in \mathbb{D}_{\min}\}.$$

For $i \in N(\mathbb{D}_{\min})$, we may identify $\phi(\{i\})$ with a point y_i , in W . In particular for each vertex y_j of $\Delta(Y)$ there exists an individual j , say, such that $\phi(\{j\}) = y_j$. Let $V = \{1, \dots, v\}$ be the *vertex group* of such players. For any individual $i \in N \setminus N(\mathbb{D}_{\min})$, let $\phi(\{i\})$ be an arbitrary point in $\text{Int}W$. Let $\{P_1, \dots, P_n\}$ be the family of Euclidean preference correspondences on W , with bliss points $\{y_i : i \in N\}$ as determined by this assignment of bliss points to individuals. Let $p = (p_1, \dots, p_n)$ be the preference profile so defined.

By Lemma 4.2.6, for $M_j \in \mathbb{D}_{\min}, C(W, P_M) = \text{Con}(\{y_i : i \in M_j\})$ may be identified with the j^{th} -face $\phi(M_j)$ of the complex. In particular, since $\dim(W) \geq v - 1$, the v distinct faces of the simplex do not intersect. Now

$$C(W, P_{\mathbb{D}_{\min}}) \subseteq \bigcap_{\mathbb{D}_{\min}} C(W, P_M) = \bigcap_{\mathbb{D}_{\min}} \phi(M) = \Phi.$$

Hence $\text{Core}(\sigma, W, N, p) = C(W, P_{\mathbb{D}_{\min}})$ is empty. ■

Corollary 4.2.2. Let σ be a non-collegial voting rule with Nakamura number $v(\sigma)$. Assume W is admissible with $\dim(W) \geq v(\sigma) - 1$. Then there exists a Euclidean preference profile $P = (P_1, \dots, P_n)$ such that $\text{Cycle}(W, P_\sigma) \neq \Phi$.

Proof. If $\text{Cycle}(W, P_\sigma) = \Phi$, then since the Euclidean profile just constructed is ldc, then by Corollary 4.1.3 we see that $C(W, P_{\mathbb{D}_{\min}}) \neq \Phi$, contradicting Theorem 4.2.2.

4.3 Smooth Preference

From now on we consider preference profiles that are representable by smooth utility functions. As in Definition 2.1.2, the preference relation p_i is *representable* by a utility function

$$u_i: W \rightarrow \mathbb{R} \text{ iff, for any } x, y \in W, xpy \Leftrightarrow u(x) > u(y).$$

A *smooth* function $u_i: W \rightarrow \mathbb{R}$ has a continuous differential

$$du_i: W \rightarrow L(\mathbb{R}^w, \mathbb{R}),$$

where $L(\mathbb{R}^w, \mathbb{R})$ is the topological space of continuous linear maps from \mathbb{R}^w to \mathbb{R} . A *smooth profile*, u , for a society N is a function

$$u = (u_1, \dots, u_n): W \rightarrow \mathbb{R}^n,$$

where each component $u_i: W \rightarrow \mathbb{R}$ is a *smooth utility function* representing i 's preference. We shall use the notation $U(W)$ for the class of smooth utility functions on W and $U(W)^N$ for the class of smooth profiles for the society N on W . Just as with preferences, we write $u_M(y) > u_M(x)$ whenever $u_i(y) > u_i(x)$, for all $i \in M$, for $M \subseteq N$. Because we use calculus techniques, we shall assume in the following discussion that W is either an open subset of \mathbb{R}^w , with full dimension, w , or that W is identical to \mathbb{R}^w . In the notation that follows, we shall delete the reference to W and N when there is no ambiguity.

The Pareto set $Pareto(M, u)$ for coalition $M \subseteq N$ is

$$Pareto(M, u) = \{x \in W : u_M(y) > u_M(x) \text{ for no } y \in W\}.$$

When σ is a binary social preference function, we write $\sigma(u)$ for the preference relation $\sigma(p)$, where $p \in O(W)^N$ is the underlying preference profile represented by u . In this case we shall write $Core(\sigma, u) = Core(\sigma, p)$.

When σ is a voting rule, with the family of σ -decisive coalitions, \mathbb{D} , then

$$Core(\sigma, u) = Core(\mathbb{D}, u) = \bigcap_{\mathbb{D}} Pareto(M, u).$$

With smooth preferences we make use of the critical and local “approximations” to the global optima set.

Instead of regarding a preference profile $p = (p_1, \dots, p_n)$ as a primitive, we define the notion of a profile of “direction gradients,” as follows.

Definition 4.3.1. Let $W \subseteq \mathbb{R}^w$, $u \in U(W)^N$, and σ be a voting rule with decisive family \mathbb{D} .

- (i) For each $i \in N$, let $p_i[u]: W \rightarrow \mathbb{R}^w$ be defined such that $p_i[u](x)$ is the *direction gradient* of u_i at x with the property that for all $v \in \mathbb{R}^w$,

$$du_i(x)(v) = (p_i[u](x) \cdot v)$$

where $(p_i[u](x) \cdot v)$ is the scalar product of $p_i[u](x)$ and v .

Let $\mathbf{p}[u]: W \rightarrow (\mathbb{R}^w)^n$ be the profile (of direction gradients) defined by $\mathbf{p}[u](x) = (p_1[u](x), \dots, p_n[u](x))$.

- (ii) For each $i \in N$, and each $x \in W$ let

$$H_i(x) = \{y \in W : p_i[u](x) \cdot (y - x) > 0\}$$

be the *critical preferred set* of i at x . Call

$$H_i: W \rightarrow W$$

the *critical preference correspondence* of i .

- (iii) For each coalition $M \subset N$, define the *critical M-preference correspondence* $H_M: W \rightarrow W$ by

$$H_M(x) = \bigcap_{i \in M} H_i(x).$$

- (iv) Define the “critical” preference correspondence $H_{\mathbb{D}}: W \rightarrow W$ of $\sigma(u)$ by

$$H_{\mathbb{D}}(x) = \bigcup_{M \in \mathbb{D}} H_M(x).$$

- (v) Define the *critical M-Pareto set* by

$$\Theta(M, u) = \{x \in W : H_M(x) = \Phi\}.$$

(vi) Define the *critical core* of $\sigma(u)$ by

$$\begin{aligned}\Theta(\sigma, u) &= \Theta(\mathbb{D}, u) = \{x \in W : H_{\mathbb{D}}(x) = \Phi\} \\ &= \bigcap_{M \in \mathbb{D}} \Theta(M, u).\end{aligned}$$

(vii) Define the *local M -Pareto set*, $L\Theta(N, u)$, by $x \in L\Theta(N, u)$ iff there exists a neighborhood V of x such that for no $y \in V$ is it the case that $u_M(y) > u_M(x)$.

(viii) Define the *local core* of $\sigma(u)$ by

$$LCore(\sigma, u) = LCore(\mathbb{D}, u) = \bigcap_{M \in \mathbb{D}} L\Theta(M, u).$$

Comment 4.3.1. We use the symbols Θ and $L\Theta$ to stand for critical and local Pareto sets, to distinguish them from the Pareto set. Note also that the direction gradient for i (given the profile u) may be written

$$p_i[u](x) = \left[\frac{\partial u_i}{\partial x_i}, \dots, \frac{\partial u_1}{\partial x_w} \right] \Big|_x$$

where x_1, \dots, x_w is a convenient system of coordinates for \mathbb{R}^w . Thus the profile $\mathbf{p}[u](x) \in (\mathbb{R}^w)^n$ of direction gradients at x may also be represented by the n by w *Jacobian matrix* at x :

$$J[u](x) = \left[\frac{\partial u_i}{\partial x_j} \right]_{\substack{i=1, \dots, n \\ j=1, \dots, w}}.$$

For convenience, from now on we drop reference to u and write $\mathbf{p}(x)$ for $\mathbf{p}[u](x)$ and $p_i(x)$ for $p_i[u](x)$. Note that $\mathbf{p}: W \rightarrow (\mathbb{R}^w)^n$ is a continuous function with respect to the usual topologies on W and $(\mathbb{R}^w)^n$. Standard results in calculus give the following.

Lemma 4.3.1. Let $u \in U(W)^N$ and let σ be a simple voting rule with decisive coalitions \mathbb{D} . Then the following sets are closed and are nested as indicated below.

For each $M \subseteq N$,

$$\text{Pareto}(M, u) \subseteq L\Theta(M, u) \subseteq \Theta(M, u).$$

Thus,

$$\begin{aligned} \text{Core}(\sigma, u) &\equiv \text{Core}(\mathbb{D}, u) = \bigcap_{\mathbb{D}} \text{Pareto}(M, u) \\ &\quad \bigcap \quad \quad \quad \bigcap \\ L\Theta(\sigma, u) &\equiv L\Theta(\mathbb{D}, u) = \bigcap_{\mathbb{D}} L\Theta(M, u) \\ &\quad \bigcap \quad \quad \quad \bigcap \\ \Theta(\sigma, u) &\equiv \Theta(\mathbb{D}, u) = \bigcap_{\mathbb{D}} \Theta(M, u). \end{aligned}$$

Definition 4.3.2. For $u \in U(W)^N$, say u satisfies the *convexity property* iff for each $i \in N$, each $x \in W$,

$$\begin{aligned} P_i(x) &= \{y \in W : u_i(y) > u_i(x)\} \\ &\subseteq \{y \in W : p_i(x) \cdot (y - x) > 0\} = H_i(x), \end{aligned}$$

where $p_i(x)$ is the direction gradient of u_i at x .

Obviously, if u satisfies the convexity property, then all the inclusions in Lemma 4.3.1 are identities. Note that the usual properties of *quasi-concavity* or *quasi-convexity* imply this convexity property (see Appendix 4.1.2). Lemma 4.3.1 allows us to determine the critical Pareto set, and thus the Pareto set for a coalition when the convexity property is satisfied.

First we define

$$\begin{aligned} \text{Pos}_m &= \{\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{R}^m : \alpha_i > 0 \text{ for } i = 1, \dots, m\} \\ \overline{\text{Pos}}_m &= \{\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{R}^m : \alpha_i \geq 0 \text{ for } i = 1, \dots, m\}. \end{aligned}$$

A vector $\alpha \in \overline{\text{Pos}}_m \setminus \{0\}$ is called *semi-positive*. Without loss of generality, a vector $\alpha \in \overline{\text{Pos}}_m \setminus \{0\}$ can be assumed to satisfy $\sum_M \alpha_i = 1$, with $\alpha_i \geq 0$, for all i . If $\alpha \in \text{Pos}_m$ we shall call α *strictly positive*. We now seek to characterize points in $\Theta(W, M, u)$.

Lemma 4.3.2 (Schofield, 2003a). Let u be a smooth profile on W for a society N . Let $M \subseteq N$ be a coalition with $|M| = m$.

- (i) Then $x \in \Theta(M, u)$ iff $\exists \alpha \in \overline{\text{Pos}}_m \setminus \{0\}$ such that $\sum_M \alpha_i p_i(x) = 0$.
(ii) $H_M: W \rightarrow W$ is ldc.

This gives us a slight generalization of Lemma 4.2.1 in the case of non-convex preferences.

Lemma 4.3.3. If σ is a voting rule, W' is a compact, convex subset of W with $\dim(W) \leq v(\sigma) - 2$, and u is smooth, then $\Theta(\sigma, u) \cap W' \neq \Phi$.

Proof. Clearly each H_i , and thus each H_M , is semi-convex. Moreover each H_M is also ldc. Then $H_{\mathbb{D}}$ is both ldc and semi-convex, and the proof follows as in Lemma 4.2.3. ■

Lemma 4.3.4. A necessary condition for $x \in \text{Core}(\mathbb{D}, u)$ is that $0 \in \text{Con}[\{p_i(x) : i \in M\}]$, for all $M \in \mathbb{D}$.

Proof. It follows from Lemma 4.3.2 that

$$x \in \Theta(M, u) \text{ if and only if } 0 \in \text{Con}[\{p_i(x) : i \in M\}].$$

Since

$$\text{Pareto}(M, u) \subseteq \Theta(M, u),$$

the result follows. ■

Notice that we assume here that W is open. If W has a non-empty boundary, then the condition for a core point is slightly more complicated. See McKelvey and Schofield (1987).

Definition 4.3.3. A profile $u \in U(W)^N$ on W is called a *Euclidean profile* iff for each i , $u_i: W \rightarrow \mathbb{R}$ is given by

$$u_i(x) = -1/2 \|x - y_i\|^2$$

where y_i is a point in $\text{Int } W$. Here $\|\cdot\|$ is the Euclidean norm in \mathbb{R}^w . The point y_i is called i 's *bliss point*. (Note in particular that $du_i(x) = (y_i - x)$.)

Lemma 4.3.5. Let $u \in U(W)^N$ be a Euclidean profile on a W , with bliss points $\{y_i : i \in N\}$.

- (i) Then

$$\text{Pareto}(M, u) = \text{Con}[\{y_i : i \in M\}]$$

for any subset M of N .

(ii) For any voting rule, σ , with decisive coalitions, \mathbb{D} , then

$$\text{Core}(\mathbb{D}, u) = \bigcap_{M \in \mathbb{D}} [\text{Con}(\{y_i : i \in M\})].$$

Proof. Without loss of generality it follows from Lemma 4.3.2 that $x \in \Theta(M, u)$ iff

$$\sum_M \alpha_i p_i(x) = 0, \quad \text{where} \quad \sum_M \alpha_i = 1.$$

Thus

$$\sum_M \alpha_i (y_i - x) = 0, \quad \text{or} \quad \sum_M \alpha_i y_i = x.$$

But a Euclidean profile satisfies the convexity property, and so

$$\Theta(M, u) = \text{Pareto}(M, u).$$

The result follows. ■

Euclidean profiles are extremely useful for constructing examples, since the core of the voting rule will be the intersection of a family of convex sets, each one of which is the convex hull of the bliss points of the members of one of the decisive coalitions.

4.3.1 Non-Convex Preference

Theorem 4.2.1 shows that as long as W is admissible with $\dim(W) \leq v(\sigma) - 2$, and preference is semi-convex and ldc then the core for σ is non-empty. Lemma 3.2.2 demonstrates, however, that for majority rule (except for the case $(n, q) = (4, 3)$) the Nakamura number is three. Thus a majority rule core can generally only be guaranteed in one dimension. In one dimension if preferences are convex then the profile is single-peaked (see Definition 3.4.1). Thus Theorem 4.2.1 may be thought of as an extension of those results which show that single-peakedness is sufficient for certain rationality properties of majority rule, and thus for the existence of a majority rule core. See Sen (1966), Sen and Pattanaik (1969), Black (1958) and the discussion in Section 3.4.

In this section we briefly examine the consequence of dropping the

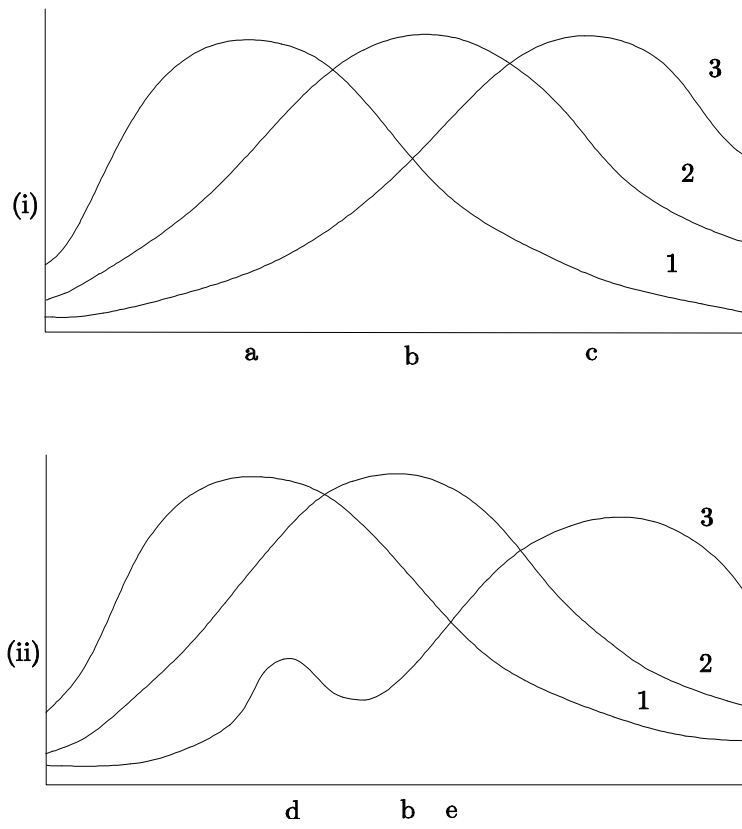


Figure 4.1: Convex and non-convex preference

convexity assumption on preference.

Example 4.3.1. (Kramer and Klevorick, 1974). Consider a situation where W is a compact interval in the real line. Let $N = \{1, 2, 3\}$ and let σ_2 be majority rule, so that

$$\mathbb{D}_{\min} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}.$$

For each $i \in N$, let u be a profile under which each individual has smooth, and convex, preference, and let a, b, c be the bliss points of 1, 2, 3 respectively. Such a situation is represented in Figure 4.1(i).

Clearly $Pareto(\{1, 2\}, u) = [a, b]$, etc., and so

$$Core(\mathbb{D}, u) = [a, b] \cap [b, c] \cap [a, b].$$

Thus the “median” bliss point $\{b\}$ is the core for σ_2 at this profile. Now consider a small perturbation of the utility function of player 3, so this player’s preference is no longer convex. Call the new smooth profile u' . With preferences as in Figure 4.1(ii) it is evident 3 now prefers d to b , so $\{1, 3\}$ both prefer d to b , and b is no longer a core point under $\sigma(u')$. Furthermore, $\{2, 3\}$ both prefer e to d , and $\{1, 2\}$ both prefer b to e . Thus there is a voting cycle

$$b\sigma(u')e\sigma(u')d\sigma(u')b.$$

Consequently, neither Walker’s Theorem (Lemma 4.1.3) nor Greenberg’s result (Corollary 4.2.1) can be used to show existence of a core. In fact it is clear that the core, for the situation represented in Figure 4.1(ii) is empty. Figure 4.1 illustrates that, even when a majority rule core exists, for a convex profile, on a one-dimensional admissible set of alternatives, there is a “small” perturbation of the profile sufficient to destroy this core. See Rubinstein (1979 and Cox (1984) for further details. Note, however, that the point b in Figure 4.1(ii) has the property that for a sufficiently small neighborhood V of b , there exists no point $y \in V$ such that $y\sigma(u')b$. Thus

$$\{b\} \in LCore(\sigma, u').$$

Even in the absence of convexity of preference, a local majority rule core will exist on an admissible subset of the real line, as long as preference is smooth and “well behaved” (Kramer and Klevorick, 1974). Notice also that Lemma 4.3.3 shows that $\Theta(\sigma, u') \neq \Phi$, and the existence of the

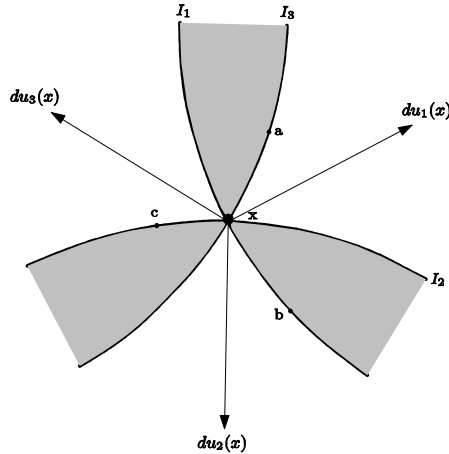


Figure 4.2: Non-convex social preference

local core $LCore(\sigma, u')$ as demonstrated under further relatively weak assumptions on u' .

4.4 Local Cycles

Section 4.2 has shown that, when W is admissible and preferences are ldc and semi-convex, then a necessary and sufficient condition for the non-emptiness of the core of $\sigma(p)$ is that the dimension of W is bounded above by $v(\sigma) - 2$. For this reason we call the integer $v^*(\sigma) = v(\sigma) - 2$ the *stability dimension* for the non-collegial voting rule, σ . Obviously a Euclidean profile is ldc and semi-convex, so this result holds for any Euclidean profile. We can illustrate the emptiness of the core, and the existence of cycles in dimension above $v^*(\sigma)$ by the following example.

Example 4.4.1. (Kramer, 1973a). To illustrate the necessity of the dimension constraint, consider again the case with $N = \{1, 2, 3\}$, and $\mathbb{D} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$ in two dimensions. That is to say, consider majority rule, σ , with $v(\sigma) = 3$ in dimension $v(\sigma) - 1$. Because we are above the stability dimension, there may be no core. Let u be a Euclidean

profile, and x a point in the interior of $Pareto(N, u)$, so that

$$\{du_i(x) : i = 1, 2, 3\}$$

are positively dependent. We can construct such a profile by choosing bliss points $\{y_1, y_2, y_3\}$, with $x \in Int[Con[\{y_1, y_2, y_3\}]]$, the interior of the convex hull of the three points. Figure 4.2 illustrates such a situation. Through the point x let

$$I_i = \{y \in W : u_i(y) = u_i(x)\}$$

be the indifference curve for player i . As the figure makes evident, it is possible to find three points $\{a, b, c\}$ in W such that

$$\begin{aligned} u_1(a) &> u_1(b) = u_1(x) > u_1(c) \\ u_2(b) &> u_2(c) = u_2(x) > u_2(a) \\ u_3(c) &> u_3(a) = u_3(x) > u_3(b). \end{aligned}$$

That is to say, preferences on $\{a, b, c\}$ give rise to a *Condorcet cycle*. Note also that the set of points $P_{\mathbb{D}}(x)$, preferred to x under the voting rule, are the shaded “win sets” in the figure. Clearly $x \in Con P_{\mathbb{D}}(x)$, so $P_{\mathbb{D}}(x)$ is not semi-convex. Indeed it should be clear that in any neighborhood V of x it is possible to find three points $\{a', b', c'\}$ such that there is voting cycle

$$a \rightarrow b \rightarrow c \rightarrow a.$$

Indeed, $Core(\mathbb{D}, u)$ is empty.

Now consider the critical preference correspondences $\{H_M : W \rightarrow W$, for $M \in \mathbb{D}_{\min}$, associated with the three minimal decisive coalitions in \mathbb{D} . Figure 4.3 shows the three preferred sets $\{H_{12}(x), H_{13}(x), H_{23}(x)\}$. As before if we write $H_{\mathbb{D}}(x)$ the union of these three sets, then we see that $x \in Con H_{\mathbb{D}}(x)$, so that $H_{\mathbb{D}}$ is not semi-convex. Indeed, we may infer that $\Theta(\mathbb{D}, u)$ is empty.

The existence of the “permutation” cycle in this example implies that all “exclusion principles,” which are sufficient to guarantee rationality properties of majority rule, must fail on this profile.

We formalize this observation in the following subsection.

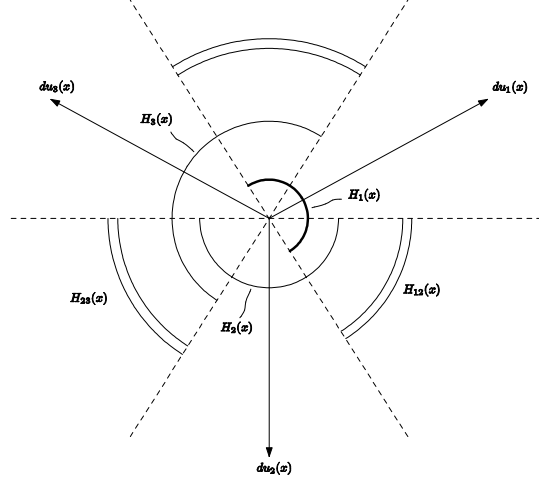


Figure 4.3: Non-convexity of the critical preference cones

4.4.1 Necessary and Sufficient Conditions

The formal definitions and proof of the existence of local cycles in dimension above $v^*(\sigma)$ are adapted from Schofield (1978). Proof of the theorem is technically difficult and we do not attempt to prove it here.

Definition 4.4.1. Let σ be a voting rule on W , with Nakamura number $v(\sigma)$, and \mathbb{D} be its family of decisive coalitions. Let $u \in U(W)^N$.

- (i) Say a point $x \in W$ belongs to the *local cycle set*, $LCycle(\sigma, u)$ iff, for any neighborhood V of x , there exists a subset of points $Z = \{x_1, \dots, x_s\}$, with $s \geq v(\sigma)$, with $Z \subseteq V$, such that $\sigma(u)$ is cyclic on Z .
- (ii) Let \mathbf{p} be the profile (of direction gradients) defined by u , so

$$\mathbf{p}: W \rightarrow (\mathbb{R}^w)^n \text{ where } \mathbf{p}(x) = (p_1(x), \dots, p_n(x)).$$

For each $M \subset N$, and each $x \in W$, define

$$\mathbf{p}_M(x) = \{y \in \mathbb{R}^w : y = \sum \alpha_i p_i(x) : i \in M, \text{ and } \alpha \in \overline{\text{Pos}_m} \setminus \{0\}\}$$

Call $\mathbf{p}_M(x)$ the (generalized) *direction gradient* for coalition M at x .

(iii) Let

$$\mathbb{D}_x = \{M \in \mathbb{D} : 0 \notin \mathbf{p}_M(x)\}.$$

Whenever $\mathbb{D}_x = \Phi$, then define

$$\mathbf{p}_\sigma(x) \equiv \mathbf{p}_{\mathbb{D}}(x) = \{0\},$$

whereas if $\mathbb{D}_x \neq \Phi$, then define

$$\mathbf{p}_\sigma(x) \equiv \mathbf{p}_{\mathbb{D}}(x) = \bigcap_{M \in \mathbb{D}_x} p_M(x),$$

where the intersection on the right is taken over the subfamily \mathbb{D}_x .

(iv) Call $\mathbf{p}_{\mathbb{D}}(x)$ the (generalized) *direction gradient* for the family \mathbb{D} at x .

Notice that by Lemmas 4.3.1 and 4.3.2 that $x \in \Theta(\sigma, u)$ iff $0 \in \mathbf{p}_M(x)$ for all $M \in \mathbb{D}$. In other words, $\mathbb{D}(x) = \Phi$, and $\mathbf{p}_\sigma(x) = \{0\}$. The relationship between $\Theta(\sigma, u)$ and $LCycle(\sigma, u)$ is summed up in the following theorem.

Theorem 4.4.1 (Schofield, 1978, 1985). Let σ be a voting rule on W , with $v(\sigma) = v$, and \mathbb{D} be its family of decisive coalitions. Let $u \in U(W)^N$. Then

(i)

$$\Theta(\sigma, u) = \{x \in W : \mathbf{p}_\sigma(x) \equiv \mathbf{p}_{\mathbb{D}}(x) = \{0\}\}$$

is a closed set.

(ii) The local cycle set is given by

$$\begin{aligned} LCycle(\sigma, u) &= \{x \in W : x \in Con[H_{\mathbb{D}}(x)]\} \\ &= \{x \in W : \mathbf{p}_{\mathbb{D}}(x) = \Phi\} \end{aligned}$$

and is an open set.

(iii) If $\dim(W) \leq v(\sigma) - 2$, then $LCycle(\sigma, u) = \Phi$.

(iv) If $\dim(W) = v(\sigma) - 1$, then

$$LCycle(\sigma, u) \subseteq \Theta(N, u).$$

Moreover, if u satisfies the convexity property, then

$$LCycle(\sigma, u) \subseteq Pareto(N, u).$$

(v) If $\dim(W) \geq v(\sigma)$, then

$$LCycle(\sigma, u) \setminus Pareto(N, u).$$

may be non empty. ■

Notice that this theorem gives a generalization of Lemma 4.2.3 and Theorem 4.2.2 in the case of smooth preferences, when we do not assume convex preferences. In the case that W' is a compact, convex subset of W , then the method of proof of Lemma 4.3.3 gives the following.

Theorem 4.4.2

(i) If W' is a compact, convex subset of W , then

$$\Theta(\sigma, u)/W' \cup LCycle(\sigma, u) \neq \Phi.$$

(ii) Moreover, if u satisfies the convexity property then

$$Core(\sigma, u)/W' \cup LCycle(\sigma, u) \neq \Phi.$$

Example 4.4.2. To illustrate this theorem, consider Example 4.4.1 again where $v(\sigma) = 3$. Suppose first that W is two-dimensional. As in Figure 4.4, let $\{1, 2, 3\}$ be the bliss points of the three players. At a point x in the interior of $Pareto(N, u)$, consider $p_\sigma(x) = \bigcap_{M \in \mathbb{D}} p_M(x)$. Clearly

$$\mathbb{D}(x) = \{\{1, 2\}, \{2, 3\}, \{1, 3\}\}.$$

Now,

$$\mathbf{p}_{12}(x) \cap \mathbf{p}_{13}(x) = p_1(x).$$

However, $p_1(x) \notin \mathbf{p}_{23}(x)$, so $x \in LCycle(\sigma, u)$. Now consider a point

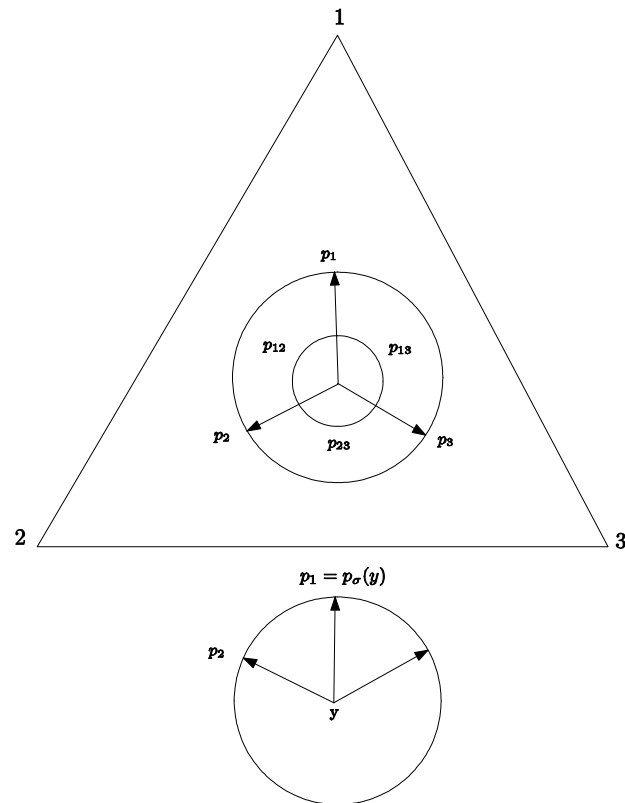


Figure 4.4: Condition for local cyclicity at a point

$y \notin Pareto(N, u)$. As the figure illustrates,

$$p_1(y) \in \mathbf{p}_{12}(x) \cap \mathbf{p}_{13}(y) \cap \mathbf{p}_{23}(y),$$

so $\mathbf{p}_\sigma(y) \neq \Phi$. Thus $y \notin LCycle(\sigma, u)$, and so

$$LCycle(\sigma, u) \subseteq Int[Pareto(N, u)].$$

On the other hand, suppose that W is three-dimensional. Then for some $y \notin Pareto(N, u)$, it will be the case that the direction gradients are linearly independent. Obviously $\mathbf{p}_\sigma(y) = \Phi$. In general, $Pareto(N, u)$ will belong to a two-dimensional subspace of W , so that $W \setminus Pareto(N, u)$ will be dense. Then $LCycle(\sigma, u)$ will also be dense in W .

The same argument can be carried out for a general voting rule, σ , in dimension $v(\sigma)$. This gives the following theorem.

Theorem 4.4.3. Let σ be a voting rule on W , with Nakamura number $v(\sigma)$, and \mathbb{D} be its family of decisive coalitions. Then, if $\dim(W) \geq v(\sigma)$, there exists a Euclidean profile $u \in U(W)^N$ such that $LCycle(\sigma, u)$ is open dense in W .

We have shown in this chapter that (in the presence of compactness and convexity) if the dimension of the “policy space” W is no greater than $v(\sigma) - 2$ then a core will exist, whereas if $\dim(W) \geq v(\sigma) - 1$ then a profile can be constructed so that the core is empty and the cycle set non-empty. Indeed, above the stability dimension $v^*(\sigma) = v(\sigma) - 2$, *local cycles* may occur, whereas below the stability dimension local cycles may not occur. In dimension $v^*(\sigma) + 1$ local cycles will be constrained to the Pareto set. By analogy with Corollary 3.3.2, for the finite case, we may infer that manipulation by coalitions can occur, but in dimension $v^*(\sigma) + 1$ they cannot lead outside the Pareto set. This will not be true, however, in dimension $v^*(\sigma) + 2$, since local cycles can “wander” far from the Pareto set.

However, there may be reasons to suppose that cycles will be restrained to the Pareto set. By Theorem 4.2.2,

$$Core(\sigma, u)/W' \cup LCycle(\sigma, u) \neq \Phi.$$

whenever W' is compact, convex and smooth preference is convex. This suggests that we define the choice function, called the heart, written

$\mathcal{H}(\sigma, u)$, which on any subset, W' of W , is defined by

$$\mathcal{H}(\sigma, u)/W' = [Core(\sigma, u) \cup [LCycle(\sigma, u) \cap Pareto(N, u)]]/W'.$$

Then, by the above results, this set will be non-empty when W' is compact, convex, and the smooth preferences are convex. The next chapter will develop this notion of the heart.

4.5 Appendix to Chapter 4

Definition 4.5.1. Topological spaces

(i) A set W is a *topological space* iff there exists a family

$$T = \{U_\alpha : \alpha \in J\}$$

of subsets of W , called *open sets*, such that

- (a) both the empty set, Φ , and W itself belong to T .
 - (b) if K is a finite subset of the index set, J , then $\bigcap_{\alpha \in K} U_\alpha$ also belongs to T .
 - (c) if K is a subset (whether finite or not) of the index set J , then $\bigcup_{\alpha \in K} U_\alpha$ also belongs to T .
 - (d) If $x \in W$ and $U_\alpha \in T$ such that $x \in U_\alpha$, then U_α is called an (open) neighborhood of x . When attention is to be drawn to the topology T , we write (W, T) for the set W endowed with the topology T .
- (ii) A set $V \subset W$ is *closed* in T iff $V = W \setminus U$, where U is an open set in T .
- (iii) A set $V \subset W$ is *dense* in T iff whenever $x \in W \setminus V$ and U is a neighborhood of x , then $U \cap V \neq \Phi$.
- (iv) An *open cover* for a topological space (W, T) is a family $S = \{U_\alpha : \alpha \in K\}$, where each $U_\alpha \in T$ such that $\bigcup_{\alpha \in K} U_\alpha = W$. S is called *finite* iff K is finite.
- (v) If $S = \{U_\alpha : \alpha \in K\}$ is an open cover for W , then a *subcover*, S' , of S is an open cover $S' = \{U_\alpha : \alpha \in K'\}$ for W where K' is a subset

of K .

- (vi) A topological space (W, T) is *compact* iff for any open cover, S , for W there exists a finite subcover, S' , of S .
- (vii) A topological space (W, T) is *Hausdorff* iff for any $x, y, \in W$ with $x \neq y$, there exist open neighborhoods U_x, U_y of x, y respectively such that

$$U_x \cap U_y = \Phi.$$

- (viii) The *Euclidean norm* $\| - \|$ on \mathbb{R}^w is given by $\|x\| = [\sum x_j^2]^{\frac{1}{2}}$ where $x = (x_1, x_2, \dots, x_w)$ in the usual coordinate system for \mathbb{R}^w .
- (ix) The *Euclidean topology* on \mathbb{R}^w is the natural topology T where $U_\alpha \in T$ if and only if for every $x \in U_\alpha$, there exists some radius, r , such that the open ball $\{y \in \mathbb{R}^w : \|y - x\| < r\} \subseteq U_\alpha$.
- (x) For any subset $W \subseteq \mathbb{R}^w$, the Euclidean topology on W is $T/W = \{U_\alpha/W : U_\alpha \in T\}$.
- (xi) A subset $W \subseteq \mathbb{R}^w$ is compact if the topological space $(W, T/W)$ is compact.
- (xii) The *interior* of $W \subseteq \mathbb{R}^w$, written $\text{Int}(W)$, is the open subset of W in \mathbb{R}^w defined as follows: $x \in \text{Int}(W)$ iff $x \in W$ and there is an open set U_α in the topology T which contains x , such that $U_\alpha \subset W$. The boundary of W is $W \setminus \text{Int}(W)$.

Definition 4.5.2. Convexity of preference

- (i) If $u_i \in U(W)$ then say u_i is *pseudo-concave* iff $u_i(y) > u_i(x)$ implies $du_i(x)(y - x) > 0$. Say u_i is *strictly pseudo-concave* iff $u_i(y) \geq u_i(x)$ and $x \neq y$ implies $du_i(x)(y - x) > 0$.
- (ii) If $u_i: W \rightarrow \mathbb{R}$ then say u_i is *quasi-concave* iff $u_i(\alpha y + (1 - \alpha)x) \geq \min(u_i(x), u_i(y))$ for all $\alpha \in [0, 1]$ and all $x, y \in W$. Say u_i is *strictly quasi-concave* iff $u_i(\alpha y + (1 - \alpha)x) > \min(u_i(x), u_i(y))$ for all $\alpha \in (0, 1)$ and all $x, y \in W$ with $x \neq y$.

Chapter 5

The Heart

The previous two chapters have shown that when W is a topological vector space of dimension w and preference is smooth on W then a core exists and cycles do not exist for the voting rule, σ , whenever w is less than or equal to a stability dimension, $v^*(\sigma)$. Although a core need not exist in dimension $v^*(\sigma) + 1$ and above, it is possible that the core may exist “sometimes” in this dimension range. If the core for $\sigma(u)$ is non-empty, and for a sufficiently small “perturbation” u' of the profile u the core for $\sigma(u')$ is still non-empty then we shall say the core is *structurally stable*.

In this chapter we examine whether a voting core can be *structurally stable* in dimension greater than $v(\sigma) - 1$.

5.1 Symmetry Conditions at the Core

Whether or not a point x belongs to the critical core, $\Theta(\sigma, u)$, or the cycle set, $LCycle(\sigma, u)$ of a voting rule σ , for the smooth profile u , is entirely dependent on the nature of the gradients $\{p_i(x) : i \in N\}$ at the point x . In Examples 4.4.1 and 4.4.2, the direction gradients at the point x satisfied the condition that $\mathbf{p}_{\mathbb{D}}(x) = \Phi$, for the family, \mathbb{D} , of σ -decisive coalitions. Because of this condition, it was possible to construct local cycles in the neighborhood of x . Similarly, if $x \in Core(\sigma, u)$ then it is necessary that $H_M(x) = \Phi$ for all $M \in \mathbb{D}$. This in turn imposes necessary conditions on the direction gradients.

Definition 5.1.1. For any vector $y \in \mathbb{R}^w$ define its dual y^* by

$$y^* \equiv \{z \in \mathbb{R}^w : (z \cdot y) > 0\},$$

where $(z \cdot y)$ means scalar product.

For example, if $u_i \in U(W)$ then, by Definition 4.3.1, the critical preferred set defined by u_i at $x \in W$ is

$$\begin{aligned} H_i(x) &= \{y \in W : (p_i(x) \cdot (y - x)) > 0\} \\ &= [(p_i(x))^* + \{x\}] \cap W. \end{aligned}$$

The set $(p_i(x))^*$ is called the *preference cone* for i at x , located at the origin. Thus $H_i(x)$ is simply the *cone*, $(p_i(x))^*$, shifted by the vector x away from the origin. Now consider a profile, $u \in U(W)^N$. Say that a coalition $M \subset N$ is *effective* at $x \in W$ iff there exists some vector $y \in \mathbb{R}^w$ such that for all $i \in M$, it is the case that $p_i(x) \in (y)^*$. Choose $\mu > 0$, μ sufficiently small, such that $x + \mu y \in W$. Then $(p_i(x) \cdot \mu y) > 0$ for all $i \in M$ and so $x + \mu y \in H_M(x)$. By Lemmas 4.3.1 and 4.3.2, if a decisive coalition is effective at a point x , then x cannot belong to $\Theta(\sigma, u)$, and therefore cannot belong to $Core(\sigma, u)$. The necessary condition on a vector $\mathbf{p}(x)$ of direction gradients at x so that $x \in Core(\sigma, u)$ can then be expressed as the requirement that no decisive coalition is effective at x .

Definition 5.1.2.

- (i) Let $\mathbf{p} = (p_1, \dots, p_n) \in (\mathbb{R}^w)^n$ be a profile of vectors in \mathbb{R}^w , for a society N of size n . For each $M \subset N$, define
- (a) $\mathbf{p}_M = \text{Con}[p_i : i \in M]$.
 - (b) $\mathbf{sp}_M = \text{Span}(\{p_i : i \in M\})$.i.e., $q \in \mathbf{sp}_M(x)$ iff there exists $\{\alpha_i \in \mathbb{R} : i \in M\}$ such that $q = \sum_M \alpha_i p_i$.
 - (c) $(\mathbf{p}_M)^* = \bigcap_{i \in M} (p_i)^* \subset \mathbb{R}^w$.

The half-space $(p_i)^*$ is called the i^{th} preference half space, and the cone $(\mathbf{p}_M)^*$ is called the *preference cone of coalition* M .

- (ii) For any non-zero vector $y \in \mathbb{R}^w$ let

$$N_{\mathbf{p}}(y) = \{i \in N : p_i \in (y)^*\}$$

be the subset of N which is *effective* for y , given \mathbf{p} .

- (iii) If \mathbb{D} is a family of subsets of N , say $\mathbf{p} \in (\mathbb{R}^w)^n$ is a \mathbb{D} -*equilibrium* iff for no $y \in \mathbb{R}^w$ does $N_{\mathbf{p}}(y) \in \mathbb{D}$.

- (iv) Given a family \mathbb{D} of subsets of N , and given any set $L \subset N$, define the set of *pivotal coalitions* (for \mathbb{D} in $N \setminus L$), written $E_L(\mathbb{D})$, as follows: $M \in E_L(\mathbb{D})$ iff $M \subset N \setminus L$ and for any disjoint partition $\{A, B\}$ of $N \setminus L \setminus M$ either $M \cup A \in \mathbb{D}$ or $M \cup B \in \mathbb{D}$.
- (v) If $\mathbf{p} \in (\mathbb{R}^w)^n$ and $L = \{i \in N : p_i = 0\}$, then say $M \in E_L(\mathbb{D})$ is *blocked* iff it is the case that $0 \in \mathbf{p}_{M^*}$ where $M^* = \{i \in N \setminus L : p_i \in \mathbf{sp}_M\}$.
- (vi) Say \mathbf{p} satisfies *pivotal symmetry* (with respect to L and \mathbb{D}) iff either $E_L(\mathbb{D}) = \Phi$ or every $M \in E_L(\mathbb{D})$ is blocked.

We now show that pivotal symmetry is a necessary condition on a profile of vectors in order that no decisive coalition be effective.

Before doing this we need to note a property of $E_L(\mathbb{D})$. It is evident that when σ is a voting rule, then if $M \in \mathbb{D}$ and $M \subset M'$ then $M' \in \mathbb{D}$. Any family of coalitions satisfying this property is called a *monotonic family*.

Lemma 5.1.1. If \mathbb{D} is a monotonic family then, for any $L \subset N$, $E_L(\mathbb{D})$ is a monotonic family of subsets of $N \setminus L$.

Proof. Suppose $M \in E_L(\mathbb{D})$. By definition if $\{A, B\}$ is a partition of $N \setminus L \setminus M$ then either $M \cup A \in \mathbb{D}$ or $M \cup B \in \mathbb{D}$. Let $M' = M \cup C$, where $C \subset N \setminus L$, and let $\{A', B'\}$ be a partition of $N \setminus (L \cup M \cup C)$. Suppose $M \cup C \cup A' \notin \mathbb{D}$. Since $\{C \cup A', B'\}$ is a partition of $N \setminus (L \cup M)$ and $M \in E_L(\mathbb{D})$ it follows that $M \cup B' \in \mathbb{D}$. But since \mathbb{D} is monotonic, $M \cup C \cup B \in \mathbb{D}$. Thus $M' \cup A \notin \mathbb{D}$ implies $M' \cup B \in \mathbb{D}$. Hence $M' \in E_L(\mathbb{D})$ and so $E_L(\mathbb{D})$ is monotonic. ■

Theorem 5.1.1. (McKelvey and Schofield, 1987). A profile

$$\mathbf{p} = (p_1, \dots, p_n) \in (\mathbb{R}^w)^n$$

is a \mathbb{D} -equilibrium only if \mathbf{p} satisfies pivotal symmetry with respect to $L = \{i \in N : p_i = 0\}$ and \mathbb{D} .

Proof. We show that a decisive coalition $M \in \mathbb{D}$ must be blocked if it is not to be effective.

Assume $E_L(\mathbb{D}) \neq \Phi$. Suppose that for some $M \in E_L(\mathbb{D})$ it is the case that $0 \notin \mathbf{p}_{M^*}$. Consider the case first of all that $\dim[\mathbf{sp}_M] = w$. Now $M^* = \{i \in N \setminus L : p_i \in \mathbf{sp}_M\}$ so $M^* = N \setminus L$. Since $E_L(\mathbb{D}) \neq \emptyset$,

some subset, R say, of $N \setminus L$ belongs to \mathbb{D} . By assumption $0 \notin \mathbf{p}_{M^*}$. But then $0 \notin \mathbf{p}_R$, and so R is effective. Hence \mathbf{p} cannot be a \mathbb{D} -equilibrium. Suppose, without loss of generality therefore, that $\dim[\mathbf{sp}_M] < w$, and so $\dim[\mathbf{sp}_{M^*}] < w$. Then there exists $\beta \in \mathbb{R}^w$ with $(\beta \cdot p_i) = 0, \forall i \in M^*$ and $(\beta \cdot p_j) \neq 0, \forall j \in N \setminus L \setminus M^*$. Let

$$\begin{aligned} A &= \{i \in N \setminus L : (\beta \cdot p_i) > 0\} \\ B &= \{i \in N \setminus L : (\beta \cdot p_i) < 0\}. \end{aligned}$$

Since $M \in E_L(\mathbb{D})$ and $E_L(\mathbb{D})$ is monotonic, either $M^* \cup A \in \mathbb{D}$ or $M^* \cup B \in \mathbb{D}$. Without loss of generality, suppose $M^* \cup A \in \mathbb{D}$. Since $0 \notin \mathbf{p}_{M^*}$, M^* is effective and so there exists $\alpha \in \mathbb{R}^w$ such that $M^* \subset N_p(\alpha)$. Indeed α can be chosen to belong to \mathbf{sp}_{M^*} . Clearly there exists $\delta > 0$ such that $((\beta + \delta\alpha) \cdot p_i) > 0$ for all $i \in A$. But for all $i \in M^*$, it is the case that $((\beta + \delta\alpha) \cdot p_i) = (\delta\alpha \cdot p_i) > 0$. Hence $M^* \cup A \subset N_p(\beta + \delta\alpha)$. Since $M^* \cup A$ is both effective and decisive, \mathbf{p} cannot be a \mathbb{D} -equilibrium. Thus if \mathbf{p} is a \mathbb{D} -equilibrium it must satisfy pivotal symmetry with respect to L and \mathbb{D} . ■

To indicate how to apply this result, suppose that we wish to examine whether a point $x \in W$ belongs to $\Theta(\sigma, u)$. By Lemma 4.3.2, this is equivalent to the requirement that the profile

$$\mathbf{p}(x) = (p_1(x), \dots, p_n(x)) \in (\mathbb{R}^w)^n$$

be a \mathbb{D}_σ -equilibrium. Let $L = \{i \in N : p_i(x) = 0\}$. By Theorem 5.1.1, $\mathbf{p}(x)$ must satisfy pivotal symmetry with respect to L and \mathbb{D}_σ . In applying this theorem, the following corollary will be useful.

