Axiom of *Choice*

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Popular Talk

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(A Brief History of Time)

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  (pardon this one contradiction…)

Overview

- What is an *Axiom*?
- Axiom of *Choice*
- A puzzle
- Impossible solution of that puzzle!

(ok... pardon this slide as well...)

What is an *Axiom*?

- a statement or proposition which is regarded as being established, accepted, or *self-evidently true*.

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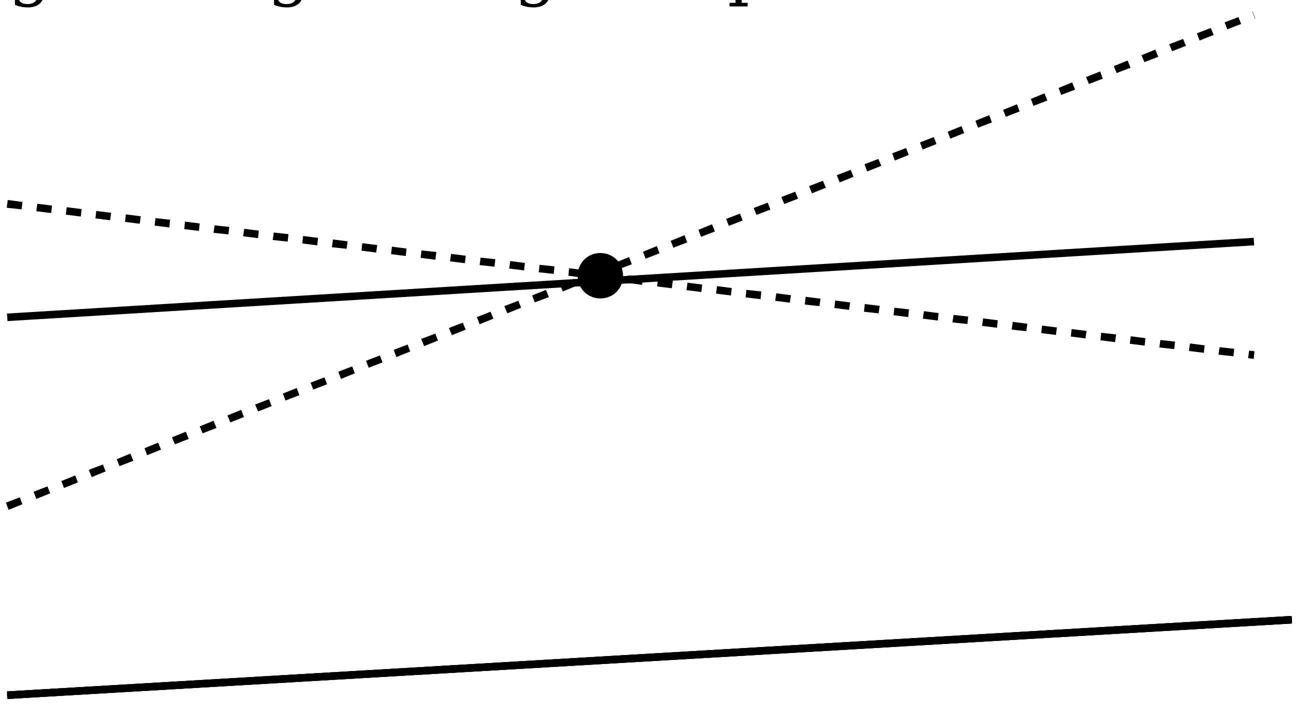
- *self evident truth* requiring no proof.
  -- Barrons' Word List #5
- ***Surprisingly, this is pretty much against the widely accepted technical interpretation!***

Parallel Line Axiom : Geometry

- Given a line and a point outside the line, there exists a unique line parallel to the given line and passing through the given point.



Parallel Line Axiom : Geometry

- The parallel postulate can neither be proved nor disproved using the other axioms of Geometry!

Parallel Line Axiom : Geometry

- So am I allowed to assume that the statement is false, without sounding illogical?