Corollary 5.1.1. Let \mathbf{p} be a profile and $L = \{i \in N : p_i = 0\}$. The \mathbf{p} is a \mathbb{D} -equilibrium only if, for each $M \in E_L(\mathbb{D})$, there exists at least one individual $j_M \in N \setminus L \setminus M$ such that $\{p_i : i \in M \cup j_M\}$ are linearly dependent.

Proof. Suppose on the contrary that $\{p_i : i \in M\}$ are linearly independent and there exists no individual $j_M \in N \setminus L \setminus M$ such that $j_M \in \mathbf{sp}_M$. But then $M^* = M$. Since $0 \notin \mathbf{sp}_M$ it follows that $0 \notin \mathbf{p}_M$. But this contradicts pivotal symmetry. Hence either $0 \in \mathbf{sp}_M$ or there exists $j_M \in N \setminus L \setminus M$ such that $j_M \in \mathbf{sp}_M$. ■

This corollary gives a useful necessary condition for equilibrium.

For any $M \subseteq N$, define the *singularity set* for M at the utility profile $u_i \in U(W)^N$ by

$$\wedge(M, u) \equiv \{x \in W : 0 \in \mathbf{sp}_M(x)\},$$

where $\mathbf{sp}_M(x) = \text{Span}(\{p_i(x) : i \in M\})$. Obviously, the critical Pareto set, $\Theta(M, u)$, satisfies the inclusion

$$\Theta(M, u) \subseteq \wedge(M, u)$$

It follows from Corollary 5.1.1 that if $x \in \Theta(\mathbb{D}, u)$, for some family, \mathbb{D} , then, for each $M \in E_L(\mathbb{D})$, there exists some individual $j_M \in N \setminus L \setminus M$ such that

$$x \in \wedge(M \cup \{j_M\}, u).$$

From now on we will focus on q -rules, and it is useful to define the family

$$E_1(\mathbb{D}) = \{M \subset N : M \in E_L(\mathbb{D}) \text{ for some } L \subset N \text{ with } |L| = 1\}.$$

We now introduce a number of integers that will prove useful in classifying q -rules.

Definition 5.1.3. For the q -rule, σ_q , with $\mathbb{D}_q = \{M \subseteq N : |M| \geq q\}$, and $n/2 < q < n$, define

$$\begin{aligned} e(\sigma_q) &\equiv (e(\mathbb{D}_q) = \min\{|M| : M \in E_L(\mathbb{D})\}) \\ e_1(\sigma_q) &\equiv e_1(\mathbb{D}_q) = \min\{|M| : M \in E_L(\mathbb{D}), \text{ where } |L| = 1\}. \end{aligned}$$

Corollary 5.1.2. If σ is a q -rule with $n/2 < q < n$, then $e(\sigma_q) = 2q - n - 1$ and $e_1(\sigma_q) = 2q - n$. For majority rule, σ_m , if n is odd then $e(\sigma_m) = 0$, and $e_1(\sigma_m) = 1$, whereas if n is even then $e(\sigma_m) = 1$, and $e_1(\sigma_m) = 2$.

Proof. Suppose that $|M| = 2q - n - 1$, and consider a partition $\{A, B\}$ of $N \setminus M$. If $|A| \geq n + 1 - q$ then $|M \cup A| \geq q$ and so $M \cup A \in \mathbb{D}_q$. On the other hand, if $|A| \leq n - q$, then $|N \setminus A| \geq q$ and so $M \cup (N \setminus M \setminus A) = M \cup B \in \mathbb{D}_q$. Thus $|M| \geq 2q - n - 1 \Rightarrow M \in E(\mathbb{D})$. Clearly if $|M| \leq 2q - n - 2$, then there exists $A \subset N \setminus M$ such that $|A| = n + 1 - q$ yet $|M \cup A| = q - 1$ so $M \cup A \notin \mathbb{D}_q$ and $|N \setminus A| = q - 1$ so $N \setminus A \notin \mathbb{D}_q$.

Thus $e(\sigma_q) = 2q - n - 1$. In similar fashion, if $p_k = 0$, let $N' = N \setminus \{k\}$ so $|N'| = n' = n - 1$. Then

$$e_1(\sigma_q) = 2q - n' - 1 = 2q - (n - 1) - 1 = 2q - n.$$

Finally majority rule σ_m , with n odd, then $q = (k + 1)$ and $n = (2k + 1)$ so

$$e(\sigma_m) = 2(k + 1) - (2k + 1) - 1 = 0, \text{ and } e_1(\sigma_m) = 1.$$

If n is even then $q = (k + 1)$ and $n = (2k)$ so $e(\sigma_m) = 2(k + 1) - 2k - 1 = 1$ and $e_1(\sigma_m) = 2$. ■

We can use this corollary to determine those conditions under which a core can be *structurally stable*. This argument depends on the idea of a topology on the set $U(W)^N$ of smooth preferences. Details of this topology can be found in Golubitsky and Guillemin (1973), Smale (1973), Hirsch (1976), and Saari and Simon (1977). Further details are in Schofield (2003a).

Definition 5.1.4. Let W be a subset of \mathbb{R}^w .

(i) A set $V \subset U(W)^N$ is open in the C^0 -topology, T_0 , on $U(W)^N$ iff for any $u \in V$, $\exists \delta > 0$ such that

$$\{u' \in U(W)^N : \|u'_i(x) - u_i(x)\| < \delta, \forall x \in W, \forall i \in N\} \subset V,$$

$\|-\|$ where is the Euclidean norm on W . Write $(U(W)^N, T_0)$ for this topological space.

(ii) For $u \in U(W)^N$, and $x \in W$, let

$$J[u](x) = \begin{bmatrix} \frac{\partial u_i}{\partial x_j} \end{bmatrix}_{\substack{i=1, \dots, n \\ j=1, \dots, w}}.$$

be the n by w Jacobian matrix of differentials. Let $\| \cdot \|_{n,w}$ be the natural norm on such matrices.

(iii) A set $V \subset U(W)^N$ is open in the C^1 -topology, T_1 , on $U(W)^N$ iff

for any $u \in V, \exists \delta_1, \delta_2 > 0$ such that

$$\left\{ \begin{array}{l} u' \in U(W)^N: \|u'_i(x) - u_i(x)\| < \delta_1, \\ \text{and } \|J[u'_i](x) - J[u_i](x)\|_{n,w} < \delta_2, \\ \forall i \in N, \forall x \in W \end{array} \right\} \subset V.$$

Write $(U(W)^N, T_1)$ for this topological space.

Comment 5.1.1.

In general if T_1 and T_2 are two topologies on a space U , then say T_2 is *finer than* T_1 iff every open set in the T_1 -topology is also an open set in the T_2 -topology. T_2 is strictly finer than T_1 iff T_2 is finer than T_1 and there is a set V which is open in T_2 but which is not open in T_1 . The C^1 -topology on $U(W)^N$ is strictly finer than the C^0 -topology on $U(W)^N$.

We shall now consider the question whether $Core(\sigma, u) \neq \Phi$ for all u in some open set in $(U(W)^N, T_1)$.

Definition 5.1.5. Let σ be a voting rule.

(i) The *stable subspace* of profiles on W for σ is

$$Stable(\sigma, W) = \{u \in U(W)^N: Core(\sigma, u) \neq \Phi\}.$$

If $u \in Stable(\sigma, W)$ and there exists a neighborhood $V(u)$ of u in the C^1 -topology such that $u' \in Stable(\sigma, W)$ for all $u' \in V(u)$, then σ is said to have a *structurally stable core* at the profile u .

(ii) If $u \in Stable(\sigma, W)$ and in any neighborhood V of u in the C^1 -topology there exist $u' \notin Stable(\sigma, W)$, then σ is said to have a *structurally unstable core* at the profile u .

Note that if σ has a structurally stable core at u , then there is a neighborhood $V(u)$ of u in $U_1(W)^N$ with $V(u) \subset Stable(\sigma, W)$ and so the interior, $Int\ Stable(\sigma, W)$, is non-empty. We now examine conditions under which a structurally stable core can or cannot occur.

Comment 5.1.2. A set $V \subset U(W)^N$ is called a *residual* set in the topology on $U(W)^N$ iff it is the countable intersection of open dense sets in the topology. It can be shown that any residual subset of $U(W)^N$ in the C^1 -topology is itself dense. Let K be a property which can be satisfied

by a smooth profile, and let

$$U[K] = \{u \in U(W)^N : u \text{ satisfies } K\}.$$

Then K is called a *generic property* iff $U[K]$ contains a residual set in the C^1 -topology.

Now consider the topology on

$$U_{con}(W)^N = \{u \in U(W)^N : u \text{ satisfies the convexity property}\},$$

induced by taking the restriction of T_1 to $U_{con}(W)^N$. Saari (1997) has shown that the core is generically empty in $(U_{con}(W)^N, T_1)$ for a q -rule if the dimension of W is sufficiently high. To sketch a proof, note first that the transversality theorem implies that if M_1, M_2 are two disjoint subsets of N , and if $w \geq \max\{|M_1|, |M_2|\}$, then the dimension of

$$\wedge(M_1, u) \cap \wedge(M_2, u)$$

is generically at most $|M_1| + |M_2| - 2 - w$. In particular, if $|M_1| + |M_2| - 2 - w < 0$, then $\wedge(M_1, u)$ and $\wedge(M_2, u)$ generically do not intersect. Moreover if $M_1 = \{i\}$ and $M_2 = \{j\}$ for singleton i, j with $i \neq j$, then again $\wedge(\{i\}, u)$ and $\wedge(\{j\}, u)$ generically do not intersect, for all w . Thus, in applying the notion of pivotal symmetry at a point $x \in W$, we may make the generic assumption that $L = \{i \in N : p_i(x) = 0\}$ has cardinality at most 1.

Definition 5.1.6. Let $u \in U(W)^N$ and let $\mathbf{p}[u]$ be the profile of direction gradients induced by u . Then a point x belongs to the *bliss core*, written $x \in BCore(\sigma, u)$ iff $x \in Core(\sigma, u)$ with $p_i(x) = 0$ for exactly one individual $i \in N$.

Corollary 5.1.3. Let σ be a q -rule. If $\dim(W) \geq 2q - n + 1 > 0$, then the property that $BCore(\sigma, u)$ is empty is generic.

Proof. By Corollary 5.1.1, if $x \in BCore(\sigma, u)$ then $x \in \wedge(M \cup \{j_M\}, u)$ for any $M \in E_1(\mathbb{D})$, and some $j_M \notin M$. By Corollary 5.1.2, $|M|$ can be assumed to be $2q - n$. By the transversality theorem, $\wedge(M, u) \cap \wedge(\{j_M\}, u)$ generically has dimension at most $(2q - n) + 1 - 2 - (2q - n + 1) < 0$. Thus $BCore(\sigma, u)$ is generically empty. ■

It immediately follows that a bliss core cannot be structurally stable in dimension greater than $2q - n + 1$.

Saari (1997) also showed that if $\dim(W) \leq 2q - n$ then a structurally stable bliss core can occur. Similar results hold for structural stability of a *non-bliss core*, where $p_i(x) = 0$ for no individual. Saari's result allows us to define the *instability dimension*, $w^*(\sigma_q)$, for a q -rule.

Definition 5.1.7. The *instability dimension*, $w^*(\sigma_m) \equiv w^*(\mathbb{D}_m)$ for majority rule with n odd is given by $w^*(\sigma_m) \equiv w^*(\mathbb{D}_m) = 2$, whereas for n even, $w^*(\sigma_m) \equiv w^*(\mathbb{D}_m) = 3$. For all other q -rules, the *instability dimension*, $w^*(\sigma_q) \equiv w^*(\mathbb{D}_q)$, is defined to be

$$2q - n + 1 + \max \left\{ \frac{(4q - 3n - 1)}{2(n - q)}, 0 \right\}.$$

Saari (1997) has shown that even the non-bliss core for a q -rule is generically empty in dimension at least $w^*(\sigma_q)$. These results can be further applied to the case of majority rule.

Definition 5.1.8.

- (i) Let M be a society of even size $m = 2k$. A *Plott partition* of M is a disjoint partition $\{N_r\}_{r=1}^k$ of non-empty subsets of N such that $|N_r| = 2$ for $r = 1, \dots, k$.
- (ii) If $n = 2k$ is even, a profile of vectors $\mathbf{p} = (p_1, \dots, p_n)$ is an *even Plott equilibrium* iff $p_i = 0$ for no $i \in N$, and there is a Plott partition $\{N_r\}_{r=1}^k$ of N , such that, for each $N_r = \{p_i, p_j\}$ there is some $\alpha > 0$ such that $p_i + \alpha p_j = 0$.
- (iii) If $n = 2k + 1$ is odd, a profile of vectors $\mathbf{p} = (p_1, \dots, p_n)$ is an *odd Plott equilibrium* iff there is exactly one $i \in N$, such that $p_i = 0$, and there is a Plott partition $\{N_r\}_{r=1}^k$ of $N \setminus \{i\}$, such that $(p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n)$ is an even Plott equilibrium.
- (iv) In either case we say \mathbf{p} is a Plott equilibrium.

Plott (1967) obtained the following theorem, which can be obtained as a corollary of Theorem 5.1.1.

Theorem 5.1.2. (Plott, 1967). Let \mathbb{D}_m be the family of decisive coalitions for majority rule.

- (i) If $\mathbf{p} = (p_1, \dots, p_n) \in (\mathbb{R}^w)^n$ is a Plott equilibrium then it must be a \mathbb{D}_m -equilibrium.
- (ii) If n is odd and \mathbf{p} is a \mathbb{D}_m -equilibrium then $p_i = 0$ for at least one $i \in N$. Moreover, if \mathbf{p} is a \mathbb{D}_m -equilibrium with $p_i = 0$ for exactly one $i \in N$ then \mathbf{p} is an odd Plott equilibrium.
- (iii) If n is even and p is a \mathbb{D}_m -equilibrium with $p_i = 0$ for no $i \in N$, then p is an even Plott equilibrium.

To illustrate, suppose n is odd, and that $L = \{i \in N : p_i = 0\}$ is empty. Then the empty set Φ is clearly pivotal. But then pivotal symmetry cannot be satisfied, and by Theorem 5.1.1, \mathbf{p} cannot be a \mathbb{D}_m -equilibrium. Thus L cannot be empty.

This theorem effectively gives necessary conditions for a profile of vectors to be a majority rule equilibrium in the case with n odd. Note, however, that if n is even and $p_i = 0$ for some $i \in N$ then p may be a majority rule equilibrium but not a Plott equilibrium. This relationship between the symmetry properties required for p to be a majority rule equilibrium and a Plott equilibrium permits an examination of the properties of the core for majority rule.

Corollary 5.1.4. Let σ be a majority rule, with $W = \mathbb{R}^w$, and $u \in U(W)^N$.

- (i) If $x \in W$ is such that the vector $\mathbf{p}(x) = (p_1(x), \dots, p_n(x))$ of direction gradients defined by u at x is a Plott equilibrium then $x \in \Theta(\sigma, u)$, the critical core.
- (ii) If n is odd and $x \in \text{Core}(\sigma, u)$ with $p_i(x) = 0$ for exactly one $i \in N$, then $\mathbf{p}(x) = (p_1(x), \dots, p_n(x))$ is an odd Plott equilibrium.
- (iii) If n is even and $x \in \text{Core}(\sigma, u)$ with $p_i(x) = 0$ for no $i \in N$ then $\mathbf{p}(x) = (p_1(x), \dots, p_n(x))$ is an even Plott equilibrium.
- (iv) Furthermore if $u \in U(W)^N$ satisfies the convexity property then $\text{Core}(\sigma, u)$ may be substituted for $\Theta(\sigma, u)$ in (i).

Note that the Plott equilibrium condition on the direction gradients at a point is neither necessary nor sufficient for the point to belong to the majority rule core. If preferences satisfy the convexity property then the

Plott conditions are *sufficient*. Even in the presence of convexity, however, the conditions are not necessary. For example, if n is even, with $p_i(x) = 0$ for one individual, then x may be a majority rule core point although $\mathbf{p}(x)$ is not a Plott equilibrium. For example, a pair of voters, $M = \{i, j\}$ will pivot, so the pivot condition $0 \in \mathbf{p}_{M^*}(x)$ can be satisfied if there is a third voter k , say, such that $0 \in \text{Con}[p_i(x), p_j(x), p_k(x)]$.

Example 5.1.1. Consider the case where $W = \mathbb{R}^2$ and each individual, i , has a Euclidean utility function with bliss point x_i . First of all let σ be the q -rule with $(n, q) = (4, 3)$. From Lemma 3.2.2 we know that the Nakamura number is $v(\sigma) = 4$ and so the stability dimension $v^*(\sigma) = v(\sigma) - 2 = 2$. By earlier results, $\text{Core}(\sigma, u)$ will be non-empty. Figure 5.1(i) represents the situation where $\mathbf{p}(x) = (p_1(x), p_2(x), p_3(x), p_4(x))$ is a Plott equilibrium, where $p_i(x) = 0$ for no $i \in N$. By Corollary 5.1.4, $\{x\} = \text{Core}(\sigma, u)$. Now consider Figure 5.1(ii). The bliss point y of player 2 belongs to $\text{Core}(\sigma, u)$, because pivotal symmetry is satisfied at that point. The first case is an example of a non-bliss core, and the second of a bliss core. By Saari's Theorem, both these cores are structurally stable.

Example 5.1.2. Now let σ be the majority q -rule with $(n, q) = (6, 4)$, and let \mathbb{D}_m be the decisive coalitions. We know that $v^*(\sigma) = 1$ and so there is no guarantee that the core is non-empty in two dimensions. Figure 5.2(i) presents a Euclidean profile, u , such that, at the point $x \in \mathbb{R}^2$, $\mathbf{p}(x)$ is an even Plott equilibrium and $x \in \text{Core}(\sigma, u)$. Figure 5.2(ii) presents a perturbation (small in the C^1 -topology) of this profile u to a new profile u' , such that $\text{Core}(\sigma, u')$ is empty. Thus the non-bliss core in Figure 5.2(i) is structurally unstable. In Figures 5.2(i) and (ii), the arcs between the bliss point pairs $\{3, 6\}$, $\{1, 4\}$ and $\{2, 5\}$ are called *median lines*. On each such line there are two bliss points. Moreover, there are two further bliss points on either side of each of these lines, giving a majority of bliss points on both sides of these medians. Another way to express this condition, for example, is that the arc $[3, 6]$ is the intersection of the two Pareto sets $\text{Pareto}(\{3, 4, 5, 6\}, u)$ and $\text{Pareto}(\{3, 2, 1, 6\}, u)$. The intersection of these median lines gives the core in Figure 5.2(i). In Figure 5.2(ii), these median lines do not intersect. Instead they bound a small triangle, which can readily be shown to be the local cycle set,

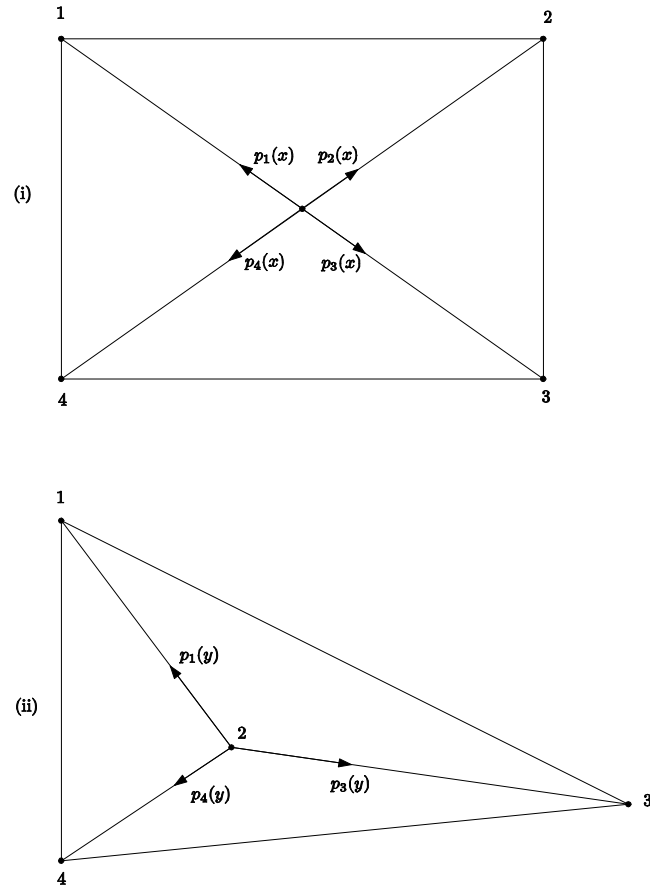


Figure 5.1: Euclidean preferences with the q rule given by $(n, q) = (4, 3)$

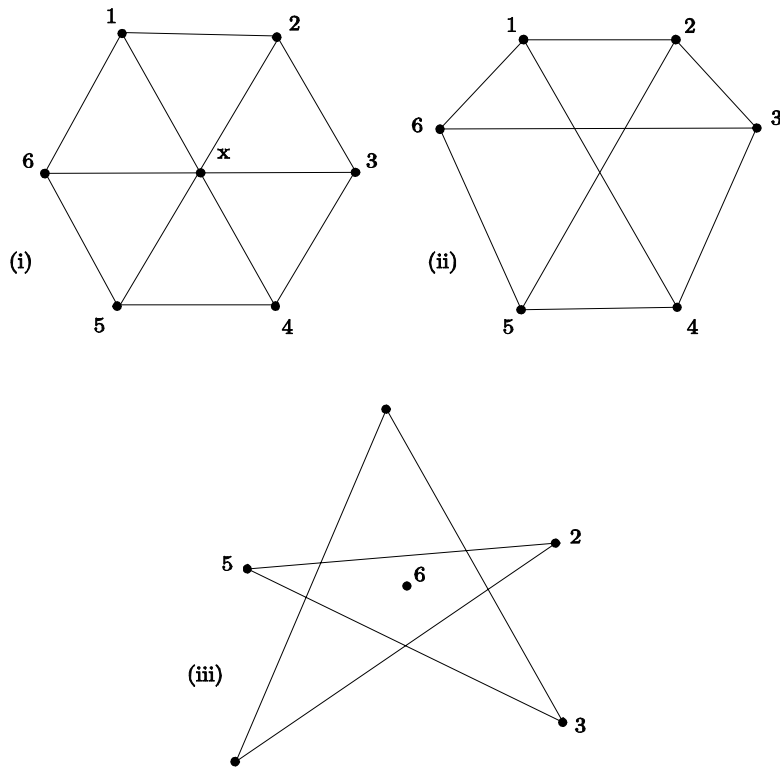


Figure 5.2: Euclidean preferences with the q rule given by $(n, q) = (6, 4)$

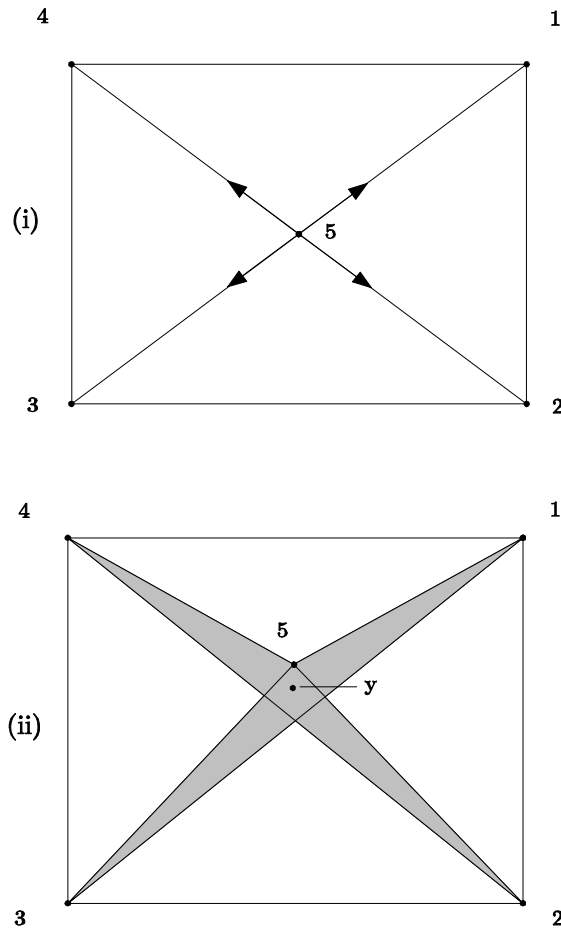


Figure 5.3: Euclidean preferences with the q rule given by $(n, q) = (5, 3)$

$LCycle(\sigma, u')$. Thus, Figures 5.2(i) and 5.2(ii) illustrate the proposition derived in the previous chapter that the heart

$$\mathcal{H}(\sigma, u) = Core(\sigma, u) \cup [LCycle(\sigma, u)]$$

is not empty. Now, consider the profile u represented by Figure 5.2(iii). It should be clear that pivotal symmetry is satisfied at the bliss point of player 6, and so the bliss core is non empty. However, the direction gradients at this point do not satisfy the Plott symmetry conditions. For a small perturbation, u' , of this profile the core is still non-empty, so the bliss core is structurally stable, as is consistent with Saari's Theorem. In three dimensions, the bliss core cannot be structurally stable.

Example 5.1.3. In the same way Figure 5.3(i) represents a Euclidean profile for the q -rule with $(n, q) = (5, 3)$. We can see that the bliss core at the bliss point of player 5 in the figure is structurally unstable. After a small perturbation, the bliss point of player 5 is not located at the intersection of the median lines. The perturbation gives Figure 5.3(ii), showing an empty core. The cycle set, $LCycle(\sigma, u')$, is non-empty and is the interior of the four pointed star, bounded by the median lines $[1, 5], [4, 5], [2, 5], [3, 5]$ and $[1, 3]$, and shown shaded in the figure.

We have defined the heart by

$$\mathcal{H}(\sigma, u) = Core(\sigma, u) \cup [LCycle(\sigma, u) \cap Pareto(N, u)].$$

We now give an alternative definition that links it to another choice concept called the *uncovered set*.

Definition 5.1.9. The uncovered set and the heart.

- (i) A *strict preference correspondence*, Q , on a set, or space, W is a correspondence $Q : W \rightrightarrows W$ such that $x \notin Q(x)$ for every $x \in W$.
- (ii) Let $Q : W \rightrightarrows W$ be a preference correspondence on the space W . As before, the *choice* of Q from W is

$$C(Q) = \{x \in W : Q(x) = \Phi\}.$$

- (iii) The covering correspondence Q^* of Q is defined by $y \in Q^*(x)$ iff $y \in Q(x)$ and $Q(y) \subset Q(x)$. Say y *covers* x . The *uncovered set*,

$C^*(Q)$ of Q , is

$$C^*(Q) = C(Q^*) = \{x \in W : Q^*(x) = \Phi\}.$$

- (iv) If W is a topological space, then $x \in W$ is *locally covered* (under Q) iff for any neighborhood Y of x in W , there exists $y \in Y$ such that (a) $y \in Q(x)$, and (b) there exists a neighborhood Y' of y , with $Y' \subseteq Y$ such that $Y' \cap Q(y) \subset Q(x)$.
- (v) The *heart* of Q , written $\mathcal{H}(Q)$, is the set of locally uncovered points in W .

In the application of these notions, the correspondence, Q , will be taken to be the correspondence $P_{\mathbb{D}}$ defined by some non-collegial family, \mathbb{D} , of decisive coalitions, given by the simple voting rule, σ , at a smooth profile u . In this case we write $C^*(\mathbb{D}, u)$ or $C^*(\sigma, u)$ for the uncovered set. It can be shown that the heart defined previously is identical to the heart as just defined in terms of this correspondence (Schofield, 1999b). Indeed, the heart can be defined in terms of the decisive coalitions \mathbb{D} and the profile $\mathbf{p}[u]$ of direction gradients. From now on we shall write either $\mathcal{H}(\mathbb{D}, u)$ or $\mathcal{H}(\sigma, u)$ for a simple voting rule, σ , with decisive coalitions, \mathbb{D} . Under fairly general conditions, if the core, $Core(\mathbb{D}, u)$, is non-empty, then it is contained in both $C^*(\mathbb{D}, u)$ and $\mathcal{H}(\mathbb{D}, u)$. It can also be shown that the heart, regarded as a correspondence $\mathcal{H}_{\mathbb{D}} \equiv \mathcal{H}(\mathbb{D}, -) : U_{con}(W)^N : \rightarrow W$ is lower hemi-continuous (see definition 4.1.1). To illustrate this, consider the profile, u , in Figure 5.2(i), where $x = Core(\mathbb{D}, u)$, so $\mathcal{H}_{\mathbb{D}}(u)$ is non-empty. Then, as Figure 5.2(ii) illustrates, in any neighborhood V of x in W , there is a neighborhood Y of u in $U_{con}(W)^N$ such that, at the profile $u' \in Y$, then $\mathcal{H}_{\mathbb{D}}(u') \cap V \neq \Phi$.

5.2 Examples of the Heart and Uncovered Set

For the case $(n, q) = (5, 3)$, Figures 5.3 and 5.4 show the cycle set is bounded by the five median lines. In Figure 5.4, for example, the heart is the symmetric pentagon generated by the five bliss points. The *yolk* is the smallest circle that touches all the median lines, and is represented by the small inner circle. McKelvey (1986) showed that the uncovered

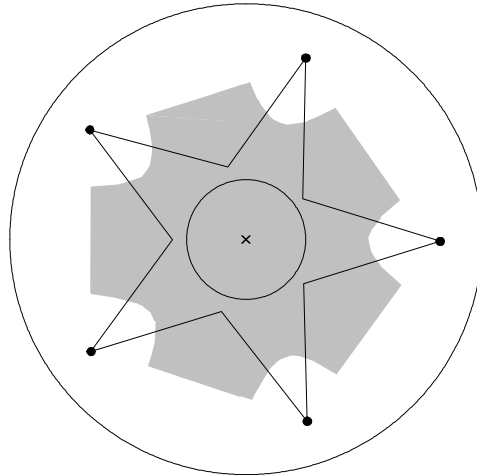


Figure 5.4: The heart, the yolk and the uncovered set

set is contained within the ball of radius 4 times the yolk radius (r), thus demonstrating that the size of the uncovered set is fairly constrained. This can be seen in Figure 5.4 where the uncovered set is the symmetric blunt pentagon lying inside the larger outer circle of radius $4r$. Figure 5.5 illustrates the heart and the yolk (under majority rule) when there is a uniform distribution of the voter preferred points on the boundary of the triangle. The lines marked M_1, M_2, M_3, M_4 are representative median lines. Figure 5.6 shows the heart and yolk when there is a uniform distribution of voter bliss points on the pentagon. Obviously, as the voter distribution becomes symmetric, then the heart collapses to a core point.

Both the heart and the uncovered set have been suggested as predictors in experimental games. Perhaps the main advantage of the heart is that it is easily computed, since it can be determined from the median lines of the political game. The next section considers the results of experiments from spatial voting games with no core.

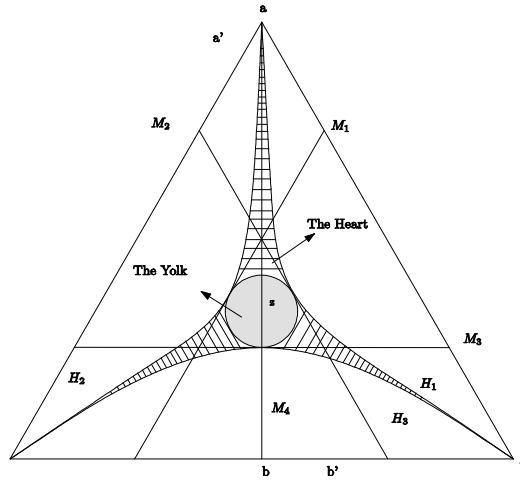


Figure 5.5: The heart with a uniform electorate on the triangle

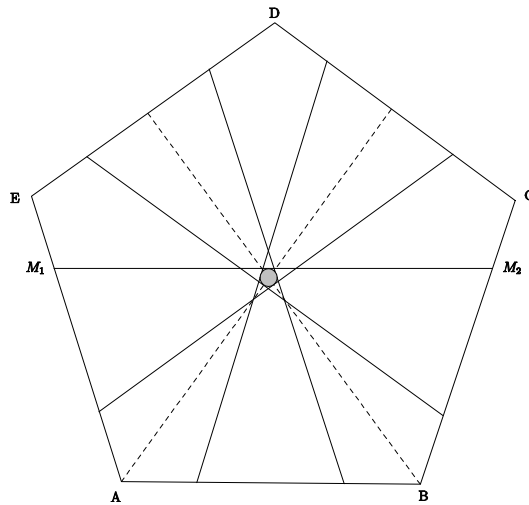


Figure 5.6: The heart with a uniform electorate on the pentagon

5.3 Experimental Results

Figures 5.7 through 5.12 present the experimental results obtained by various authors for the q -rule with $(n, q) = (5, 3)$. In Figure 5.7 there is a core point at the bliss point of player 1. As the figure suggests, all experimental outcomes clustered near that point. The experiments in Figures 5.8 to 5.12 are more interesting, since the core is empty in all five figures. Four of these experiments were designed to test an equilibrium notion called the *competitive solution* (McKelvey, Ordeshook and Winer, 1978; Ordeshook and McKelvey, 1978; McKelvey and Ordeshook, 1982).¹³ The competitive solution in these figures is denoted CS. The cycle set, or *heart* as it is dubbed here, is again the pentagon generated by the voter bliss points. All observations in Figure 5.8 lie in this set. Figures 5.9 and 5.10 present the results of the experiment carried out by Laing and Olmstead (1996). See also Laing and Slotznick (1987). Approximately 24 out of 30 of the data points lie in the heart. In Eavey's two experiments presented in Figures 5.11 and 5.12 (Eavey, 1996), 15 out of 20 of the data points lie in the heart. We may say the "success rate" of this notion is about 80 percent. These observations are only meant to suggest that the heart has some intrinsic merit. One advantage of $\mathcal{H}_{\mathbb{D}}$ as a "solution theory" is that it will "converge to the core," in the sense that if $Core(\mathbb{D}, u)$ is non-empty, for some $u \in U_{con}(W)^N$, and u' converges to u , in the C^1 -topology, then $\mathcal{H}_{\mathbb{D}}(u')$ converges to $Core(\mathbb{D}, u)$. To say the heart, $\mathcal{H}_{\mathbb{D}}(u')$, converges to $Core(\mathbb{D}, u)$ just means that if $x \in Core(\mathbb{D}, u)$, then there is a sequence of points $\{x' : x' \in \mathcal{H}_{\mathbb{D}}(u')\}$ converging to x , as u' converges to u .

¹³See McKelvey and Ordeshook (1990) for a survey of these experimental results on Committees.

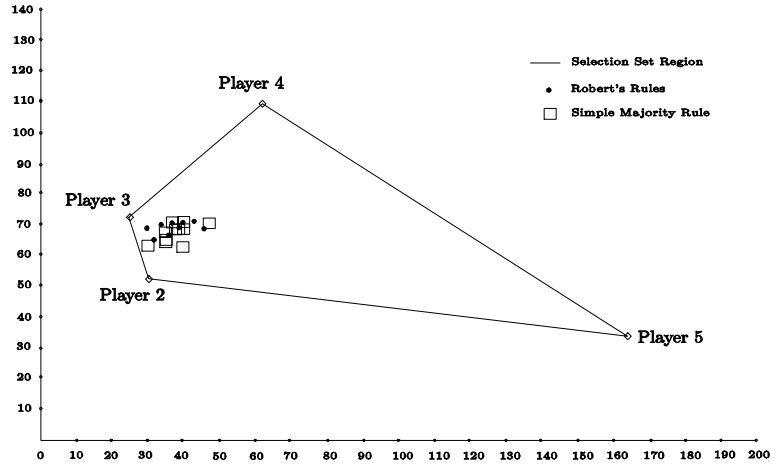


Figure 5.7: Experimental results of Fiorina and Plott (1978)

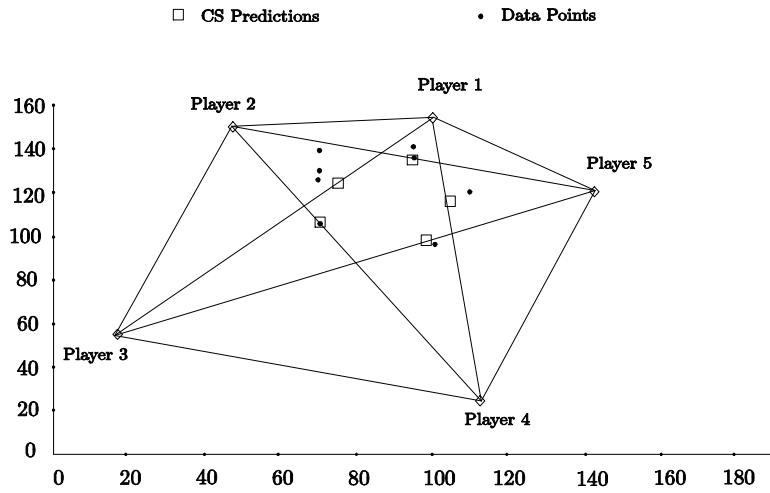


Figure 5.8: Experimental results of McKelvey and Ordeshook (1978)

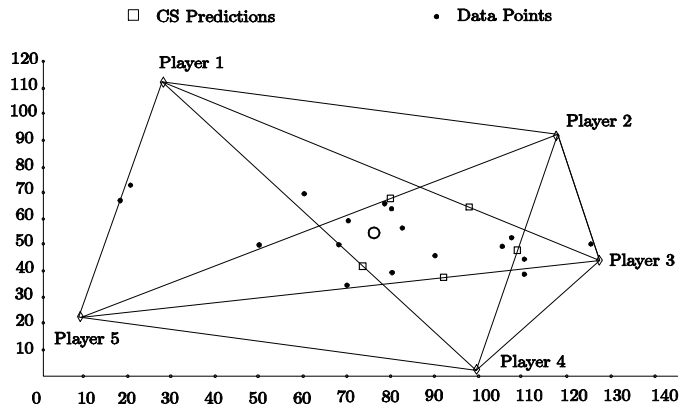


Figure 5.9: Experimental results of Laing and Olmstead (1978)

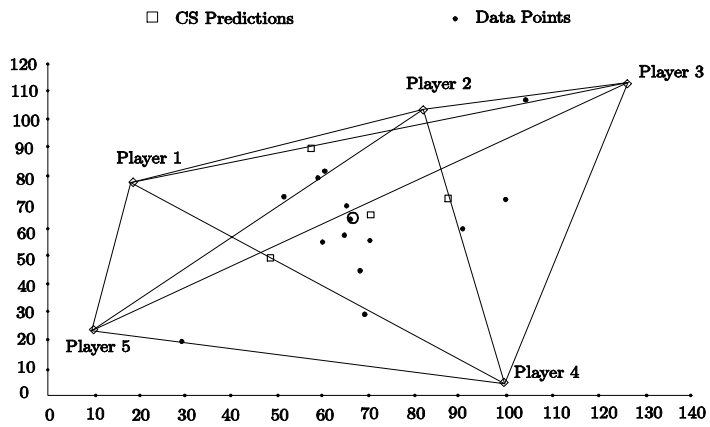


Figure 5.10: Experimental results of Laing and Olmstead (1978)

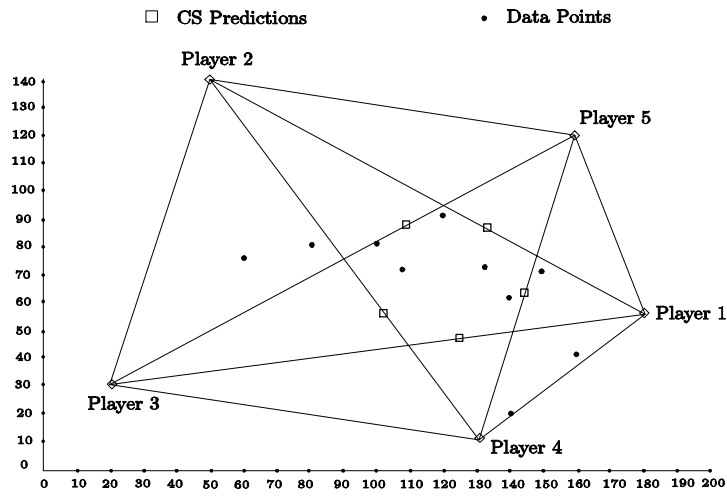


Figure 5.11: Experimental results of Eavey (1991)

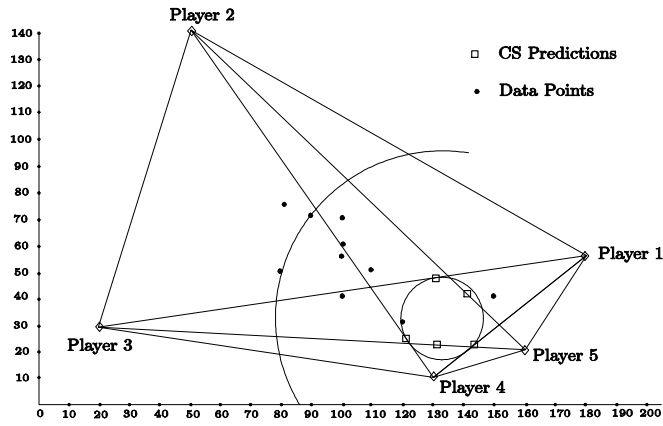


Figure 5.12: Experimental results of Eavey (1991)

Chapter 6

A Spatial Model of Coalition

6.1 Empirical Analyses of Coalition Formation

Empirical work in the 1970s on coalition formation in multiparty systems tended to be based on cooperative game-theoretical notions (Riker, 1962) or on models of policy bargaining in a one-dimensional space (Downs, 1957). More precisely, under the assumption that parties seek perquisites (such as portfolios, ministries), it is natural to suppose that minimal winning (MW) coalitions form. Here we change terminology from the previous chapters, and use the term minimal *winning* to mean a coalition that controls a majority of the seats, but may lose no party and still be winning. On the other hand, if policy is relevant, then an obvious notion is that of a minimal connected winning (MCW) coalition (Axelrod, 1970; de Swaan, 1973). A MCW coalition is a group of parties that controls at least a majority of the seats, and also comprises parties that are adjacent to one another in the one-dimensional space.

Much of the research on the characteristics of multiparty governments in European democracies was based on the construction of typologies designed to distinguish between different qualitative features of the various political systems (Duverger, 1954; Sartori, 1966; Rokkan, 1970). This research concentrated on an empirical relationship between the duration of multiparty coalition governments and the *fragmentation*¹⁴ of the polity (Taylor and Herman, 1971; Herman and Sanders,

¹⁴Fragmentation can be identified with the effective number of parties (Laakso and Taagepera, 1979). That is, let H (the Herfindahl index) be the sum of the squares of the relative seat shares and $n_s = H^{-1}$ be the *effective number*.

1977; Warwick, 1979). For example, Table 6.1 (from Schofield, 1995) lists the average duration of government in twelve European polities for the period 1945-1987 together with the average effective number in each polity. The relationship between effective number and duration is quite weak.

Table 6.1. Duration (in months) of government, 1945–1987

Country	Average duration	Effective number n_s
Luxembourg	45	3.5
Ireland	39	2.6
Austria	38	2.2
Germany	37	2.9
Iceland	34	3.7
Norway	32	3.2
Sweden	28	3.2
Netherlands	27	4.5
Denmark	26	4.5
Belgium	22	4.0
Finland	15	5.0
Italy	13	3.5
Average	26	3.7

Other empirical work on coalition type (Taylor and Laver, 1973) compared the notions of MW and MCW in order to model coalition formation. One problem with these two notions was the occurrence of minority and surplus coalitions. (A minority coalition is a non-winning coalition, while a surplus coalition is supra-winning, able to lose a party and still be winning.) Herman and Pope (1973) had observed that minority coalitions seemed to contradict the logic behind the MW notion. Table 6.2 presents data from Laver and Schofield (1990) and Schofield (1993) indicating that out of the 218 coalition governments in these 12 European countries in the period in question, over 70 were minority. Although 31 of these had some sort of implicit support, at least 40 were unsupported minorities. Moreover of the 46 surplus governments that formed, only nine were MCW.

Table 6.2. Frequency of coalition types, by country, 1945–1987

Country	One party controls legislative majority	No Party Controls a Legislative Majority					Total
		Minimal winning (MW)		Non-MW			
		MCW and MW	MW not MCW	MCW not MW	Surplus not MCW	Minority	
Austria	6	5	1	–	–	1	13
Belgium	1	7	8	–	4	2	22
Denmark	–	2	–	–	–	18	20
Finland	–	4	1	–	17	10	32
Germany	2	9	1	–	–	–	12
Iceland	–	6	4	–	2	2	14
Ireland	4	–	3	–	–	5	12
Italy	4	–	3	6	8	14	35
Luxembourg	–	8	1	–	1	–	10
Netherlands	–	4	2	3	5	3	17
Norway	4	3	–	–	–	8	15
Sweden	1	5	–	–	–	10	16
Total	22	53	24	9	37	73	218

Dodd (1974, 1976) attempted to account for minority and surplus governments in terms of the degree of conflict of interest. (Conflict of interest is calculated in terms of inter-party policy differences on a one-dimensional scale.) His theory was that party systems with high fragmentation, as indicated by N_s , would give rise to minority government when conflict of interest was high. Conversely with high fragmentation but low conflict of interest, surplus governments were expected. While this theory was attractive, it failed to explain why both minority and surplus governments were common in both Finland and Italy (Table 6.2). Table 6.3 shows average duration of government by coalition type. While there is some indication that MW coalitions are longer lived than minority coalitions, it is not clear how exactly fragmentation, coalition type and duration are related.