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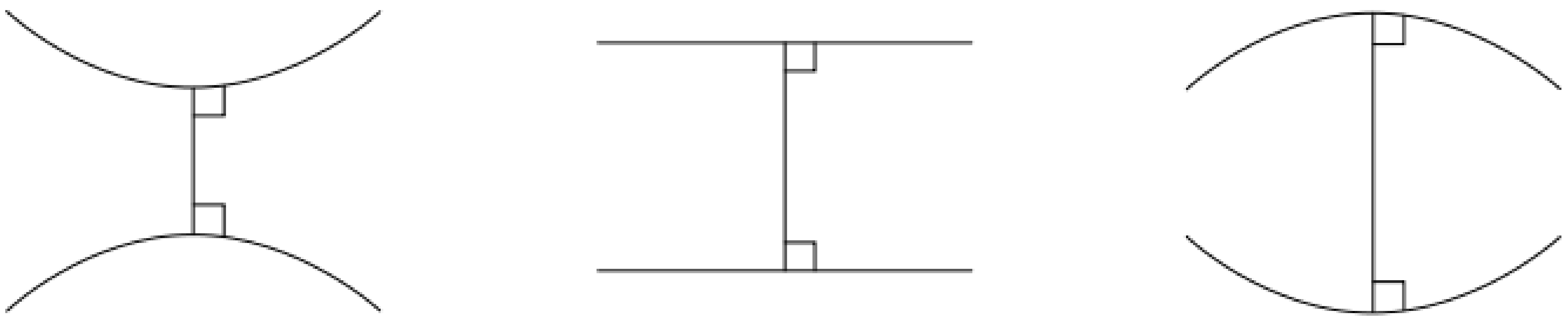
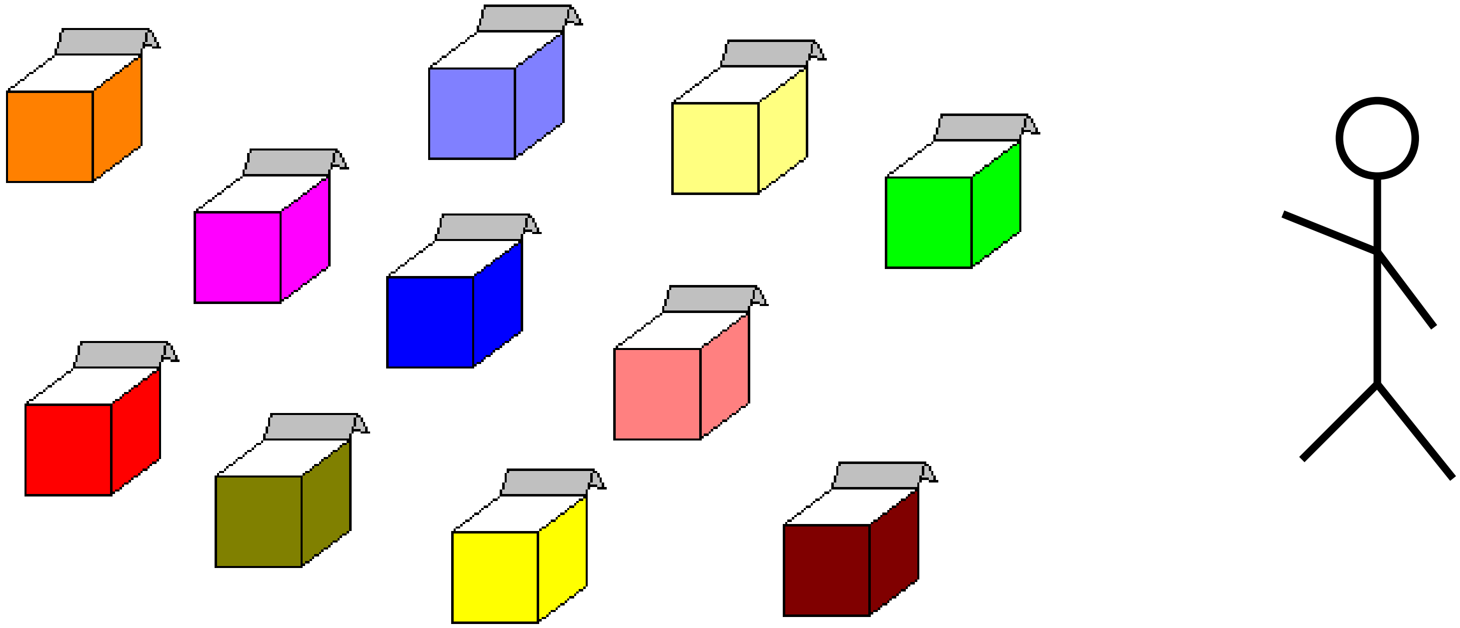