A fully-developed formal theory of coalition would connect the nature of the electoral system, the motivations of parties concerning policy and prerequisites, and the process of government formation, in a way which makes sense of the empirical phenomena. A number of attempts have

been made to model the motivations of parties in a game-theoretic framework. For example, one class of models is based on the Downsian framework, where parties compete via policy declarations to the electorate in order to maximize the number of seats they obtain (Shepsle, 1991). However, since parties are assumed in these models to be indifferent to policy objectives, viewing policy solely as a means to gain power, symmetry would suggest that one equilibrium would be the situation where all parties declare the same position. Indeed, the next chapter considers such a model, and shows that there are necessary and sufficient conditions for convergence of this kind.

Table 6.3. Duration of European coalitions, 1945-1987

Country	Single-party majority	Minimal winning	Surplus majority	Minority	Total
Luxembourg	n.a.	47	5	n.a.	45
Ireland	49	42	n.a.	30	39
Austria	46	40	24	20	38
Germany	n.a.	33	49	n.a.	37
Iceland	n.a.	39	40	8	34
Norway	48	37	n.a.	24	32
Sweden	24	24	n.a.	30	28
Netherlands	n.a.	31	34	4	27
Denmark	n.a.	43	n.a.	22	26
Belgium	46	25	12	7	22
Finland	n.a.	19	15	10	15
Italy	n.a.	17	17	9	13
Total	45	33	21	19	26

However, such electoral models do not make clear the relationship between seat (or vote) maximization and membership of government. There is generally no guarantee that the party gaining the most seats will become a member of the governing coalition. This chapter will use the idea of the *core*, developed in the previous chapter, to argue that a dominant party, located at the center of the policy space, can control the formation of government. Instead of assuming that the “political game” is a

constant sum or based on a one-dimensional policy space, we shall consider situations where the policy space may have two or more dimensions, and government results from bargaining between three or more parties. The political game is divided into two components. In the post-election phase, the “positions” of the parties are assumed to be given, as is the distribution of seats. This distribution defines a set of winning coalitions. Given the set of winning coalitions, and party positions, we use the theory presented in Chapter 5 to examine the “political heart.” Under some circumstances, the heart will consist of a single policy point, the “core.” If the “core” is stable under small perturbations in the positions of the parties then it is said to be “structurally stable.” If a party’s position is at the “structurally stable core,” then we shall call this party the “core party.” Under these circumstances, it is argued that the “core party” may form a minority government. If the heart is not given by a point, then it will comprise a domain in the policy space, “the cycle set.” Using the arguments presented in Chapter 5, we can infer that the “cycle set” will be “bounded” by the preferred positions of a particular set of parties. The bounding “proto-coalitions” form the basis for coalitional bargaining. This model of the heart can then be used to describe, heuristically, the general pattern of coalition formation.

The pre-election calculations of parties involve calculations over the relationship between party position, electoral response, and the effect that the resulting party positioning and parliamentary strength has on coalition bargaining. Schofield and Sened (2006) propose that these calculations are based on beliefs by the political actors that can be represented by a “selection” from the heart correspondence. More formally, let party positions be given by the vector, \mathbf{z} , and suppose, for convenience, that the parties have Euclidean preferences derived from this vector of bliss points. Let $\mathbb{D}(\mathbf{z})$ be the set of winning or decisive coalitions that occur after the election, as a result of the declarations of the vector, \mathbf{z} , of positions. Since preferences and decisive coalitions are now specified, we can use the symbol, $\mathcal{H}(\mathbb{D}(\mathbf{z}), \mathbf{z})$, to denote the “post election heart.” Then beliefs of the political agents can be represented by a mapping $g : W^n \rightarrow \mathbb{W}$, where the selection $g(\mathbf{z})$ is a lottery with support, $\mathcal{H}(\mathbb{D}(\mathbf{z}), \mathbf{z})$ in W . This lottery, $g(\mathbf{z})$, in the space, \mathbb{W} , of all lotteries in W , specifies what party leaders expect to occur as a result of the choice of a vector $\mathbf{z} \in W^n$ of party positions, and the outcome, $\mathbb{D}(\mathbf{z})$. This generates a “pre-election

game,” where the utilities of the agents over the choice of positions are induced by their beliefs or expectations of post election outcomes. Determining whether this pre-election game has equilibria is extremely difficult. As a first step, the next section will examine the post-election behavior of parties to gain some insight into the nature of coalition bargaining.

Although scholars are in fair agreement concerning the positions of parties in a one-dimensional policy space (Laver and Schofield, 1990), party positions in two dimensions are much more difficult to ascertain. Empirical models can be constructed on the basis of multi-dimensional data on party policy positions that have been derived from the content analysis of party manifestos in European polities,¹⁵ and more recently in Israel.¹⁶ Using factor analysis it is generally possible to reduce these data to two dimensions giving a tractable description of the main political issues in these countries.

Using these estimates of party position, we can then determine whether the core is empty, and if it is, deduce the location of the “cycle set” or heart. Because the Nakamura number for these weighted voting rules will be three, we can infer that the core will always be non-empty if the policy space is one-dimensional and preferences of the parties are single peaked. In particular, the core in the one-dimensional situation corresponds to the position of the median party or legislator (that is the politician or party who is positioned so that there is a majority neither to the left nor to the right). Note in particular that any MCW coalition must contain the median party. More importantly, any one-dimensional policy-based model of coalition formation predicts that the median legislator will necessarily belong to the governing coalition. Thus the spatial model implies that the median party can effectively control policy outcomes if the policy space is uni-dimensional.

In two dimensions, it is possible for a core to occur in a structurally stable fashion, but as shown below it will generally be necessary that the core party is dominant in terms of its seat strength. As in the one-

¹⁵The original manifesto group (Budge, Robertson and Hearl, 1987) used a 54-category policy coding scheme to represent party policy in 19 democracies. The more recent work (Budge, Klingemann, Volkens and Bara, 2001) covers 25 countries. See also Benoit and Laver (2006) who use expert estimates.

¹⁶Schofield and Sened (2005a,b, 2006).

dimensional case, the core party will be able to veto any coalitional proposal. As a consequence we expect this party to belong to the government. On the other hand, if the core is empty then no party can have a veto of this kind, and it is natural to expect greater uncertainty in coalition outcomes. In such a situation, for any incumbent coalition and policy point, there is always an alternative coalition that can win with a new policy point. This it can do by seducing some members of the incumbent coalition away, by offering them a higher policy payoff than they can expect if they remain loyal to the original coalition. However, because the heart will be bounded by a small number of median arcs, we can identify these arcs with a set of minimal winning coalitions. It is suggested that bargaining between the parties will result in one or other of these coalition governments. In this chapter we shall use the estimated positions and relative sizes of the parties, together with the concepts of the core and heart to suggest a categorization of different types of bargaining environments, distinguishing between *unipolar*, *bipolar* and *triadic* political systems.

In *left unipolar* systems such as Norway, Sweden, Denmark and Iceland, there is typically one larger party and three or four smaller parties. The larger party may be able to dominate coalition politics, and form a minority government with or without the tacit support of one of the other parties. In *triadic* systems, such as Austria and Germany (where typically there are two large and one or two small parties) most coalition cabinets are both minimal winning and minimal connected winning. *Center unipolar* systems, such as Belgium, Luxembourg and Ireland, typically have two large and at least two other small parties. Minority or surplus coalitions are infrequent and governments are usually minimal winning coalitions. In *bipolar systems*, such as the Netherlands and Finland, there are typically two large and a number of smaller parties. Finally, Italy (until the election of 1994) had a strongly dominant party, the Christian Democrats. This party was in every coalition government, and relatively short-lived governments were very common (Mershon, 2002). By 1994, the dominance of the Christian Democrat party had evaporated (Giannetti and Sened, 2004).

This typology is only meant to be indicative. As we discuss the various polities, it is quite clear that under proportional representation, the number of parties and their relative strengths can change in radical ways,

inducing complex changes in the possibility of a core and in the configuration of the heart.

6.2 A Spatial Model of Legislative Bargaining

As in the previous chapters, we assume that each party chooses a preferred position (or *bliss point*) in a *policy space* W . From now on we shall denote the parties as $N = \{1, \dots, j, \dots, n\}$ and the vector of bliss points as $\mathbf{z} = (z_1, \dots, z_n)$. After the election we denote the number of seats controlled by party, j , by s_j and let $\mathbf{s} = (s_1, \dots, s_n)$ be the vector of parliamentary seats. We shall suppose that any coalition with more than half the seats is winning, and denote the set of winning coalitions by \mathbb{D} . This assumption can be modified without any theoretical difficulty. For each winning coalition M in \mathbb{D} there is a set of points in W such that, for any point outside the set there is some point inside the set that is preferred to the former by all members of the coalition. Furthermore, no point in the set is unanimously preferred by all coalition members to any other point in the set. This set is the Pareto set, $Pareto(M)$, of the coalition, as introduced previously. If the conventional assumption is made that the preferences of the actors can be represented in terms of Euclidean distances, then this compromise set for a coalition is simply the convex hull of the preferred positions of the member parties. (In two dimensions, we can draw this as the area bounded by straight lines joining the bliss points of the parties and including all coalition members.) Since preferences are described by the vector, \mathbf{z} , we can denote this as $Pareto(M, \mathbf{z})$. Now consider the intersection of these compromise sets for all winning coalitions. If this intersection is non-empty, then it is a set called the *Core* of \mathbb{D} at \mathbf{z} , written $Core(\mathbb{D}, \mathbf{z})$. At a point in $Core(\mathbb{D}, \mathbf{z})$ no coalition can propose an alternative policy point that is unanimously preferred by every member of some winning coalition.

In general, $Core(\mathbb{D}, \mathbf{z})$ will be at the preferred point of one party. The analysis of McKelvey and Schofield (1987), presented in the previous chapter, obtained pivotal symmetry conditions that are necessary at a core point. Clearly a necessary and sufficient condition for point x to be in $Core(\mathbb{D}, \mathbf{z})$ is that x is in the Pareto set of every minimal winning coalition. As shown in Chapter 5, the symmetry conditions depend on

certain subgroups called pivot groups. Alternatively, we can determine all median lines given by the pair (\mathbb{D}, \mathbf{z}) . To illustrate these conditions, consider the configuration of party strengths after the election of 1992 in Israel. (The election results in Israel for the period 1988 to 2003 are given in Table 6.4.) The estimates of party positions in Figure 6.1 were obtained from a survey of the electorate carried out by Arian and Shamir (1995), complemented by an analysis of the party manifestos (details can be found in Schofield and Sened, 2006).

As Figure 6.1 indicates, all median lines go through the Labor party position, so given the configuration of seats and positions, we can say Labor is the *core party* in 1992. Another way to see that the Labor position, z_{lab} , is at the core is to note that the set of parties above the median line through the Labor-Tsomet positions (but excluding Labor) only control 59 seats out of 120. When the party positions are such that the core does indeed exist, then any government coalition must contain the core party. When the core party is actually at a core position then it is able to influence coalition bargaining in order to control the policy position of the government. Indeed, if we assume that parties are only concerned to control policy, then the party at the core position would be indifferent to the particular coalition that formed. The ability of the core party to control policy implies a tendency for core parties to form minority governments, since they need no other parties in order to fulfil their policy objectives. In fact, in 1992, Rabin first created a coalition government with Shas, and then formed a minority government without Shas.

We have emphasized that in two dimensions the core can be empty. To see the consequences of this, consider the configuration of party positions in Israel after the election of 1988, as presented in Figure 6.2, again using the seat allocations from Table 6.4. In this case there is a median line through the Tzomet, Likud positions, so the coalition of parties above this line is winning. It is evident that the Labor does not belong to the Pareto set of the coalition including Likud, Tzomet and the religious parties. Indeed, it can be shown that the symmetry conditions necessary for the existence of a core are nowhere satisfied. In this case, there are cycles of different coalitions, each preferred by a majority of the legislature to some other coalition policy in the cycle.

Table 6.4. Knesset seats

Party	1988	1992	1996	1999	2003
Labor (LAB)	39	44	34	28	19 ^a
Democrat (ADL)	1	2	4	5	2 ^a
Meretz (MRZ)	–	12	9	10	6
CRM, MPM, PLP	9	–	–	–	3
Communist (HS)	4	3	5	3	3
Balad	–	–	–	2	3
<i>Left Subtotal</i>	<i>53</i>	<i>61</i>	<i>52</i>	<i>28</i>	<i>36</i>
Olim	–	–	7	6	2 ^b
III Way	–	–	4	–	–
Center	–	–	–	6	–
Shinui (S)	2	–	–	6	15
<i>Center Subtotal</i>	<i>2</i>	<i>–</i>	<i>11</i>	<i>18</i>	<i>17^b</i>
Likud (LIK)	40	32	30	19	38 ^b
Gesher	–	–	2	–	–
Tzomet (TZ)	2	8	–	–	–
Israel Beiteinu	–	–	–	4	7
<i>Subtotal</i>	<i>42</i>	<i>40</i>	<i>32</i>	<i>23</i>	<i>45</i>
Shas (SHAS)	6	6	10	17	11
Yahadut (AI, DH)	7	4	4	5	5
Mafdal (NRP)	5	6	9	5	6
Moledet (MO)	2	3	2	4	–
Techiya (TY)	3	–	–	–	–
<i>Right Subtotal</i>	<i>23</i>	<i>19</i>	<i>25</i>	<i>31</i>	<i>22</i>
Total	120	120	120	120	120

^aADL, under Peretz, combined with Labor, to give 21 seats.

^bOlim joined Likud to give 40 seats, and the right 47 seats.

The heart, $\mathcal{H}(\mathbb{D}, \mathbf{z})$, given the seat strengths and party positions, is the star-shaped figure, bounded by the five median lines. It is reasonable to conclude, in the absence of a core party, that a coalition government will be based on a small number of minimal winning coalitions. Notice that this inference provides a good reason to consider using a two-dimensional rather than a *one*-dimensional model of policy bargaining. In

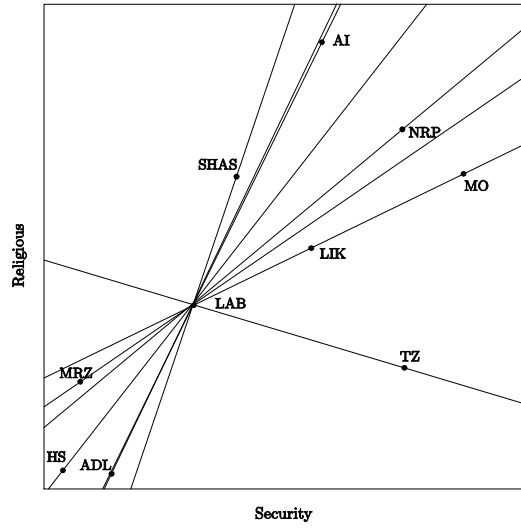


Figure 6.1: The core in the Knesset in 1992

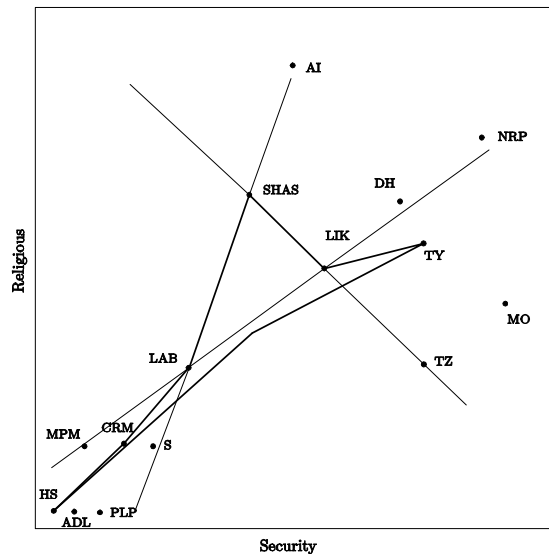


Figure 6.2: The heart in the Knesset in 1988

a single-dimensional model there will always be a core party (since there will always be a party to which the median legislator belongs). Moreover it can happen that this *median* core party is small in size. For example, in Figure 6.2, if there were only the single security dimension, then the Shas position would be the median, and it could be concluded that Shas could form a minority government. In fact this did not occur. In two dimensions, if a core does exist then it must be at the position of the largest party. We can therefore deduce that in 1992, only Labor could be a core party.

We can compare the heart in Figure 6.2 with other solution notions, such as the *yolk* and uncovered set (McKelvey, 1986; Cox, 1987; Banks, Bordes and Le Breton, 1991; Bordes, Le Breton and Salles, 1992). The *yolk* in Figure 6.2 is the smallest circle that just touches the median lines that bound the heart. Its theoretical justification is in term of a process of policy amendments with a fixed agenda. While such a process is appropriate for policy making in the U.S. Senate, it does not seem relevant for coalition bargaining over government formation. The *uncovered set* is also centrally located like the heart, and can be shown to be the support of mixed strategy equilibria in models of elections (Banks, Duggan and Le Breton, 2002, 2006). The work of Banks and Duggan (2000) considers bargaining between political parties when the party positions and seat strengths are given. Their analysis suggests that the outcome can be described as a *lottery* across a set of points on the boundary of the heart.

Figure 6.3 shows the positions of the parties after the election of 1996, together with an estimate of the electoral distribution, based on the survey data obtained by Arian and Shamir (1999), while Figure 6.4 gives a schematic representation of the heart, based on party positions after 2003. The figure shows Labor with 21 seats, after Am Ehad, with 2 seats, joined Labor in 2003, while Likud has 40 seats after being joined by Olim, with 2 seats. Although Barak, of Labor, became Prime Minister in 1999, he was defeated by Ariel Sharon, of Likud, in the election for prime minister in 2001. The set denoted the heart in this figure represents the coalition possibilities open to Sharon after 2003.

The figure can be used to understand the consequences after Sharon seemingly changed his policy on the security issue in August 2005, by pulling out of the Gaza Strip. First, Likud reacted strongly against this change in policy. Then, in the first week of November 2005, Amir Peretz,

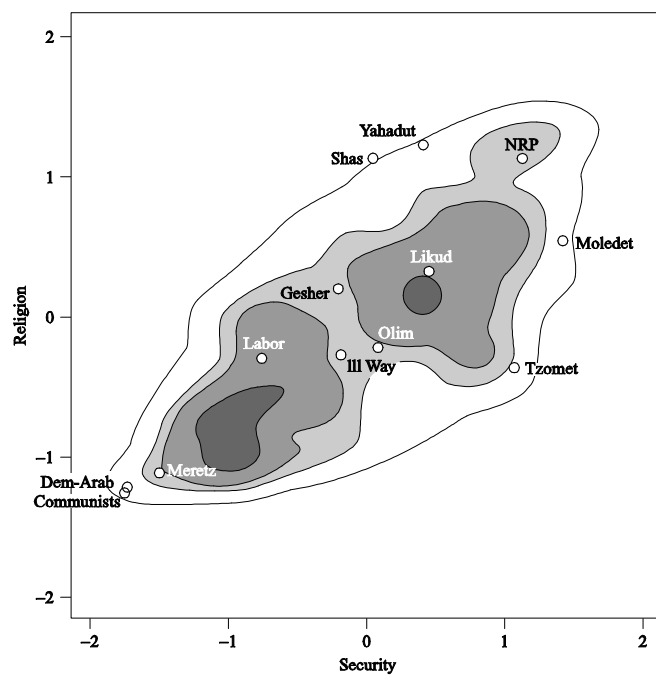


Figure 6.3: Party positions in the Knesset in 1996

a union activist, and leader of Am Ehad, won an election against Shimon Peres for leadership of the Labor Party.

Sharon then left the Likud Party and allied with Peres, the former leader of Labor and other senior Labor Party members, to form the new party, *Kadima* (“Forward”). We can infer that the coalition of Sharon and Peres positioned Kadima at the origin of the policy space, as shown in Figure 6.5. This figure gives estimates of party positions at the March 28, 2006, election to the Knesset. Because of Sharon’s stroke in January 2006, Ehud Olmert had taken over as leader of Kadima, and was able to take 29 seats. Likud only took 12 seats, while the four parties on the upper right of the figure won 38 seats. One surprise of the election was the appearance of a Pensioners’ party with 7 seats. As Figure 6.5 indicates, the parties on the right (even with the Pensioners’ Party) do not have the required 61 seats for a majority, so Kadima is located at the structurally stable core position. Even though Kadima is estimated to be a core party, Olmert needed the support of Labor to be able to deal with the complex issue of fixing a permanent border for Israel. The débâcle in Lebanon severely weakened the Kadima–Labor coalition, and in October 2006, the 61 members of the coalition voted to bring Israel Beiteinu into the government. Barak then won the election for the Labor Party leadership on 12 June, 2007, and became Minister of Defense in the government on 18 June. In November, Olmert proposed a land-for-peace proposal, possibly involving the separation of Jerusalem. It appears that Sharon’s change of policy has led to a fundamental transformation in the political configuration, from the coalition structure without a core (that had persisted since 1996), to a new configuration, associated with the center, core party, Kadima.

6.3 The Core and the Heart of the Legislature

In this section we shall use the results of Chapter 5 on pivotal symmetry to examine more formally the situation when a party can occupy a structurally stable core position.

6.3.1 Examples from Israel

Example 6.3.1. Consider again the election of 1992 in Israel. Table 6.4

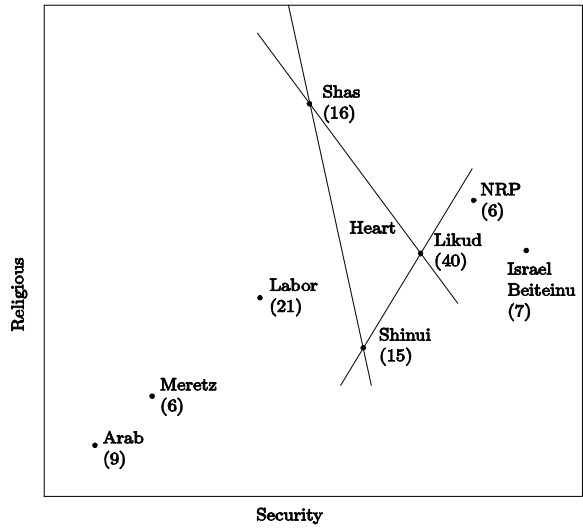


Figure 6.4: The configuration of the Knesset after the election of 2003

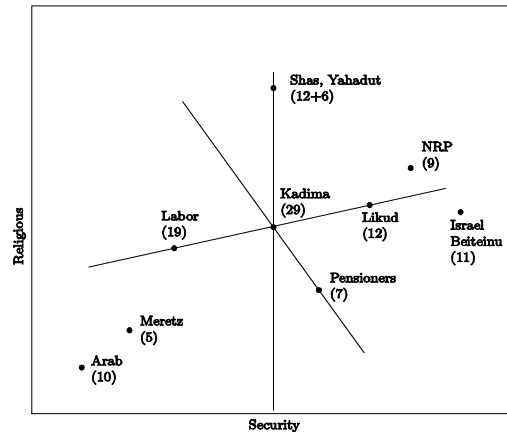


Figure 6.5: The configuration of the Knesset after the election of 2006

shows that, after this election, the coalition $M_1 = \{\text{Labor, Meretz, Democrat Arab, Communist Party}\}$ controlled 61 seats while the coalition M_2 of the remaining parties, including Likud, controlled only 59 seats out of 120. Thus the decisive structure in 1992 may be written \mathbb{D}_{1992} and includes the decisive coalitions

$$\{M_1, M_2 \cup \text{Labor}, M_2 \cup \text{Meretz}\}.$$

Since $M_1 \cap M_2$ is empty, the Nakamura number is three, and a core can only be guaranteed in one dimension. To formally examine the pivotal symmetry condition at the Labor position, let $L = \text{Labor}$, and consider whether $M = \text{Likud}$ pivots at this position. Take the disjoint partition $\{A, B\}$ of the parties other than Labor and Likud, where $A = \{\text{Democrat Arab, Communist Party, Meretz, Shas}\}$, with 23 seats, and $B = \{\text{parties on the right excluding Shas}\}$ with 21 seats. Now *Likud* has 32 seats, so clearly neither $\text{Likud} \cup A$ nor $\text{Likud} \cup B$ is a decisive coalition. Thus *Likud* does not pivot. Indeed, any pivotal coalition in $E_{\text{Labor}}(\mathbb{D}_{1992})$ must contain at least two parties. It is easy to see in Figure 6.1 that any pivotal coalition containing say Likud, Shas and NRP is *blocked at the Labor position by an opposing coalition*, such as Meretz and ADL. Moreover, this blocking is not destroyed by a small perturbation of the party positions. For this reason, the core at the Labor position is structurally stable.

Example 6.3.2. Now consider the election of 2003, where Likud is the largest party. It is obvious that Labor together with Meretz and the parties on the left pivot together. For the Likud position to be a core it is necessary that this “proto-coalition” be blocked. But Figure 6.4 indicates that this proto-coalition is not blocked at the the Likud position. Consequently, the Likud position was not at a core position.

Definition 6.3.1. Let \mathbb{D} be a set of decisive coalitions given by a set of weights, or seat strengths $[s(1), \dots, s(i), \dots, s(n)]$.

- (i) Party j is *weakly dominant* in \mathbb{D} iff for any $k \neq j$, and any proto-coalition $M \subseteq N \setminus \{j\} \setminus \{k\}$ with $[M \cup \{k\}] \in \mathbb{D}$, then $[M \cup \{j}] \in \mathbb{D}$.
- (ii) Party j is *dominant* in \mathbb{D} iff, for every $k \neq j$, there exists a proto-coalition $M \subseteq N \setminus \{j\} \setminus \{k\}$ such that $[M \cup \{j}] \in \mathbb{D}$, yet $[M \cup \{k\}] \notin \mathbb{D}$.

- (iii) Party j is *strongly dominant* in \mathbb{D} if j is *dominant*, and for any $k \neq j$, there is a partition $\{A, B\}$ of $N \setminus \{j\} \setminus \{k\}$ such that $[A \cup \{j\}] \in \mathbb{D}$ implies $[A \cup \{k\}] \notin \mathbb{D}$ and $[B \cup \{j\}] \in \mathbb{D}$, implies $[B \cup \{k\}] \notin \mathbb{D}$.

As the examples suggest, if there is a dominant party then it can occupy a structurally stable core position in two dimensions if it is also strongly dominant. To see this, suppose j is strongly dominant, and consider whether party $k \neq j$ pivots at the position z_j . Take a partition $\{A, B\}$ of $N \setminus \{j\} \setminus \{k\}$ such that $[A \cup \{k\}] \notin \mathbb{D}$ and $[B \cup \{k\}] \notin \mathbb{D}$. Clearly k does not pivot. If there is a third party l such that $\{j, l\}$ pivots then there will exist a set of positions $\{z_1, \dots, z_n\}$ such that $\{j, l\}$ is blocked. Moreover, by the results of the previous chapter, $\{j, l\}$ can be blocked in a structurally stable fashion. To see that no party $k \neq j$ can occupy the structurally stable position, note that either $[A \cup \{j\}] \in \mathbb{D}$ or $[B \cup \{j\}] \in \mathbb{D}$. Thus $\{j\}$ pivots. To block $\{j\}$ there must be some other party l , say, with gradient $p_l(z_k)$ opposite to $p_j(z_k)$ at $p_k(z_k) = 0$. But this Plott symmetry condition is structurally unstable.

We now examine the calculations by parties over policy positions, on the basis of these ideas.

6.3.2 Examples from the Netherlands

Example 6.3.3. Consider the elections of 1977 and 1981 in the Netherlands. Table 6.5 gives the election results together with the National Vote Shares, while Table 6.6 gives the sample survey estimates of the vote shares, based on Rabier and Inglehart (1981) Euro-barometer voter survey. Figure 6.6 gives estimates of the positions of the four main parties: Labor (PvdA), the Christian Democratic Appeal (CDA), Liberals (VVD) and the Democrats '66 (D'66). These estimates were obtained using data from the middle-level Elites Study (ISEIUM, 1983). The background to the figure gives the estimate of the distribution of voter bliss points derived from the Euro-barometer survey. This example is discussed in more detail in Chapter 7 below.

A coalition $\{CDA, VVD\}$ with 77 seats formed in December 1977, and lasted 41 months until the election in 1981. After the second election, a short lived "surplus coalition," $\{PvdA, CDA, D'66\}$, with 109 seats first formed and then collapsed to a minority coalition, $\{CDA, D'66\}$. The increase of seats for the D'66 between the elections meant that the me-

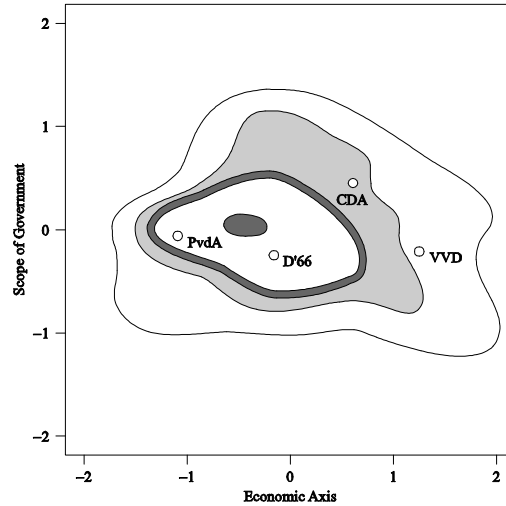


Figure 6.6: Party positions in the Netherlands in 1977

dian lines, and therefore the heart, changed. Using the estimates of party position from Figure 6.6 for both 1977 and 1981, and assuming Euclidean preferences but ignoring the small parties, we see that in 1977, the heart is bounded by the three median arcs, $[z_{PvdA}, z_{CDA}]$, $[z_{PvdA}, z_{VVD}]$ and $[z_{CDA}, z_{VVD}]$ while in 1981, the heart is bounded by $[z_{PvdA}, z_{CDA}]$, $[z_{PvdA}, z_{VVD}]$ and $[z_{CDA}, z_{D66}]$. Notice that we can infer that the minority coalition $\{CDA, D'66\}$ had the implicit support of the VVD.

Let us now perform the thought experiment of moving party positions to determine whether a core can exist. If seat strengths are unchanged, then after 1977 there is no possible core position. For example, movement by the PvdA to a position inside the convex set $[z_{CDA}, z_{D66}, z_{VVD}]$ does not put it at a core position. In the same way, a move by the CDA to the interior of $[z_{PvdA}, z_{D66}, z_{VVD}]$ is also not a core. On the other hand, with the seat strengths after 1981, such moves would have given core positions. This thought experiment raises the question of the choice of party positioning, since the logic of legislative bargaining would suggest that any party at the core could guarantee membership of government (Banks and Duggan, 2000). However, this thought experiment is inadequate, be-

cause moves by the parties would change their seat strengths.

Table 6.5. Seats and votes in the Netherlands

Party	1977 Seats	1981 Seats	1977 Vote %	1981 Vote %
Labor (PvdA)	53	44	33.8	28.3
Christian Appeal (CDA)	49	48	31.9	30.8
Liberals (VVD)	28	26	17.9	17.3
Democrat (D'66)	8	17	5.4	11.1
Sub-total	138	135	89.0	87.5
Communist (CPN)	2	3	1.7	2.1
Radicals (PPR)	3	3	1.7	2.0
Small parties (SGP, RPF)	7	9	7.6	8.4
Total	150	150	100	100

To examine the consequence of such moves, Schofield, Martin, Quinn and Whitford (1998) used the Rabier Inglehart (1981) Euro-barometer survey data to construct a multinomial logit (MNL) model of the election. The model ignored the small parties and used vote intentions from the sample to construct a stochastic vote model. This is discussed in more detail in Chapter 7. It was found appropriate to add in what are called valence values for the four parties. Valence is simply an exogenous component of voter evaluation based on subjective estimates of the quality of the party leaders.

Table 6.6. Estimated vote shares and valences in the Netherlands

Party	Estimated vote %	Sample vote%	Model %	Valences
PvdA	35.3	36.9	38.6	1.596
CDA	29.9	33.8	31.9	1.403
VVD	24.2	18.9	21.7	1.015
D'66	10.6	10.4	7.8	0
Total	100.0	100.0	100.0	

Table 6.6 gives the *sample* vote shares for the four major parties (these excluded the small parties, so the four shares sum to 100 percent). The

MNL model gave the *estimated* shares given in Table 6.6. Using the empirical estimates of these valences, normalized by setting the valence of the D'66 to zero, increased the statistical significance of the model. The valences are shown in the table. As discussed in Chapter 7, simulation of the model shows that the electoral origin is a *Nash equilibrium* of a simple vote-maximizing model. Because the PvdA has the highest estimated valence, the simulation shows that when all parties are at the origin, then the PvdA would have received 38.6 percent of the vote. (The estimated vote shares when all parties are at the origin is given by the column labeled *Model* in Table 6.6). Because the model did not include the small parties, we can infer that the PvdA would have obtained 53 seats when all parties are at the electoral origin. Similarly we obtain estimates for the CDA of 45 seats, the VVD of 29 seats and the D'66 of 11 seats. Notice that these estimated vote shares at the origin are very close to the National vote shares in 1977 as well as the sample vote shares. If parties are concerned to maximize vote shares, and can compute the outcome of adopting policy positions at the electoral origin, then they can also estimate that overall vote shares would be quite insensitive to such movements. Indeed, the decisive structure resulting from this Nash equilibrium will be identical to the one resulting after the 1977 election.

Example 6.3.4.

We now consider a more complicated situation, that of the Dutch Parliament after the recent election of November 2006. Table 6.7 shows the party strengths while Figure 6.7 shows the party positions as estimated by Shikano and Linhart (2007). The coalition government of {CDA, VVD, D'66} had broken up on 29 June, 2006 over the so-called "Ayaan Hirsi Ali affair." She had become a Member of Parliament for the VVD, but was stripped of Dutch nationality by the Minister for Integration and Alien Affairs because of allegations that she lied on her application for asylum. When her nationality was reinstated, on the basis of a document exonerating this Minister (a member of the VVD), the D'66 pulled out of the coalition, leading to a minority caretaker government of {CDA, VVD} with only 72 seats, out of 150, installed on 7 July.

After the election in November 2006, a coalition {CDA, PvdA, CU}, with 80 seats, was formed on 7 February, 2007. Although this coalition

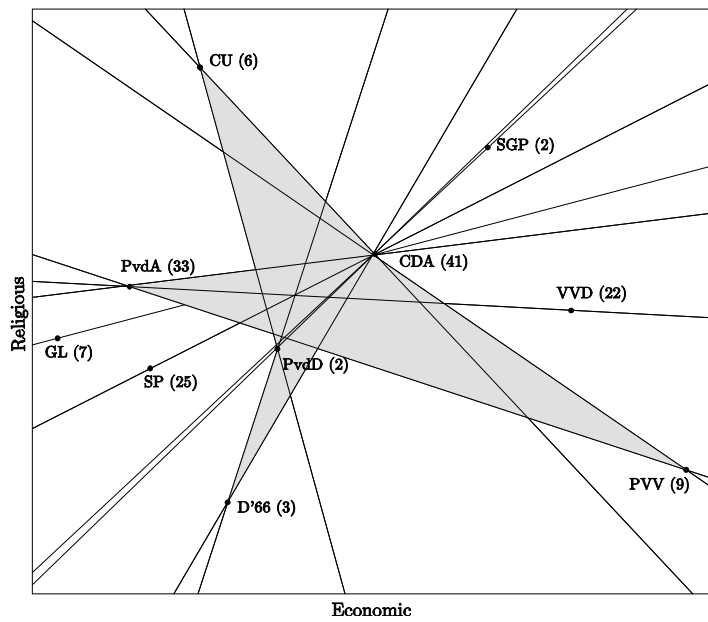


Figure 6.7: The Dutch Parliament in 2006

might seem fairly unusual, being a combination of parties with a religious basis and the labor party, it is compatible with the notion of the heart. The heart is the star shaped figure bounded by a small number of median lines in the two-dimensional policy space generated by the economic axis (dimension 1) and the religious axis (dimension 2). These medians can be associated with various coalitions:

- (i) Two coalitions involving the CDA associated with the median arcs $\{[z_{CU}, z_{CDA}], [z_{CDA}, z_{PVV}]\}$
- (ii) Three coalitions involving the PvdA associated with the median arcs $\{[z_{PvdA}, z_{CDA}], [z_{PvdA}, z_{PVV}], [z_{PvdD}, z_{CU}]\}$
- (iii) Two probably unlikely coalitions involving the parties on the left of the economic axis associated with the arc $[z_{PvdD}, z_{D66}]$, and on the right, associated with the arc $[z_{CDA}, z_{D66}]$.

Table 6.7. Seats in the Dutch Parliament, 2003 and 2006

Party	2006	2003
Labor (PvdA)	33	42
Labor for Animals (PvdD)	2	
Green Party (GL)	7	8
Liberals (VVD)	22	28
Left Liberals (D'66)	3	6
Socialists (SP)	25	9
Protestant Party (SGP)	2	2
Christian Union (CU)	6	3
Christian Appeal (CDA)	41	44
Freedom Party (PVV)	9	
Lijst Pim Fortuyn		8
Total	150	150

As Shikano and Linhart (2007) note, with 10 parties there are over 500 possible winning coalitions. While the heart does not give a precise prediction of which coalition will form, it provides clues over the complex bargaining calculations that policy-motivated party leaders are faced with when attempting to form majority coalitions in polities based on proportional representation (PR). In particular, because of the conflict that the affair generated between the VVD and D'66, the {CDA, PvdA,

CU} coalition is one of the few possible viable coalitions. Even so, it took over six months of negotiation before the coalition parties could agree.

There are some general points that can be made on the basis of an examination of politics in the Netherlands. Obviously, under PR there is little incentive for parties to coalesce. On the contrary, parties may well fragment. The strengths of the parties may fluctuate as a result of local events. Such a fluctuation is compatible with the electoral model presented in the next chapter, since the model suggests that this fluctuation is due to shifting valences—the perceptions of the competence of the parties on the part of the electorate. Ignoring the possibility of fragmentation of parties, the policy locations of parties seem quite stable over time, suggesting that the electoral response is primarily due to changes in valence. Computing the relationship between valence, activists and party location is an extremely complicated theoretical problem. On the other hand, if we take the post-election positions and strengths of the parties as given then we can use the concept of the heart to gain some insight into the nature of post-election bargaining over government formation in polities based on proportional representation. The next section outlines a typology based on this idea.

6.4 Typologies of Coalition Government

The previous examples suggest that parties do not appear to adopt Nash-equilibrium positions based on a simple vote-maximizing game. Because of this, Chapter 7 considers a more general electoral model, where each party is dependent on activist support. In this model parties gain support from activists, as long as the party position is chosen in response to activist demands. We can interpret this to mean that the party implicitly has policy preferences. However, since there may well be many potential activist groups in a polity, we may expect a number of parties to respond to activist demands. In Chapter 7 we discuss the simpler case of plurality rule, as in Britain, where there will tend to be no more than three parties. In polities using electoral systems based on proportional representation (PR) there appears to be no rationale forcing activist groups to coalesce. In the following discussion of legislative politics we shall use estimates of party positions, and examine the nature of the core, or heart, under

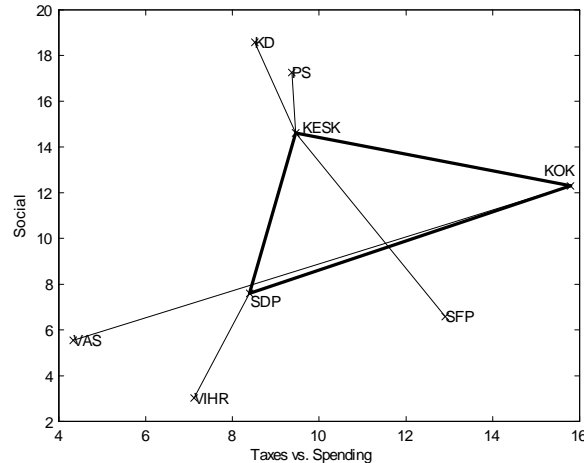


Figure 6.8: Finland in 2003

the assumption that the party positions are essentially determined by exogenous activist groups influencing the position adopted by the party. If the reasoning presented in the previous section is valid, then we should expect minority governments in situations where there is a core party.

6.4.1 Bipolar Systems

The Netherlands. As in the previous examples of the Netherlands, there are two weakly dominant parties, namely the Labor Party (PvdA) and the Christian Democratic Appeal (CDA) or its predecessor, the Catholic People's Party (KVP). For this reason, we shall use the term *bipolar*. However since the CDA/KVP or D'66 tends to be located at the median on the economic dimension, there is little possibility that the PvdA can occupy the two-dimensional core. In the period 1945–1987, there were only three minority governments (in 1966–1967 for 5 months, 1972–1973 for 10 months, and 1982 for 6 months). Figure 6.7 makes clear the bipolarity of the polity, as coalitions lie essentially on either side of the line separating the upper right of the figure from the left.

Finland. The pattern of coalition formation in Finland is quite com-

plex. The effective seat number is approximately 5.5, reflecting the fact that there are generally four larger parties – the Left Alliance (VAS) with 19 seats out of 200 in 2003, the Social Democrat Party (SDP) with 53 seats in 2003, the Center Party (KESK) with 55 seats and the Conservative Party or National Coalition (KOK) with 40 seats. In addition there are a number of small parties: the Green League (Vihreä Liito, VIHF) with 14 seats, the Christian Democrats (Kristillisdemokraatit, KD) with 7 seats, Perussuomalaiset (PS) with 3, and the Swedish People's Party (SFP) with 8. On the usual left–right scale either SDP or KESK is at the median (Laver and Schofield, 1990). The system is *bipolar*, just as with the Netherlands, since the heart is determined by the SDP and KESK, both of which can be weakly dominant parties. Figure 6.8 indicates that the heart is based on the triad of {SDP, KESK, KOK}, with a median line dividing the right–wing parties from those of the left.¹⁷

6.4.2 Left Unipolar Systems

Table 6.2 suggests that the frequent minority governments in the period 1945–1987 in Denmark, Sweden and Norway were based on core parties on the left of the policy space.

Denmark. The political system has a high degree of fragmentation (the effective number increased from about 3.8 in the late 1940s to 7.0 in 1970). The largest party is the Social Democrat Party (SD) with 30–40 percent of the seats, and the Liberals (or Venstre, V) with 20 to 30 percent. The SD is the only dominant party. The SD was in 13 out of 21 governments in the period 1945–1987, while Venstre was a member of the remaining governments.

Governments without a clear majority are typical in Denmark, though tacit support is often provided by small parties. The pattern that emerges is one of SD minority governments with support of the radical liberals (RV), Socialist People's Party (SF) or Communist Party (DKP) alternating with governments consisting of the Venstre and the Conservative People's Party (KF).

¹⁷The estimates of party positions given by Benoit and Laver (2006) are used to construct the figures in the rest of this chapter .

Table 6.8. Elections in Denmark, 1957 and 1964

Party		Seats	
		1957	1964
Communists	DKP	6	–
Socialist People's Party	SF	–	10
Social Democrats	SD	70	76
Radical Venstre	RV	14	10
Venstre or Liberals	V	45	38
Conservative Party	KF	30	36
Justice Party	RF	9	–
Others		1	5
Total		175	175
Actual governments:		1957 to 1960: {SD, RV, RF}	
		1960 to 1964: {SD, RV}	
		1964 to 1968: SD minority.	

For example, Table 6.8 gives the election results for 1957 and 1964. Because the parties on the right controlled more than a majority of the seats in 1957, we can infer that the core is empty. In 1964, the right coalition gained only 84 seats, and the core SD formed a minority government.

Note however that the Danish system became more fragmented, so that the possibility of a core declined. Figure 6.9 gives the estimates of positions in 2001, including those of new parties: the Center Democrats (CD), the Christian Peoples Party (KrF), Danish People's Party (DF) and the Red-Greens (or Enhedslisten (Enh)). The figure shows the median lines. The heart is the star shaped set given in the figure, generated by the SD, DF, KF and V positions.

In the election 2001, the effective number was over 6.5, and a coalition of {V, KF} formed, controlling 72 seats, out of 179. This coalition gained 70 seats in 2005, and stayed in power.

It would seem that the major party positions may have changed very little over time, but there is a clear indication of an increase in fragmentation.

Sweden. The dominance of the Social Democratic Party (SAP) in Sweden was quite pronounced, since it typically obtained just less than

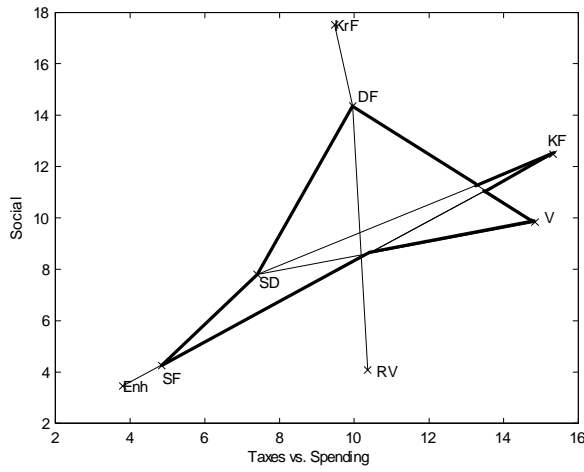


Figure 6.9: Denmark in 2001

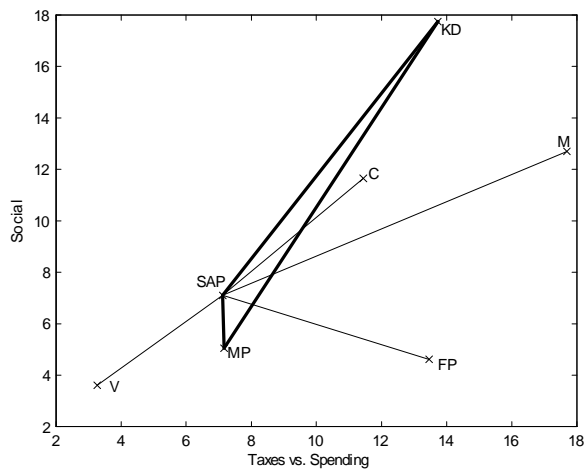


Figure 6.10: Sweden in 2002

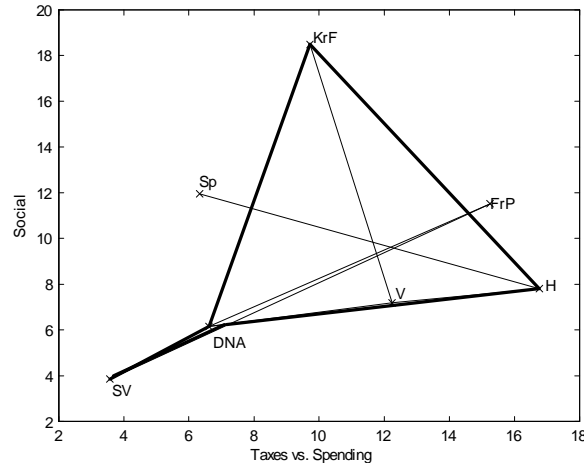


Figure 6.11: Norway in 2001

50 percent of the vote, until 1970. This implied that the only coalition excluding the SAP was a counter coalition of four other parties on the right, making the SAP a natural core party.