Hyperbolic Euclidean Elliptic

Axiom of *Choice*

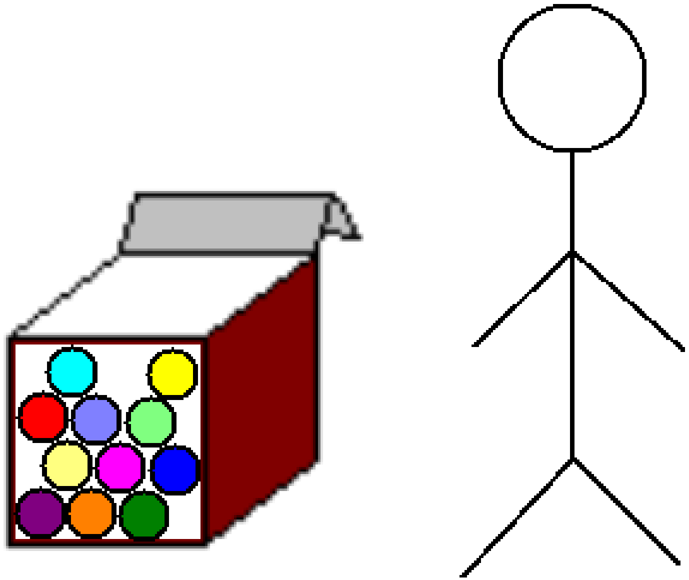
- Given a *set* of ***non-empty boxes***, it is possible to ***choose*** an object from each box.



  – Any Questions ?

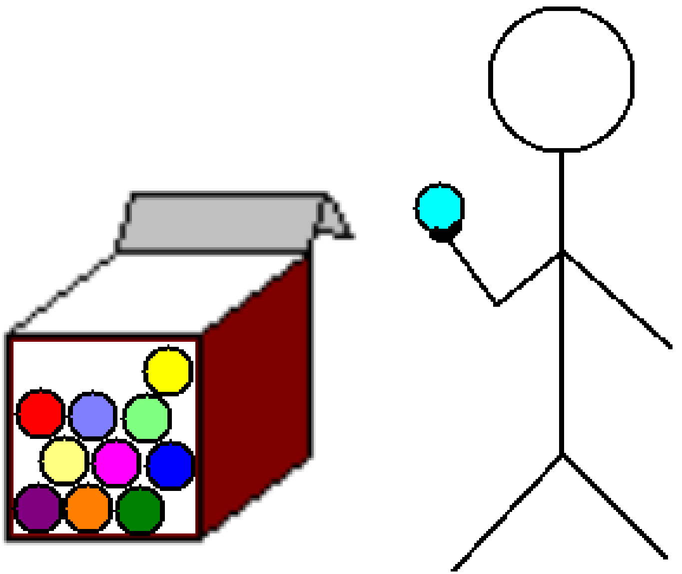
So what's the fuss?

- Given ***one non-empty box***, is it possible to ***choose*** an object from that box?