In contrast, Figure 6.10 shows the political configuration in 2002. The heart is a triangle bounded by the positions of the Christian Democrats (KD, with 33 seats, out of 349), the SAP (with 144) and the Green Party (MP, with 17 seats). The parties outside the heart are the Center Party (C, with 22 seats), the Moderate Party (M, with 55 seats), the Liberal People's Party (FP, with 48 seats) and the Left Party (V for Vansterpartiet, with 30 seats). Thus, in 2002, the SAP, the Left Party and the Greens together took 53 percent of the vote and 191 seats out of 349. In the 2006 election, the four parties of the right (KD, M, FP and C) formed a pre-election coalition, gained 48 percent of the vote and 178 seats, and were able to form the government.

Norway. The Labor Party (Det Norske Arbeiderpartie or DNA) occupies a position similar to that of the SAP in Sweden. Indeed the DNA has often been the strongly dominant party. Until 1961 it controlled a majority of the seats. The Socialist Left Party (SV) took only 2 seats (out

of 155) in 1977 but jumped to 17 seats in 1989, and in the recent election in September 2005, took 15 out of 169. After the election of 1981, the three parties on the right (Center Party, Sp; Christian People's Party, KrP; and Conservatives, H) controlled a majority. From 1989, the radical right-wing populist Progress Party (FrP), founded by Anders Lange, grew rapidly, gaining 38 seats in 2005. After the 1989 and 1993 elections the DNA was essentially at the core position with a plurality of the vote and was able to form a minority government. In 1997, however, the DNA lost a couple of seats, and the DNA leader, Jagland, stepped down, leading to the formation of a minority right-wing coalition, led by Bondevik of the KrP, together with the Center Party. The unwillingness of the three right-wing parties to form a coalition with the FrP led to the minority right-wing coalition from 1997 to 2005. In the 2005 election, the Center Party switched, forming a Red-Green coalition with the DNA and the SV. This alliance took 87 seats out of 169, and was able to form the first majority coalition in Norway since 1985. (See Strom, 1991, for an earlier discussion of minority coalitions in Norway.) Note, however, that if the parties on the right could agree to form a coalition with the Progress Party, then the heart is the set bounded by the positions of the DNA, the Sp and the Liberals (V), making the Sp a pivot party between coalitions of the left and right. See Figure 6.11.

6.4.3 Center Unipolar Systems

Belgium. Belgium is an interesting example with respect to the theoretical prediction about the core. In the period up to the late 1960s, the political configuration based on three parties meant that the core was empty and minimal winning coalition governments the rule. However, after 1970, increasing political fragmentation resulting from conflicts over language and regional autonomy led to the replacement of the three-party system with a multiparty system generated by the federalist-unitary dimension. The entrance of new parties, including the nationalist Voksunie (VU) in 1954, the Rassemblement Wallon (RW) and the Francophone Democratic Front (FDF), increased the effective seat number (to 6.0 by 1971).

The centrist Christelijke Volkspartij (CV) was almost continually in power until the election of 1999, when it lost its plurality status, gain-

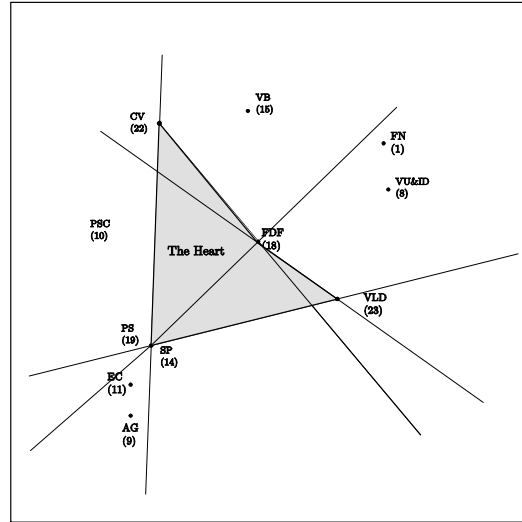


Figure 6.12: The heart in Belgium in 1999

ing only 22 seats out of 150, in comparison to the 23 seats of the Flemish Liberal and Democrat Party (VLD). A coalition of six parties with 94 seats formed the government: VLD, the two wings of the Socialist Party (Parti Socialiste, or PS, and the Socialistische Partij, SP, with 33 seats between them), the Free Democrat Party (FDF) with 18 seats, and 20 seats from two other small, green parties (Ecolo, EC, and Agalev, AG). Figure 6.12 shows the party positions, on the assumption that the two socialist parties (PS and SP) were at the same position. The heart illustrates the various coalition possibilities. The Volksunie had split into a nationalist wing (VU&ID) and a more federalist component, the Flemish Block (VB). These parties, together with the National Front (FN) are shown to be positioned outside the heart.

In 2003, the CV renamed itself the Christian Democratic and Flemish Party (CD&V) and won 21 seats while the FDF was renamed the Reformist Movement (MR) and won 24 seats. The green parties only won four seats. The other small parties were the New Flemish Alliance (N-VA) with 1 seat and the Humanistic Democratic Center (CDH) with 8. The Flemish Socialist Party (SP) formed an alliance with Spirit (Sp), a

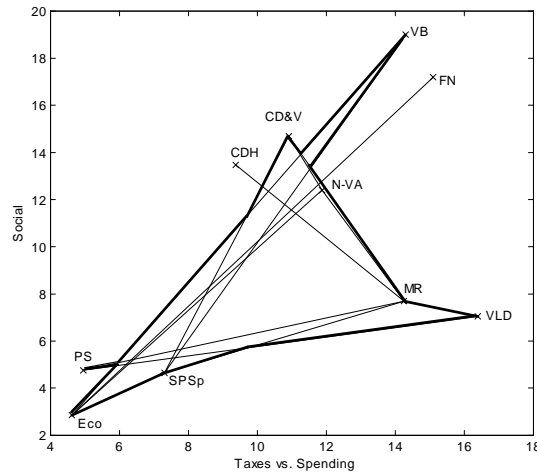


Figure 6.13: The heart in Belgium in 2003

small offshoot of the VU, and together they won 23 seats. Assuming that the two parties, PS and SPSp, were at distinct positions gives the heart as shown in Figure 6.13. This illustrates the more complex coalition possibilities as a result of the increasing fragmentation that occurred between 1999 and 2003. In the election of 10 June, 2007, the CD&V went from 21 seats to 30 (out of 150), becoming the largest party in the Chamber of Representatives. After a month of negotiation, King Albert II asked the leader of the CD&V, Yves Leterme, to be *formateur* of a coalition government, Yves Leterme. Leterme found this impossible, and resigned from the task on 23 August. At the time of going to press (7 November, 2007), Belgium had been without a government for a record 149 days.

Luxembourg. The largest party is the Christian Social Party (CSV) with about one-third of the seats, followed by the Luxembourg Socialist Workers' Party (LSAP) with between one-quarter and one-third of the seats. The smaller Democratic Party (DP) generally gains just less than one-fifth of the seats. The heart is clearly based on the triad of the positions {LSAP, CVP, DP}, and governments tend to be associated with

pairwise minimal winning coalitions: {LSAP, DP} in 1974–1979, {CVP, DP} in 1979–1984 and 1999–2004, and {LSAP, CVP} after the election of 2004.

Ireland. Ireland is especially interesting because it has a dominant center–right party (Fianna Fáil) and unlike Belgium or Luxembourg, there have been a number of minority (Fianna Fáil) governments. To see the complexity of the bargaining possibilities, consider Table 6.9 which lists the seat strengths after February 1987 in the Dáil Eireann.

Table 6.9. Party and faction strengths in the Dáil Eireann, 1987

<i>Left</i>	
Workers' Party (WP)	4
Democratic Socialist Party	1
Labor (LB)	12
Tony Gregory (Left wing Independent)	1
Sean Treacy (Ex-Labor Independent)	1
Ceann Comhairle: Neil Blaney (Independent, NB)	1
<i>Center</i>	
Fine Gael (FG)	51
Fianna Fáil (FF)	81
Progressive Democrats (PD)	14
Total	166

A coalition of Fine Gael and Labor¹⁸ had collapsed in January 1987, and Garret Fitzgerald became Toaiseach, leading a caretaker minority Fine Gael government. Clearly the natural minimal winning coalitions are {Fianna Fáil, Progressive Democrats} with 94 seats, {Fianna Fáil, Fine Gael}, {Fianna Fáil, Labor} with 93, and an unlikely coalition of Fianna Fáil with the far left parties. Figure 6.14 indicates the median lines based on estimates of the party positions and that of the independent, Neil Blaney at NB. We may infer that Fianna Fáil was indeed a core party, suggesting a minority government. This is precisely what occurred. Sean Treacy became Ceann Comhairle (Chairman) of the Dail. Tony Gregory

¹⁸Throughout this book we use the U.S. spelling, labor, for these parties in Ireland and the United Kingdom, rather than the English spelling Labour.

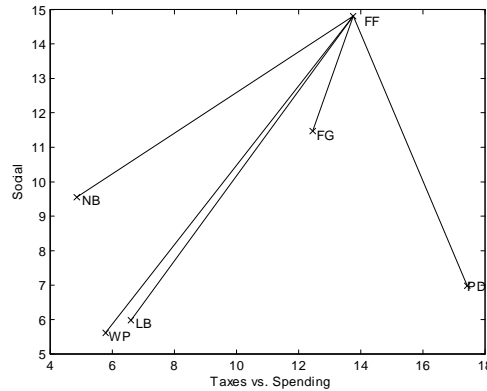


Figure 6.14: Ireland in 1987

abstained and Haughey, leader of Fianna Fáil (with the support of Neil Blaney) had 82 votes out of 164, with Treacy casting the deciding vote for the government.

After the 2002 election, Fianna Fáil obtained 82 seats, out of 166, while the other party strengths were: Fine Gael (FG, 31), Labor (LB, 21), Progressive Democrats (PD, 8), Greens (GR, 6), Sinn Féin (SF, 5), with 14 seats going to the Socialist Party and Independents. Bertie Ahern, leader of Fianna Fáil, formed a coalition with the Progressive Democrats and Rory O'Hanlon was unanimously elected Ceann Comhairle at the first meeting of the 29th Dáil on 6 June 2002.

In the May 2007 election, Fianna Fáil won 78 seats, while the Progressive Democrats only won 2 seats, not enough to form a majority coalition. Fine Gael increased its strength to 51, with only 5 seats going to independents, while Sinn Féin won 4 seats and Labor won 20 seats. Figure 6.15 suggests the nature of the heart. The medians through the FG position are based on the assumption that the Ceann Comhairle would be a member of Fianna Fáil and the five independents' positions are between FG and LB. The Greens joined the government for specific policy objectives and cabinet positions, and a coalition of Fianna Fáil, the Greens and the Progressive Democrats, together with four of the independents, elected John

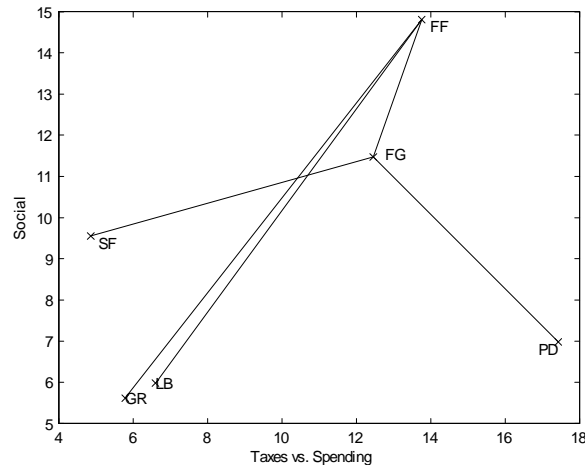


Figure 6.15: Ireland in 2007

O'Donoghue to be Ceann Comhairle by 90 to 75 on 14 June, making Ahern, of Fianna Fáil, the Toiseach of the Dáil Eireann.

6.4.4 A Right Unipolar System

Iceland. To some extent Iceland is a mirror image of the three Scandinavian political systems. The largest party is the right-wing Independence Party (IP) which took 22 seats out of 63 in the 2003 election. At the center are two parties: the Progressive Party (PP) with 12 seats in 2003 and a Liberal Party (F) with 4 seats in 2004. On the left is the Social Democratic Alliance (SDA) with 20 seats, and the Left-Green Movement (G) with 4. The heart is given by the triad of positions {SDA, PP, IP} indicating the likelihood of minimal winning coalitions. David Oddsson, the leader of IP, served as Prime Minister from 1991 to 1995, in alliance with SDA, and then from 1995 to 2004 in alliance with the PP. Oddsson was succeeded in September 2004 by Halldor Asgrimsson of the PP. A coalition government of the IP, under Geir Haarde, with the PP, was formed in June 2006. Figure 6.16 gives an estimate of the heart in 2003.

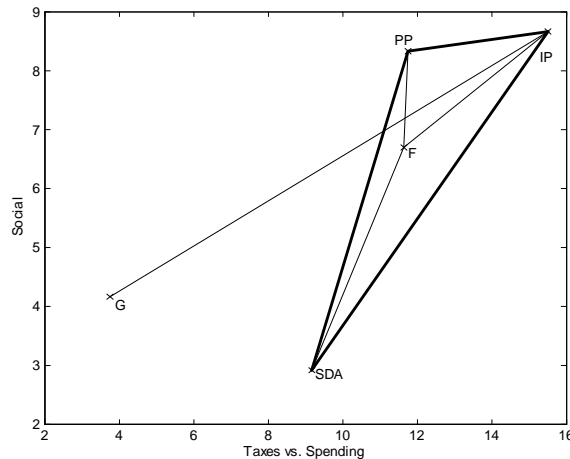


Figure 6.16: Iceland in 2003

6.4.5 Triadic Systems

Austria. In Austria the large parties are the Social Democrat Party (SPO) and People's Party (OVP). Until 1959 the Communists (KPO) had roughly four seats, while the Freedom Party (FPO, but called the League of Independents before 1956), generally won between 6 and 11 seats up until 1979. The OVP won majorities in 1945 (with 85 seats) and in 1966 (with 84 seats). The SPO, under Bruno Kreisky, gained majorities in the elections of 1971, 1975 and 1979, and between 1983 to 1986 formed a coalition with the FPO. From 1986 until 1999 the grand SPO–OVP coalition governed. From 1995 to 1999, partly under the leadership of Jorg Haider, the FPO increased in strength from 41 to 52 seats, making it an obvious coalition partner for the OVP (also with 52 seats out of 183). Surprisingly, the FPO gained a slightly larger proportion of the vote than the OVP. Various controversies over the FPO leadership led to a new election in 2002. Haider had resigned the leadership of the FPO in 2000, and, in the 2002 election, the FPO strength fell to 18 seats, while the OVP jumped to 79 seats. For the first time since 1966, the OVP gained

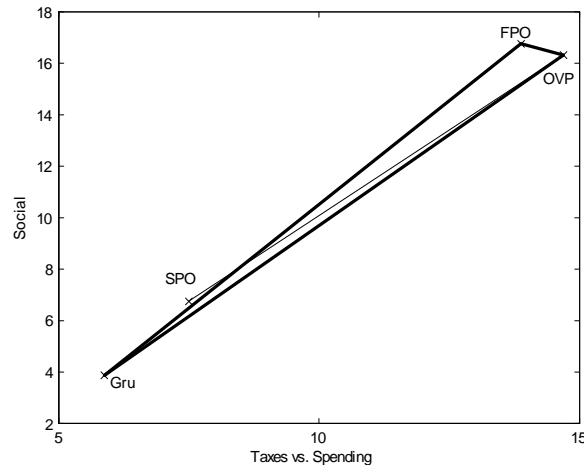


Figure 6.17: Austria in 2006

a higher proportion of the vote than the SPO (presumably because of the collapse of the FPO). In 2005, Haider formed a new party, the “Alliance for the future of Austria,” BZO), which only gained 7 seats in the 2006 election. The OVP, with 66 seats, then formed a coalition with the FPO (with its 21 seats), against the SPO, with its 68 seats and the Greens (Gru) with its 21 seats. Figure 6.17 shows the heart for the election of 2006. Assuming that the BZO is located at the FPO position, the heart is based on the triad {SPO/Gru, OVP, FPO}.

Germany. Figure 6.18 shows the heart for the election of 2002 in Germany, where the Christian Democrats (CDU/CSU) gained 248 seats, the Social Democrat Party (SPD) gained 251 seats, and the Free Democrat Party (FDP) gained 47 seats. The Greens (GRU) with 55 seats formed a minimal winning coalition with the SPD until the September 2005 election. As the figure indicates, the {SPD, GRU} median line is one of the boundaries of the heart, and so this coalition is a natural one to form. After the September 2005 election, however, the Greens gained 51 seats against 61 for the FDP and 54 for the Party of Democratic Socialism (PDS). Since the CDU only gained 225 seats in contrast to 222 for the

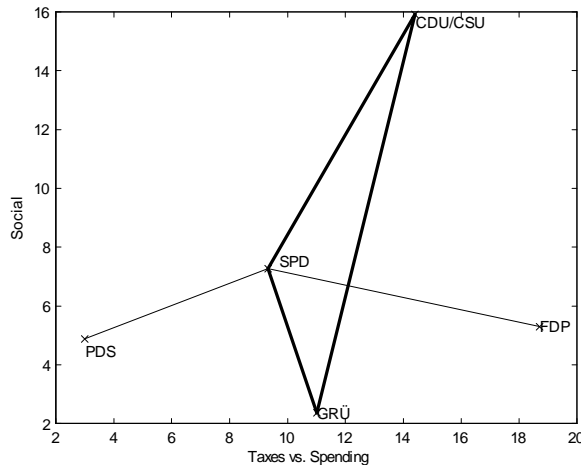


Figure 6.18: Germany in 2002

SPD there was an impasse. The coalition {PDS, GRU, SPD} is now possible, causing a contraction of the heart. Eventually Angela Merkel, of the CDU, became Chancellor, leading the grand CDU/CSU/SPD coalition.

6.4.6 A Collapsed Core

Italy. Italy needs a category of its own, as it was originally a center unipolar system, where the dominant party, the Christian Democrat Party (DC), was in a uniquely powerful position until the 1994 election. The DC went from 206 seats (out of 630) in 1992 to 33 in 1994. Until 1987 the DC controlled about 40 percent of the seats, with the Communist Party (PCI) and Socialists (PSI) controlling just less than 30 percent each. The smaller parties include the Social Democrats (PSDI), Republicans (PRI), Liberals (PLI), Monarchists (PDIUM) and Neofascists (MSI). Aside from the first two governments in 1946 and 1947, the Communists never belonged to a coalition government. The DC was strongly dominant, and the only party able to position itself at a structurally stable core in a two-dimensional policy space, as indicated in Figure 6.19 (based on Giannetti

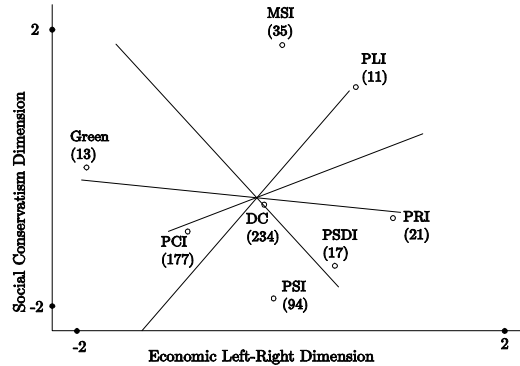


Figure 6.19: The core in Italy in 1987

and Sened, 2004).

The persistence of the Pentapartito coalition (1979–1989) comprising a coalition of DC, PSI, PRI, PLI and the PSDI is further evidence that the core was non-empty. To control the distribution of government perquisites, the DC maintained a grand, anti-PCI coalition. Schofield (1993) suggested that corruption associated with these perquisites eventually led to an anti-DC coalition based on new parties such as the Northern League and the Greens. Mershon (1996a,b, 2002), Giannetti and Sened (2004) and Schofield and Sened (2006) discuss the dramatic changes in Italian politics that occurred in the period 1992–1996. Figure 6.20 indicates the quite new Italian configuration based on the positions of the parties in 2001: the Alleanza Nazionale (AN, 24 seats), Democratici di Sinistra (DS, 31 seats), Forza Italia (FI, 62 seats), La Margherita (Marg, 27 seats) and Rifondazione Comunista (RC, 11 seats).

6.5 Concluding Remarks

Each of these 12 European countries, together with Israel, discussed in this chapter, displays very complex characteristic features.

Although this chapter has suggested a typology of these polities based on the qualitative features of the core and the heart, it is evident that the suggested typology does not give a full account of the subtle aspects of

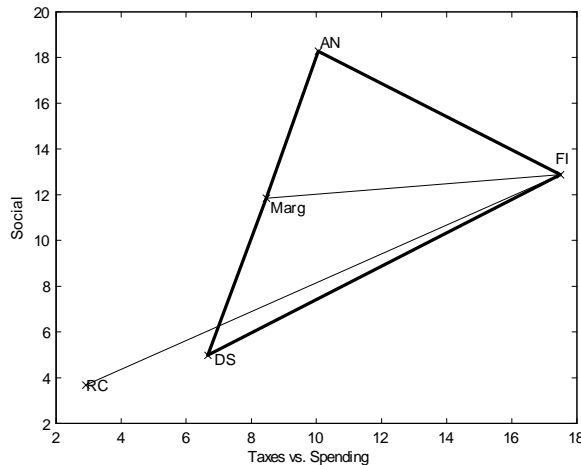


Figure 6.20: Italy in 2001

coalitional bargaining. The key features of this typology is the degree of fragmentation, and the extent of centrality (i.e., whether a dominant party occupies the core position). What is remarkable, however, is the degree to which each country exhibits a pattern of coalition government that is consistent, in some sense, over time. It is hardly surprising that comparative scholars have found these patterns to be of such great theoretical interest. Estimating party positions, and attempting to model coalition bargaining between the parties is a major challenge for comparative research. Recent work by Benoit and Laver (2006) on estimating party position for a large number of political configurations is a significant advance, and their estimates have proved invaluable as a means to estimate the legislative heart in these polities.

The purpose of the spatial analysis presented in this chapter is to give some insight into the complexities of multiparty bargaining. The typology presented here has used the theory developed in the previous chapters, based on the existence of core parties and on the heart as an indication of the bargaining domain when the core is empty. Some countries are characterized by the existence of a dominant party, able to attain enough

seats to be strongly dominant and command the core position. In the bipolar polities there are two potentially dominant parties, each one of which may be able to gain enough seats on occasion to control the core. Increasing fragmentation may make it less likely that a core party can exist. As the configuration of the heart becomes more complex, then bargaining over government will also become more complex. It is hardly surprising that fragmentation will be associated with less durable government (see King, Alt, Burns and Laver, 1990).

The main theoretical point that emerges is that the configuration of the heart in these polities suggests that there is hardly any centripetal tendency towards an electoral center. It is consistent with this analysis that activist groups will tend to pull the parties away from the center. Indeed, we can follow Duverger (1954) and Riker (1953) and note that under proportional electoral methods, there is very little motivation for interest groups to coalesce. Consequently, the fragmentation of interest groups will lead to a degree of fragmentation in the polity. Fragmentation may be mitigated by the electoral system (especially if there is a relatively high electoral requirement which determines whether a party will obtain some legislative representation. However, even when there is a degree of majoritarianism in the electoral system (as in Italy in recent years) this may have little effect on reducing fragmentation.

Clearly if one party dominates coalition policy for a long period of time then there will be a much higher degree of stability than indicated purely by government duration. However, as the situation in Italy circa 1994 suggests, if there is a core party facing little in terms of real political opposition, then corruption may become persistent. For democratic polities, there may be an element of a quandary associated with the choice of an electoral system. If it is based on proportional representation then there may be the possibility of dominance by a centrally located party. Alternatively, there may be coalitional instability resulting from a fragmented polity and a complex configuration of parties. Another way of expressing, in simplified form, the difference between proportional representation and plurality rule is this: under proportional electoral methods, bargaining to create winning coalitions occurs *after* the election. Under plurality rule, if interest groups do not form a coalition *before* the election, then they can be obliterated, creating a pressure to coalesce.

Popper (1945, 1988) may have had this in mind, when he argued that

“first past the post” or plurality rule forces political agents to form opposing coalitions, associated with majoritarian parties. In a sense, this tendency requires society to make a decision, one way or the other, about the best option. Indeed, it is possible that plurality rule induces political leaders to be more willing to make risky decisions than they would under proportional rule (Schofield, 2006a; Falaschetti, 2007).

The next three chapters present a model of elections under plurality and proportional rule, based on the logic of activist groups.

6.6 Appendix

Table 6.10. Recent elections in Europe

Country	Party name	Party	Vote %	Seats	Date
Austria	Freedom Party	FPO	10	18	2002
	The Greens	Gru	9.5	17	2002
	Austrian People's Party	OVP	42.3	79	2002
Belgium	Social Democratic Party	SPO	36.5	69	2002
	Christian Democratic	CD&V	13.2	21	2003
	Humanist Democ. Centre	CDH	5.5	8	2003
	Ecolo	Eco	3.1	4	2003
	National Front	FN	2	1	2003
	Groen!	Gro!	2.5	0	2003
	Reformist Movement	MR	11.4	24	2003
	New Flemish Alliance	NVA	3.1	1	2003
	Socialist Party	PS	13	25	2003
	Soc. Party-Anders-Spirit	SPSp	14.9	23	2003
	Flemish Block	VB	11.6	18	2003
	Liberals and Democrats	VLD	15.4	25	2003
Denmark	Centrumdemokraterne	CD	1.8	0	2001
	Dansk Folkeparti	DF	12	22	2001
	Enhedslisten	Enh	2.4	4	2001
	Fremskridtspartiet	FrP	0.6	0	2001
	Konservative Folkeparti	KF	9.1	16	2001
	Kristeligt Folkeparti	KrF	2.3	4	2001
	Radikale Venstre	RV	5.2	9	2001
	Socialdemokratiet	SD	29.1	52	2001
	Socialistisk Folkeparti	SF	6.4	12	2001
	Venstre	V	31.3	56	2001
Finland	Kristillisdemokraattit	KD	5.3	7	2003
	Suomen Keskusta	KESK	24.7	55	2003
	Kansallinen Kokoomus	KOK	18.5	40	2003
	Perussuomalaiset	PS	1.6	3	2003
	Sosialidemokraattinen	SDP	22.9	53	2003
	Svenska Folkepartiet	SFP	4.6	8	2003
	Vasemmistoliitto	VAS	9.9	19	2003
Germany	Vihreä Liitto	VIHR	8	14	2003
	Dem.Union./ Soc.Union.	CDU/CSU	38.5	248	2002
	Communist Party	DKP	0	0	2002
	German People's Union	DVU	0	0	2002
	Free Democratic Party	FDP	7.4	47	2002
	Green Party	GRÜ	8.6	55	2002
	National Dem. Party	NPD	0.45	0	2002
	Democratic Socialist	PDS	4.3	2	2002
	Republicans	Rep	0.1	0	2002
	Social Democratic Party	SPD	38.5	251	2002
Iceland	Rechtsstaatlicher	Schil	0.3	0	2002
	Framsóknarflokkurinn	PP	17.7	12	2003
	Sjálfstæðisflokkurinn	IP	33.7	22	2003
	Frjálslyndi Flokkurinn	F	7.4	4	2003
	Nytt afl	N	1	0	2003
Ireland	Samfylkingin (alliance)	SDA	31	20	2003
	Vinstrihreyfing -	G	8.8	5	2003
	Fianna Fáil	FF	41.5	81	2002

Country	Party Name	Party	Vote %	Seats	Date	
Italy	Fine Gael	FG	22.5	31	2002	
	Greens	GR	3.8	6	2002	
	Labor	LB	10.8	21	2002	
	Progressive Democrats	PD	4	8	2002	
	Sinn Fein	SF	6.5	5	2002	
	Independents		9.5	13	2002	
	Alleanza Nazionale	AN	12	24	2001	
	Democratici di Sinistra	DS	16.6	31	2001	
	Forza Italia	F	29.4	62	2001	
	Federazione dei Verdi	Green	1.1	0	2001	
	Italia dei Valori	It.Val.	3.9	0	2001	
	Lega Nord	LN	3.9	0	2001	
	Soc. Fiamma Tricolore	MSFT	0.4	0	2001	
	La Margherita	Marg	14.5	27	2001	
	Comunisti Italiani	PDCI	1.7	0	2001	
	Lista Pannella Bonino	Pann	2.3	0	2001	
	Rifondazione Comunista	RC	5	11	2001	
	Unione di Centro	UDC	3.2	0	2001	
	Netherlands	Socialisti Democratici	SDI	1.1	0	2001
Christian Democratic		CDA	26.5	41	2006	
Labor		PvdA	21.2	33	2006	
Socialists		SP	16.6	25	2006	
Liberals		VVD	14.7	22	2006	
Freedom Party		PVV	5.9	9	2006	
Green Party		GL	4.6	7	2006	
Christian Union		CU	4	6	2006	
Left Liberals		D'66	2	3	2006	
Labor for Animals		PvdD	1.8	2	2006	
Protestant Party		SGP	1.6	2	2006	
Norway		Det Norske Arbeiderparti	DNA	24.3	43	2001
		Fremskrittspartiet	FrP	14.7	26	2001
		Høyre	H	21.2	38	2001
	Kristelig Folkeparti	KrF	12.5	23	2001	
	Rød Valgallianse	RV	1.2	0	2001	
	Sosialistisk Venstreparti	SV	12.4	22	2001	
	Senterpartiet	Sp	5.6	10	2001	
	Venstre	V	3.9	2	2001	
	Sweden	Centerpartiet	C	6.1	22	2002
		Folkpartiet Liberalerna	FP	13.3	48	2002
Kristdemokraterna		KD	9.1	33	2002	
Moderata Samlingspartiet		M	15.2	55	2002	
Miljöpartiet de Gröna		MP	4.6	17	2002	
Sosialdemokratiska		SAP	39.8	144	2002	
Vänsterpartiet		V	8.3	30	2002	
UK	Labor Party	LAB	35.3	356	2005	
	Conservative Party	CON	32.3	198	2005	
	Liberal Democrat	LIB	22.1	62	2005	
N.Ireland	Social Dem. and Labor	SDLP	17.5	3	2005	
	Sinn Fein	SF	24.3	5	2005	
	Ulster Unionist Party	UU	17.7	1	2005	
	Democratic Union	DU	33.7	9	2005	

Chapter 7

A Spatial Model of Elections

7.1 Political Valence

The models of coalition bargaining discussed in the previous chapter suggest that even when there is no majority party then a large, centrally located party, at a “core” position in the policy space, will be dominant. Such a core party can, if it chooses, form a minority government by itself and control policy outcomes.¹⁹ If party leaders are aware of the fact that they can control policy from the core, then this centripetal tendency should lead parties to position themselves at the center. Moreover, the “mean voter theorem,” based on a stochastic model of election and on vote maximization, suggests that the electoral origin will be a Nash equilibrium.²⁰ These two very different models of political strategy suggest that parties will tend to locate themselves at the electoral center.

Yet, contrary to this intuition, there is ample empirical evidence that party leaders do not necessarily adopt centrist positions. For example, Budge, Robertson and Hearl (1987) and Laver and Hunt (1992), in their study of European party manifestos, found no evidence of a strong centripetal tendency. The electoral models for Italy and Israel presented in Giannetti and Sened (2004) and Schofield and Sened (2006) estimated party positions in various ways, and concluded that there was no general

¹⁹In addition to the arguments in the previous chapter, see Schofield, Grofman and Feld (1989); Laver and Schofield (1990); Sened (1995); Banks and Duggan (2000); Schofield and Sened (2006).

²⁰Adams (1999a,b, 2001); Adams and Merrill (1999a,b, 2005, 2006); Lin, Enelow and Dorussen (1999); Banks and Duggan (2005); McKelvey and Patty (2006).

indication of policy convergence by parties. As the previous chapter has suggested, the only clear indications of parties adopting very centrist positions were the examples of the Christian Democrats in Italy, up until the election of 1992, and Kadima in Israel at the election of 2006. This chapter examines the evidence for Israel,²¹ Turkey,²² the Netherlands,²³ and Britain²⁴ to determine if the non-convergence noted previously can be accounted for by a stochastic electoral model that includes “valence” (Stokes, 1992).

These empirical models have all entailed the addition of heterogeneous intercept terms for each party. One interpretation of these intercept terms is that they are valences or party biases, derived from voters’ judgements about characteristics of the candidates, or party leaders, which cannot be ascribed to the policy choice of the party. One may conceive of the valence that a voter ascribes to a party leader as a judgement of the leader’s quality or competence.²⁵ This idea of valence has been utilized in a number of recent formal models of voting.²⁶

The chapter considers a general valence model based on activist support for the parties.²⁷ This activist valence model presupposes that party activists donate time and other resources to their party. Such resources allow a party to present itself more effectively to the electorate, thus increasing its valence. Since activists tend to be more radical than the average voter, parties are faced with a dilemma. By accommodating the political demands of activists, a party gains resources that it can use to enhance its valence, but by adopting the radical policies demanded by activists, the party may appear too extreme and lose electoral support. The party must therefore balance the electoral effect against the activist

²¹Schofield and Sened (2006).

²²Schofield and Ozdemir (2007).

²³Schofield, Martin, Quinn and Whitford (1998); Quinn, Martin and Whitford (1999).

²⁴Schofield (2005a,b).

²⁵Stokes (1963) used the term *valence issues* to refer to those that “involve the linking of the parties with some condition that is positively or negatively valued by the electorate.” As he observes, “in American presidential elections... it is remarkable how many valence issues have held the center of the stage.” Stokes’s observation is validated by recent empirical work on many polities, as well as by a study on the psychology of voting (Westen, 2007).

²⁶Ansolahehere and Snyder (2000); Groseclose (2001); Aragones and Palfrey (2002, 2005).

²⁷Aldrich (1983a,b, 1995); Aldrich and McGinnis (1989).

valence effect. Theorem 7.2.1 presents the requisite balance condition between electoral and activist support. Since valence in this model is affected by activist support, it may exhibit “decreasing returns to scale” and this may induce concavity in the vote-share functions of the parties. Consequently, when the concavity of the activist valence is sufficiently pronounced then a pure strategy Nash equilibrium (PNE) of the vote-maximizing game will exist. However, Theorem 7.2.1 indicates that there is no reason for this equilibrium to be one where all parties adopt centrist positions.

In some polities, activists’ valence functions will be sufficiently concave so that only one PNE will exist. However, computation of PNE is extremely difficult and as a first step this chapter concentrates instead on conditions for existence of “local pure strategy Nash equilibria” (LNE). Recent analyses of elections in Israel have used simulation techniques to examine the nature of these local equilibria (Schofield and Sened, 2006).

The next section of this chapter presents a characterization of LNE for the stochastic electoral activist model, in terms of the Hessians of the vote-share functions of the parties. Throughout it is assumed that the stochastic errors have the Type I extreme value (Gumbel) distribution, Ψ . The formal model based on Ψ parallels the empirical models based on multinomial logit (MNL) estimation (Dow and Endersby, 2004). Theorem 7.2.2 specializes to the simpler case when only exogenous valence is relevant, so that the activist valence functions are zero. For the case of fixed or exogenous valence, Theorem 7.2.2 shows that the model is classified by a “convergence coefficient,” c , which is a function of all the parameters of the model.

A sufficient condition for the existence of a convergent LNE at the electoral mean is that this coefficient is bounded above by 1. When the policy space is of dimension w , then the necessary condition for existence of a PNE at the electoral mean, and thus for the validity of the “mean voter theorem” (Hinich, 1977; Lin, Enelow and Dorussen, 1999), is that the coefficient is bounded above by w . It is shown that the convergence coefficient is (i) an increasing function of the maximum valence difference (ii) an increasing function of the spatial parameter, β , giving the relative importance of policy difference, and (iii) an increasing function of the electoral variance of the distribution of voter preferred points.

When the necessary condition fails, then parties, in equilibrium, will

adopt divergent positions. In general, parties whose leaders have the lowest valence will take up positions furthest from the electoral mean. Moreover, because a PNE must be a local equilibrium, the failure of existence of the LNE at the electoral mean implies non-existence of such a centrist PNE. The failure of the necessary condition for convergence has a simple interpretation. If the variance of the electoral distribution is sufficiently large in contrast to the expected vote-share of the lowest valence party at the electoral mean, then this party has an incentive to move away from the origin towards the electoral periphery. Other low valence parties will follow suit, and the local equilibrium will be one where parties are distributed along a “principal electoral axis.” The general conclusion is that, with all other parameters fixed, then a convergent LNE can be guaranteed only when the convergence coefficient, c , is “sufficiently” small. Thus, divergence away from the electoral mean becomes more likely the greater is β , the valence difference and the variance of the electoral distribution.

To illustrate the theorem, empirical studies of voter behavior, for Israel, in the election of 1996, and for Turkey, in the elections of 1999 and 2002, are used to show that the condition on the empirical parameters of the model, necessary for convergence, was violated. The equilibrium positions obtained from the formal result under vote maximization were found to be comparable with, though not identical to, the estimated positions. In Israel, for example, the two highest valence parties were symmetrically located on either side of the electoral origin, while the lowest valence parties were located far from the origin.

Since vote maximization is a natural assumption to make for political competition, the result suggests that there are essentially two non-centrist political configurations:

If there are two or more dimensions of policy, but there is a principal electoral axis associated with higher electoral variance, then all parties will tend to be located on, or close to this axis. In particular, if there are two competing high valence parties, then they will locate themselves at vote-maximizing positions on this axis, but on opposite sides of the electoral mean. Low valence parties will be situated on this axis, but far from the center. The unidimensionality of the resulting configuration will give a centrist party on the axis the ability to control government and thus policy. As the discussion of Israel in the previous chapter suggests, there may be a core party, as in 1992 and 2006, and the party will be able

to control the core because of the high exogenous valence of the party's leader.

If both policy dimensions are more or less equally important, with low covariance between the electoral distributions on both axes, then there will be no principal axis and parties can locate themselves throughout the policy space. Again high valence parties will tend to position themselves nearer the mean. To construct a winning coalition, one or other of the high valence, centrist parties must bargain with more "radical" low valence parties. In those situations where there is no core party, then a number of government coalitions are possible, and the range of possible outcomes is suggested by the legislative heart, as presented in the previous chapter.

In contrast to the examples from Israel and Turkey, the empirical evidence from the Netherlands in 1981 indicates that the eigenvalues of the Hessians of the vote-share functions at the joint electoral origin were all negative. In other words, the joint origin was an LNE for the stochastic model with exogenous valence. Clearly this is compatible with the mean voter theorem. This inference does not rule out the existence of other non-convergent LNE, but no other local equilibria were found by simulation (Quinn and Martin, 2002).

For a two-dimensional stochastic model of the 1997 election in Britain, it was found that the estimated position of the Conservative Party was incompatible with the results with exogenous valence. However, this model did provide an explanation for the position of the centrist Liberal Democrat Party. The results of Theorem 7.2.2 with activist valence are then used to explain the changes in positions of the two larger parties in Britain between the elections of 1992 and 1997. Indeed, the empirical model suggests that as the exogenous valence of the Labor Party leaders increased in the 1990s, then the party's activists became less important. This provides an explanation why the party could become more centrist on the economic axis. On the other hand, as the valences of the leaders of the Conservative Party fell in the same period, then the influence on the party of anti-Europe activists increased. This suggests why the party adopted an anti-European Union position. While these observations are particular to Britain, they appear applicable to any polity such as the U.S., where activist support is important.

Although the equilibrium for the exogenous valence models for Is-

rael and Turkey correctly predict non-convergence, they do not accurately predict the positions of all the parties. Moreover, the analyses of the Netherlands and Britain strongly suggest that the valence model will more accurately reflect party positions if the notion of valence is extended to include the influence of activists. The more general inference is that parties are located in political niches which they inhabit in a balance between activist influence and electoral preferences. This more complex model is elaborated in the following two chapters.

The next section of this chapter presents the formal model and statement of the theorems. Section 7.3 gives the empirical applications, while Section 7.4. briefly comments on the idea of the balance solution when there are two or more opposed activist groups for each party. An empirical appendix gives details on the electoral model for Turkey.

7.2 Local Nash Equilibrium with Activists

We consider a model of competition among a set, N , of parties. The electoral is an extension of the multiparty stochastic model of Lin, Enelow and Dorussen (1999), but modified by inducing asymmetries in terms of valence. The basis for this extension is the extensive empirical evidence that valence is a significant component of the judgements made by voters of party leaders. There are a number of possible choices for the appropriate model of multiparty electoral competition. The simplest one, which is used here, is that the utility function for party, j , is proportional to the vote-share, E_j , of the party. With this assumption, we can examine the conditions on the parameters of the stochastic model which are necessary for the existence of a pure strategy Nash equilibrium (PNE). Because the vote-share functions are differentiable, we use calculus techniques to estimate optimal positions. We can then obtain sufficient conditions for the existence of local pure strategy Nash equilibria (LNE). Clearly, any PNE will be an LNE, but not conversely. Additional conditions of concavity or quasi-concavity are sufficient to guarantee existence of PNE.

The key idea underlying the formal model is that party leaders attempt to estimate the electoral effects of party declarations, or manifestos, and choose their own positions as best responses to other party declarations, in order to maximize their own vote-share. The stochastic model essen-

tially assumes that party leaders cannot predict vote response precisely, but can estimate an expected vote-share. In the model with valence, the stochastic element is associated with the weight given by each voter, i , to the average perceived quality or valence of the party leader.

Definition 7.2.1. The Stochastic Vote Model $\mathbb{E}(\boldsymbol{\lambda}, \boldsymbol{\mu}, \beta; \Psi)$ with Activist Valence. Voters are characterized by a distribution, $\{x_i \in W : i \in V\}$, of voter bliss points for the members of the electorate, V , of size v . We assume that W is an open, convex subset of Euclidean space, \mathbb{R}^w , with w finite. Each of the parties in the set $N = \{1, \dots, j, \dots, n\}$ chooses a policy, $z_j \in W$, to declare. Let $\mathbf{z} = (z_1, \dots, z_n) \in W^n$ be a typical vector of party policy positions.

Given \mathbf{z} , each voter, i , is described by a vector

$$\begin{aligned} \mathbf{u}_i(x_i, \mathbf{z}) &= (u_{i1}(x_i, z_1), \dots, u_{ip}(x_i, z_n)), \text{ where} \\ u_{ij}(x_i, z_j) &= \lambda_j + \mu_j(z_j) - \beta \|x_i - z_j\|^2 + \epsilon_j \\ &= u_{ij}^*(x_i, z_j) + \epsilon_j. \end{aligned}$$

Here $u_{ij}^*(x_i, z_j)$ is the observable component of utility. The term, λ_j , is the fixed or *exogenous valence* of agent j , while the function $\mu_j(z_j)$ is the component of valence generated by activist contributions to agent j . The term β is a positive constant, called the *spatial parameter*, giving the importance of policy difference defined in terms of the Euclidean norm, $\|\cdot\|$, on W . The vector $\epsilon = (\epsilon_1, \dots, \epsilon_j, \dots, \epsilon_n)$ is the stochastic error, whose multivariate cumulative distribution will be denoted by Ψ .

It is assumed that the exogenous valence vector

$$\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_n) \text{ satisfies } \lambda_n \geq \lambda_{n-1} \geq \dots \geq \lambda_2 \geq \lambda_1.$$

Voter behavior is modeled by a probability vector. The probability that a voter i chooses party j at the vector \mathbf{z} is

$$\rho_{ij}(\mathbf{z}) = \Pr[u_{ij}(x_i, z_j) > u_{il}(x_i, z_l)], \text{ for all } l \neq j]. \quad (7.1)$$

$$= \Pr[\epsilon_l - \epsilon_j < u_{ij}^*(x_i, z_j) - u_{il}^*(x_i, z_j)], \text{ for all } l \neq j]. \quad (7.2)$$

Here \Pr stands for the probability operator generated by the distribution assumption on ϵ .

The *expected vote-share* of agent j generated by the model $\mathbb{E}(\boldsymbol{\lambda}, \boldsymbol{\mu}; \beta; \Psi)$ is

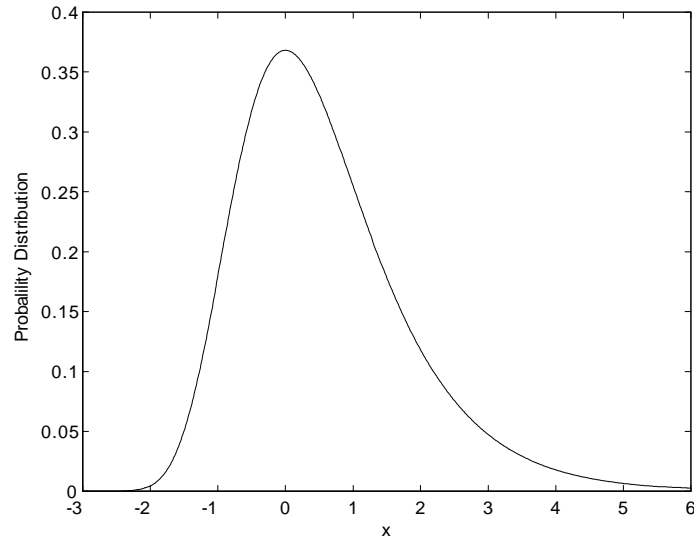


Figure 7.1: The Gumbel distribution

$$E_j(\mathbf{z}) = \frac{1}{v} \sum_{i \in V} \rho_{ij}(\mathbf{z}). \quad (7.3)$$

The differentiable function $E : W^n \rightarrow \mathbb{R}^n$ is called the *party profile function*.

The most common assumption in empirical analyses is that Ψ is the *Type I extreme value (or Gumbel) distribution*. The theorems in this chapter are based on this assumption.

Definition 7.2.2. The Type I Extreme Value Distribution, Ψ .

The cumulative distribution, Ψ , has the closed form

$$\Psi(x) = \exp[-\exp[-x]],$$

with probability density function

$$\psi(x) = \exp[-x] \exp[-\exp[-x]]$$

and variance $\frac{1}{6}\pi^2$ (see Figure 7.1).

The difference between the Gumbel and normal (or Gaussian) distributions is that the latter is perfectly symmetric about zero.

With this distribution assumption, it follows, for each voter i , and party j , that

$$\rho_{ij}(\mathbf{z}) = \frac{\exp[u_{ij}^*(x_i, z_j)]}{\sum_{k=1}^n \exp u_{ik}^*(x_i, z_k)}. \quad (7.4)$$

This implies that the model satisfies the independence of irrelevant alternative property (IIA): for each individual i , and each pair, j, k , the ratio

$$\frac{\rho_{ij}(\mathbf{z})}{\rho_{ik}(\mathbf{z})}$$

is independent of a third party l . (See Train, 2003: 79.)

In this stochastic electoral model it is assumed that each party j chooses z_j to maximize V_j , conditional on $\mathbf{z}_{-j} = (z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_p)$.

Definition 7.2.3. Equilibrium Concepts.

(i) A strategy vector $\mathbf{z}^* = (z_1^*, \dots, z_{j-1}^*, z_j^*, z_{j+1}^*, \dots, z_n^*) \in W^n$ is a *local strict Nash equilibrium* (LSNE) for the function $E : W^n \rightarrow \mathbb{R}^n$ iff, for each party $j \in N$, there exists a neighborhood W_j of z_j^* in W such that

$$E_j(z_1^*, \dots, z_{j-1}^*, z_j^*, z_{j+1}^*, \dots, z_p^*) > E_j(z_1^*, \dots, z_j, \dots, z_n^*) \\ \text{for all } z_j \in W_j - \{z_j^*\}.$$

(ii) A strategy vector $\mathbf{z}^* = (z_1^*, \dots, z_{j-1}^*, z_j^*, z_{j+1}^*, \dots, z_n^*)$ is a *local weak Nash equilibrium* (LNE) iff, for each agent j , there exists a neighborhood W_j of z_j^* in W such that

$$E_j(z_1^*, \dots, z_{j-1}^*, z_j^*, z_{j+1}^*, \dots, z_n^*) \geq E_j(z_1^*, \dots, z_j, \dots, z_n^*) \\ \text{for all } z_j \in W_j.$$

(iii) A strategy vector $\mathbf{z}^* = (z_1^*, \dots, z_{j-1}^*, z_j^*, z_{j+1}^*, \dots, z_n^*)$ is a *strict or weak, pure strategy Nash equilibrium* (PSNE or PNE) iff W_j can be replaced by W in (i) and (ii) respectively.

- (iv) The strategy z_j^* is termed a “local strict best response,” a “local weak best response,” a “global weak best response,” a “global strict best response,” respectively to $\mathbf{z}_{-j}^* = (z_1^*, \dots, z_{j-1}^*, z_{j+1}^*, \dots, z_n^*)$.

Obviously if \mathbf{z}^* is an LSNE or a PNE it must be an LNE, while if it is a PSNE then it must be an LSNE. We use the notion of LSNE to avoid problems with the degenerate situation when there is a zero eigenvalue to the Hessian. The weaker requirement of LNE allows us to obtain a necessary condition for \mathbf{z}^* to be an LNE and thus a PNE, without having to invoke concavity. Of particular interest is the joint mean vector

$$x^* = \frac{1}{v} \sum_{i \in V} x_i. \quad (7.5)$$

We first transform coordinates so that in the new coordinate system, $x^* = 0$. We shall refer to $\mathbf{z}_0 = (0, \dots, 0)$ as the *joint origin*.

Theorem 7.2.1 below shows that $\mathbf{z}_0 = (0, \dots, 0)$ will generally not satisfy the first-order condition for an LSNE, namely that the differential of E_j , with respect to z_j be zero. However, if the activist valence function is identically zero, so that only exogenous valence is relevant, then the first-order condition will be satisfied. On the other hand, Theorem 7.2.2 shows that there are necessary and sufficient conditions for \mathbf{z}_0 to be an LSNE in a model without activist valence. A corollary of Theorem 7.2.2 gives these conditions in terms of a “convergence coefficient” determined by the Hessian of party 1, with the lowest valence.

It follows from (7.4) that for voter i , with bliss point, x_i , the probability, $\rho_{ij}(\mathbf{z})$, that i picks j at \mathbf{z} is given by

$$\rho_{ij}(\mathbf{z}) = [1 + \sum_{k \neq j} \exp(f_{jk})]^{-1}, \quad (7.6)$$

where $f_{jk} = \lambda_k + \mu_k(z_k) - \lambda_j - \mu_j(z_j) + \beta \|x_i - z_j\|^2 - \beta \|x_i - z_k\|^2$.

Theorem 2.2.2 below shows that the first-order condition for \mathbf{z}^* to be an LSNE is that it be a *balance solution*.

Definition 7.2.4: The Balance Solution for the Model $\mathbb{E}(\boldsymbol{\lambda}, \boldsymbol{\mu}; \beta; \Psi)$.

- (i) Let $[\rho_{ij}(\mathbf{z})] = [\rho_{ij}]$ be the matrix of voter probabilities at the vector \mathbf{z} ,

and let

$$[\alpha_{ij}] = \frac{\rho_{ij} - \rho_{ij}^2}{\sum_k^v (\rho_{kj} - \rho_{kj}^2)}$$

be the matrix of coefficients. The *balance equation* for z_j^* is given by expression

$$z_j^* = \frac{1}{2\beta} \frac{d\mu_j}{dz_j} + \sum_{i=1}^v \alpha_{ij} x_i. \quad (7.7)$$

- (ii) The vector $\sum_i \alpha_{ij} x_i$ is called the *weighted electoral mean* for party j , and can be written

$$\sum_{i=1}^v \alpha_{ij} x_i = \frac{d\mathcal{E}_j^*}{dz_j}. \quad (7.8)$$

- (iii) The balance equation for party j can then be rewritten as

$$\left[\frac{d\mathcal{E}_j^*}{dz_j} - z_j^* \right] + \frac{1}{2\beta} \frac{d\mu_j}{dz_j} = 0. \quad (7.9)$$

- (iv) The bracketed term on the left of this expression is termed the *marginal electoral pull of party j* and is a gradient vector pointing towards the weighted electoral mean. This weighted electoral mean is that point where the electoral pull is zero. The vector $\frac{d\mu_j}{dz_j}$ is called *the marginal activist pull for party j* .
- (v) If the vector $\mathbf{z}^* = (z_1^*, \dots, z_j^*, \dots, z_n^*)$ satisfies the set of balance equations, $j = 1, \dots, n$, then call \mathbf{z}^* the *balance solution*.

Theorem 7.2.1 is proved in Schofield (2006b).

Theorem 7.2.1. Consider the electoral model $\mathbb{E}(\boldsymbol{\lambda}, \boldsymbol{\mu}; \beta; \Psi)$ based on the Type I extreme value distribution, and including both exogenous and activist valences. The first-order condition for \mathbf{z}^* to be an LSNE is that it is a balance solution. If all activist valence functions are highly concave, in the sense of having negative eigenvalues of sufficiently great magnitude, then the balance solution will be a PNE.

When the valence functions $\{\mu_j\}$ are non-zero, then it is the case that generically \mathbf{z}_0 cannot satisfy the first-order condition. Instead the vector $\frac{d\mu_j}{dz_j}$ “points towards” the position at which the activist valence is maximized. When this marginal or gradient vector, $\frac{d\mu_j}{dz_j}$, is increased, (if activists become more willing to contribute to the party) then the equilibrium position is pulled away from the weighted electoral mean of party j , and we can say the “activist effect” for the party is increased. On the other hand if the activist valence functions are fixed, but the exogenous valence, λ_j , is increased, or the exogenous valence terms $\{\lambda_k : k \neq j\}$ are decreased, then the vector $\frac{d\mathcal{E}_j^*}{dz_j}$ increases in magnitude, and the equilibrium is pulled towards the weighted electoral mean. We can say the “electoral effect” is increased.

The second-order condition for an LSNE at \mathbf{z}^* depends on the negative definiteness of the Hessian of the activist valence function. If the eigenvalues of these Hessians are negative at a balance solution, and of sufficient magnitude, then this will guarantee that a vector \mathbf{z}^* that satisfies the balance condition will be an LSNE. Indeed, this condition can ensure concavity of the vote-share functions, and thus of existence of a PNE.

In the case that all activist valence functions $\{\mu_j\}$ are identically zero, we write the electoral model as $\mathbb{E}(\boldsymbol{\lambda}; \beta; \Psi)$. Then by Theorem 7.2.1, the coefficients, α_{ij} , are independent of i . Thus, when there is only exogenous valence, the balance condition gives

$$z_j^* = \frac{1}{v} \sum_{i=1}^v x_i. \quad (7.10)$$

By a change of coordinates we can choose $\sum x_i = 0$. In this case, the marginal electoral pull is zero at the origin and the joint origin $\mathbf{z}_0 = (0, \dots, 0)$ satisfies the first-order condition.

To characterize the variation in voter preferences, we represent in a simple form the covariance matrix, ∇_0 , given by the distribution of voter bliss points.

Definition 7.2.5. The Electoral Covariance Matrix, ∇_0 .

Let $W = \mathbb{R}^w$ be endowed with a system of coordinate axes $r =$

$1, \dots, w$. For each coordinate axis let $\xi_r = (x_{1r}, x_{2r}, \dots, x_{vr})$ be the vector of the r^{th} coordinates of the set of v voter bliss points. The scalar product of ξ_r and ξ_s is denoted by (ξ_r, ξ_s) .

(i) The symmetric $w \times w$ electoral covariance matrix about the origin is denoted ∇_0 and is defined by

$$\nabla_0 = \frac{1}{v} [(\xi_r, \xi_s)]_{r,s=1,\dots,w}^{r=1,\dots,w}.$$

(ii) Let $(\sigma_r, \sigma_s) = \frac{1}{v}(\xi_r, \xi_s)$ be the electoral covariance between the r^{th} and s^{th} axes, and $\sigma_s^2 = \frac{1}{v}(\xi_s, \xi_s)$ be the electoral variance on the s^{th} axis, with

$$\sigma^2 = \sum_{s=1}^w \sigma_s^2 = \frac{1}{v} \sum_{s=1}^w (\xi_s, \xi_s) = \text{trace}(\nabla_0)$$

the total electoral variance.

At the vector $\mathbf{z}_0 = (0, \dots, 0)$ the probability $\rho_{ij}(\mathbf{z}_0)$ that i votes for party j is independent of i , and is given by

$$\rho_j = \left[1 + \sum_{k \neq j} \exp[\lambda_k - \lambda_j] \right]^{-1}. \quad (7.11)$$

Definition 7.2.6: The Convergence Coefficient of the Model $\mathbb{E}(\boldsymbol{\lambda}; \beta; \Psi)$.

(i) The coefficient A_j for party j is

$$A_j = \beta(1 - 2\rho_j). \quad (7.12)$$

(ii) The characteristic matrix for party j is

$$C_j = [2A_j \nabla_0 - I], \quad (7.13)$$

where I is the w by w identity matrix.

(iii) The convergence coefficient of the model is

$$c(\boldsymbol{\lambda}; \beta; \Psi) = 2\beta[1 - 2\rho_1]\sigma^2 = 2A_1\sigma^2. \quad (7.14)$$

At the vector $\mathbf{z}_0 = (0, \dots, 0)$ the probability $\rho_{ij}(\mathbf{z}_0)$ that i votes for party j is independent of i , and is given by (7.11). Thus, if all valences are identical then $\rho_j = \frac{1}{n}$, for all j , as expected. The effect of increasing λ_j , for $j \neq 1$, is clearly to decrease ρ_1 , and therefore to increase A_1 , and thus $c(\boldsymbol{\lambda}; \beta; \Psi)$. Schofield (2007a) proves the following theorem.

Theorem 7.2.2. The necessary condition for the joint origin to be an LSNE in the model $\mathbb{E}(\boldsymbol{\lambda}; \beta; \Psi)$ is that the characteristic matrix

$$C_1 = [2A_1 \nabla_0 - I]$$

of the party 1, with lowest valence, has negative eigenvalues. ■

Theorem 7.2.2 immediately gives the following corollaries:

Corollary 7.2.1. Consider the model $\mathbb{E}(\boldsymbol{\lambda}; \beta; \Psi)$. In the case that X is w -dimensional, then the necessary condition for the joint origin to be an LNE is that $c(\boldsymbol{\lambda}; \beta; \Psi) \leq w$. ■

Ceteris paribus, an LNE at the joint origin is “less likely” the greater are the parameters β , $\lambda_p - \lambda_1$ and σ^2 .

Corollary 7.2.2. In the two-dimensional case, a sufficient condition for the joint origin to be an LSNE for the model $\mathbb{E}(\boldsymbol{\lambda}; \beta; \Psi)$ is that $c(\boldsymbol{\lambda}; \beta; \Psi) < 1$. ■

It is evident that sufficient conditions for existence of an LSNE at the joint origin in higher dimensions can be obtained using standard results on the determinants, $\{\det(C_j)\}$, and traces, $\{\text{trace}(C_j)\}$, of the characteristic matrices.

Notice that the case with two parties of equal valence immediately gives a situation with $2\beta[1 - 2\rho_1]\sigma^2 = 0$, irrespective of the other parameters. However, if $\lambda_2 > \lambda_1$, then the joint origin may fail to be an LNE if $\beta\sigma^2$ is sufficiently large.

Corollary 7.2.3. In the case that W is w -dimensional and there are two parties, with $\lambda_2 > \lambda_1$, then the joint origin fails to be an LNE if $\beta > \beta_0$ where

$$\beta_0 = \frac{w[\exp(\lambda_2 - \lambda_1) + 1]}{2\sigma^2[\exp(\lambda_2 - \lambda_1) - 1]}. \quad (7.15)$$

Proof. This follows immediately using $\rho_1 = [1 + \exp[\lambda_2 - \lambda_1]]^{-1}$. ■

It follows that if $\lambda_2 = \lambda_1$ then $\beta_0 = \infty$. Since β is finite, then the necessary condition for an LNE must be satisfied.

Example 7.2.1. We can illustrate these corollaries, in the case the necessary condition fails, by assuming that W is a compact interval, $[-a, +a] \subset \mathbb{R}$. Suppose further that there are three voters at $x_1 = -1, x_2 = 0$ and $x_3 = +1$. Then $v^2 = \frac{2}{3}$. Suppose that $\lambda_2 > \lambda_1$ and $\beta > \beta_0$ where β_0 is as above. Then \mathbf{z}_0 fails to be an equilibrium, and party 1 must move z_1 away from the origin, either towards x_1 or x_3 .

To see this suppose $\lambda_2 = 1$ and $\lambda_1 = 0$, so $\beta_0 = 1.62$. If $\beta = 2.0$, then the condition fails, since we find that at $\mathbf{z}_0 = (0, 0)$, for each i ,

$$\rho_{i1}(\mathbf{z}_0) = [1 + [\exp(1)]]^{-1} = 0.269,$$

so $E_1(\mathbf{z}_0) = 0.269$.

Now consider $\mathbf{z} = (z_1, z_2) = (+0.5, 0)$. We find

$$\rho_{11} = [1 + [\exp(3.5)]]^{-1} = 0.029,$$

while

$$\rho_{21} = [1 + [\exp(1.5)]]^{-1} = 0.182$$

and

$$\rho_{31} = [1 + [\exp(1 - 1.5)]]^{-1} = 0.622.$$

Thus the vote share for 1 is

$$E_1(\mathbf{z}) = \frac{1}{3}[0.029 + 0.182 + 0.622] = 0.277.$$

Hence candidate 1 can slightly increase vote-share by moving away from the origin. Obviously the joint origin cannot be an equilibrium. The gain from such a move is bigger the greater is $\lambda_2 - \lambda_1$ and β .

Example 7.1.2. in Two Dimensions. Now consider the two-dimensional case with $x_1 = (-1, 0), x_2 = (0, 0), x_3 = (+1, 0), x_4 = (0, 1), x_5 = (0, -1)$.

It follows that $\xi_1 = (-1, 0, +1, 0, 0)$, and $\xi_2 = (0, 0, 0, +1, -1)$. The electoral covariance matrix is then

$$\nabla_0 = \frac{1}{5} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad (7.16)$$

so $\sigma^2 = \frac{4}{5}$. The crucial condition for a local equilibrium at the origin is that the Hessian of the vote-share function of player 1 has negative eigenvalues. The Hessian is given by the matrix

$$\frac{2\beta[1 - 2\rho_1]}{5} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \quad (7.17)$$

The necessary condition is that the trace of this matrix is negative. In fact, because of the symmetry of the example, the necessary condition on each eigenvalue becomes $2\beta[1 - 2\rho_1]\frac{2}{5} \leq 1$. This condition fails if $2\beta[1 - 2\rho_1]\frac{2}{5} > 1$, in which case both eigenvalues will be positive. Thus, if $\beta > \beta_1$, where

$$\beta_1 = \frac{5}{4(1 - 2\rho_1)} = \frac{5[\exp(\lambda_2 - \lambda_1) + 1]}{4[\exp(\lambda_2 - \lambda_1) - 1]},$$

then the electoral origin is a minimum of the vote-share function of player 1. Thus player 1 can move away from the origin, *in any direction*, to increase vote-share.

Schofield (2007a) and Schofield and Sened (2006) show that typically there will exist a “principal” high variance electoral axis. Simulation of empirical models with exogenous valence and n parties shows that the lowest valence player will move away from the origin on this axis when $c(\lambda; \beta; \Psi) > w$. In this case, an LSNE will exist, but not at the electoral origin, and will satisfy the condition $\|z_1^*\| > \|z_n^*\|$. In other words, in equilibrium, the highest valence party will adopt a position closer to the electoral origin, while parties with lower valence will move to the electoral periphery.

These two simple examples provide the justification of the assertion made in the second section of this chapter that when λ_2 and λ_1 are substantially different, in terms of v^2 and β , then the joint origin becomes unstable. Note, however, that the joint origin will be an equilibrium as long as λ_2 and λ_1 are similar, or σ^2 and β are small enough.

In the following section we consider models for Israel, Turkey, the Netherlands and Britain. For Israel and Turkey, the vector of estimated

party valences and the other estimated parameters were such that electoral origin could not be an LNE. Indeed, the pattern of party positions in Israel in 1996 can be shown to be similar to a non-convergent LNE, based on the empirical parameters of the vote-maximizing game with endogenous valence. The details of the empirical model for Turkey in the elections of 1999 and 2002 are presented in an Appendix to this chapter. Again, convergence could not be expected. On the other hand, for the Netherlands and Britain, it is shown that the parameters of the models imply that the sufficient conditions for an LSNE at the joint origin, in the model $\mathbb{E}(\lambda; \beta; \Psi)$, was satisfied. Indeed the eigenvalues were sufficiently negative so as to imply that the joint origin was the unique PSNE. Since the parties did not appear to be positioned at the origin, the likely explanation is that the activist valence functions were significant.

7.3 Empirical Analyses

7.3.1 Elections in Israel

Figure 6.3 in the previous chapter showed the estimated positions of the parties in the Israel Knesset, and the electoral distribution, at the time of the 1996 election. Table 7.1 presents summary statistics of this election.

The table also shows the valence estimates, based on a multinomial logit model using the assumption of a Type I extreme value distribution on the errors.²⁸ The two dimensions of policy deal with attitudes to the PLO (the horizontal axis) and religion (the vertical axis). The policy space was derived from a voter survey (obtained by Arian and Shamir, 1999) and the party positions from analysis of party manifestos (Schofield, Sened and Nixon, 1998; Schofield and Sened, 2006). Using the formal analysis, we can readily show that the *convergence coefficient* of the model greatly exceeds 2 (the dimension of the policy space). Indeed, one of the eigenvalues of the Hessian of the low valence party, the NRP (also called Mafdal), can be shown to be positive. Indeed it is obvious that there is a principal component of the electoral distribution, and this axis is the eigenspace of the positive eigenvalue. It follows that low

²⁸This estimated model correctly predicts 63.8 percent of the voter choices. The log marginal likelihood of the model was -465 .

valence parties should position themselves close to this principal axis, as illustrated in the simulation given below in Figure 7.2.

Table 7.1. Vote shares and seats in the Knesset

Party	1996 National %	1996 Sample %	1996 Seats	1996 Valence
Others Left	7.3	0	9	
Meretz	7.6	6.0	9	0
Labor	27.5	44.0	34	4.15
Olim, Third Way	9.0	1.8	13	- 2.34
Likud	25.8	43.0	30	3.14
Shas	8.7	2.0	10	- 2.96
NRP (Mafdal)	8.0	5.1	9	- 4.52
Moledet	2.4	1.8	2	- 0.89
Others Right	3.7	0	4	

In 1996, the lowest valence party was the NRP with valence -4.52 . The spatial coefficient is $\beta = 1.12$, so to use Theorem 7.2.2, we note that the valence difference between the NRP and Labor is $4.15 - (-4.52) = 8.67$, while the difference between the NRP and Likud is $3.14 - (-4.52) = 7.66$. Since the electoral variance on the first axis is 1.0, and on the second axis it is 0.732, with covariance 0.591, we can compute the characteristic matrix of the NRP at the origin as follows:

$$\rho_{NRP} \simeq \frac{1}{1 + e^{4.15+4.52} + e^{3.14+4.52}} \simeq 0.$$

$$A_{NRP} \simeq \beta = 1.12.$$

$$C_{NRP} = 2(1.12) \begin{pmatrix} 1.0 & 0.591 \\ 0.591 & 0.732 \end{pmatrix} - I = \begin{pmatrix} 1.24 & 1.32 \\ 1.32 & 0.64 \end{pmatrix}.$$

(As in Definition 7.2.6, I is the 2 by 2 identity matrix.)

From the estimate of C_{NRP} it follows that the two eigenvalues are 2.28 and -0.40 , giving a saddlepoint, and a value of 3.88 for the convergence coefficient. This exceeds the necessary upper bound of 2. The major eigenvector for the NRP is (1.0, 0.8), and along this axis the NRP vote-share function increases as the party moves away from the origin. The minor, perpendicular axis associated with the negative eigenvalue is given

by the vector $(1, -1.25)$. Figure 7.2 gives one of the local equilibria in 1996, obtained by simulation of the model. The figure makes it clear that the local equilibrium positions of all parties lie close to the principal axis through the origin and the point $(1.0, 0.8)$. In all, five different LNE were located. However, in all equilibria, the two high valence parties, Labor and Likud, were located close to the positions given in Figure 7.2. The only difference between the various equilibria was that the positions of the low valence parties were perturbations of each other.

It is evident that if the high valence party occupies the electoral origin, then all parties with lower valence can compute that their vote-share will increase by moving up or down the principal electoral axis. In seeking local maxima of the vote shares all parties other than the highest valence party should vacate the electoral center. Then, however, the first-order condition for the high valence party to occupy the electoral center would not be satisfied. Even though this party's vote-share will be little affected by the other parties, it too should move from the center. The simulation illustrated in Figure 7.2 make it clear that there is a correlation between a party's valence and the distance of the party's equilibrium position from the electoral mean. A similar analysis is given in Schofield and Sened (2006) for the elections of 1992 and 1988.

The simulation for 1996 is compatible with the formal analysis: low valence parties, such as the NRP and Shas, in order to maximize vote shares must move far from the electoral center. Their optimal positions will lie either in the "north-east" quadrant or the "south-west" quadrant. The vote-maximizing model, without any additional information, cannot determine which way the low valence parties should move.

In contrast, since the valence difference between Labor and Likud was relatively low, their local equilibrium positions are close to, but not identical to, the electoral mean. Intuitively it is clear that once the low valence parties vacate the origin, then high valence parties, like Likud and Labor, should position themselves almost symmetrically about the origin, and along the principal axis.

Clearly, the configuration of equilibrium party positions will fluctuate as the valence differences of the parties change in response to exogenous shocks. The logic of the model remains valid, however, since the low valence parties will be obliged to adopt relatively "radical" positions in order to maximize their vote shares.

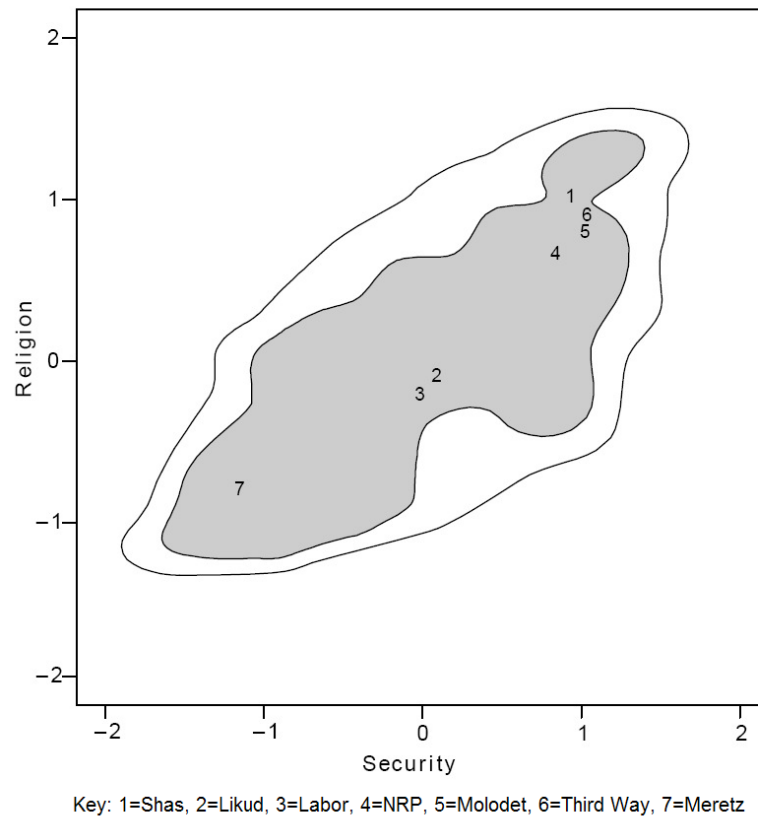


Figure 7.2: A local Nash equilibrium in the Knesset in 1996

There is a disparity between the estimated party positions in 1996 given in Figure 6.3 and the simulated equilibrium positions given in Figure 7.2. The two religious parties, Shas and Yahadut, are estimated to be far from the principal axis, seeming in contradiction to the prediction of the stochastic model. Moreover, the high valence parties, Labor and Likud appear further from the origin than suggested by the simulation. This disparity may be accounted for by modifying the assumption that valence is exogenous, and by allowing for the influence of activists on party position. This version of the model is discussed further in Chapter 8.

7.3.2 Elections in Turkey 1999-2007

In empirical analysis it is difficult to estimate the activist valence functions. However, it is possible to use socio-demographic variables as proxies. Instead of using (7.1) as the estimator for voter utility, we can use the expression

$$u_{ij}(x_i, z_j) = \lambda_j - \beta \|x_i - z_j\|^2 + \theta_j^T \eta_i + \varepsilon_j. \quad (7.18)$$

where the k -vector θ_j represents the effect of the k different socio-demographic parameters (class, domicile, education, income, etc.) on voting for party j while η_i is a k -vector denoting the i^{th} individual's relevant "socio-demographic" characteristics. We use θ_j^T to denote the transpose of θ_j so $\theta_j^T \eta_i$ is a scalar. When β and $\{\lambda_j\}$ are assumed zero then we call the model pure *socio-demographic* (SD). When $\theta_j^T \eta_i$ are assumed zero then the model is called *pure spatial*, and when all parameters are included then the model is called *joint*. The differences in log marginal likelihoods for two models then gives the Log Bayes' factor for the pairwise comparison.²⁹ This technique can then be used to determine which is the superior model. We can use this model to explain Turkish election results in 1999 and 2002, given in Tables 7.2 and 7.3. Figures 7.3 and 7.5 show the electoral distributions (based on sample surveys of sizes 635 and 483, respectively) and estimates of party positions for 1999 and 2002.

Minor differences between these two figures include the disappear-

²⁹Since the Bayes' factor for a comparison of two models is simply the ratio of marginal likelihoods, the log of the Bayes factor is the difference in log likelihoods.

ance of the Virtue Party (FP) which was banned by the Constitutional Court in 2001, and the change of the name of the Kurdish party from HADEP to DEHAP. The most important change is the appearance of the new Justice and Development Party (AKP) in 2002. This latter party obtained about 35 percent of the vote and 363 seats out of 550 seats in 2002. Figure 7.4 presents an estimate of the heart in 1999. In 1999, a DSP minority government formed, supported by ANAP and DYP. This only lasted about 4 months, and was replaced by a DSP-ANAP-MHP coalition. During the period 1999–2002, Turkey experienced two severe economic crises. As Tables 7.2 and 7.3 show, the vote shares of the parties in the governing coalition went from about 53 percent in 1999 to 15 percent in 2002. In 2002, AKP obtained a majority and so was a core party. Tables 7.5 and 7.6, in the empirical Appendix, present the results of a MNL estimation of these elections.³⁰ The estimations include various socio-demographic characteristics such as religious orientation. Note in particular that one of the socio-demographics is “Alevi.” Alevi belong to a non-Sunni religious/cultural community living in Turkey and in some parts of Iran, and comprise 13-15 percent of the Turkish population. They are closer to Shia Islam than Sunni Islam but the majority of Shia and Sunni do not regard Alevi as Muslims. Alevi tend to support “Kemalism” and the secular state and vote for CHP, leading to a Sunni-Alevi tension in the society.

The Log Bayes’ factors reported in the Appendix show that the joint MNL model is superior to all others. It is noticeable that the valences of the ANAP and MHP dropped from 1999 to 2002. In 1999, the estimated λ_{ANAP} was -0.114 , whereas in 2002 it was -0.567 , while λ_{MHP} fell from 2.447 to 1.714 . The estimated valence, λ_{AKP} , of the new Justice and Development Party (AKP) in 2002 was 1.968 , which we might ascribe to the disillusion of most voters with the other parties, as well as the charisma of Recep Tayyip Erdogan, leader of the AKP.³¹ Notice that the β coefficient was 0.456 in 1999, and 1.445 in 2002, suggesting that electoral preferences over policy had become more intense. The es-

³⁰The estimation is based on a factor analysis of a sample survey conducted by Veri Arastirma for TUSES.

³¹Abdullah Gul became Prime Minister after the November 2002 election because Erdogan was banned from holding office. Erdogan took over as Prime Minister after winning a by-election in March 2003.

timated convergence coefficient, c , given in Definition 7.2.6, is 2.014 for 1999 and 6.48 in 2002, giving a formal reason why convergence should not occur in these elections.

Table 7.2 Turkish election results 1999

Party Name	.	% Vote	Seats	% Seats
Democratic Left Party	DSP	22.19	136	25
Nationalist Action Party	MHP	17.98	129	23
Virtue Party	FP	15.41	111	20
Motherland Party	ANAP	13.22	86	16
True Path Party	DYP	12.01	85	15
Republican People's Party	CHP	8.71	-	-
People's Democracy Party	HADEP	4.75	-	-
Others	-	4.86	-	-
Independents	-	0.87	3	1
Total			550	
Convergence Coefficient=2.014				

Table 7.3 Turkish election results 2002

Party Name		% Vote	Seats	% Seats
Justice and Development Party	AKP	34.28	363	66
Republican People's Party	CHP	19.39	178	32
True Path Party	DYP	9.54	-	-
Nationalist Action Party	MHP	8.36	-	-
Young Party	GP	7.25	-	-
People's Democracy Party	DEHAP	6.22	-	-
Motherland Party	ANAP	5.13	-	-
Felicity Party	SP	2.49	-	-
Democratic Left Party	DSP	1.22	-	-
Others and Independents	-	6.12	9	2
Total	-		550	
Convergence coefficient = 6.48				

To compute the convergence coefficients, we proceed as follows.

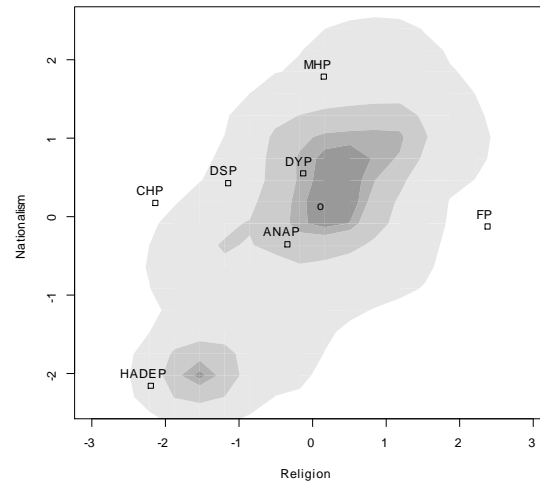


Figure 7.3: Party positions and voter distribution in Turkey in 1999

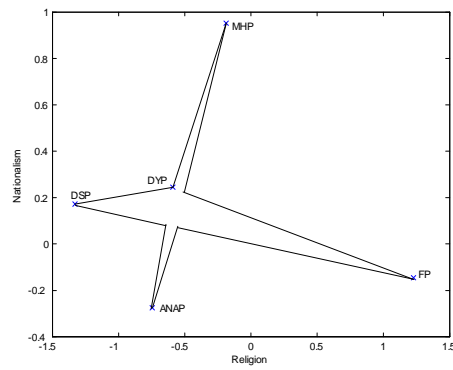


Figure 7.4: The heart in Turkey in 1999

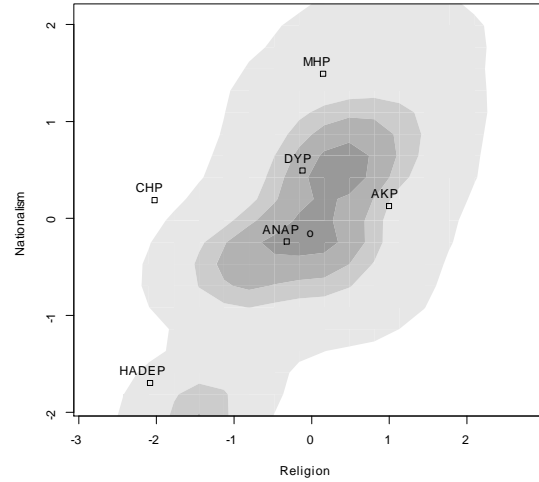


Figure 7.5: Party positions and voter distribution in Turkey in 2002

The 1999 Election The empirical model given in Table 7.5 estimates the electoral variance on the first axis (religion) to be 1.20 while on the second axis (nationalism) the electoral variance is 1.14, with the covariance between the two axes equal to +0.78.

The electoral covariance matrix is the 2 by 2 matrix

$$\nabla_0 = \begin{bmatrix} 1.2 & 0.78 \\ 0.78 & 1.14 \end{bmatrix}.$$

As Table 7.5 shows, the β coefficient is 0.456, while the party with the lowest valence is CHP with $\lambda_{CHP} = -0.673$.

When all parties are located at the origin, the probability, ρ_{CHP} , that a voter chooses CHP is equal to

$$[1 + \exp(0.559) + \exp(1.136) + \exp(1.688) + \exp(0.059) + \exp(3.1)]^{-1} = 0.028.$$

The characteristic matrix of the CHP is

$$C_{CHP} = [2\beta(1 - 2\rho_{CHP})(\nabla_0) - I].$$

$$\begin{aligned}\text{Now } \beta(1 - 2\rho_{CHP}) &= 0.456 \times 0.944 \\ &= 0.4305\end{aligned}$$

$$\text{Thus } C_{CHP} = \begin{bmatrix} 0.033 & +0.674 \\ 0.674 & -0.019 \end{bmatrix}.$$

Again, I is the 2 by 2 identity matrix. The Theorem also shows that the *necessary* condition for convergence is that the convergence coefficient,

$$c = \beta[1 - 2\rho_{CHP}]trace(\nabla_0)$$

is bounded above by 2.

But $trace(\nabla_0) = 2.34$, so

$$c = 2 \times 0.4305 \times 2.34 = 2.014.$$

Since $c > 2.0$, we know that one eigenvalue of C_{CHP} must be negative. It is easy to show that the eigenvalues of C_{CHP} are $+0.679$ and -0.664 . The eigenvector corresponding to the positive eigenvalue is $(+1.04, +1.0)$ while the second minor eigenvector is $(+1.0, -1.04)$.

The first eigenvector corresponds to the principal electoral component, or eigenspace, aligned at approximately 45 degrees to the religion axis. On this principal axis, the vote share of the CHP increases as it moves away from the electoral origin. The minor, perpendicular axis is aligned at right angles to the first, and on this axis, the vote share of the CHP decreases as it moves away from the origin. Clearly the origin is a saddlepoint, and we can expect all parties to align themselves close to the principal axis. Many of the parties are so aligned in Figure 7.4. The fact that some of the parties are located off this axis can be attributed to the influence of activists.

The 2002 Election The empirical model given in Table 7.6 estimates the electoral variance on the first axis (religion) to be 1.18 while the electoral variance is on the second axis 1.15, with the covariance between the two axes equal to 0.74.

Thus

$$\nabla_0 = \begin{bmatrix} 1.18 & 0.74 \\ 0.74 & 1.15 \end{bmatrix}$$

with $trace(\nabla_0) = 2.33$.

As Table 7.6 shows, the β coefficient is 1.445, while the party with the lowest valence is ANAP with $\lambda_{ANAP} = -0.567$.

Because we use the Type I distribution, when all parties are located at the origin, the probability, ρ_{ANAP} , that a voter chooses ANAP is given by

$$\begin{aligned} & [1 + \exp(2.535) + \exp(1.67) + \exp(3.163) + \exp(2.281)]^{-1} \\ & = 0.019, \end{aligned}$$

The Hessian of the vote share function of ANAP (when all parties are at the origin) is

$$\begin{aligned} C_{ANAP} &= [2\beta(1 - 2\rho_{ANAP})(\nabla_0) - I]. \\ \text{Now } \beta(1 - 2\rho_{ANAP}) &= 1.39, \text{ so} \\ C_{CHP} &= \begin{bmatrix} 2.28 & 2.06 \\ 2.06 & 2.20 \end{bmatrix}. \end{aligned}$$

The convergence coefficient is now given by

$$\begin{aligned} c &= \beta[1 - 2\rho_{ANAP}]trace(\nabla_0) \\ &= 2 \times 1.39 \times 2.33 = 6.48. \end{aligned}$$

This greatly exceeds the upper bound of 2.0 for convergence to the electoral origin. The major eigenvalue for the ANAP is 4.30, with eigenvector $(+1.10, +1.0)$ while the minor eigenvalue is 0.18, with orthogonal eigenvector $(-1.0, +1.10)$. In this case, the electoral origin is a minimum of the vote share function of ANAP. As before, the first eigenvector corresponds to the principal electoral component, or eigenspace, alligned at approximately 45 degrees to the religion axis. On both principal and minor axis, the vote share of ANAP increases as it moves away from the electoral origin, but because the major eigenvalue is much larger than the minor, we can expect some of the parties in equilibrium to adopt positions far from the principal electoral axis. Figure 7.5 is consistent with this inference.

In the 2007 election, the Kurdish Party (now called the Freedom and Solidarity Party, DTP) contested the election as independents, and thus were not subject to the 10 percent cut-off, and were able to win 24 seats. As Table 7.4 shows, the AKP took 46.6 percent of the vote and 340 seats, reflecting the continuing high valence of Erdogan. Abdullah Gul, Erdogan's ally in the AKP and a practising Muslim who has been Turkey's foreign minister for over four years, was elected as the country's 11th

president on 28 August, despite strong opposition from the army and militant secularists.³²

Table 7.4 Turkish election results 2007

Party Name		% Vote	Seats	% Seats
Justice and Development Party	AKP	46.6	340	61.8
Republican People's Party	CHP	20.9	112	20.3
Nationalist Movement Party	MHP	14.3	71	12.9
Democrat Party	DP	5.4	-	-
Young Party	GP	3.0	-	-
Felicity Party	SP	2.3	-	-
Independents	-	5.2	27 ³³	4.9
Others	-	2.3	-	-
Total		100	550	100

7.3.3 Elections in the Netherlands

First, we consider a multinomial logit (MNL) model for the elections of 1977 and 1981 in the Netherlands (Schofield, Martin, Quinn and Whitford, 1998; Quinn, Martin and Whitford, 1999) using data from the middle level Elites Study (ISEIUM, 1983) coupled with the Rabier and Inglehart (1981) Euro-barometer voter survey. There are four main parties: Labor (PvdA), Christian Democratic Appeal (CDA), Liberals (VVD) and Democrats (D'66), with approximately 35 percent, 35 percent, 20 percent and 10 percent of the popular vote. Table 6.6 in Chapter 6 gave the National Vote shares for the parties in 1977 and 1981, as well as the sample vote-share from the Eurobarometer survey. The table also gave the valences for an MNL model based on the positions of the parties as shown in Figure 7.6. This figure gives the estimated positions of the parties based on the middle level Elites Study. As in Figure 6.3, an estimate of the density contours of the electoral distribution of voter bliss points is also shown, based on the voter survey.

The empirical model estimated exogenous valences, which were normalized, by choosing the D'66 to have exogenous valence $\lambda_{D66} = 0$. The

³²*The Independent*, 29 September 2007.

³³Twenty-four of these "independents" were in fact members of the DTP—the Kurdish Freedom and Solidarity Party.

other valences are $\lambda_{VVD} = 1.015$, $\lambda_{CDA} = 1.403$ and $\lambda_{PvdA} = 1.596$. To compute the D'66 Hessian, we note that the electoral variance on the first axis is $\sigma_1^2 = 0.658$, while on the second it is $\sigma_2^2 = 0.289$. The covariance (σ_1, σ_2) is negligible.

The spatial coefficient $\beta = 0.737$ for the model with exogenous valence. Thus the probability of voting for each of the parties, as well as the Hessians when all parties are at the origin, can be calculated as follows:

$$\begin{aligned}\rho_{D66} &= \frac{1}{1 + e^{1.015} + e^{1.403} + e^{1.596}} = 0.078. \\ A_{D66} &= 0.737 \times 0.844 = 0.622. \\ \text{Hence } C_{D66} &= 2A_{D66}\nabla_0 - I \\ &= (1.24) \begin{pmatrix} 0.658 & 0 \\ 0 & 0.289 \end{pmatrix} - I \\ &= \begin{pmatrix} -0.18 & 0 \\ 0 & -0.64 \end{pmatrix}, \\ \text{so } c &= 2 \times 0.622 \times 0.947 = 1.178.\end{aligned}$$

Although the convergence coefficient exceeds 1.0, so the sufficient condition, given by Corollary 7.2.2 is not satisfied, the necessary condition of Corollary 7.2.1 is satisfied, and the eigenvalues for the Hessian of D'66 can be seen to be negative. By Theorem 7.2.2, the joint origin is an LSNE for the stochastic model with exogenous valence.

In a similar way, we can compute the other probabilities, giving

$$(\rho_{D66}, \rho_{VVD}, \rho_{CDA}, \rho_{PvdA}) = (0.078, 0.217, 0.319, 0.386).$$

This vector can be identified as the expected vote shares of the parties when all occupy the electoral origin. Note also that these expected vote shares are very similar to the sample vote shares

$$(\mathcal{S}_{D66}^*, \mathcal{S}_{VVD}^*, \mathcal{S}_{CDA}^*, \mathcal{S}_{PvdA}^*) = (0.104, 0.189, 0.338, 0.369),$$

as well as the average of the national vote shares in the two elections.

$$(E_{D66}^*, E_{VVD}^*, E_{CDSA}^*, E_{PvdA}^*) = (0.094, 0.199, 0.356, 0.352).$$

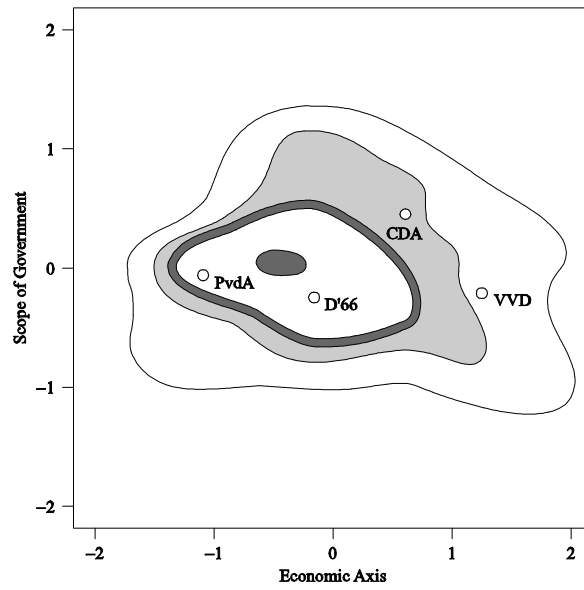


Figure 7.6: Party positions in the Netherlands

These national vote shares can be regarded as approximations of the expected vote shares. Quinn and Martin (2002) performed a simulation of the empirical model and showed that the joint origin was indeed a PSNE for the vote-maximizing model with the exogenous valence values estimated by the MNL model. Moreover, the positions given in Figure 7.6 could not be an LSNE of the stochastic model with exogenous valence alone. This conflict between the predicted equilibrium positions of the model and the estimated positions suggest that the activists for the parties played an important role in determining the party positions. Although we do not have data available on the activist valences for the parties, these empirical results indicate that Theorem 7.2.1 is compatible with the following two hypotheses:

- (i) the party positions given in Figure 7.6 are a close approximation to the actual positions of the parties;
- (ii) each party was at a Nash equilibrium position in an electoral contest involving a balance for each party between the centripetal marginal electoral pull for the party and the centrifugal marginal activist pull on the party.

We now examine this possibility further in the case of recent elections in Britain.

7.3.4 The Election in the United Kingdom in 1997

Figure 7.7 shows the estimated positions of the parties, based on a survey of Party MPs in 1997 (Schofield, 2005a,b). In addition to the Conservative Party (CONS), Labor³⁴ Party (LAB) and Liberal Democrat Party (LIB) responses were obtained from Ulster Unionists (UU), Scottish Nationalists (SNP) and Plaid Cymru (PC). The first axis is economic, the second axis concerned attitudes to the European Union (pro to the “south” of the vertical axis). The electoral model with exogenous valence was estimated for the election in 1997.

For 1997, $(\lambda_{con}, \lambda_{lab}, \lambda_{lib}, \beta)_{1997} = (+1.24, 0.97, 0.0, 0.5)$ so

$$\rho_{lib} = \frac{e^0}{e^0 + e^{1.24} + e^{0.97}} = \frac{1}{7.08} = 0.14.$$

³⁴We use the U.S. spelling for this party.