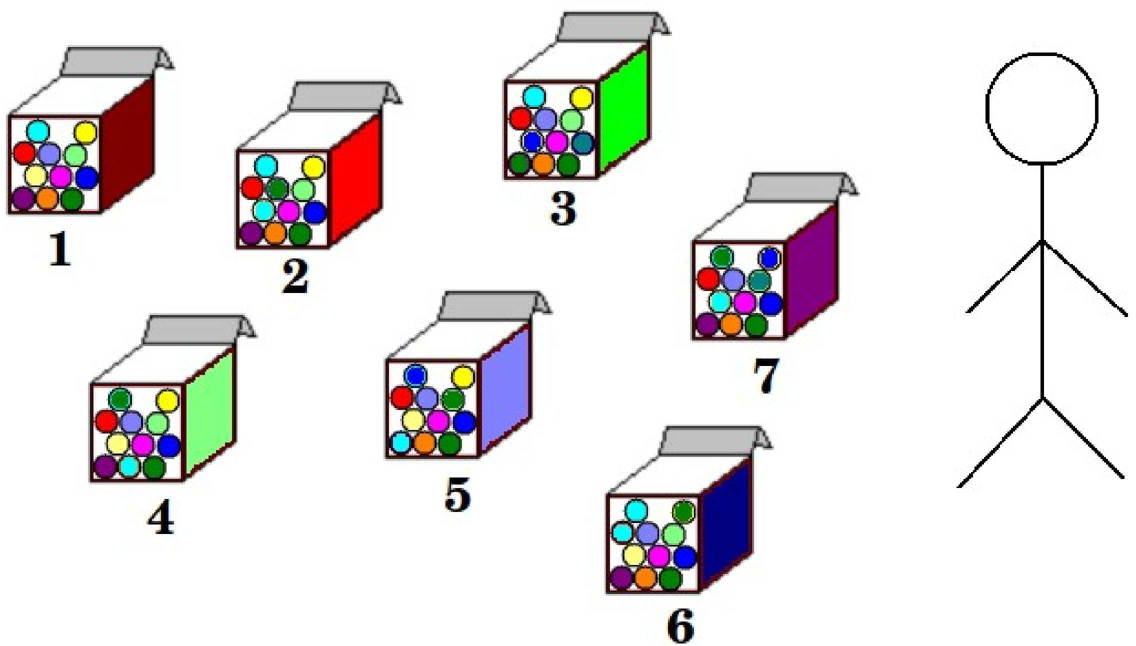
So what's the fuss?

- Yes!! Give me a box.
  I can choose one element from it...



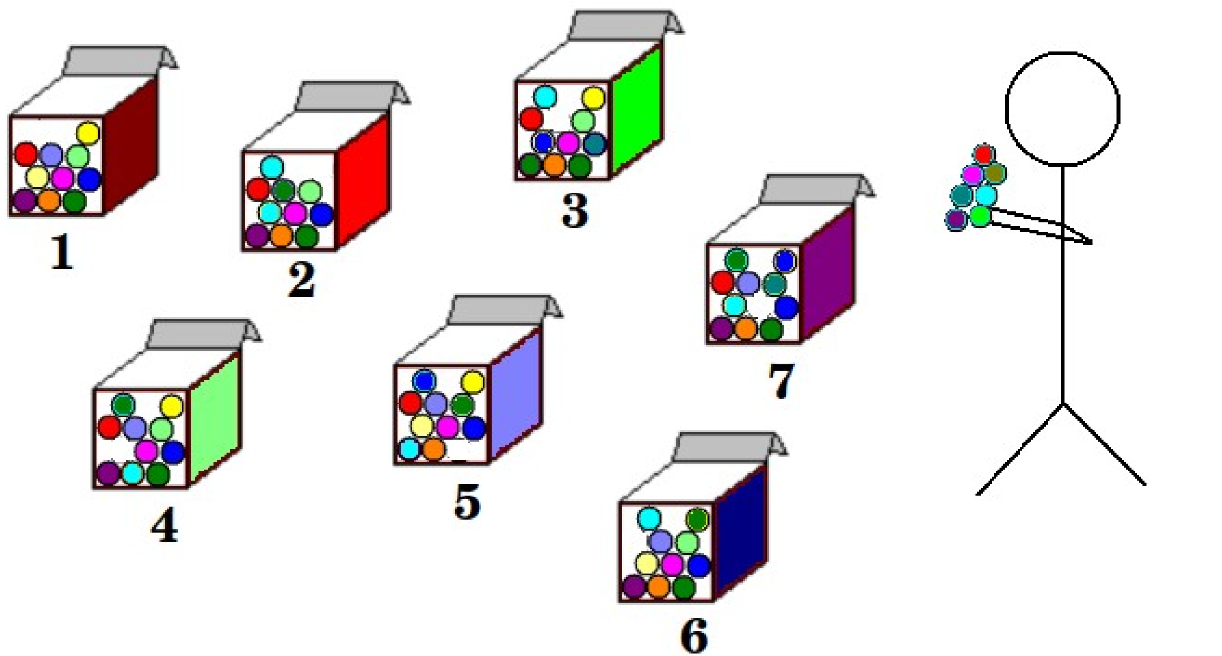
So what's the fuss?