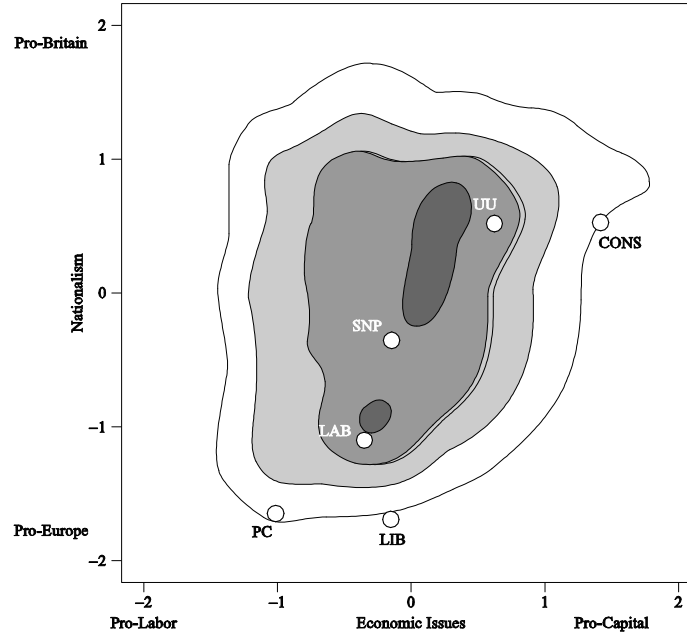


Figure 7.7: Party positions in the United Kingdom

Since the electoral variance is 1.0 on the first economic axis and 1.5 on the European axis, we obtain

$$A_{lib} = \beta(1 - 2\rho_{lib}) = 0.36 \text{ and}$$

$$C_{lib} = (0.72) \begin{pmatrix} 1.0 & 0 \\ 0 & 1.5 \end{pmatrix} - I = \begin{pmatrix} -0.28 & 0 \\ 0 & +0.08 \end{pmatrix}.$$

The convergence coefficient can be calculated to be 1.8. Although the necessary condition is satisfied, the origin is clearly a saddlepoint for the Liberal Democrat Party. Note that the second “European” axis is a “principal electoral axis” exhibiting greater electoral variance. This axis is the eigenvector associated with the positive eigenvalue. Because the covariance between the two electoral axes is negligible, we can infer that, for each party, the eigenvalue of the Hessian at the origin is negative on the first or minor “economic” axis. According to the formal model with

exogenous valence, all parties should have converged to the origin on this minor axis. Because the eigenvalue for the Liberal Democrat Party is positive on the second axis, we have an explanation for its position away from the origin on the Europe axis in Figure 7.7. However there is no explanation for the location of the Conservative Party so far from the origin on both axes. Schofield (2005a,b) offers a model (based on an earlier version of Theorem 7.2.2) where the falling exogenous valence of the Conservative Party leader increases the marginal importance of two opposed activist groups in the party: one group “pro-capital” and one group “anti-Europe.” The next section comments on this observation.

7.4 The Influence of Activists

The empirical analysis discussed in the previous section showed that overall Conservative valence dropped from 1.58 in 1992 to 1.24 in 1997, while the Labor valence increased from 0.58 to 0.97. These estimated valences include both exogenous valence terms for the parties and the activist component. Recent studies of these elections³⁵ suggest that when Tony Blair took over from John Smith as leader of the Labor Party, then the exogenous valence, λ_{lab} , of the party increased up to the 1997 election. Conversely, the exogenous valence, λ_{con} , for the Conservatives fell. Since the coefficients in the equation for the electoral pull for the Conservative Party depend on $\lambda_{con} - \lambda_{lab}$, Theorem 7.2.1 implies that the effect would be to increase the marginal effect of activism for the Conservative Party, thus pulling the optimal position away from the party’s weighted electoral mean. The opposite conclusion holds for the Labor Party, since increasing $\lambda_{lab} - \lambda_{con}$ has the effect of reducing the marginal activist effect.

Indeed, it is possible to include the effect of two potential activist groups for the Labor Party: one “pro-Europe,” called E, and one “pro-labor,” called L. The optimal Labor position will be determined by a ver-

³⁵For an empirical analysis of the election of 2005 see Clarke, Sanders, Stewart and Whiteley (2006). For discussion of the nature of party competition in Britain from the early 1980s to the present see Whiteley (1983); Clarke, Stewart and Whiteley (1997, 1998); Clarke, Sanders, Stewart and Whiteley (2004); Seyd and Whiteley (1992, 2002); Whiteley and Seyd (2002); Whiteley, Seyd and Billingham (2006).

sion of the balance equation

$$\left[\frac{d\mathcal{E}_{lab}^*}{dz_{lab}} - z_{lab}^* \right] + \frac{1}{2\beta} \left[\frac{d\mu_{lab,L}}{dz_{lab}} + \frac{d\mu_{lab,E}}{dz_{lab}} \right] = 0 \quad (7.19)$$

which equates the “electoral pull” against the two “activist pulls,” generated by the two different activist functions, $\mu_{lab,L}$ and $\mu_{lab,E}$. In the same way, if there are two activist groups for the Conservatives, generated by functions $\mu_{con,C}$ and $\mu_{con,B}$ centered at C and B respectively, then we obtain a balance equation:

$$\left[\frac{d\mathcal{E}_{con}^*}{dz_{con}} - z_{con}^* \right] + \frac{1}{2\beta} \left[\frac{d\mu_{con,C}}{dz_{con}} + \frac{d\mu_{con,B}}{dz_{con}} \right] = 0. \quad (7.20)$$

Since the electoral pull for the Conservative Party fell between the elections, the optimal position, z_{con}^* , will be one which is “closer” to the locus of points that generates the greatest activist support. This locus is where the joint marginal activist pull is zero. This locus of points can be called the “activist contract curve” for the Conservative Party.

The next chapter develops an activist model of this kind, where preferences of different activists on the two dimensions may accord different saliences to the policy axes. The “activist contract curves” for the two parties will be catenaries that depend on the ratios of the saliences that different activists have on the two dimensions.

According to Theorem 7.2.1, the reason the Labor Party under Blair was able to move to a position closer to the origin between the elections of 1992 and 1997 was that his increasing valence reduced the importance of pro-labor activists in the party. On the other hand, the declining valences of the Conservative Party leaders, first William Hague, and then Iain Duncan Smith, increased the importance of the marginal activist effect for the party. This appears to have the effect of obliging the party to move to the fairly extreme position shown in Figure 7.7. Figure 7.8 gives a schematic representation of the balance loci of the two parties and the change in party positioning over recent decades.

It remains to be seen whether the new leader, David Cameron, can gain high enough valence in contrast to the new leader of Labor, Gordon Brown, to overcome the apparent dominant influence of anti-Europe activist sentiment in the party.

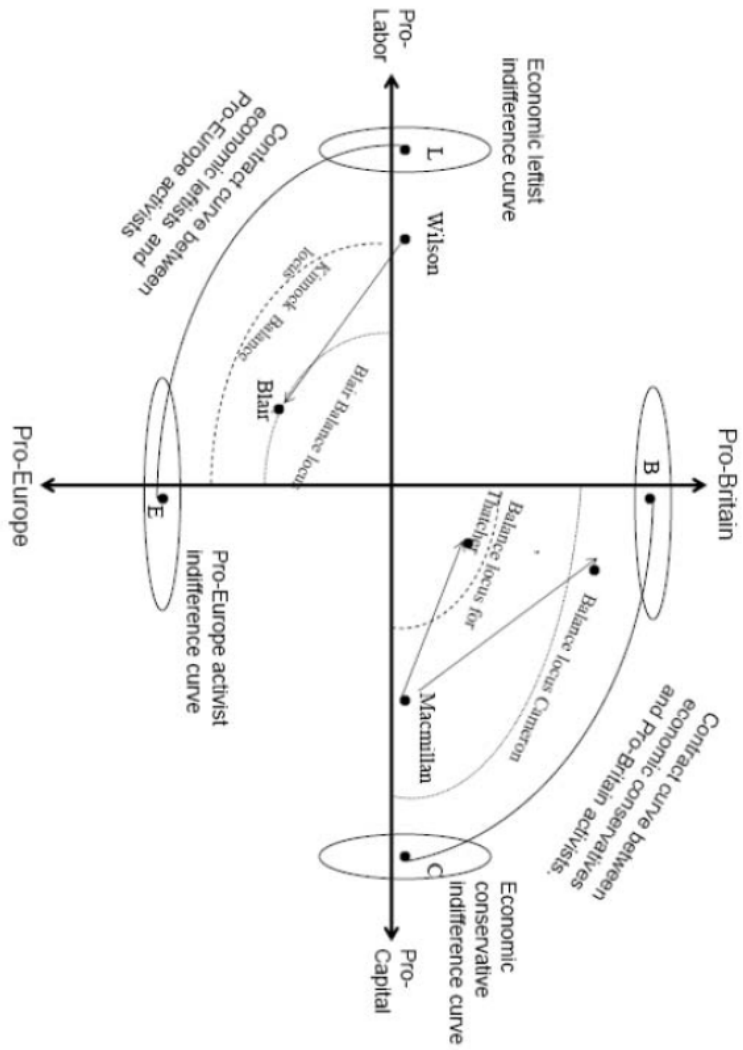


Figure 7.8: Balance loci for parties in Britain

7.5 Concluding Remarks

The above discussion of the possible role of activists is developed only for the case of two parties and two potential activist groups for the two parties. The model is developed further in the next two chapters. Theoretically it should be possible to carry out the analysis for any number of parties, and an arbitrary number of potential interest groups.

7.6 Empirical Appendix

Table 7.5. Multinomial Logit Analysis of the 1999 Election in Turkey,
(normalized with respect to DYP)

	Party	Posterior		95% Confidence Interval	
		Mean	Std Dev	Lower Bound	Upper Bound
Spatial Distance	β	0.456	0.104	0.243	0.648
λ (valence)	ANAP	-0.114	0.727	-1.513	1.227
	CHP	-0.673	0.770	-2.166	0.786
	DSP	0.463	0.720	-0.930	1.825
	FP	1.015	0.878	-0.709	2.755
	HADEP	-0.610	1.230	-3.004	1.803
	MHP	2.447	0.669	1.167	3.664
Age	ANAP	0.001	0.012	-0.021	0.023
	CHP	-0.009	0.013	-0.033	0.016
	DSP	-0.008	0.012	-0.031	0.014
	FP	-0.023	0.014	-0.050	0.003
	HADEP	-0.053	0.023	-0.103	-0.014
Education	MHP	-0.044	0.012	-0.067	-0.022
	ANAP	0.006	0.065	-0.115	0.130
	CHP	0.106	0.063	-0.012	0.232
	DSP	0.077	0.058	-0.024	0.197
	FP	-0.129	0.081	-0.285	0.018
Urban	HADEP	0.144	0.097	-0.038	0.335
	MHP	-0.060	0.061	-0.175	0.070
	ANAP	0.531	0.367	-0.156	1.279
	CHP	0.354	0.395	-0.374	1.078
	DSP	0.582	0.359	-0.147	1.271
Kurd	FP	0.417	0.416	-0.418	1.183
	HADEP	0.264	0.634	-0.918	1.497
	MHP	-0.201	0.378	-0.922	0.593
	ANAP	1.132	0.924	-0.410	3.138
	CHP	1.715	0.911	0.194	3.637
Soc. Econ. Status	DSP	-0.102	1.083	-2.650	2.098
	FP	1.116	0.972	-0.733	3.024
	HADEP	5.898	0.926	4.290	7.904
	MHP	0.063	0.933	-1.751	2.148
	ANAP	0.080	0.165	-0.302	0.394
Alevi	CHP	0.163	0.176	-0.195	0.499
	DSP	-0.010	0.158	-0.322	0.333
	FP	0.120	0.179	-0.230	0.458
	HADEP	-0.119	0.264	-0.598	0.384
	MHP	0.168	0.159	-0.147	0.469
	ANAP	-0.697	0.972	-2.687	1.168
	CHP	3.089	0.693	1.965	4.715
	DSP	0.934	0.729	-0.383	2.423
	FP	0.346	0.939	-1.374	2.007
	HADEP	1.355	0.972	-0.332	3.605
	MHP	-0.873	0.925	-3.225	0.676

$v=635$

Log marginal likelihood = -1177.938

Table 7.6 Multinomial Logit Analysis of the 2002 Election in Turkey
(normalized with respect to DYP)

	Party	Posterior		95% Confidence Interval	
		Mean	Std Dev	Lower Bound	Upper Bound
Spatial Distance	β	1.445	0.143	1.180	1.723
λ (valence)	AKP	1.968	0.667	0.708	3.432
	CHP	1.103	0.797	-0.579	2.615
	DEHAP	2.596	1.246	-0.254	5.049
	MHP	1.714	0.889	-0.021	3.426
	ANAP	-0.567	0.880	-2.487	1.133
Age	AKP	-0.031	0.011	-0.052	-0.010
	CHP	-0.019	0.013	-0.045	0.005
	DEHAP	-0.060	0.024	-0.110	-0.014
	MHP	-0.067	0.017	-0.103	-0.034
Education	ANAP	-0.004	0.014	-0.031	0.022
	AKP	-0.070	0.062	-0.185	0.045
	CHP	-0.007	0.068	-0.136	0.115
	DEHAP	-0.142	0.108	-0.365	0.079
	MHP	-0.048	0.079	-0.202	0.106
Urban	ANAP	-0.078	0.076	-0.237	0.064
	DYP	0.050	0.406	-0.770	0.844
	CHP	0.121	0.443	-0.744	1.001
	DEHAP	-1.138	0.688	-2.426	0.236
	MHP	-0.570	0.536	-1.649	0.504
Kurd	ANAP	0.661	0.479	-0.228	1.628
	AKP	2.086	1.105	0.203	4.596
	CHP	1.251	1.171	-0.891	3.839
	DEHAP	5.996	1.208	3.960	8.945
	MHP	1.595	1.312	-0.960	4.258
Soc. Econ. Status	ANAP	1.603	1.199	-0.535	4.358
	AKP	0.142	0.160	-0.160	0.457
	CHP	0.198	0.191	-0.196	0.560
	DEHAP	-0.217	0.281	-0.755	0.301
	MHP	0.317	0.204	-0.083	0.703
Alevi	ANAP	0.214	0.209	-0.182	0.613
	AKP	-0.249	0.983	-2.125	1.743
	CHP	2.567	0.817	1.111	4.489
	DEHAP	0.377	1.045	-1.519	2.540
	MHP	-0.529	1.410	-3.565	2.292
	ANAP	1.392	0.931	-0.323	3.560

 $v=483$

Log marginal likelihood = -731.94

Table 7.7 Log Bayes factors for model comparisons in 1999

\mathbb{E}_2	Joint	Spatial	Socio-Dem.
Joint	na	6.13*	31.34**
\mathbb{E}_1 Spatial	-6.13	na	25.20**
Socio-Dem.	-31.34	-25.20	na

** Strong support for \mathbb{E}_1 . * Positive support for \mathbb{E}_1 .

Table 7.8 Log Bayes factors for for model comparisons in 2002

\mathbb{E}_2	Joint	Spatial	Socio-Dem.
Joint	na	5.17*	58.17***
\mathbb{E}_1 Spatial	-5.17	na	52.99***
Socio-Dem.	-58.17	-52.99	na

***: Extremely strong support for \mathbb{E}_1 . * Positive support for \mathbb{E}_1 .

Chapter 8

Activist Coalitions

The analysis presented in Chapter 7 allows for comparative analysis of the model over a range of parameters, including the dimension and nature of the policy space, the importance of policy, the variation in voter's average perception of the relative quality of the various candidates, and the number of parties. Theorem 7.2 covers the specific situation when activist valence is identically zero, so that only exogenous valence is relevant. This theorem provides the necessary and sufficient conditions for convergence of all parties to the electoral center. It is proposed here that in Argentina in 1989, the necessary condition failed, leading to divergence of the positions of the two major parties. Once the parties were seen to adopt different positions, then activists were motivated to provide resources to the party most attractive to them. Such support then tends to drive the parties further apart.

Theorems 7.2.1 and 7.2.2 in the previous chapter suggest that PNE for a vote-maximizing game need not exhibit convergence of party position, particularly when activist influence is pronounced. This chapter applies Theorem 7.1 to an examination of how a party's equilibrium position will be affected when it responds to different activists groups with contradictory agendas. As the intensity of support from a group of activists increases, the party leader will consider the benefits of moving along a "balance locus" between them and an opposed group of activists. In particular, when there are two dimensions of policy, these strategic moves by the parties in response to activist support will induce a rotation of the party positions. These transformations bring about a change in the most salient dimension of policy, thus inducing a political "realignment."

The model is applied to the case of Argentinian elections in 1989–

1995, because of the deep transformations that occurred in a very short period of time. Indeed, between 1989 and 1995, Argentina's polity experienced: (i) the saliency of a new dimension, namely the value of its currency, (ii) a sharp change in the population's perception of the relative "quality" of the two major parties, the PJ and the UCR, and (iii) the emergence of a potent activist group, in the form of the recently privatized firms and their political allies.

8.1 Activist Support and Valence

To develop the model, consider competition between two parties, 1 and 2, in a policy space with $w = 2$, where party 1 has traditionally been on the left of the economic (x) axis, and party 2 on the right of the same axis. The model examines the effect of the second (y) axis of policy by using the work presented by Miller and Schofield (2003), based on "ellipsoidal" utility functions of potential activist groups. In the application to the Argentine polity presented in the next section, the y -axis will represent policy in support of a hard or a soft currency.

Consider the first-order equation $\frac{d\mu_1}{dz} = 0$, for maximizing the total valence of 1 when there are two activist groups, L, H, whose preferred points are, say, L , H , and whose utility functions are u_L and u_H . The contributions of the groups to party 1 are Σ_L and Σ_H . The model uses the following set of assumptions.

Assumption 8.1.1. Valence Functions.

- (i) The total activist valence for 1 can be decomposed into two components

$$\mu_1(z_1) = \mu_L(\Sigma_L(z_1)) + \mu_H(\Sigma_H(z_1)). \quad (8.1)$$

where μ_L, μ_H are functions of Σ_L, Σ_H , respectively.

- (ii) The contributions Σ_L, Σ_H can be written as functions of the utilities of the activist groups, so

$$\Sigma_L(z_1) = \Sigma_L(u_L(z_1)) \quad \text{and} \quad \Sigma_H(z_1) = \Sigma_H(u_H(z_1)). \quad (8.2)$$

Note that there is no presumption that these functions are linear.

(iii) The gradients of the contribution functions are given by

$$\frac{d\Sigma_L}{dz} = \alpha_L^*(z) \frac{du_L}{dz} \quad \text{and} \quad \frac{d\Sigma_H}{dz} = \alpha_H^*(z) \frac{du_H}{dz}. \quad (8.3)$$

The coefficients $\alpha_L^*(z), \alpha_H^*(z) > 0$, for all z , and are differentiable functions of z .

(iv) The gradients of the two valence functions satisfy

$$\frac{d\mu_L}{dz} = \alpha_L^{**}(z) \frac{d\Sigma_L}{dz} \quad \text{and} \quad \frac{d\mu_H}{dz} = \alpha_H^{**}(z) \frac{d\Sigma_H}{dz}, \quad (8.4)$$

where again the coefficients $\alpha_L^{**}(z), \alpha_H^{**}(z) > 0$, for all z , and are differentiable functions of z .

Under these assumptions, the first-order equation becomes

$$\frac{d\mu_1}{dz} = \left[\alpha_L(z) \frac{du_L}{dz} + \alpha_H(z) \frac{du_H}{dz} \right] = 0 \quad (8.5)$$

where $\alpha_L(z), \alpha_H(z) > 0$. Since these are assumed to be differentiable functions of z , this equation generates the smooth one-dimensional *contract curve* associated with the utility functions of the activist groups.

The solution to the first-order equation will be a point on the contract curve that depends on the various coefficient functions $\{\alpha_L^*, \alpha_L^{**}, \alpha_H^*, \alpha_H^{**}\}$. Note that these various activist coefficients are left unspecified. They are determined by the response of activist groups to policy positions.

Assumptions 8.1.1.(i)-(iv) are quite natural. They posit that the utility gradient of the activist group dictates the gradient of each contribution function, which in turn gives the direction of most rapidly increasing valence for party 1.

To apply this analysis, suppose that an economic activist, situated on the left of the economic axis, with preferred point $L = (x_l, y_l)$ has a utility function $u_L(x, y)$ based on the “ellipsoidal cost function,” u_L , of the form

$$u_L(x, y) = A - \left(\frac{(x - x_l)^2}{a^2} + \frac{(y - y_l)^2}{b^2} \right). \quad (8.6)$$

Assuming that $a < b$ means that such an activist is more concerned with economic policy than currency issues. We also suppose that a hard cur-

rency activist with preferred point $H = (x_h, y_h)$ has a utility function u_H of the form

$$u_H(x, y) = B - \left(\frac{(x - x_h)^2}{e^2} + \frac{(y - y_h)^2}{f^2} \right). \quad (8.7)$$

Assuming that $f < e$ means that such an activist is more concerned with currency policy than with issues on the x -axis. The contract curve generated by these utility functions is given by the equation

$$\left[\alpha_L \frac{du_L}{dz} + \alpha_H \frac{du_H}{dz} \right] = 0 \quad (8.8)$$

with $\alpha_L \geq 0, \alpha_H \geq 0$, but $(\alpha_L, \alpha_H) \neq (0, 0)$. Using this expression it can be shown that the “contract curve,” between the point (x_l, y_l) and the point (x_h, y_h) , generated by the utility functions is given by the equation

$$\frac{(x - x_l)}{a^2} \frac{b^2}{(y - y_l)} = \frac{(x - x_h)}{e^2} \frac{f^2}{(y - y_h)}. \quad (8.9)$$

This can be rewritten as

$$\frac{(y - y_l)}{(x - x_l)} = \gamma_1 \frac{(y - y_h)}{(x - x_h)} \text{ where } \gamma_1 = \frac{b^2 e^2}{a^2 f^2} > 1. \quad (8.10)$$

This “contract curve” between the two activist groups, centered at L and H , is a *catenary*, whose curvature is determined by the “salience ratios” $(\frac{b}{a}, \frac{e}{f})$ of the utility functions of the activist groups. By (8.10), this catenary can be interpreted as the closure of the one-dimensional locus of points given by the first-order condition for maximizing the total valence $\mu_1(z_1) = \mu_L(\Sigma_L(z_1)) + \mu_H(\Sigma_H(z_1))$, generated by the contributions (Σ_L, Σ_H) offered by the two groups of activists.

This locus is called *the activist catenary for 1*. Note that while a position of candidate 1 on this catenary satisfies the first-order condition for maximizing the total valence function it need not maximize vote-share. In fact, maximization of vote-share requires considering the marginal electoral effect. From Theorem 7.2.1, the first-order condition is given by the *balance equation* for candidate 1:

$$\left[\frac{d\mathcal{E}_1^*}{dz_1} - z_1^* \right] + \frac{1}{2\beta} \left[\alpha_L(z_1^*) \frac{du_L}{dz_1} + \alpha_H(z_1^*) \frac{du_H}{dz_1} \right] = 0. \quad (8.11)$$

The coefficient functions, $\{\alpha_L, \alpha_H\}$, depend on the various gradient coefficients introduced under Assumption 8.1.1, and are explicitly written as functions of z_1^* . For fixed z_2 , the locus of points satisfying this equation is called the *balance locus* for 1. It is also a one-dimensional smooth catenary, and is obtained by shifting the contract curve for the activists (who are centered at L and H) towards the weighted electoral mean of party 1. Notice, for example, that if $\alpha_H^*(z_1)$, the coefficient that determines the willingness of the currency activist group to contribute, is high, then this group will have a significant influence on the position of candidate 1. Obviously, the particular solution z_1^* on this balance locus depends on the second-order condition on the Hessian of the vote function E_1 , and this will depend on the various coefficients and on $\frac{d\mathcal{E}_1^*}{dz_1}$. Moreover, by Theorem 7.2.1, the weighted electoral mean of 1 depends on the weighted electoral coefficients

$$[\alpha_{i1}] = \left[\frac{\rho_{i1}(1 - \rho_{i1})}{\sum_{k=1}^v (\rho_{k1}(1 - \rho_{k1}))} \right] \quad (8.12)$$

and thus on the valence functions as well as the location of the opposition candidate. Candidate 1 can, in principle, determine the best response to z_2 by trial and error. By the implicit function theorem, we can write $z_1^*(z_2)$ for the *best response*, or solution to the balance equation for 1, at fixed z_2 .

In the same way, if there are two activist groups for party 2, centered at $R = (x_r, y_r)$ and $S = (x_s, y_s)$ with utility functions based on ellipsoidal cost functions, with

$$u_R(x, y) = G - \left(\frac{(x - x_r)^2}{g^2} + \frac{(y - y_r)^2}{h^2} \right), \quad g < h \quad (8.13)$$

$$\text{and } u_S(x, y) = K - \left(\frac{(x - x_s)^2}{r^2} + \frac{(y - y_s)^2}{s^2} \right), \quad r > s, \quad (8.14)$$

then the “contract curve” between the point (x_r, y_r) and the point (x_s, y_s)

is given by the equation

$$\frac{(y - y_r)}{(x - x_r)} = \gamma_2 \frac{(y - y_s)}{(x - x_s)} \quad (8.15)$$

where

$$\gamma_2 = \frac{h^2 r^2}{g^2 s^2}. \quad (8.16)$$

As before, this contract curve gives the first-order condition for maximizing the valence function

$$\mu_2(z_2) = \mu_R(\Sigma_R(z_2)) + \mu_S(\Sigma_S(z_2)) \quad (8.17)$$

and can be identified with the *activist catenary for 2*, given by

$$\left[\alpha_R(z) \frac{du_R}{dz} + \alpha_S(z) \frac{du_S}{dz} \right] = 0. \quad (8.18)$$

Again, this expression is derived from the utility functions u_R and u_S for the activist groups located at R and S respectively. The locus of points on which vote-share is maximized is given by the *balance locus for 2*:

$$\left[\frac{d\mathcal{E}_2^*}{dz_2} - z_2^* \right] + \frac{1}{2\beta} \left[\alpha_R(z_2^*) \frac{du_R}{dz_2} + \alpha_S(z_2^*) \frac{du_S}{dz_2} \right] = 0. \quad (8.19)$$

As before, this locus is obtained by shifting the activist contract curve for 2, to adjust to the electoral pull for the party. The coefficients will be determined by the second-order condition on E_2 .

Assumption 8.1.2. Concavity of Valences. The contribution functions Σ_L, Σ_H are assumed to be concave in z_1 , and the contribution functions Σ_R, Σ_S are assumed concave in z_2 . It is further assumed that the valences $\mu_L, \mu_H, \mu_R, \mu_S$ are concave functions of $\Sigma_L, \Sigma_H, \Sigma_R, \Sigma_S$ respectively.

These assumptions imply that the total activist valence functions

$$\mu_1(z_1) = \mu_L(\Sigma_L(u_L(z_1))) + \mu_H(\Sigma_H(u_H(z_1))) \quad (8.20)$$

and

$$\mu_2(z_2) = \mu_R(\Sigma_R(u_L(z_2))) + \mu_S(\Sigma_S(u_S(z_2))) \quad (8.21)$$

are concave functions of z_1, z_2 , respectively.

These assumptions appear natural because (i) the utility functions of the activist groups for both 1 and 2 are concave in z , and (ii) the effect of contributions on activist valence can be expected to exhibit decreasing returns.

In this case of two activist groups for each of two parties, the pair of positions (z_1^*, z_2^*) satisfying the above balance loci gives the *balance solution* specified in Theorem 7.2.1. This theorem, together with the above assumptions, can then be used to obtain a sufficient condition for existence of PNE. Indeed, once the parameters of the activist groups are determined, then existence and location of the LNE can be ascertained. Indeed, as the Theorem asserts, if the activist functions are sufficiently concave, then the LNE will in fact be PNE. The same technique can be used when there are more than two activist groups for each candidate.

As noted above, we can write $z_1^*(z_2)$ for the locus of points satisfying the balance equation for 1 at fixed z_2 . This balance locus given by the function $z_1^*(z_2)$ will lie in a domain bounded by the contract curve of the activists who contribute to party 1. A similar argument gives the balance locus $z_2^*(z_1)$, which again will lie in a domain bounded by the contract curve of the activists who contribute to 2. Both $z_1^*(z_2)$ and $z_2^*(z_1)$ can be regarded as solution submanifolds of W^2 , where $z_i^*(z_j) \in W^2$ iff z_i^* is a best response to z_j . Then these two solution submanifolds are generically two-dimensional submanifolds of W^2 . Transversality arguments can be used to show that these will generically intersect in a zero-dimensional vector, or set of vectors (Schofield, 2003a). There may be many first-order solutions, but the assumption of sufficient concavity of the total valence functions gives a balance solution which is a PNE. The same argument can be carried out for an arbitrary number of parties (Schofield, 2001).

8.2 Argentina's Electoral Dynamics: 1989–1995

The main contenders in Argentina's 1989 presidential election were Carlos Menem, the candidate of the PJ (*Partido Justicialista*) and Eduardo Angeloz, the candidate of the UCR (*Union Civica Radical*).

Angeloz had the disadvantage of coming from the same political party as the president in office, forced to call an election in 1989 because of hyperinflation. Angeloz's platform was located in the center-right of the economic axis of the political space. His most important proposal was the so-called "red pen," to reduce the size of the state apparatus in an attempt at fiscal austerity.

Menem was a charismatic, populist candidate, but lacked a sound political platform. His platform, such as it was, included a universal rise in salaries (*salario*) and an emphasis on the productive sector (*revolucion productiva*). This platform, clearly located in the left of the economic axis, gave Menem broad support from the working class, and constituted the key to his electoral victory.

Surprisingly, once in office Menem adopted policies that were the opposite of his electoral promises, including the liberalization of trade, the privatization of several state companies, a freeze of public salaries and the deregulation of the markets. Also, in 1991 Menem established a currency board, the so-called “Convertibility Plan,” which succeeded in controlling hyperinflation. This provided the basis for four years of macroeconomic stability and growth.

However, the Convertibility Plan proved to be vulnerable to both exogenous “contagion” and fiscal imbalances and led to a progressive appreciation of the Argentinean currency. This currency appreciation created both losers and winners in the polity. Among the latter were the recently privatized firms (seeking to maximize the value of their assets and profits denominated in dollars) and most of the upper middle class, who came to enjoy the benefits of inexpensive imported goods. The losers consisted of the export-oriented sector, together with many small and medium-sized firms and their employees, who could not survive the appreciation of the peso and the liberalization of the economy.

Menem was re-elected in 1995, with a manifesto promising to maintain the administration’s economic policy. Because he had broken the electoral promises of 1989, Menem lost about 15 percent of the leftist votes. However, because of the high standard of living achieved during Menem’s administration, the gain in upper-middle-class votes compensated for the working-class defection. In 1995 Menem took approximately 50 percent of the vote, with about 70 percent of his share coming from the traditional working-class constituency of the PJ, and about 30 percent from other constituencies, mainly voters who had previously chosen the UCR (Gervasoni, 1997). The win for Menem was partly due to his newly acquired vote from the middle class, and partly due to the relative success of a new party, the FREPASO (with 29 percent). Minor parties took 4.6 percent and the UCR suffered a great defeat (with only 17 percent of the vote).

This sequence of events is at odds with the electoral models used to analyze elections. First, the policy positions of the main parties in 1989 seems to contradict the “mean voter theorem.” This theorem predicts the convergence of the candidates to the electoral center. Second, Menem's policy positions in 1989 and 1995 were very different. In particular, the position that won him the election in 1995 led to considerable economic benefits for a constituency that was opposed to him in 1989.

The aim of this chapter is two-fold. The first intention is to use the theory presented above to explain a “paradox,” which is contrary to generally accepted theory: As discussed in the previous chapter, actual political systems generally display *divergence* rather than *convergence*. The case of the presidential elections in Argentina in the 1989 and 1995 is a good example of such divergence. The second aim to present a theory of the kind of “political realignment” (Sundquist, 1973) that occurred between these elections in Argentina, with a view to understanding such realignments more generally. Perhaps more importantly, the model suggests that there may well be a high degree of contingency over whether a populist leader or a right-wing political candidate comes to power in presidential polities that resemble Argentina in the distribution of electoral preferences. Such polities include many in Latin America, as indicated by recent events in Mexico and Bolivia.

Prior to the election of 1989, Argentina was under the administration of the UCR and in the grip of hyperinflation. Carlos Menem, the candidate for the opposition party, PJ, adopted a populist platform well to the left of the electoral center on the traditional left–right axis. Menem proposed typical redistributive policies in favor of the working class coupled with incentives to the “productive sector” of the economy. In contrast, the platform proposed by Angeloz, of the UCR, focused on fiscal discipline and a reduced role of the state. Thus, a one-dimensional policy space seems a reasonable approximation to Argentina in 1989.³⁶

The results in Chapter 7 suggest that there are two different cases depending on the parameters of the model.

First, suppose that the convergence coefficient $c = 2\beta(1 - 2\rho)\sigma^2$ is

³⁶This standard, unidimensional, model of voting has been widely used in the recent literature. For example, see Osborne and Slivinski (1996); Bueno de Mesquita, Morrow, Siverson and Smith (2003); Acemoglu and Robinson (2005); and Herrera, Levine and Martinelli (2005).

bounded above by the dimension of the policy space, $w = 1$. In this case, we say that the *critical condition* is satisfied. If the exogenous valences are very similar (with $|\lambda_{PJ} - \lambda_{UCR}|$ close to zero), then the vote-share, ρ , of both parties will be close to $\frac{1}{2}$, and c will be close to 0. With just two parties, the critical condition is given by

$$\beta \leq \beta_0 \equiv \frac{[\exp(\lambda_{PJ} - \lambda_{UCR}) + 1]}{2\sigma^2[\exp(\lambda_{PJ} - \lambda_{UCR}) - 1]}. \quad (8.22)$$

Note that if λ_{PJ} approaches λ_{UCR} , then β_0 approaches ∞ so the critical condition is always satisfied.

For the sake of exposition we consider only two parties, but a similar critical condition can be obtained for an arbitrary number of parties. In fact, in 1989 three candidates contested the election. Angeloz obtained 37 percent of the votes, Menem 47 percent, and Alsogaray, a rightist candidate, about 6 percent.

Consider Figure 8.1, in which we assume a distribution of voter bliss points whose mean is the electoral origin. The left–right axis is termed the “labor–capital” axis in the figure. The vertical axis may be ignored by the moment. The estimated strategies of the PJ and UCR in the 1989 election are represented by the points $PJ89^*$ and $UCR89^*$, respectively. Prior to the election, we may suppose that $|\lambda_{PJ} - \lambda_{UCR}|$ was indeed close to zero. In a model without activists there would be no reason for either party to vacate the center. Notice, however, that a perturbation in the valences of the parties, so that $|\lambda_{PJ} - \lambda_{UCR}| \neq 0$, will induce a move by the low valence party away from the origin whenever $\beta > \beta_0$.

In this second situation let us assume that $\lambda_{PJ} > \lambda_{UCR}$. By Theorem 7.2.2, if both the electoral variance σ^2 and the spatial coefficient β are large enough, then the low valence party, the UCR, should retreat from the origin, in order to increase its vote-share. Thus the position near to $UCR89^*$ is compatible with Theorem 2.

However, because $\lambda_{PJ} > \lambda_{UCR}$, it follows that if the UCR could not obtain electoral support from activists then it would lose the election. The consequence will be that both PJ and UCR should move further apart, in opposite directions away from the electoral origin, to obtain increasing support from the left activists, at L (for the PJ) and from the conservative activists at R (for the UCR). The vote-maximizing equilibrium ($PJ89^*$, $UCR89^*$) results from these centrifugal moves to balance the

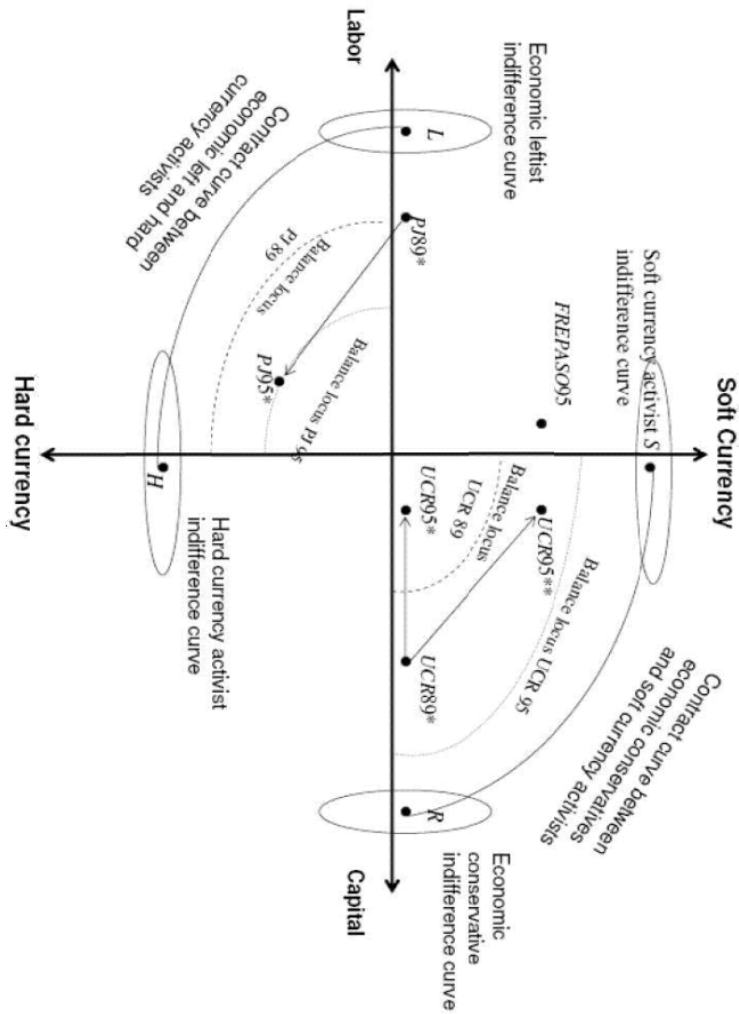


Figure 8.1: Activists in Argentina

attraction of the weighted electoral mean and the influence of the activists. Menem's higher valence, together with populist support from the left activists at L gave him the electoral victory.

The point L can be taken to be the preferred policy of the working-class "syndical" leaders, who provided key support for Menem's 1989 electoral victory. Because the choices of the syndical leaders were followed by a large part of the Argentinean working class, the effect of this support, represented by the valence function μ_L , was pronounced. This explains why Menem's strategy against a discredited UCR was far to the left, as indicated in Figure 8.1. This analysis seems to be a fairly accurate description of Argentina's polity for the election of 1989. We now use the model to analyze the events after 1989, leading up to the 1995 election.

The main issue is whether Menem's drastic and successful repositioning after the 1989 election can be explained by this model. Until hyperinflation was defeated, any debate regarding the optimal real exchange rate was fruitless. Thus, it was not until Menem's Convertibility Plan stabilized the level of prices that the currency issue gained significant saliency. Because the Convertibility Plan was successful against hyperinflation through fixing the nominal rate of exchange of the Argentinean peso in a 1-to-1 ratio to the American dollar, the currency issue naturally ended up focusing on the Convertibility Plan itself. The Convertibility Plan became most salient during the Mexican crisis, popularly known as "Tequila," in December 1994. Because the next presidential election (in which Menem would seek his re-election) was scheduled for May 1995, the issue dominated the electoral debate. The vertical axis in Figure 8.1 represents the policy options in this new axis, which will be called the "currency dimension."

Two groups gained from the Convertibility Plan. The European firms that won most of the privatization concessions of Argentinean companies benefited from the progressive appreciation of the peso after 1991 via the increased value in their assets and profits. Though these originated in Argentina, they were denominated in dollars. The upper middle class benefited from this policy too, since it enjoyed a consumption boom of foreign goods and the reappearance of credit after so many years of high inflation.

The main losers from the Convertibility Plan were those small and

medium entrepreneurs, and their employees, who could not overcome the difficulties associated with the appreciation of the Argentinean currency and the liberalization of the economy.³⁷

Among all groups affected by the value of the currency, the privatized companies had the greatest potential as an effective activist group. This was a consequence of their small number, their large pool of financial resources and their lobbying power. On the other hand, any attempt at activism against the Plan by either small and medium entrepreneurs and their employees had to overcome their collective action problem.

Consider again the positions $PJ89^*$ and $UCR89^*$ on either side of, and approximately equidistant from, the electoral origin as in Figure 8.1. The figure also gives balance loci for the PJ and UCR in 1989 and 1995. These balance loci can be derived from the four different activist groups centered at L , H , R and S .

To apply this model developed previously, consider a move by Menem along the balance locus from position $PJ89^*$ to position $PJ95^*$. By such a move, Menem would certainly gain the support of the activists located at H , while losing some of the political contributions of erstwhile supporters located at L . While Σ_L would fall, Σ_H would increase. Because of the higher marginal gain of the hard currency activists, we expect $\mu_L + \mu_H$ to increase. This reasoning is reinforced by the assumption of concavity of each activist valence function, since this implies that $\frac{d\mu_H}{dz_{PJ}}$ would be positive and high, and $\frac{d\mu_L}{dz_{PJ}}$ would be negative, but of low modulus, as the PJ position moves along the balance locus away from L .

Menem's overall exogenous valence was high in 1995 for two reasons. First, a large proportion of the electorate still regarded Menem as the guardian of the working-class interests, mainly because his party, the *Partido Justicialista*, was associated with the iconic figures of Juan Domingo and Evita Peron, revered by the working class. Second, the absolute success of the Convertibility Plan in controlling hyperinflation had the effect of increasing Menem's exogenous valence because he appeared to be the only politician who could solve what appeared to be the most difficult problem facing the country. The increased exogenous va-

³⁷The rate of unemployment peaked at 18 percent in 1995, the year in which the Tequila affected the Argentinean economy.

lence shifted the balance locus for Menem towards the origin, while the emergence of the hard currency activist group, in turn, induced Menem to move along the one-dimensional balance locus, from $PJ89^*$ to $PJ95^*$. These effects are illustrated in Figure 8.1.³⁸

Conversely, the exogenous increase in $\lambda_{PJ} - \lambda_{UCR}$ shifted the UCR balance locus towards the contract curve between the activist positions, R and S . The model suggests that this change would imply an optimal position for the UCR at a position such as $UCR95^{**}$ in Figure 8.1. Indeed, the drop in UCR valence led to a search for disaffected voters in the “north-west” region of the figure. A centrist position for the UCR, say at $UCR95^*$, would not cause centrist voters to choose the UCR with high probability (because of the higher exogenous valence of Menem). This suggests that $(PJ95^*, UCR95^{**})$ is a local equilibrium, in the sense that each position is a best response to the opposition position. With the assumption of sufficient concavity, this pair of positions would be a PNE.

Two candidate slates were opposed to one another in the UCR primaries for 1995. The Storani–Terragno slate adopted a position similar to $UCR95^{**}$ in the figure, which the model suggests is an optimal response to $PJ95^*$. The other slate, Massaccesi–Hernandez, adopted the position $UCR95^*$ in the figure. The model suggests that this was not a best response. Because of its low exogenous valence, severely aggravated by events between 1989 and 1995, the UCR could not win with such a centrist position.

The Massaccesi–Hernandez slate won the primaries, so the UCR position can be represented by $UCR95^*$. The UCR suffered a historical defeat, obtaining only 17 percent of the vote. Moreover, the candidate Jose Octavio Bordon, the candidate for a new party, FREPASO, outperformed UCR, with 29 percent of the vote. His position, denoted $FREPASO95$ in Figure 8.1, was close to $UCR95^{**}$, although somewhat to the left on the economic axis.

The electoral data for the 1989 and the 1995 elections are consistent with the change of electoral support for Menem implied by the model. Among the voters with low to moderate income, Menem’s support decreased from 63 percent to 59 percent. Among the voters of middle and

³⁸Seligson (2003) and Szusterman (1996) discuss the electoral platforms of PJ, UCR and FREPASO in the 1995 election. Their estimates and those presented in Figure 8.1 are broadly consistent.

upper middle income, it increased from 40 percent to 49 percent and from 38 percent to 47 percent, respectively. Finally, among the upper class voters, it increased from 13 percent to 42 percent (Gervasoni, 1997). Together, the vote proportions gave Menem 50 percent of the overall vote.

It is crucial for this analysis that there were indeed two dimensions of policy. If the distribution of voter bliss points displayed high covariance between the two axes in Figure 8.1, then the contract curves between R and S and between L and H would be degenerate. To test the validity of this assumption, Schofield and Cataife (2007) examined a data set on Argentinean presidential elections for the period 1983–1999.³⁹ This data set contained the following information:

- (i) actual vote in the presidential elections of 1989 and 1995 and intention of vote for 1999.
- (ii) the voter socio-demographical variables.
- (iii) responses to several issue questions regarding the opinion of the “subject” on particular policy issues. These included the subject’s degree of agreement regarding the Convertibility Plan at the time of the survey.

Applying factor analysis techniques to the issue questions gave an estimate of the position (or bliss point) of each voter in a space of reduced dimension.

One of the fundamental premises of the model presented here is that the Convertibility Plan emerged around 1995 as a new dimension in the Argentinean polity. A principal-components factor model, based on the Arromer survey data, was used to test this premise.

Using the 10 issue questions in the survey, four factors were obtained and given the following interpretations:

Factor 1 represented the standard “economic redistribution” dimension (whether extra social assistance should be provided, whether food and education should be taxed, etc.).

Factor 2 reflected attitudes to the Convertibility Plan.

An additional factor 3, representing the dimension associated with “eco-

³⁹The data set Arromer [TOP045(1998) in the Roper Center Archive] is based on a national poll conducted by the survey organization Graciela Romer & Asociados, with face to face methodology, and sample size of 1,203 respondents.

conomic structural reforms” (labor market flexibility, privatization and other related policies), was not salient for the 1995 election, since by that time most of the structural reforms had been already implemented. A factor 4 representing the standard “social” dimension (human rights, order vs. freedom, etc.) but had little salience, particularly after 1990 when the main policy issue on human rights was abruptly ended by President Menem’s pardon for those responsible for the violations of human rights during the dictatorships of the 1970s.

Factors 1 and 2 can be interpreted as the orthogonal axes of the underlying policy space defining the election of 1995. Factor 1 corresponds to the Labor/Capital axis, which can be called *econ*, while factor 2 corresponds to the currency axis, which can be called *cur*. This preliminary analysis of the formal model seems to capture the essence of the Argentinean political economy circa 1995. Figure 8.2 presents an estimate of the distribution of voter positions in the space of reduced dimension given by factors 1 and 2. The central electoral domain shows the estimated probability density function of the voter distribution, while the dots are individual bliss points outside the central domain. Clearly the electoral covariance matrix ∇_0 generated by these data exhibits little covariance, implying that ∇_0 has insignificant off-diagonal terms.