- Given a ***finite number of non-empty boxes***, is it possible to ***choose*** an object from each box?



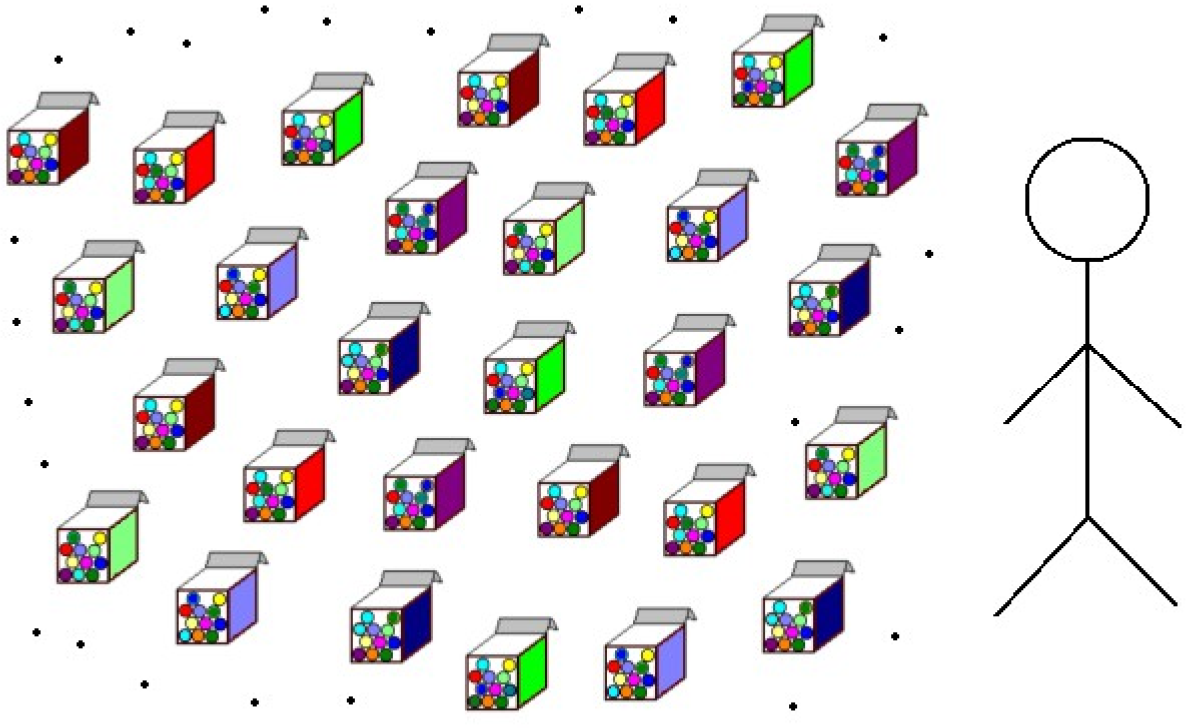
So what's the fuss?

- Yes!! Give me finite number of boxes.
  I can go to each box and choose an element from that…