The figure suggests that the electoral variance σ_{econ}^2 on the labor/capital axis slightly exceeded the variance σ_{cur}^2 on the currency axis, while the covariance $(\sigma_{cur}, \sigma_{econ})$ was 0. If indeed $|\lambda_{PJ} - \lambda_{UCR}|$ were close to zero, then, for the exogenous valence model, Corollary 7.2.2 implies convergence to the electoral mean. Because of the lack of evidence for convergence, it can be assumed that $\lambda_{PJ} > \lambda_{UCR}$. Theorem 7.2.2, for the model with exogenous valence, suggests that the Hessian, C_{UCR} , at the joint origin, has two positive eigenvalues, corresponding to a minimum of the vote-share function for the UCR. The model with exogenous valence alone then gives local equilibrium positions for the PJ and the UCR on opposite sides of the economic axis, with the PJ position closer to the origin than the UCR position. The estimate of both the PJ and UCR positions in 1995 is at odds with this inference.

This suggests that the estimated locations of $(PJ95^*, UCR95^{**})$ in Figure 8.1 are indeed compatible with the activist valence model, and that Menem was able to use his high exogenous valence to take advantage of the saliency of the currency issue.

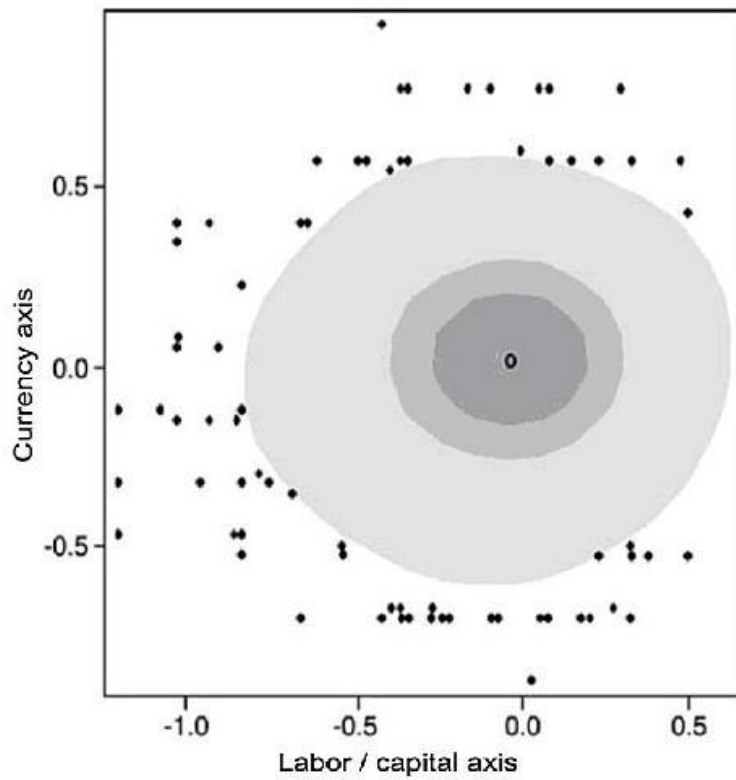


Figure 8.2: The voter distribution in Argentina in 1989

8.3 Concluding Remarks

This chapter has presented a general activist model of elections and used it to analyze the complex Argentinean polity in 1989–1995.

The success of the Convertibility Plan in controlling hyperinflation changed three of the political variables in the Argentinean polity: (i) it critically altered the relative valence of the two main parties, (ii) it introduced a second dimension, and (iii) it created a strong activist group.

These changes are compatible with the model proposed here, and can be used to account for the seeming paradox of non-convergence of the parties to the electoral origin.

The model implies that the changes in the political variables led to new equilibrium strategies of the candidates. The higher-valence candidate adopted a position that was supported by upper-middle-class voters. The low-valence candidate miscalculated, and moved closer to the electoral center and suffered a damaging defeat. Part of this defeat was due to the emergence of a third party, which adopted a position close to the estimated optimal strategy for the low-valence party.

In this model, the Convertibility Plan, the fundamental cause of these exogenous changes, was the result of a clever electoral strategy adopted by Menem. Cataife and Schofield (2007) suggest that the creation of the Convertibility Plan was due to the alignment of interest between three different actors: (i) Argentina's upper middle class, (ii) money-motivated domestic politicians and (iii) the U.S. Department of Treasury (representing the interests of the U.S. government).

Because politicians need to win office in order to pursue their ultimate goals, and because the upper middle class provided activist support, the model presented here gives a framework with which to understand this political realignment. Elaborating the model to examine the game between activists and candidates would also involve a more detailed analysis of the, implicit contract between candidates and activists and the compatibility of these political motivations with foreign interests.

Indeed, a similar model can be used to examine the ability of political leaders to remain in office in non-democratic regimes (Schofield and Levinson 2008). The only difference to the model would be a modification of the vote-share function, to reflect the nature of regime support.

Chapter 9

Coalitions in the United States

9.1 Convergence or Divergence

As discussed in the previous chapters, models of elections tend to give two quite contradictory predictions about the nature of political competition. In two-party competition, if the “policy space” involves two or more independent issues, then the majority rule core will be empty. This implies that if there are two candidates who desire to win, then, generically there will exist no “pure strategy Nash equilibrium.” In such a game it can be presumed that instability of some kind is possible. That is to say, whatever position is picked by one party there always exists another policy point which will give the second party a majority over the first.

On the other hand, the earlier electoral models based on the work of Hotelling (1929) and Downs (1957) suggest that parties will converge to an electoral center (at the electoral *median*) when the policy space has a single dimension. Although a pure strategy Nash equilibrium generically fails to exist in competition between two agents under majority rule in high enough dimension, there will exist mixed strategy equilibria whose support is located near to the electoral center.⁴⁰ These various and contrasting theoretical results are in need of resolution: will democracy tend to generate centrist compromises, or can it lead to disorder?

Partly as a result of these theoretical difficulties with the “deterministic” electoral model, and also because of the need to develop empirical models of voter choice, Chapter 7 focused on “stochastic” vote models.

⁴⁰See McKelvey (1986); Banks, Duggan and Le Breton (2002, 2006).

A formal basis for such models is provided by the notion of “quantal response equilibria” (McKelvey and Palfrey, 1995). In such models, the behavior of each voter is characterized by a vector of choice probabilities determined by the candidate positions.⁴¹ A standard result in this class of models is that all parties converge to the electoral origin when the parties are motivated to maximize vote-share (or plurality in the two-party case).⁴² The predictions as regards convergence are at odds with the general perception that the principal parties in the United States implement very different policies when in office.

The focus of the last two chapters was the apparent paradox that actual political systems display neither *chaos* nor *convergence*. The key idea is that the convergence result need not hold if there is an asymmetry in the electoral perception of the “quality” of party leaders. The early empirical work by Poole and Rosenthal (1984) on U.S. presidential elections included these valence terms and noted that there was no evidence of candidate convergence.

This chapter applies the valence model presented in Chapters 7 and 8 where activists provide crucial resources of time and money to their chosen party, and these resources are dependent on the party position.⁴³ The party then uses these resources to enhance its image before the electorate, thus affecting its overall valence. Although activist valence is affected by party position, it does not operate in the usual way by influencing voter choice through the distance between a voter’s preferred policy position, say x_i , and the party position. Rather, as party j ’s activist valence, $\mu_j(z_j)$, increases due to increased contributions to the party in contrast to the support $\mu_k(z_k)$ received by party k , then (in the model) all voters become more likely to support party j over party k .⁴⁴

The problem for each party is that activists are likely to be more extreme than the typical voter. By choosing a policy position to maximize activist support, the party will lose centrist voters. The party must there-

⁴¹See the earlier studies by Hinich (1977); Enelow and Hinich (1984, 1989); Coughlin (1992). For an empirical comparison of deterministic and probabilistic models, see Burden (1997).

⁴²See McKelvey and Patty (2006); Banks and Duggan (2005).

⁴³For convenience, it is assumed that the activist valence, $\mu_j(z_j)$, of party j is only dependent on the policy position, z_j , and not on $z_k, k \neq j$, but this is not a crucial assumption. A more general version of the model is briefly considered in Chapter 10.

⁴⁴In other words, it is not the source of the resources that matters, just the amount.

fore determine the “optimal marginal condition” to maximize vote-share. Theorem 7.2.1 gave this as a (first-order) *balance condition*. Moreover, because activist support is denominated in terms of time and money, it is reasonable to suppose that the activist function will exhibit decreasing returns. For example, in an extreme case, a party that has no activist support at all may benefit considerably by a small policy move to favor a particular interest group. On the other hand, when support is very substantial, then a small increase due to a policy move will little affect the electoral outcome. For this reason it is reasonable to assume that the functions themselves are concave, so their Hessians are everywhere negative-definite. Theorem 7.2.1 pointed out that when these functions are sufficiently concave, then the vote-maximizing model will exhibit a Nash equilibrium.

As discussed in the next section for the two-party case, when the convergence condition of Theorem 7.2.1 is violated, then electoral logic will force one of the parties to move away from the origin so as to increase vote-share. Such a move, by creating asymmetry between the parties, is the trigger for differential activist support for the parties.

It is intrinsic to the model that voters evaluate candidates not only in terms of the voters’ preferences over intended policies, but also in terms of judgements over their capacity to carry out these policies. While the model presented in Chapters 7 and 8 is relatively simple, the analysis of the trade off between preference and judgement proved to be quite complex. This chapter is devoted to a discussion of U.S. politics in the light of the model in order to suggest possible lines of research over the formation of electoral judgements and over the logic of activist support.

In the next section, the activist vote-maximizing model is used in an informal fashion to explain the partisan realignment that has occurred in U.S. politics since 1960. This section has the additional purpose of suggesting that neither chaos nor convergence is a likely phenomenon. However, it is suggested that electoral success is highly contingent on the changing judgements of incumbent party leaders, and on the willingness of activists to support the parties. While the chaos indicated by pure social choice theory may not occur, elections may well be highly unpredictable.

9.2 Activist Support for the Parties

To illustrate the stochastic model, consider Figure 9.1, and suppose first that the economic dimension alone is relevant for political policy making. As before, it will be assumed that there is an electoral distribution of voter bliss points, whose mean is taken as the electoral origin. Ignoring activism for the moment, then the results of Chapter 7 show that there are two very different possibilities, depending on the parameters of the model. There is a “convergence coefficient” (labeled c) defined by the valences, λ_{dem} and λ_{rep} , together with the variance of the electoral distribution and the spatial coefficient, β . If the valences are sufficiently similar (as expressed by an inequality given in terms of c), then both parties will position themselves at the electoral origin, and both will gain about 50 percent of the vote. In this case, potential activists are unlikely to be motivated to contribute to the parties. As long as $c \leq 1$, then this convergent situation is stable.

On the other hand, if the valences differ, with $\lambda_{dem} > \lambda_{rep}$, say, and if the electoral variance and β are both sufficiently large, so $c > 1$, then the lower valence candidate will vacate the origin in order to increase vote-share. For purposes of exposition, it can be supposed that conservative economic activists have the preferred position, E. If the Republican candidate moves away from the origin, to a position similar to R, then economic conservative activists would be induced to support this candidate. The asymmetry induced by this support will cause liberal economic activists at L to support the Democratic candidate. Then R will be pulled further towards E, while D will be pulled towards L. Moreover, if the marginal effect of activists for the Republicans is greater than for the Democrats, then the optimal candidate positions, R and D, will satisfy $|R| > |D|$. This model implies that once the convergent equilibrium is destroyed because of some exogenous change in parameters, and activists become motivated to support the appropriate parties, then convergence can never be recreated.

Note that in terms of the model, there is no reason why R should be to the right, and L to the left. However, once the move is made in one direction or the other, then activist support will tend to reinforce the left–right positioning of the parties.

This simple marginal calculation becomes more interesting when there

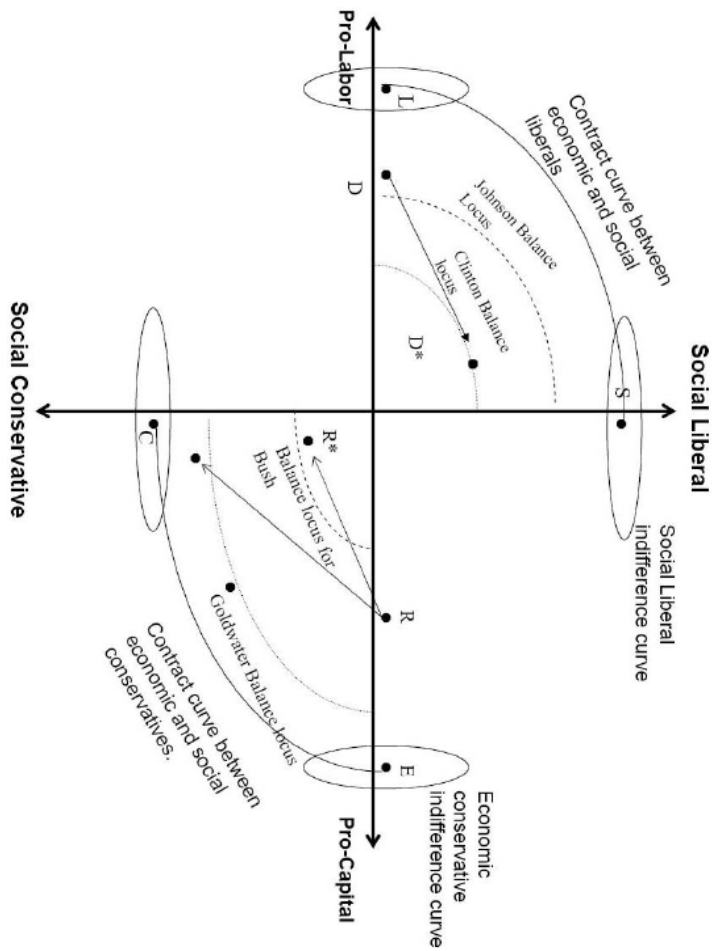


Figure 9.1: Activists in the United States

is a second “social” dimension of policy. Consider the initial positions R and D, on either side of, and approximately equidistant from the origin, as in the figure. Both Social Conservatives, represented by C, and Social Liberals represented by S, would be indifferent between both parties. A Democratic candidate by moving to position D^* will benefit from activist support of the social liberals, but will lose some support from the liberal economic activists. Note that the figure is based on the assumption, used in Chapter 8, that activists are characterized by ellipsoidal indifference contours, reflecting the different saliences they put on the policy axes. The “contract curve” between the two activist groups, centered at L and S, represents the set of conflicting interests or “bargains” that can be made between these two groups over the policy to be followed by the candidate. Chapter 8 showed that when activist groups have different saliences, then this contract curve is a *catenary* whose curvature is determined by the eccentricity of the utility functions of the activist groups. This contract curve is called *the Democratic activist catenary*. Using u_L and u_S to denote the utility functions of the two representative pro-Democrat activists, then, as in (8.18) in Chapter 8, this catenary is a one-dimensional curve given by:

$$\left[\alpha_L \frac{du_L}{dz} + \alpha_S \frac{du_S}{dz} \right] = 0. \quad (9.1)$$

Here α_L, α_S are parameters determined by the activist calculations over support for the party. A move by the Democratic candidate to a position on the contract curve in Figure 9.1 will maximize the total contributions to the candidate. Of course, this position will depend on the relative willingness of the two activist groups to contribute. In the same way, the *Republican activist catenary* is given by the contract curve between economic conservative activists, positioned at E, and social conservative activists, positioned at C. Denoting the utility functions of these activists by u_E, u_C , respectively, then again by (8.18) the contract curve between these two activists is given by

$$\left[\alpha_E \frac{du_E}{dz} + \alpha_C \frac{du_C}{dz} \right] = 0. \quad (9.2)$$

We can again make the assumption that the marginal contributions of the activists of both parties affect the two parties activist valences μ_{dem} and μ_{rep} by the marginal equations

$$\frac{d\mu_{dem}}{dz} = \left[\alpha_L(z) \frac{du_L}{dz} + \alpha_S(z) \frac{du_S}{dz} \right], \quad (9.3)$$

$$\frac{d\mu_{rep}}{dz} = \left[\alpha_E(z) \frac{du_E}{dz} + \alpha_C(z) \frac{du_C}{dz} \right], \quad (9.4)$$

at the positions $z = z_{dem}$ or z_{rep} as appropriate. The four coefficients in these equations parametrize the willingness of activists to contribute, as well as the effects these contributions have on party valences. These assumptions allow us to determine the first-order conditions for maximizing vote share. By Theorem 7.2.1, the first-order condition for the candidate positions (z_{dem}^*, z_{rep}^*) to be a Nash equilibrium in the vote-share maximizing game is that they satisfy the *balance equations*. Thus, for each party $j = dem$ or rep , there is a weighted electoral mean for party j , given by the expression

$$\frac{d\mathcal{E}_j^*}{dz_j} = \sum_i \alpha_{ij} x_i, \quad (9.5)$$

and which is determined by the set of voter preferred points $\{x_i\}$. Notice that the coefficients $\{\alpha_{ij}\}$ for candidate j will depend on the position of the other candidate, k . The *balance equation* for each j is given by:

$$\left[\frac{d\mathcal{E}_j^*}{dz_j} - z_j^* \right] + \frac{1}{2\beta} \left[\frac{d\mu_j}{dz_j} \right] = 0. \quad (9.6)$$

As in the previous chapters, the locus of points satisfying this equation is called the *balance locus* for the party. It is also a catenary obtained by shifting the appropriate activist catenary towards the weighted electoral mean of the party. The gradient vector $\frac{d\mu_j}{dz_j}$ is called *the marginal activist pull for party j* (at the position z_j^*) representing the marginal effect of the activist groups on the party's valence. The first bracketed term in this equation is the *marginal electoral pull of party j* (at z_j^*), and this pull is zero at the weighted electoral mean. At any other point, it is a vector pointing towards the weighted electoral mean.

To illustrate, the pair of positions (D*, R*) in Figure 9.1 can be taken to represent party positions that maximize vote shares. Indeed, Theorem 7.2.1 asserts that the degree of concavity of the activist functions may be sufficient to guarantee that this pair of positions is a *pure strategy Nash*

equilibrium. The locations of the balance loci for the two parties depend on the difference between the exogenous valences, λ_{dem} and λ_{rep} . In particular if $\lambda_{dem}-\lambda_{rep}$ is increased for some exogenous reason, then the relative marginal activist effect for the Republicans becomes more important, while for the Democrats it becomes less important. Theorem 7.2.1 suggests that these PNE will fluctuate considerably from one election to another, as they will depend in a subtle way on the candidates' exogenous valences and the response of the various activist groups to the party candidates.

The positioning of R* in the electoral quadrant labeled "Conservatives" in Figure 9.1 and of D* in the liberal quadrant is meant to indicate the realignment that took place after the election victory of Kennedy over Nixon in 1960. By 1964 Lyndon Johnson had moved away from a typical New Deal Democratic position, D, to a position comparable to D*. By doing so, he brought about a transformation that eventually lost the south to the Republican Party. According to the model just presented, such a move by Johnson was a rational response to civil rights demands, by increasing activist support. Although the support by the social liberals would be small initially at position L, the rate of increase of support (associated with a move along the Democratic balance locus) would be large in magnitude. Conversely the initial rate of the loss of support from Labor would be relatively small. A move by Johnson away from L along the catenary would thus lead to a substantial increase in overall activist support. Moreover, the empirical analysis in Schofield, Miller and Martin (2003) suggests that λ_{dem} for Johnson was large in contrast to λ_{rep} for Goldwater. This suggests that Goldwater's dependence on activist support was greater than Johnson's. This is reflected in Figure 9.1, where the balance locus for Goldwater is shown to be further from the electoral origin than the balance locus for Johnson.

Thus the magnitude of $\frac{du_C}{dz}$ provides an explanation why socially conservative activists responded so vigorously to the new Republican position adopted by Goldwater, and came to dominate the Republican primaries in support of his proposed policies. These characteristics of the balance solution appear to provide an explanation for Johnson's electoral landslide in 1964.

The response by Republican candidates after this election, while taking advantage of the political realignment, has brought about something

of a dilemma for both parties. This can be seen by considering in detail the balance condition at the position R^* , in Figure 9.1. At that point the two activist gradient vectors, $\frac{du_E}{dz}$, $\frac{du_C}{dz}$, will point away from the electoral origin. Because the distance from E is significant, the marginal contribution from economic conservative activists will be negative as the Republican position moves down the catenary. Further movement down the Republican catenary, in response to social conservative activism, will induce some activists located near E to recalculate the logic of their support. Indeed, members of the business community, who can be designated “cosmopolitans,” and who are economically conservative but relatively liberal in their social values, are likely to be concerned about the current policy choices of the Republican President.

In parallel, a Democratic position further along the Democrat catenary, particularly one associated with a Democratic candidate who has exogenous valence higher than the Republican opponent, would bring into being a new gradient vector associated with activist support from cosmopolitan economic activists for the Democratic candidate. Small moves by such a candidate would induce a significant increase in contributions.

The dynamic logic of this electoral model is that both parties will tend to move in a clockwise direction as they attempt to maximize electoral response by obtaining support from their respective activist groups. The model suggests that eventually the Democratic candidate will be located close to S while the Republican candidate will be close to C. From then on, populists will dominate the Republican Party and cosmopolitans will dominate the Democrat Party.

9.2.1 Realignment and Federalism

The model just presented suggests that realignment of party positions from (D,R) about 1960 to positions close to (S,C) in 2006 takes about two generations. Indeed, Schofield, Miller and Martin (2003) suggest that a slow realignment has been going on since at least the election of 1896. The positions of future Republican presidential candidates may closely resemble the position of the Democrat candidate, William Jennings Bryan, in 1896, while the positions of future Democratic candidates may come to resemble the position of Bryan’s opponent, the Republican

candidate, William McKinley.

The reason such a realignment may take many decades to come into being is due to the power of party activists who support the existing realignment at a given time. Consider the New Deal party alignment, based on economic conservatism/liberalism. The Republican activists were small business, professionals, and middle-class people who felt they had a lot to lose by more government regulation/redistribution. Most of the New Deal supporters were northern labor and southern agrarian interests who had something to gain by challenging the McKinley northeastern/business coalition. Many of these Democratic activists resisted the realignment that eventually occurred in the 1960s, when the Democratic Party sponsored civil rights and other socially liberal legislation. This caused the Republicans to react with their southern strategy, attracting socially conservative voters who were alienated by this version of liberalism.

The question remains however: why didn't this realignment occur in the 1950s or the late 1940s? The slow pace was not because there were no advocates. There were always voices calling for the Democratic Party to be more liberal on the social dimension. The ability to suppress divisive social issues reached its limits in 1948, when a fiery anti-segregation convention speech by Hubert Humphrey ignited northern liberals; it also led southern segregationists to walk out and form the States' Rights Democratic Party, nominating Strom Thurmond as their presidential candidate. Although Truman overcame this split to retain the White House, the frightening prospect of losing the Solid (Democratic) South forced Adlai Stevenson to relegate the race issue to the sidelines for two more elections, in the hope of keeping the New Deal coalition alive. The New Deal coalition limped into its third decade by suppressing social policy differences among economic liberals. The slow pace was also not due to lack of incentive for Republicans: many Republican strategists realized they had a lot to gain if the Democratic Party split on the issue of civil rights.

However, the New Deal economic activists (both Democratic and Republican) all had something to lose by allowing a realignment based on the social dimension. The typical northern Democratic activist – perhaps a member of a labor union – was in a strong position of power in the party. The Democratic leadership in Congress knew that they had to consult the unions on major legislation, and defer to them in large part.

Similarly, southern farmers knew that they were getting a flow of benefits from their pivotal position in the Democratic Party. They also knew that they would have to forego that position of party dominance if the Democrats ever became serious about civil rights. Thus, New Deal activists had every reason to try to maintain control of the party machinery (caucuses, the nomination process, etc.) in order to prevent what many nevertheless regarded as inevitable: the eventual emergence of a strong civil rights bill.

The same thing could be said of the traditional G.O. P. activists of the New Deal era. They did not have the same control of the legislative process since they were out of power in Congress most of the time, but they still had a lot to lose by a party realignment that emphasized social, rather than economic dimensions of policy. Of course, there were many north-eastern Republicans who were liberal on civil rights, who supported civil rights legislation, and who hated the Goldwater revolution. The Goldwater revolution was a revolution in the Republican Party precisely because it took power away from the traditional G.O. P. activists and handed it over to western (and eventually southern) business and ideological interests. The traditional G.O. P. activists hated the outcome. The business of the Republican Party became something quite different from the agenda of the Eisenhower period: the fight for fiscal conservatism and low taxes.⁴⁵

In short, party realignment takes time because the position of a party is not simply controlled by vote-maximizing politicians. Policy choice is constrained by party elites who control party machinery and would rather lose a few elections than change the orientation of the party.

Federalism also has an effect. The Goldwater revolution in the G.O. P. started in the 1960s, and was resisted by socially liberal north-eastern Republicans. By 1980, they had definitively lost control of the national Republican Party. However, they had not lost control of the local Republican parties in states such as Pennsylvania and Maine. As a result, social liberals from the north-east (like Jeffords and Chaffee, for example) re-

⁴⁵Nichols 2007 has noted that it was the Republican, Dwight Eisenhower, who pushed through the Civil Rights Act of 1957. Civil rights legislation had been blocked for 82 years because Southern Democrats and conservative Republicans had been able to stop the required two-thirds majority to effect cloture, bringing an end to a filibuster. The vote for the bill included 37 Republican votes.

mained a diminishing minority voice in the national Republican Party even as the Republican Party came to stand for huge deficits, opposition to abortion, and constitutional amendments to ban gay marriage. The 2006 election saw the defeat of a few more of these socially liberal Republicans. This has made the Republican Party somewhat more socially conservative than it had been before. In 2006, all positions of congressional leadership in the G.O. P. are held by southerners and westerners. In contrast, the Democratic Party is now solidly in control of what used to be the heart of the Republican Party: the north-east. Federalism in U.S. politics therefore acts to slow down the pace of realignment.

The following two sections of this chapter attempt to draw out some inferences about the past and present from the simple activist model just presented.

9.3 Coalitions of Enemies

9.3.1 The New Deal Coalition

The classic example of an unnatural coalition of enemies was the New Deal coalition. It is often forgotten, from the perspective of the 21st century, just how problematic the New Deal coalition was. Prior to Al Smith's presidential race in 1928, the Democratic Party was a socially conservative, agrarian party that regularly lost presidential elections. William Jennings Bryan was the party's nominee three times, and he epitomized the anti-immigrant, anti-urban, bias of his nativist supporters.⁴⁶

Al Smith's nomination in 1928 offered the hope that the Democrats could suppress the social differences between the urban, Catholic and union immigrants of the North and the rural, Protestant, white nativist voters of the South. With the impetus provided by the Great Depression, Franklin Roosevelt was able to make this coalition work by emphasizing the anti-business, pro-government economic liberalism of both southern farmers and northern labor. At the same time, he realized that social issues such as race had to be suppressed as far as possible – or they would

⁴⁶See Kazin 2006.

split the New Deal Democratic coalition down the middle.

But the tensions dividing the social liberal and social conservative wings of the Democratic Party could not survive the 1960s, when the civil rights movement, the anti-war movement, urban riots, the rulings of a libertarian Supreme Court, and the women's movement all moved social issues to the forefront. President Kennedy was in the uncomfortable position of being forced to choose between racial liberals and the traditional South – a choice he postponed making as long as possible. In June 1963, shortly after the Birmingham protests, Kennedy committed the Democratic Party to a strong civil rights bill – despite anticipating that the South's electoral votes would no longer go to the Democratic Party.

After the Civil Rights Act passed in 1964, Lyndon Johnson told an aide he was afraid that, in signing the bill, he had just given the South to the Republicans “for your lifetime and mine.”⁴⁷ After observing the Vietnam peace movement, urban riots, recreational drug use, sexually explicit television, and the women's liberation movement, millions of social conservatives never again voted Democratic, despite their history of support for the economic liberalism of the New Deal. Indeed, 1964 was the last presidential election in which the Democrats earned more than 50 percent of the white vote in the United States.

9.3.2 The Creation of the Republican Coalition

It was also in the 1964 election that the first tentative steps toward the current Republican coalition were made. Goldwater saw that opposition to the federal government was a concern shared by Republicans of the west and southern opponents of civil rights. In a 1961 speech in Atlanta, he offered up states' rights as the basis for a coalition between anti-integration and anti-regulation forces. “We are not going to get the Negro vote as a bloc in 1964 and 1968, so we ought to go hunting where the ducks are. [School integration is] the responsibility of the states. I would not like to see my party assume it is the role of the federal government to enforce integration in the schools.”⁴⁸ Goldwater followed through on this coalitional strategy (and made himself the first choice of the white South) by voting against the Civil Rights Act of that year, and by joining in

⁴⁷Branch (1998: 404).

⁴⁸Bass and Thompson (1998: 196-197).

the South's condemnation of the national government, hippies, Vietnam protestors, and do-gooders soft on communism. The Mississippi delegation walked out of the 1964 Democratic convention almost to a man, in favor of Goldwater.

Figure 9.1 locates Goldwater in the center of the Conservative quadrant – conservative on both economic and social issues. By making social issues salient, Goldwater was able to attract many populists to his cause. In 1964, for the first time since Reconstruction, Mississippi, Alabama, Georgia, South Carolina and Louisiana all cast their electoral college votes for the Republican Party.

By all accounts, the most able speech of the Goldwater campaign was given by Ronald Reagan, who had opposed the 1964 Civil Rights Act and would oppose the Voting Rights Act of 1965. This position helped cement the transformation of the G.O. P. from the nationalist party of Lincoln to the party of states' rights – a transformation that made possible a coalition of business interests, western sagebrush rebels and southern populists.

The election had implications for the long run, as was revealed by Goldwater's explicit courting of Strom Thurmond. As the segregationist States' Rights candidate in 1948, the author of the defiant Southern Manifesto and the filibusterer of both the 1957 Civil Rights Act and the 1964 Act, Thurmond had a great deal of influence with the white southern electorate. Goldwater not only talked Thurmond into supporting his presidential candidacy (which most politicians of the Deep South were doing that year), he also talked Thurmond into officially switching parties. Thurmond became the first of many successful southern legislators to make the switch, and was, in 1968, in a position to help deliver the mid-South to Nixon.⁴⁹

In *The Emerging Republican Majority* (1969), Republican strategist Kevin Phillips analyzed the long-term implications of the new linkage between Western civil libertarians and southern social conservatives. He argued that a strong dose of southern populism would make the Republicans the majority party, by gaining the support of millions of voters, South and North, who felt threatened by the federal government and its sponsorship of civil rights programs. In a comment that rang as true in

⁴⁹Carter (2000: 329-330).

2004 as 1969, Phillips noted that the newly “Populist” Republican Party could “hardly ask for a better target than a national Democratic Party aligned with Harvard, Boston, Manhattan’s East Side, Harlem, the *New York Times* and the liberal Supreme Court.”⁵⁰

Thus, Phillips anticipated a strange resolution to one of the oldest feuds in American politics between agrarian populists (especially southern agrarian populists) and Northeastern financial interests. Populists and big business had been at loggerheads at least since the time of Andrew Jackson. In 1896, Bryan’s “Cross of Gold” speech symbolized the opposition of southern and Midwestern agriculture to eastern financial interests. Similarly, Bryan’s fundamentalist attack on evolution at the Scopes trial symbolized the commitment to conservative social ideology among many of the populists.⁵¹ Bryan’s populism did exhibit a degree of isolationism, as illustrated by his resignation from the position of Secretary of State, in protest against President Woodrow Wilson’s policies during the First World War. Gray (2007: 112) has argued that both U.S. isolationism and interventionism have had a basis in evangelical beliefs, and quotes Woodrow Wilson’s address to Congress in 1919 on the question of membership of the League of Nations. Wilson declared that “nothing less depends upon this decision, nothing less than liberation and salvation of the world.” In recent years, this moral, evangelical interventionism has come to prominence in the Republican Party. This kind of interventionism appears immune to questions of rational, strategic calculation.⁵²

If anyone had argued prior to 1960 that Wall Street and populists would happily join hands within the same political party, both sides would have laughed at the idea. But as Phillips foresaw, Republicans could (and did) form a marriage of convenience between populists and economic conservatives, in opposition to the federal government as sponsor of the social change catalyzed by the civil rights, women’s, consumer and environmental movements.

Since Phillips wrote his book, and especially since the Reagan election of 1980, the Republican Party has managed to maintain a coalition that includes both Populists and pro-business interests – Bryan and McKinley.

⁵⁰Phillips (1969: 239). See also Phillips (2006).

⁵¹Admittedly, the Scopes trial took place many years after Bryan’s presidential efforts, but Bryan’s role suggests he was consistent in his social conservatism.

⁵²See also Sammon (2007).

It has done so by simultaneously serving the economic interests of business while advancing the social agenda of the social conservative wing of the party. Each new manifestation of social change – more sexually explicit movies, the issue of gay marriage, court limitations on prayer in schools – served to tighten the link between populists and the Republican Party.

At first, purely symbolic gestures were sufficient to keep social conservatives happy in the coalition; traditional pro-business Republicans had little real commitment to the social conservative agenda, and they were still in command. Reagan offered himself as a hero of social conservative values, but seemed to care a great deal more about dismantling the economic regulatory machinery of the New Deal than advancing family values. The conservative commentator David Frum complained that Reagan could have ended affirmative action programs “with a few signatures,” but never did (Frum, 1994: 72). Lasch (1991: 515) claimed that “Reagan made himself the champion of ‘traditional values,’ but there is no evidence he regarded their restoration as a high priority. What he really cared about was the revival of the unregulated capitalism of the twenties: the repeal of the New Deal.” George Bush the elder was especially suspect to social conservatives. He seemed to embody the tolerant cosmopolitanism of his father and other New England Republican liberals.

The question has become, “Who controls the Republican Party—social conservatives with their belief in moral interventionism, or the proponents of business?”

9.3.3 Social Conservatives Ascendant in the G.O. P.

In the early 1990s, social conservatives became more than a group of citizens to whom presidential candidates could appeal once every four years. Social conservative activists began to penetrate Republican state organizations, with or without an invitation from established party officials and candidates. Kansas is a case in point.

In 1991, an anti-abortion program called “Operation Rescue” temporarily closed down Wichita’s abortion clinics. This energized social conservatives, who not only participated in larger and larger mass meetings, but also organized to take over local politics by electing pro-lifers

as precinct committee women and men, by taking control of the local party machinery out of the hands of the established moderate forces.⁵³ They mobilized fundamentalist Christian churches, and turned out in unheard of numbers at Republican primaries. The Kansas state legislature went Republican in 1992, and moderate Republicans fought back with an organization called the Mainstream Coalition. The intra-party fight has continued until the present day.

Social conservatives succeeded in capturing much of the party machinery in other states as well, and they played a prominent role in the ranks of the Republican freshmen who helped capture Congress in 1994. This development was not met with glee by traditional big-business Republicans. Only a few months after the Republican victory in Congress, *Fortune* magazine ran a cover story reflecting big business's new sense of alienation from the G.O. P. The premise was that "corporate America [is] losing its party" – to social conservatives.⁵⁴ *Fortune* interviewed corporate executives who expressed strong concern about the "growing clout of the Christian conservative movement within the G.O. P." Fifty-nine percent of the CEOs agreed that "a woman should be able to get an abortion if she wants one, no matter what the reason." Big business, especially eastern business, was run by a well-educated intelligentsia. They had little in common with the Christian evangelicals who were upset by issues such as prayer in schools, the teaching of evolution, and gay marriage. These economic conservatives saw the key positions in the House and Senate Republican leadership going, in the 1990s, to southerners whose first loyalty was to social conservatism, and they foresaw a time when the economic agenda of the Republican Party would take second place to that of the social conservatives. *Fortune* wrote that, "if the Republican National Committee published a tabloid newspaper, the headline heralding the dawn of the Newt Gingrich era might well blare: G.O. P. to big business: drop dead."⁵⁵

Despite mounting tension, the Republican coalition seemed to find a way to reconcile the diverse interests of social and business conservatives. In 2000, the Bush campaign managed to mobilize both sides of the coalition once again. The big business community was happy with

⁵³Frank (2004: 94).

⁵⁴Kirkland (2005).

⁵⁵Ibid.

Bush's choice of Dick Cheney, who had been their first choice for president in 1996. They liked Cheney's inclusive procedure for setting the Bush energy policy. Pro-business Republicans might be alarmed by the growing power of the Christian right in the party, but they could live with it as long as they received the Bush tax cuts and the loosening of business regulation. Most of the Christian right might not benefit by tax breaks for millionaires, but they could live with it as long as the Bush administration moved in the direction of an anti-abortion Supreme Court, with Bush himself leading the fight against gay marriage.

By 2006, however, the Republican Party faced some hard choices. With the wholesale political success of the Bush tax plan, and the recent appointment of two conservative judges to the Supreme Court, each segment of the Republican coalition has begun to ask, "But what have you done for me lately?" Further advances in the Republican agenda are going to be a lot more stressful for the Republican coalition, because both sides care about the same issues – and they don't agree.⁵⁶

9.3.4 Stem Cell Research

The most striking example of the instability of the Republican Party is a dramatic appeal by Republican John Danforth, retired U.S. Senator from Missouri, and advocate of stem cell research. Danforth warned that his Republican Party has been transformed into "the political arm of conservative Christians."

It is worth giving a long quote from John Danforth (2005) in which he expresses his concerns over the pressure of religious power blocs:

When government becomes the means of carrying out a religious program, it raises obvious questions under the First Amendment. But even in the absence of constitutional issues, a political party should resist identification with a religious movement. While religions are free to advocate for their own sectarian causes, the work of government and those who engage in it is to hold together as one people a very diverse country. At its best, religion can be a uniting influence, but in practice, nothing is more divisive. For politicians

⁵⁶As an illustration, in late September, 2007, a group of "pro-family" leaders" declared that they were willing to back any third party candidate for president who would join them in asserting the sanctity of human life.

to advance the cause of one religious group is often to oppose the cause of another.

Take stem cell research. Criminalizing the work of scientists doing such research would give strong support to one religious doctrine, and it would punish people who believe it is their religious duty to use science to heal the sick.

During the 18 years I served in the Senate, Republicans often disagreed with each other. But there was much that held us together. We believed in limited government, in keeping light the burden of taxation and regulation. We encouraged the private sector, so that a free economy might thrive. We believed that judges should interpret the law, not legislate. We were internationalists who supported an engaged foreign policy, a strong national defense and free trade. These were principles shared by virtually all Republicans.

But in recent times, we Republicans have allowed this shared agenda to become secondary to the agenda of Christian conservatives. As a senator, I worried every day about the size of the federal deficit. I did not spend a single minute worrying about the effect of gays on the institution of marriage. Today it seems to be the other way around.

The historic principles of the Republican Party offer America its best hope for a prosperous and secure future. Our current fixation on a religious agenda has turned us in the wrong direction. It is time for Republicans to rediscover our roots.⁵⁷

As this quote suggests, there are many potential economic advantages to be gained from medical advances, particularly those resulting from stem cell research. Acquiescence to the policy demands of social conservatives means these gains will be forgone.

Danforth has close ties to the educational and business elites in Missouri, who foresee immense advantage to stem cell research. Stem cell research promises to be a powerful engine for economic development in the very near future. Aging baby-boomers are spending a lot of money on health care, and would be willing to spend a lot more for the potential treatments that might result from a decade of stem cell research.

⁵⁷Danforth (2005).

Pro-business Republicans, therefore, cannot afford to sit back and let the social conservative wing have its way. So Danforth's attack on the social conservative "control" of the Republican Party focused on that issue. "Republicans in the General Assembly have advanced legislation to criminalize stem cell research."⁵⁸ The legislation is supported by Missouri Right to Life and opposed by the Missouri Chamber of Commerce and Industry. Senator Jim Talent was one of the prominent Republicans caught in the cross-fire. He had originally supported a federal bill to oppose certain aspects of stem cell research, but then very publicly reversed himself. The reversal immediately led to an attack by the anti-abortion Republicans that Danforth found himself opposing.

9.3.5 Immigration

One might object that the stem cell crisis is a crisis only among Republican elites: Danforth vs. Buchanan. How deeply divided are Republican voters by this issue?

The immigration issue has become increasingly important to the entire bloc of Republican voters. The Hispanic-American protests around the country, on 1 May, 2006, were front-page news, and the latest round of anti-immigrant feeling stoked the fires of social conservatives. An Arizona Republican commented on the pro-immigrant protests: "I was outraged. You want to stay here and get an education, get benefits, and you still want to say 'Viva Mexico'? It was a slap in the face."⁵⁹

Most social conservatives in the country are wage-earners, so the economic impact of competing with Mexican immigrants was, no doubt, a factor in their hostility to immigrants as well. Said a construction worker, "They should all be ejected out of the country. They are in my country and they are on my job, and they are driving down wages."⁶⁰

As an anti-immigrant backlash grew among social conservatives, dozens of Republican legislators promised to oppose the moderate temporary-worker measure in Congress. The authors of the temporary-worker measure, however, were also Republican – pro-business Republicans who felt that immigration kept America's businesses supplied with cheap labor.

⁵⁸Ibid. See also Phillips (2006) for the dangers of "radical religion."

⁵⁹Kirkpatrick (2006).

⁶⁰Ibid.

Their proposed plan was basically a version of the Reagan amnesty plan in 1986, and supported by the business community. Today, social conservatives regard the Reagan amnesty as a mistake. Senator John Cornyn of Texas was referring to the Reagan plan when he said, “This compromise would repeat the mistakes of the past, but on a much larger scale because 12 million illegal immigrants would still be placed on an easier path to citizenship.”⁶¹ The much more widespread Republican opposition to a similar plan 20 years after the Reagan amnesty is evidence of the increased mobilization and influence of social conservative activists in the Republican Party. Both immigration and stem cell research point to the difficulties in maintaining the successful Republican coalition of recent decades.

9.4 The Changing Political Equilibrium

Chapter 7 has shown that, given voters and activists with different preferences in two-dimensional space, there can exist a pure strategy Nash equilibrium for vote-maximizing candidates. At any given time, each party candidate adopts a policy position to balance the centrifugal pull of party activists with the centripetal pull of vote maximization, while also seeking a best response to the position adopted by the other party’s candidate.

The location of the candidates in equilibrium is not exactly at the center of the electoral distribution, because of the need to seek resources from party activists, who generally are located far from the center. The more inherently likable a candidate is in the eyes of the public (i.e., the higher the candidate’s exogenous valence), the less dependent will the candidate be on the support of activists, and the closer to the electoral center will be the equilibrium position. For example in Figure 9.1, the “balance locus” for Goldwater gives the possible set of policy positions that he could choose so as to maximize the electoral consequences of activist support from economic and social conservatives. Because Goldwater had a low valence due to the electoral perception of his extremism, his balance locus was far from the center, illustrating his need to get compensating activist support. Bush, on the other hand, has been seen in both

⁶¹Free Internet Press (2006).

election years as a personally attractive candidate. Consequently, he can afford to take a more centrist position, as shown in the “balance locus for Bush” in Figure 9.1. The same factor would explain a more centrist balance locus for Clinton in the 1992 election.

The “balance locus” for each Republican candidate can be represented as an arc, revealing possible trade-offs between social and economic conservatives. As the proportion of resources from social activists increases relative to those from economic activists, candidates in equilibrium will move toward the axis of social activism. Just as Democratic candidates were pulled toward the social liberal position (S in Figure 9.1) during the tumult of the 1960s and 1970s, so Republican candidates have increasingly moved toward the social conservative position (C in Figure 9.1) during the 1980s and 1990s. This kind of movement necessarily creates resentment between different factions of the party, especially in the face of conflict over issues like stem cell research and immigration discussed above.

What is the rational best response of Democrats to this movement in the Republican Party? Will the Democratic equilibrium position adjust to these conflicts in the Republican Party? There have been various prescriptions for the Democratic Party, and recent events reveal which of these is most likely to be implemented. The first prescription is a “Return to the New Deal,” emphasizing traditional economic liberalism, and the class differences between labor and business.⁶² The problem with this strategy is that it is a losing strategy as long as the Republicans retain the votes of many of the populists of the lower-left quadrant in Figure 9.1. It seems unlikely that social conservatives – now fully engaged on the issues of teaching evolution, abortion, and immigration – are ever going to return to the party of civil rights, the Kennedy Immigration Reform of 1965, and *Roe vs. Wade*. Without the populists, economic liberalism mobilizes only the “liberal” voters of the upper-left hand quadrant in Figure 9.1.

The second strategy is to soften Democratic social liberalism. There is a lot to be said for this argument, from an electoral perspective. As social policy has come to dominate partisan debate, the median position on social policy is theoretically the winning position. However, the party’s

⁶²This is advocated by Frank (2004: 243-246).

position is the result of a “balance” between the centripetal pull of the electorate and the centrifugal tugs of activists who supply the resources necessary for an effective campaign. Moving too far toward the center on social policy runs the risk of losing the base of affirmative action supporters, gay rights supporters, and supporters of women’s rights. Since the 1960s, these groups have replaced economic liberals as the primary source of activist support for the party.

As mentioned earlier, the optimal balance for vote maximization depends on party activism (the marginal contribution rate) and on the non-policy attractiveness (or valence) of the candidate. A candidate with personally attractive qualities, such as integrity or charisma, can afford to move nearer the center of the electorate than a less attractive candidate. A candidate with lower valence is more dependent on the resources mustered by party activists, and consequently must move out from the center toward the more extreme policy positions advocated by those activists. So the ability of the Democrats to pick up votes by moving toward the social policy center is contingent on the “quality” of the candidate, and is constrained by Democratic Party activists. These activists are increasingly motivated to contribute on the basis of social issues. Indeed, in the last few years, the issue of war has become important.

The Democratic best response to the increasing power of social conservatives in the Republican Party must be to seek the support of the social liberals who are increasingly disaffected in the Republican Party. This involves a move toward the social liberal axis along what is marked as “the Clinton balance locus” in Figure 9.1.

As populists demonstrate that they are in control in the Republican Party, cosmopolitan voters in the upper-right-hand quadrant are increasingly alienated. A vote-maximizing Democratic candidate will inevitably see the political advantage of picking up Republicans who believe in teaching evolution, stem cell research, and a relatively open policy toward immigrants.

Many Republican employers see immigration as being a constructive force in the American economy, and are opposed to the hard line taken by House Republicans. Further, G.O. P. opposition to stem cell research will alienate millions of educated, professional economic conservatives who are personally concerned about the health benefits of stem cell research. Many of these voters have an economic stake in stem cell research, either

as stockholders in bio-engineering firms, as professionals in the health care industry, etc. Employers, stockholders in bio-engineering, health care professionals – they are likely to be economic conservatives who have probably voted Republican all their life. What would it take to get them to vote Democratic? Perhaps not much, if social conservatives continue to be intransigent on issues like immigration and stem cell research. As suggested in the next section, an inevitable party dynamic will pull many of those voters into the Democratic Party, bringing activists and candidates along with them.

9.4.1 Party Dynamics

Whether undertaken as a consciously chosen strategy or not, the Democratic Party is going to move to the right, on the economic dimension, while staying strictly liberal on the social dimension. It will happen as a result of already-existing pressures driving socially liberal economic conservatives out of the Republican Party.

The public perception of each party is determined by the composition of each party's activists. Voters for each party make decisions about whether to become activists based on the relative location of the two parties, as determined by the existing mass of activists. If moderate social liberals leave the Republican Party for the Democratic Party, then the social policy differences between the two parties becomes even more salient, and more motivating to social policy activists.

Furthermore, as social policy activists are sorted into the two parties, this has enormous implications for potential candidates. Party activists, for example, have enormous influence in primary elections, where they constitute a larger proportion of voters who actually turn out, and where they have a big impact on voter mobilization.

The increasing dominance of social conservatives in the G.O. P. is best seen in the consequences of their influence on Republican Party primaries. One consequence will be challenges to Republican moderate (socially liberal) incumbents. A second consequence will be that Republican moderates who hope to hold office will simply switch to the Democratic Party. Both of these have the further effect of increasing the polarization of the two parties along the social dimension, while decreasing the economic policy differences between the two parties.

9.4.2 Party Challenges

While most spatial models of party competition assume a monolithic party actor, the view presented here is that the party is composed of multiple groups – activists of different ideological stripes, and party candidates and leadership who would like to respond to the centripetal force of the electoral center. The primary task facing the Republican Party is the same as it is for every majority party under plurality rule – to keep the coalition together in the face of conflict among diverse supporters. Just as L. B. Johnson confronted increasing conflict between civil rights supporters and the southern elite, Republicans face increasing conflict between moderate pro-business Republicans and hard-core social conservative activists. The outcome is likely to be as significant a transformation in the Republican Party as the one that occurred in the Democratic Party 40 years ago.

Republicans like Danforth call for the return of the Republican Party to its position in earlier decades: fiscally responsible, supportive of business (even in immigration policy), civil libertarian on issues like abortion. But “return” is something never seen in partisan realignments. One might as well call for a return to the Republican Party of Teddy Roosevelt or Lincoln. Social conservative activists, having captured the state and local machinery of the Republican Party, have no incentive to give it up, even in the interest of greater Republican success.

A case in point is Randy Graf, in Arizona’s 8th district, which includes the relatively liberal voters of Tucson as well as the conservative ranchers of Cochise County. The seat had been held for 22 years by moderate Republican Jim Kolbe, until he announced his retirement in 2006. The 2006 election was recognized early on to be one in which the Democrats had an opportunity to take back the House; as a result, the Republican National Committee hoped to keep this seat by running another moderate Republican – Steve Huffman. In fact, the RNC donated at least \$122,000 to Huffman’s campaign.

However, Randy Graf, a former golf pro, challenged the RNC’s candidate. Graf was a founding member of the Minutemen, a conservative anti-immigration group whose members took it upon themselves to patrol the border. He was also anti-abortion and against same-sex marriage. While Huffman had the support of the national party, Graf had the whole-

hearted support of the Minutemen and other social conservatives, who felt immigration was directly linked to crime, drugs, and the destruction of American identity. Graf claimed that there was “real fury with Washington,” and ran with the slogan, “change can’t wait.” Graf’s supporters called Huffman a RINO – Republican in Name Only, a favorite term used by social conservatives trying to drive social moderates out of the Republican Party. He angrily denounced the Republican Party for supporting his moderate opponent. He won the primary with 42 percent of the vote against Huffman’s 38 percent.

The Republican National Committee continued to worry about whether Graf could be elected. In the face of polls indicating a large loss for Graf, the RNC canceled \$1 million in financial support for Graf’s ads. Democrat Gabrielle Giffords found it easy to stake out a centrist, winning position, and took the House seat away from the Republicans with a 54 percent to 42 percent vote.

This election illustrates that parties are not monolithic. Elements of the party – the candidates and the national committees – are more concerned about electability than ideological purity. The National Republican Party would rather support a Republican candidate who is close to his districts’ median voter than a social conservative who could lose a House seat for the party. However, warring activists of different stripes are not generally willing to make ideological sacrifices in the interest of their parties’ candidates. At least in Tucson, the Republican Party moved incrementally toward the social conservatives, despite the leadership’s attempts to apply the brakes!

Conflict within the party played a similar role in the defeat of Lincoln Chaffee. Chaffee is a fiscal conservative and social liberal in the mold of New England Republicans going back to the Civil War – much like Prescott Bush, one-time Senator from the neighboring state of Connecticut. Chaffee was the only Republican senator to vote against the Bush tax cuts and against the Iraq war; he also let it be known he was going to write in the name of the president’s father in protest against Bush’s policies.

These actions invoked a conservative challenge in the Republican primary – from Stephen Laffey, Mayor of Cranston, and pro-lifer. In this case, Chaffee was able to beat back the conservative challenger, at some cost to his election resources. Alienating the conservative wing of the

Republican Party left him more vulnerable to a popular Democrat. The Democrat, Sheldon Whitehouse, defeated Chaffee by 53 percent to 47 percent. The result was that Senate Republicans became more homogeneously conservative on social issues; the prospect of the G.O. P. “returning” to the Republicanism that Chaffee, Jeffords and Danforth remember became even more remote.

In Rhode Island and Arizona, primary challenges by conservatives helped undermine the strategy of Republican Party leaders, who hoped to maintain control of Congress by offering strong, unified support for moderates. The challenges by Graf in Arizona and Laffey in Rhode Island illustrate in a striking way that the increasing power of social conservatives is an autonomous force for repositioning the Republican Party.

9.4.3 Party Switches

As social conservatives come to dominate the machinery in a given state or electoral district, socially moderate Republican candidates may be forced into the Democratic Party simply because they can no longer hope to win a Republican primary. The normal ambition of politicians transforms socially liberal Republicans into moderate Democrats. And once again, the result is increased party polarization on the social dimension and decreased party differences on the economic dimension.

A case in point is John Moore, a long-time executive with Cessna Aircraft in Wichita; a pro-business conservative, he was nevertheless unlikely to win a Republican primary for any state-wide position due to his “softness” on social issues. He consequently converted in 2002, and was elected as the Democratic lieutenant governor.

Moore retired in 2006, and the open position brought about an even more dramatic development. Mark Parkinson officially switched parties in time to run for the lieutenant governor’s position. Parkinson is a former Republican Party Chairman for the state of Kansas, and was elected lieutenant governor on the Democratic ticket. Others in Kansas are going the same route. In 2004, Republican Cindy Neighbor switched parties to run for the state legislature, opposed to a social conservative who had defeated her in the primary in 2004. She was elected in 2006.⁶³

Nor are these ballot box conversions limited to ambitious Kansas mod-

⁶³Milburn (2006).

erates. Perhaps the most striking and visible conversion was that of Jim Webb of Virginia. Webb is a much-decorated Vietnam war veteran who had been Reagan's Secretary of the Navy. As recently as 2000, he supported Republican George Allen to be the U.S. Senator from Virginia. In 2006, he was a Democrat running against Allen. Traditional New Deal Democrats were aghast; but Webb defeated Allen, and his presence in the party moves the Democratic center of gravity to the right on economic policy.

Each such switch makes further switches more likely. While Kansas has been seen as a state in which the Democratic Party is all but defunct, the conversion of a small number of socially moderate Republicans to the Democratic Party could easily restore a healthy two-party competition in Kansas. But in the process, each individual conversion *changes what it means to be a Democrat*. Increasingly, a Democrat is an economic moderate or conservative who is strongly liberal on social issues – not (as in the New Deal) a strong economic liberal whose Democratic affiliation is a response to class conflict.

These observations are not meant to advocate any particular strategy for either party. Rather, they suggest that partisan change continues to have a certain inevitability about it, despite the fond wishes of entrenched party activists. Each partisan realignment has occurred *despite the opposition of existing party activists*.

Populist Democrats in the 1930s, who had supported Bryan in the past, were suspicious of the ethnic industrial laborers that the New Deal brought into the party. In the same way, traditional Republican activists were aghast when their candidate Rockefeller was booed for criticizing Goldwater-style radicalism at the 1964 convention.⁶⁴

Partisan realignment is a dynamic process because of the destabilizing influence of vote-maximizing candidates who see opportunities to win elections even at the cost of generating some hostility within the ranks of the pre-existing activist cadres. As a result, partisan identities are always changing, even though there is a tendency to see them as fixed and immutable. The Republican Party in 1868 was the post-civil war party of racial equality through strong national government. The Republican Party in 1948 was the party of the balanced budget and civil libertarian-

⁶⁴Branch (1998: 402).

ism. Neither of these identities proved to be immutable, and the current identities of both parties are again in flux.

The departure of even a small number of pro-business social liberals from the Republican Party – like Jeffords of Vermont or Parkinson in Kansas – has inevitable effects on both parties. Each such departure increases the proportion of social conservatives in the Republican Party, making it easier for social conservatives to dominate both the party primaries and the activists who give the party its image to the nation. This in turn makes it even more difficult for social liberals to hope for a successful career within the G.O. P. Voters, as well as activists and candidates, adjust. If they are concerned about women's rights or the separation of church and state, they are less likely to vote as Republican and more likely to shift to independent or Democratic status.

At the same time, symmetrical adjustments are made in the Democratic Party. Just as Strom Thurmond's conversion to the Republican Party helped trigger a long list of similar conversions by socially conservative Democrats, so each socially liberal Republican who converts to the Democratic Party makes the social issue that triggered the conversion a more salient aspect of the Democratic identity. The Democratic Party and Republican Party become internally more homogeneous as regards economic policy, and more polarized with respect to social policy.

Thus, as social polarization increases between the parties, the economic differences will slowly disappear. As pro-business social liberals join the Democratic Party, it will become increasingly difficult to imagine that party going back to a New Deal identity. Just as the New Deal Democratic Party consisted of segregationists and labor unions united on an anti-business platform, the emerging Democratic Party will find itself united at a social liberal position, with a centrist position on economic policy. The proportion of Democrats who adopt a traditional anti-business stance will be reduced. A simple electoral calculus by candidates will tend to move them to a Clinton-style moderate position on economic policy – advocating (among other things), a more inclusive policy toward immigrants, and a more enthusiastic commitment to stem cell and related medical research.

Such a redefinition of the Democratic Party will serve as a catalyst for further change by the Republicans. The potential for these transformations is indicated by Tables 9.1 and 9.2 in the Appendix to this chapter.

These tables show the cross-cutting of the issues of stem cell research and immigration by the yea and nay votes in the Senate on the relevant bills. As Table 9.1 indicates, the pattern of Republican senatorial votes on stem cell research and immigration reform (on 25 May and 18 July, 2006) reveals some strong clustering, with 15 economic conservatives voting in favor of both measures, and a larger cluster of 27 social conservatives (most of them southerners) voting against both measures. The 32 Republican Nay voters were joined by 4 Democrats in the 18 July roll call. On 7 June, 2007, 38 Republicans were joined by 11 Democrats, and the independent Sanders of Vermont in voting against cloture (the cessation of discussion on the immigration bill which needed 60 Aye votes). This vote was essentially repeated on 28 June, killing the proposal until after 2008. From the perspective presented in this chapter, immigration reform involves both economic interests (pro-business) and the social axis (civil rights), and the policy proposals associated with the bill can be located in the upper right quadrant. This inference gives a rationale why the Republicans were overwhelmingly opposed, and were joined by a significant number of democrats.⁶⁵

9.5 The Future of Republican Populism

William Jennings Bryan, of course, was an anti-business radical as well as a social conservative. Bryan's social position is now ascendant in the social conservative in the Republican Party. Is it possible to imagine a day in which the Republican Party adopts Bryan's economic radicalism as well as his social conservatism?

In the short run, the move by pro-business social liberals from the Republican Party to the Democratic Party will make both parties look

⁶⁵The ability of the Republican Senators to block legislation was further illustrated on July 18, 2007, when the attempt failed on cloture on discussion of a bill "To provide for a reduction and transition of United States forces in Iraq." The attempt only obtained 52 aye votes (47 Democrats, 4 Republicans and the independent Sanders of Vermont). The Democrat Party leader, Reid of Nevada, voted Nay with the Republicans so as to leave open the option of reconsideration of the proposal later. Even with the support of four social moderate Republicans, attempts at cloture on a somewhat different bill on Iraq and an immigration bill involving high school graduates both failed in mid September 2007.

more moderate on economic policy. In the long run, the same dynamic could actually make the Republican Party more blue-collar than the Democrats. Social conservatives in the Republican Party are already quite insistent that the Democratic Party is the party of privilege and elitism. The populist rhetoric adopted by the Republican Party has pictured the Democratic Party as the home of overpaid professors, bureaucrats and social technicians. Democrats are seen as “limousine liberals” who want to indulge themselves in expensive pro-environmental policy, and who have nothing to lose when wages collapse to the levels of Third World countries.

If the Democratic Party continues to pick up social liberals like Jeffords and Parkinson (either by conscious strategy or just because they have nowhere else to go), then professionals and business leaders in the party will balance the beleaguered unions. These new elements of the party will be on the side of a balanced budget, open immigration, and accommodation with business (especially in the new computer and biotech industries). Most difficult for traditional Democrats will be the support for free trade among the new Democrats. The economic liberals in the Democratic Party will feel increasingly isolated and alienated. Listening to the populist rhetoric of Republican activists and politicians, blue-collar workers may come to expect the Republican Party to represent their economic interests, in addition to their social conservative positions.

Some Republican politicians, already accepting the social values of blue collar workers, will decide to represent the economic aspirations of their constituents as well. Why not, if professional and business elites are already heading for the door? Indeed, the role model for the complete 21st-century reincarnation of Bryan is already visible: Patrick Buchanan.

In *The Great Betrayal*, Buchanan sounds like the epitome of the anti-business populist. He blames the business elite for crucifying the working men and women of the country on a cross of free trade. NAFTA, signed by Clinton with Republican votes, is “not sustainable. NAFTA puts U.S. blue-collar workers into competition for manufacturing jobs with Mexican workers who earn 10 percent of their wages. . . . American employers now hang over the head of their workers this constant threat: accept reduced pay, or we go to Mexico!”⁶⁶

⁶⁶Buchanan (1998: 309).

As Buchanan has demonstrated, it would be a simple matter to take up the populist economic policy along with the rhetoric. Buchanan does not hesitate to point to “corporate executives” as being complicit in *The Great Betrayal*: Having declared free trade and open borders to be America’s policy, why are we surprised that corporate executives padlocked their plants in the Rust Belt and moved overseas? Any wonder that Nike president Philip Knight is the fifth-richest man in America, with \$5.2 billion, while his Indonesian workers make thirty-one cents an hour?⁶⁷

Like Phillips in 1969, Buchanan sees the future of the Republican Party in “the new populism” with both an economic and social agenda: protectionist, and anti-immigration, and anti-capitalist as well as anti-abortion. Buchanan is the model for the Republican incarnation of Bryan.⁶⁸ In the long run, a Buchanan-style Republican could complete the cycle by forming a new “New Deal” between rural social conservatives and economic liberals.

Changes in party identity will not happen quickly or without a great deal of pain caused by the political dislocation. These transformations will not necessarily result in a single, “realigning” election. However, as long as Americans understand politics to consist of more than a single dimension generated by economic ideology, then no majority coalition in the U.S. polity can be immune from the kind of tension that will eventually lead to its replacement. A second dimension, revolving around race, ethnicity, immigration, and religious values, has always been latent, even when it was suppressed as during the New Deal. The multidimensionality of U.S. politics is now apparent in contemporary life. Everything that is known about multidimensional two-party politics suggests that an inevitable dynamism accompanies American politics. The future decades will reveal the impact of today’s ongoing transformation.

9.6 Concluding Remarks

The main purpose of this chapter has been to show that convergence need not be expected in party competition. Since activist support is of key im-

⁶⁷Ibid., p. 16.

⁶⁸Buchanan (1999) expresses isolationist views that appear similar to those we may ascribe to Bryan.

portance in elections in the United States, it can be expected that party candidates will move away from the electoral origin towards the activist catenaries. There is substantial evidence that it is necessary to employ a two-dimensional policy model with both economic and social axes in order to understand American politics. This implies that party success requires the formation of coalitions among actors who have conflicting policy preferences over at least one dimension of policy. A successful party coalition is a “coalition of enemies.” This can be seen in the remarks by Republican John Danforth, quoted previously, clearly revealing the increasing hostility between evangelical Christians and the business community within the G.O. P.

Since at least the Reagan era, the winning Republican coalitions have consisted of social conservatives (located at *C* in Figure 9.1) and economic conservatives (located at *E*). However, many of the most salient issues of the 21st century threaten to split this Republican coalition. Stem cell research is important to the pro-business conservatives, but anathema to social conservatives. Mexican immigration helps business by keeping labor prices low, but is regarded with hostility by social conservatives (many of them wage-earners) who see it as a threat to their own livelihood as well as to traditional American values.

The model developed in this chapter incorporates just this kind of intra-party tension to explain the paradox of non-convergence in American party politics. Social conservative activists use new issues like stem cell research and immigration to pull the Republican Party along the catenary toward their bliss point at *C*. Republicans like Danforth see this move as a forfeiture of traditional pro-business ideals: “limited government,” “keeping light the burden of taxation and regulation.” They exert a contrary “tug” toward *E*. The combined effects of these two activist vectors must be balanced by an electoral pull toward the center to create a party equilibrium that is characterized by divergent party locations.

The equilibria of the Democratic and Republican parties are of course inter-dependent. To the extent that social conservatives are successful in moving the Republican Party location toward their bliss point, the model clearly indicates the Democratic best response. The cosmopolitans – like Danforth – have the most to lose by the increasing commitment of the G.O. P. to socially conservative causes. Many of these are professionals or business executives, with sympathy for women’s rights, environmen-

talism, and civil liberties, who have always supported Republicans after consulting their pocketbook. But if the Republican Party is seen as the party of strict immigration control (and thus higher wages) and restrictions on stem cell research (and thus foregone opportunities for growth in new biotechnology industries), then they may change their support.

The model indicates that, as the Republican Party moves toward *C*, then disaffected cosmopolitans may be increasingly tempted to contribute resources to pull the Democrats toward *S*. A Democratic Party candidate who maintains traditional social liberal positions (in support of African-American voters and womens' rights) while moving toward a moderate position in economic policy could obtain significant resources from activists located in that quadrant. Just as the Reagan Republicans constructed a winning majority by picking up social conservatives, so may a Democratic candidate find it possible to construct a new majority by attracting economic conservatives who are also social liberals. Taking an emphatic stand in favor of stem cell research and in favor of immigrants' rights could be the kind of signal needed to accomplish this goal.

The strategic problem facing the Democratic Party is to take advantage of the growing rift in the Republican Party. While many strategists urge the Democrats to return to strong New Deal economic liberalism, this strategy would do nothing to attract disaffected cosmopolitan Republicans, and on the contrary would do a great deal to drive them back into a forced coalition with the social conservatives. Nor would a reversion to New Deal liberalism result in a resurrection of the New Deal coalition. Now that issues such as abortion, gay marriage, and the Iraq war have entered into political consciousness, southern and Midwestern conservatives are not going to return to the Democratic Party by promises of protecting unions or increases in the minimal wage.

The two-dimensional nature of U.S. politics forces one to the conclusion that the next winning strategy for Democrats is to look to the suburbs, for moderates who support teaching of evolution, are eager for the health benefits of stem cell research, and offended by the effect of the Iraq war on civil liberties. Such a strategy does not require Democrats to compete for economic conservatives with tax cuts, but it no doubt requires a moderate economic policy that leaves middle-class voters feeling

comfortable voting Democratic.⁶⁹ It may mean de-emphasizing southern voters, while searching for the growing numbers of anti-war, pro-environment voters in the West.

As the formal model suggests, the other determinant of party location is party valence. The relative party valence determines whether the trade-offs between party activists occur close to the origin, or at some distance from it. The party with the lower exogenous valence is forced to move out in order to attract what non-centrist support it can from committed policy activists. Current poll results indicate a shrinking valence for the Republican Party. In May 2006, only 37 percent of Americans thought the Republican Party came closer to sharing their moral values (compared to 50 percent for Democrats), and only 22 percent of Americans thought the Republican Party was more likely to protect their civil liberties (compared to 62 percent for Democrats).⁷⁰ These figures indicate that $\lambda_{dem} - \lambda_{rep}$ is increasing, forcing the Republican Party to move out to appeal to its activist base. From the standpoint of this model, the ideal Democratic candidate would be a well-liked economic moderate who is clearly differentiated from the Republicans on the basis of social (not economic) policy.

Current events illustrate this chapter's focus on the use of non-policy electoral "valence" and the policy demands of party activists to resolve the paradoxical non-convergence of the two-party system in the United States. The formal model demonstrates how a divergent equilibrium emerges as each party attempts to balance the centripetal pull of electoral politics against the centrifugal pulls from distinct coalitions of party activists. Exogenous shocks, and the emergence of newly salient issues, such as stem cell research, can change the parameters that determine the exact location of a party equilibrium. The ability of each party to maintain a fragile coalition of party activists with quite different agendas may vary, creating opportunities for the other party to attract increasingly disaffected activists.

Schofield (2006a) suggests that extreme conflict over new issues may

⁶⁹See Frank (2004); Micklethwaite and Wooldridge (2004); Schaller (2006); Wallis (2005).

⁷⁰*New York Times*, 10 May 2006, p. A18. Obviously the situation in Iraq had an effect on these perceptions. The recent book by Gore (2007) gives a possible reason for these changes.

fragment the activist groups in a society. The result may be that party leaders and candidates become more extreme in the positions they adopt. This would provide an underlying logic to what has seemed a paradox in U.S. politics. Although the electorate seems to be no more polarized than in the past, the parties do seem more polarized.⁷¹

⁷¹See Fiorina (2005) and McCarty, Poole and Rosenthal (2006).

9.7 Appendix: Republican Senator Votes

Table 9.1. Votes of Republican senators on immigration and stem cell research

		Stem cell	
		Yea	Nay
Immigration Reform 2006	Yea	Bennett UT	Brownback KS
		Chafee RI	Craig ID
		Coleman MN	DeWine OH
		Collins ME	Graham SC
		Domenici NM	Hagel NE
		Frist TN	Martinez FL
		Gregg NH	McConnell KY
		Lugar IN	Voinovich OH
		McCain AZ	
		Murkowski AK	
		Smith OR	
		Snowe ME	
		Specter PA	
		Stevens AK	
		Warner VA	
Total	15	8	
	Nay	Alexander TN	Allard CO
		Burr NC	Allen VA
		Cochran MS	Bond MO
		Hatch UT	Bunning KY
		Lott MS	Burns MT
			Chambliss GA
			Coburn OK
			Cornyn TX
			Crapo ID
			DeMint SC
			Dole NC
			Ensign NV
			Enzi WY
			Grassley IA
			Hutchison TX
			Inhofe OK
			Isakson GA
	Kyl AZ		
	Roberts KS		
	Santorum PA		
	Sessions AL		
	Shelby AL		
	Sununu NH		
	Talent MO		
	Thomas WY		
	Thune SD		
	Vitter LA		
Total	5	27	

Notes: (Yea, Yea) = 15 pro-business, (Nay, Nay) = 27 social conservatives.
Names in bold from southern states of the old Confederacy.

Chapter 10

Final Remarks

10.1 The Madisonian Scheme of Government

In order to provide a motif for the topics discussed in the book, it is worth giving a fuller version of the quote from Madison that opened the introductory chapter.

[I]t may be concluded that a pure democracy, by which I mean a society, consisting of a small number of citizens, who assemble and administer the government in person, can admit of no cure for the mischiefs of faction. A common passion or interest will . . . be felt by a majority of the whole . . . and there is nothing to check the inducements to sacrifice the weaker party Hence it is, that such democracies have ever been spectacles of turbulence and contention; have ever been found incompatible with personal security, or the rights of property; and have in general been as short in their lives, as they have been violent in their deaths. . . .

A republic, by which I mean a government in which the scheme of representation takes place, opens a different prospect

The two great points of difference between a democracy and republic, are first, the delegation of the government, in the latter, to a small number of citizens elected by the rest; secondly, the greater number of citizens and greater sphere of country, over which the latter may be extended. . . .

[I]t may well happen that the public voice pronounced by the representatives of the people, will be more consonant to the public good, than if pronounced by the people themselves. . . .

[I]f the proportion of fit characters be not less in the large than in the small republic, the former will present a greater option, and consequently a greater probability of a fit choice . . .

[A]s each representative will be chosen by a greater number of citizens in the large than in the small republic, it will be more difficult for unworthy candidates to practice with success the vicious arts, by which elections are too often carried; and the suffrages of the people . . . will be more likely to centre on men who possess the most attractive merit and the most diffusive and established characters.

The other point of difference is, the greater number of citizens and extent of territory which may be brought within the compass of republican, than of democratic government; and it is this . . . which renders factious combinations less to be dreaded in the former, than in the latter. . . . Extend the sphere, and you take in a greater variety of parties and interests; you make it less probable that a majority of the whole will have a common motive to invade the rights of other citizens. . . .

Hence it clearly appears, that the same advantage, which a republic has over a democracy . . . is enjoyed by a large over a small republic—is enjoyed by the union over the states composing it. . . .

The influence of factious leaders may kindle a flame within their particular states, but will be unable to spread a general conflagration through the other states. . . .

In the extent and proper structure of the union, therefore, we behold a republican remedy for the diseases most incident to republican government.⁷²

In these final remarks, I shall try to relate Madison's justification for the republican scheme of representation that he made in *Federalist X* to the social choice theory presented in the earlier chapters of the book and the empirical work of the later chapters.

The key to my understanding of a general theory of social choice is that any polity must, on occasion, face difficult choices over what I call *constitutional quandaries*. Simply put, a quandary is a choice situation where all possible options appear extremely unpleasant, and laden with

⁷²James Madison, *Federalist X* (1787), in Rakove (1999: 164-166).

risk and uncertainty.⁷³ The constitutional feature of the quandary refers to the likelihood that opinion as regards the correct choice will typically be highly heterogenous. The actual choice will depend on the political mechanisms used by the society, and thus on the constitutional rules that govern political choice.

The results from social choice theory indicate that when preferences or opinions are sufficiently heterogenous, then disorder or *chaos* can ensue. The process of social decision-making is denoted by correspondence, Q , so $Q(x)$ is the set of outcomes that can come about from x , as determined by whatever social rule or political process is used by the society. The idea of *social chaos* is that there are conditions under which, starting from almost any x , it is possible to reach almost *any* possible outcome $y \in Q^t(x)$ by reiterating the social rule. In contrast we can identify the *core* or *social equilibrium*, y , to be some “stationary” outcome such that $Q(y)$ is empty. An even stronger equilibrium notion is that of an *attractor* of Q : that is a single outcome y with $y = Q^t(x)$, which results from any x , after a sufficient number of iterations of the rule. As Chapter 4 has shown, for any voting procedure, Q , without a dictator, oligarchy or collegium, with the capacity to control or restrain social choice, then as the dimension of W increases then so does the likelihood of voting chaos.⁷⁴

While these results focused on voting rules, it seems just as likely that chaos can ensue in a society where war is a possibility. When war, or intense and unrestrained conflict dominates, then we can expect chaos, and unpredictability.⁷⁵ Indeed, this aspect of social choice theory suggests that it is possible that any society can fall into disorder when there is no collegium.⁷⁶ As Keynes (1936) observed, dictators often come to power

⁷³Iraq in late 2007 is such a constitutional quandary for the United States.

⁷⁴As described in the earlier chapters, a voting rule is characterized by a family of winning coalitions, \mathbb{D} . A *collegium* is a group of voters that belongs to every coalition in \mathbb{D} , but need not itself be winning. An *oligarchy* is a winning collegium, while a *dictator* is a single agent oligarchy.

⁷⁵Bates *et al.* (2003) have documented over 400 instances of countries falling into political instability in the period 1955 to 2002.

⁷⁶The classic example of a fall into chaos is France, from the first meeting of the Estates General in May 1789, through the execution of King Louis XVI in January 1793, followed by the Terror. This chaos ended when Napoleon took power by coup d'état in November 1799, later being crowned emperor on 2 December, 1804.

when the people are in fear of chaos.

Madison also made an observation about the need for collegial power to induce stability:

for the harmony of that [British] Empire, it is evident I think that without the royal negative or some equivalent controul [*sic*] the unity of the system would be destroyed. The want of some such provision seems to have been mortal to the [ancient] Confederacies.⁷⁷

Federalist X indicates that Madison certainly viewed direct democracy as subject to chaos. Since a legislative assembly can be understood as a direct democracy, social choice theory provides a formal basis for Madison's argument about direct democracy and about what he called the "mutability" of the legislature.

The first method of mitigating chaos is to impose the concentration of power implied by the existence of a dictator, oligarchy, or collegium. Because a dictator can control every choice, it is very unlikely that such a degree of concentration of power can ever actually occur. However, we can use the term *autocrat* for one who controls the levers of power of the polity, and has at least the ability to declare war without being constrained by some strong form of political veto.⁷⁸

Schofield (2006a) suggests that a pure autocrat is likely to be willing to take extreme risks.⁷⁹ This further suggests that an autocrat will be more risk-taking than an oligarchy or a collegium. In the current situation where the Republican members of the Senate are able to block legislation that is contrary to the policy preferences of the President, then he does have some autocratic power, and I shall use the term *weak autocrat* to characterize his power. In his writings, Madison expressed concern that the president would gain autocratic power, and to avoid this, the Congressional super-majority counter-veto was devised. Thus Congress can

⁷⁷Letter to Jefferson, 24 October 1787, in Smith (1995: 498). Here, Madison refers to the collegial power of the sovereign by the term "the royal negative."

⁷⁸Saddam Hussein may not have had the total power that formally characterizes a dictator, but he certainly was an autocrat in Iraq. Putin also appears to have gained autocratic power in Russia. See Jack (2004) and Remington (2005, 2006). On 3 November 2007, Pervez Musharaff moved to consolidate autocratic power in Pakistan.

⁷⁹No formal reason is given, but the historical examples in Kennedy (1987) and the discussion in Kershaw (2007) of the choices by Axis leaders in World War II lends some support to the hypothesis.

be interpreted as having collegial power. However, because Congress may be factionalized, it can, as Madison expected, exhibit what he called “mutability”—a degree of disorder or incoherence in the laws that are passed. My understanding of the U.S. Constitution is that the Presidential veto was designed to overcome Congressional mutability.

It is evident that there may be a tendency for U.S. presidents to display some of the degree of risk preference that characterizes autocrats. I judge that Congress will generally be risk-averse, which is why, I believe, power to declare war resides in Congress. From this perspective, the weak autocracy that I ascribe to the president is an important feature of the U.S. Constitution because risk taking is an essential component of presidential power.⁸⁰ Moreover, Congressional risk-avoidance, or mutability, has the effect of delaying the resolution of fundamental constitutional quandaries. Typically, a *quandary* can only be faced if there is a risk-taking leader capable of forcing resolution.⁸¹ At the same time, the purpose of the Congressional veto, aside from restraining any tendency to full autocracy, is to cause the president to temper his risk-preference with caution.⁸²

In Chapter 5, the concept of the heart, $\mathcal{H}(Q)$, was introduced for any method of decision-making, Q . In contrast to the possibility of chaos, it was suggested that the outcomes of political decision-making in a mature democratic society would lie in the heart of the legislature, and this domain would be relatively constrained by the legislative or constitutional rules of veto and counter-veto, and by the preferences of the legislators. This notion was used in Chapter 6 to estimate the range of possible outcomes in various legislatures based on electoral rules using some form of proportional representation.⁸³ In principle, this concept of the heart can be used as a technical device to model the balance of power and prefer-

⁸⁰Many writers since Schlesinger (1973) have used the term “Imperial presidency” for the weak autocracy of the president. See also Savage (2007).

⁸¹Chapter 9 discusses examples of presidential risk-taking by Eisenhower in 1957 and Johnson in 1964, both of which entailed a conflict over civil rights between the president and Congress.

⁸²A good example of this is the caution displayed by Franklin D. Roosevelt in late 1941 as he moved the country to a war-footing, paying attention to public opinion and the concerns of Congress (Kersaw, 2007).

⁸³Note, however, in any political game that is essentially zero-sum, as in Iraq at present, the heart will be very extensive and may represent a chaotic situation.

ence between President and Congress.

This first method of attempting to create democratic order depends on balancing power and risk-taking.⁸⁴ A second, quite different method of mitigating disorder depends on basing choice on judgement rather than preference. Madison was very clear in *Federalist X* that *representative democracy* involves the choice of a *person*. His argument there suggests that he believed that the voters in the Republic could make a sound choice for the Chief Magistrate if their *judgements* were not contaminated by *preferences*. As Madison argues in *Federalist X*, the choice should be for those who “possess the most attractive merit.” It is important for the constitutional balance between President and Congress that the president be elected by a method that gives what Madison called “a probability of a fit choice.” Since presidential choices will be made in the context of uncertainty and risk, it is vital that the electorate use their judgement in making a fit choice for President. Madison clearly hoped that the selection of the President would be founded on judgement, rather than preference, and that this selection would not exhibit the democratic disorder that he believed could render the legislature ineffective.

It can be argued that Madison’s argument in *Federalist X* was partly based on his reading of Condorcet’s *Essai* of 1785.⁸⁵ In constrained situations where we may assume that judgements predominate, and voters evaluate the options in a clear-sighted fashion, then their choice of Chief Magistrate may indeed be well formed in the way that both Condorcet and Madison thought possible.

My reading of *Federalist X* is that Madison hoped that, because the election of the Chief Magistrate involved the selection of a person, rather than an option (as in the passage of a law), judgement rather than preference or interest would predominate. On the basis of the work on the electoral model presented in the earlier chapters, I now consider a more general electoral model that incorporates both judgement and preference.

⁸⁴Madison’s argument for the necessity of this balance is set out in *Federalist LI*.

⁸⁵Condorcet’s Jury theorem in the *Essai* refers to the probability that a jury makes a correct choice on the basis of majority rule. Schofield (2006a) argues that Madison received work by Condorcet from Jefferson in Paris, and acknowledged receipt on 6 September 1787 (Smith, 1995: 492). This suggests that Madison adapted Condorcet’s idea during the Fall of 1787, while writing *Federalist X* for publication on 22 November 1787.

It is entirely possible, however, that beliefs or judgements in the electorate can be transformed in a chaotic fashion. Nonetheless, it is possible that electoral belief can be associated with what may be called a *core belief*. Since a *core*, in social choice theory, is an unbeaten alternative, we can, by analogy, use the notion of a *core belief* to refer to a belief that has general acceptance in the society.⁸⁶

The next section presents a general “Madisonian” model of election, that, in principle, is applicable to any democratic polity.

10.2 Preferences and Judgements

As in earlier chapters, we assume that individuals have preferences that be represented in terms of utility functions on some “policy” space W . This space characterizes both voter interests, and possible eventualities. The space W embodies preferences over economic factors – labor, capital and land – with an additional political axis involving civil rights or religion.

The interests or beliefs of the population or electorate, V (of size v), are described by a set $\{x_i\}$ of “ideal points,” one for each voter, i . An individual’s ideal point in the space, W , is used to describe or represent that voter’s interests. In the empirical models discussed in Chapters 7 and 8, the ideal points were obtained from surveys.

The set of options, N , of size n , is a set $\{z_j\}$, each one being a point in W . In the situation of an election, each element of N is a declaration of intended or proposed policy. There is one for each candidate, j . While it is usual to conceive of each z_j as simply a point, we can easily allow z_j to involve various possibilities, associated with differing probabilities of occurrence.

The most general form of the model is one where the “latent utility” u_{ij} of voter i for candidate j has the form

$$u_{ij}(x_i, z_j) = \mu_{ij}(x_i, z_j) - A_{ij}(x_i, z_j) + \theta_j^T \eta_i + \varepsilon_{ij}.$$

⁸⁶Of course, a core belief can change rapidly. A core belief in the U.S. electorate *circa* 2003 was that there existed a close connection between al-Qaeda and Saddam Hussein, and this belief provided the justification for the invasion of Iraq. This belief has turned out to be false. By late 2007 a core belief was forming that the U.S. should leave Iraq.

Here $\theta_j^T \eta_i$ models the effect of the socio-demographic characteristics, η_i , of voter i in making a political choice. That is, θ_j is a k -vector specifying how the various socio-demographic variables appear to influence the choice for option j , so $\theta_j^T \eta_i$ is simply a number indicating the influence of i 's socio-demographic characteristics on the propensity to choose j .

The term $A_{ij}(x_i, z_j)$ is a way of representing the “preference disagreement” between the interests of voter i and the j^{th} option. In Chapter 7, it was assumed that the function $A_{ij}(x_i, z_j) = \beta \|x_i - z_j\|^2$ was a quadratic function of the distance between x_i , the preferred position (or bliss point) of voter i and z_j , the declared policy of candidate j . Here $\| - \|$ is the Euclidean norm and $\beta > 0$ is the spatial parameter.

This model is stochastic because $\{\varepsilon_{ij}\}$ is a set of possibly correlated *disturbances*. The term $\mu_{ij}(x_i, z_j)$ is the perception by a voter, i , with beliefs or interests, x_i , of the “valence” of the option presented by the candidate j . This valence is a way of modeling the non policy *judgement* by voter i of the quality of candidate j .

In the general model, the probability, ρ_{ij} , that voter i chooses option j is

$$\rho_{ij} = \Pr[u_{ij}(x_i, z_j) > u_{ij}(x_i, z_k) \text{ for all } k \neq j].$$

We can apply this model in various ways.

First, consider the pure preference based “non-stochastic” or deterministic case where all $\varepsilon_{ij} = 0$, and where valence is zero.

The early chapters on social choice theory imply that if decision making is binary (pitting one option against another), and based on majority rule, or more generally on a non-collegial voting mechanism, then disorder (in terms of the non-existence of a core) can ensue as long as the dimension of W is sufficiently large. This theory seems applicable to the case of legislative decision-making, as discussed in Chapter 6.

In the stochastic model of elections we have assumed that $\{\varepsilon_{ij}\} \neq 0$, with $\{\varepsilon_{ij}\} = \{\varepsilon_j\}$ independent of i , and also pairwise independent and identically distributed. This model can be interpreted as focusing on the “beliefs” or judgements of the participants. In particular, if the spatial coefficient, β , is 0 then this is a situation of pure “belief aggregation.” Individuals will choose among the various options with probability determined by the valence judgement that they have made. By analogy with the social choice notion, a *core belief* is one held by a majority of

the society. Condorcet (1785) argued essentially that a core belief would tend to be a correct belief. Roughly speaking, *Condorcet's Jury Theorem* asserts that, in a binary choice situation, the probability that a majority selects the true outcome will be greater than the probability that a typical individual will select the truth. Rae (1969) and Schofield (1972a,b) used a version of the theorem to argue that an uncertain society should choose majority rule since this rule maximizes the probability of a fit choice.

Since judgement will vary in a society, the simplest way to model the variation in beliefs in the society is by supposing that all the valence terms, $\mu_{ij}(x_i, z_j)$ are zero, but that judgement is determined solely by the socio-demographic characteristics, η_i , of the individual. While this would induce some variation in judgements, voting models based on this assumption perform poorly in predicting voter choice. Moreover, the jury theorem depends on the condition of voter (pairwise) "statistical independence" which is a very strong assumption. Indeed, this assumption would not be satisfied in the case that voter judgements were based solely on socio-demographic characteristics.⁸⁷

This suggests considering a more general case with $\beta \neq 0$, so both interests and judgements are involved. As discussed in Chapters 7 and 8, such a model requires considering how the various candidates respond to the pattern of electoral preferences and beliefs. Chapter 7 considered the simplest case of *exogenous or intrinsic valence*, where $\mu_{ij}(x_i, z_j) = \lambda_j$, for all i . The model showed that we can expect "convergence" to the electoral mean only in the case that the differences between the candidate valences $\{\lambda_j\}$ are "low" (in comparison to the product of β and the electoral variance, as expressed by the upper bound on the convergence coefficient, c). Convergence implies that the mean electoral preferred point corresponds to a core belief about social policy. Indeed, when all parties converge to the electoral mean, then we can view this point as the "social attractor."

However, as the empirical examples of Israel and Turkey showed, when the valence differences are sufficiently high, then parties will adopt different policy positions on the principal electoral axis. Parties with the highest electoral valence will position themselves closer to the electoral

⁸⁷For example, individual preferences in Iraq are very probably determined by such sociodemographic characteristics, and there is little likelihood of the Jury Theorem being valid in this context.

center, and depending on the electoral regime, will either gain an electoral majority, or will be at the legislative core. While there may be a core in this situation, it will not be an attractor, since there can exist groups in the society with divergent political beliefs. Thus existence of a core belief depends in a delicate way on the distribution of the intrinsic valences or judgements of the electorate, and the response of political agents to these beliefs.

It may also be the case that the social decision is the consequence of what is called a *belief cascade*. As more individuals decide that option z_s , say, is superior, then other voters will, in turn, be swayed to form a judgement in favor of z_s . Statistical independence will not be satisfied under the conditions of a belief cascade, though it is possible to conceive of situations where a belief cascade can lead to a justified outcome. Examples given in Schofield (2006a) suggest that opposing belief cascades can be generated in a population, particularly in a situation of social quandary.

The results of Chapter 7 seem most relevant for electoral systems based on proportional representation (PR). The examples also suggest that it is necessary to extend the electoral model beyond exogenous valence, by assuming $\mu_{ij}(x_i, z_j) = \lambda_j + \mu_j(z_j)$, for all i . This additional component of belief, $\mu_j(z_j)$, called *activist valence*, is a function of the behavior of activists, but is again independent of i . This model assumes that valence is directly influenced by the contributions of activists, and these are indirectly determined by the policy positions adopted by candidates or party leaders. Note that as $\mu_j(z_j)$ increases, then the probability that a voter chooses option j also increases.

As discussed in Chapter 8, the model implies that parties balance the centrifugal tendency associated with activist support against the centripetal attraction of the electoral center. However, when the electoral rule is PR, then even small parties can gain representation in the legislature. This suggests that activist groups with diverse preferences may support parties that are far from the electoral center. Indeed, the logic of the model suggests that activist groups have little incentive to coalesce under PR. Thus, these diverse activist groups will support a heterogeneous set of parties. The data presented in Chapter 6 indicates that PR tends to result in a fragmented legislature. The legislative heart was used in this chapter to estimate the range of policy outcomes resulting from coalition bargaining in the legislature. It was only in a few cases, such as Israel

in 2006 and Turkey in 2007, that the valence of the dominant party was sufficiently high to allow it to adopt a centrist position from which it was able to control the legislative core.

It is obviously possible to extend this model, by allowing an individual's beliefs, or valence, to be a function of the voter's ideal point. While this is theoretically possible, the determination of political equilibria would be very difficult. On the other hand, incorporating socio-demographic characteristics in the model does allow for this kind of variation in beliefs.

One of the long-standing puzzles arising from the study of U.S. politics is why precisely the plurality or majoritarian feature of the U.S. electoral system generates a two party structure rather than the fragmented political configurations that appear to hold under PR.⁸⁸ The discussion in Chapter 9 leads to the conjecture that the plurality electoral system of the United States gives greater power to activists than is the case in politics based on PR. Activist groups seem to face increasing returns to size, because of the plurality electoral system, and this forces coalescence.⁸⁹ If this hypothesis is valid, then it would provide a reason why political competition under plurality leads to the two-party system. Because there generally will exist two dimensions of policy, the various activist groups will tend to coalesce into at most four separate and opposed groups, and there will be incentives for these four groups to form just two pre-election activist coalitions.⁹⁰ However, even in the United States, conflict within such activist coalitions at times of constitutional quandary can lead to the collapse or transformation of the two party coalitions, bringing about a realignment of the political system.

Figure 10.1 presents estimates of the positions of possible presidential candidates in the 2008 election. We may interpret these positions as indicative of the range of possible activist groups supporting the various

⁸⁸Riker (1953, 1964, 1987); Duverger (1954); Filippov, Ordeshook and Shvetsova (2004).

⁸⁹Notice that the the model of Chapter 8 made use of concavity, or decreasing returns, with respect to the effect of the resources of the groups, *once they have formed*. This is quite different from the possible increasing returns to scale, induced by a plurality electoral system.

⁹⁰Chapter 9 noted that strong ideological disagreement within one of the two activist coalitions can lead to its collapse and the entry of a third party candidate.

candidates. It is likely that by November 2008, these will have coalesced into just two candidate positions.

Although the changes wrought by such reconfiguration may be the result of transformations in the beliefs of the electorate, the hope that Madison had over the “probability of a fit choice” will depend on the ability of the electorate to ascertain the quality of candidates through their accurate judgement of the worth of past decisions and the validity of the proposals for dealing with the future. Madison’s concern about the “influence of factious leaders” is as relevant as ever.

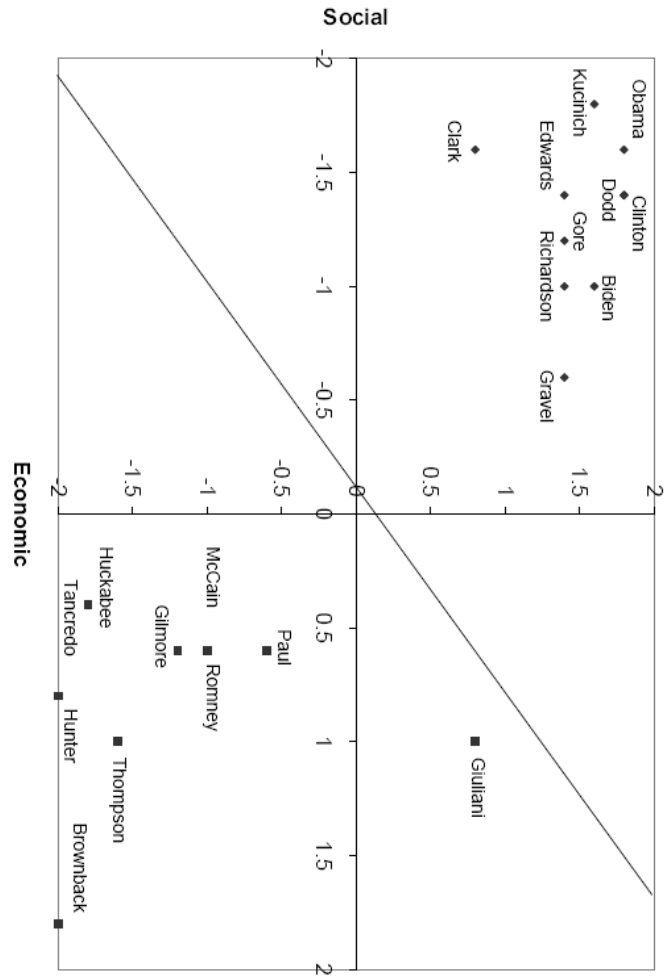


Figure 10.1: Estimated positions of possible candidates for the U.S. presidency

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