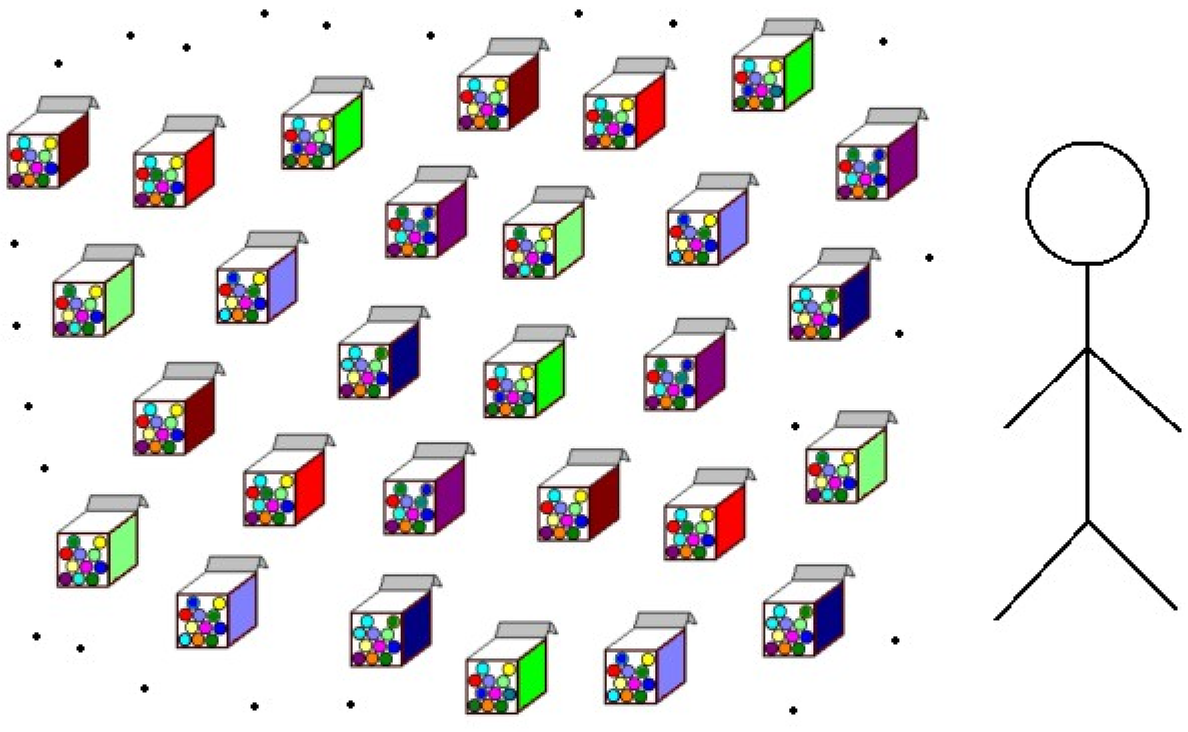
So what's the fuss?

- Given an ***infinite number of non-empty boxes***, is it possible to ***choose*** an object from each box?



So what's the fuss?

- I just can't go to every box and pick one object from each... :-/

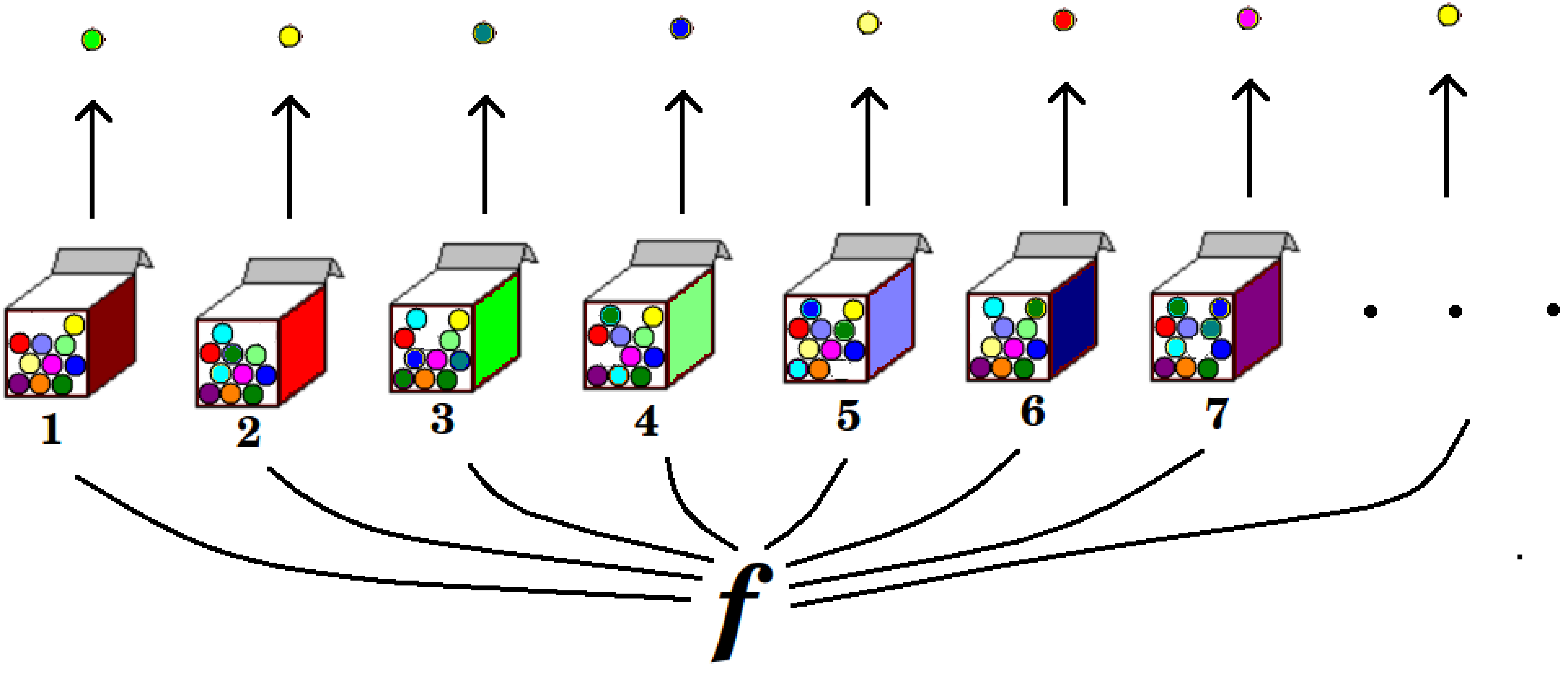


So what's the fuss?

- **Question :**
  - What is the *Math* way of representing infinite number of objects? And
  - How to deal with them?

So what's the fuss?

- **Answer :** *Sets* and *Functions*
  Use a *function* to ***choose*** an object from each box.



Example of a Choice Function

- **Question :**
  Given a collection of boxes such that,
  Box_1 contains 1, 2, ... , 10
  Box_2 contains 11, 12, ... , 20
  ....
  Box_n contains 10(n-1)+1, 10(n-1) + 2, ... , 10(n)
  ....
  ....
- Example, Choice function : $\boldsymbol{f}(Box_n) = 10(n-1) + 3$;
- So we get {3, 13, 23, ... }.

Does there always exist a Choice Function ?

- "The Axiom of Choice is necessary to select a set from an infinite number of *socks*, but not an infinite number of *shoes*."

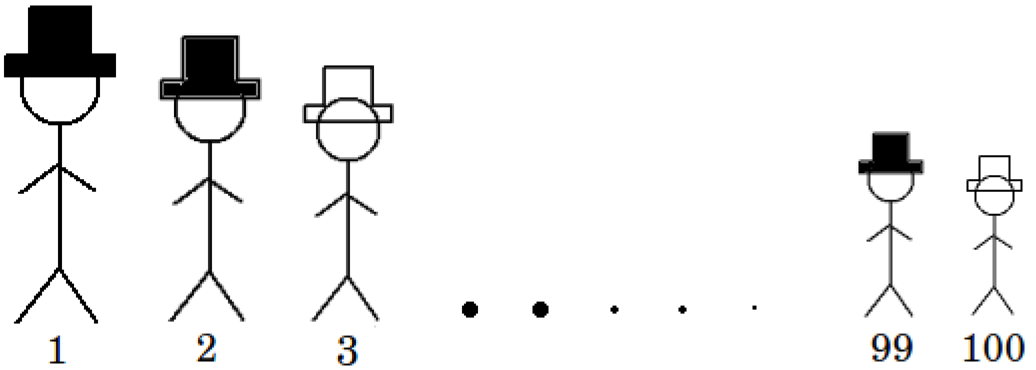
— Bertrand Russell

Ok.. So what?

- ***Banach Tarski Paradox :*** It is possible to divide a sphere into finitely many pieces and put them back together to get two spheres!

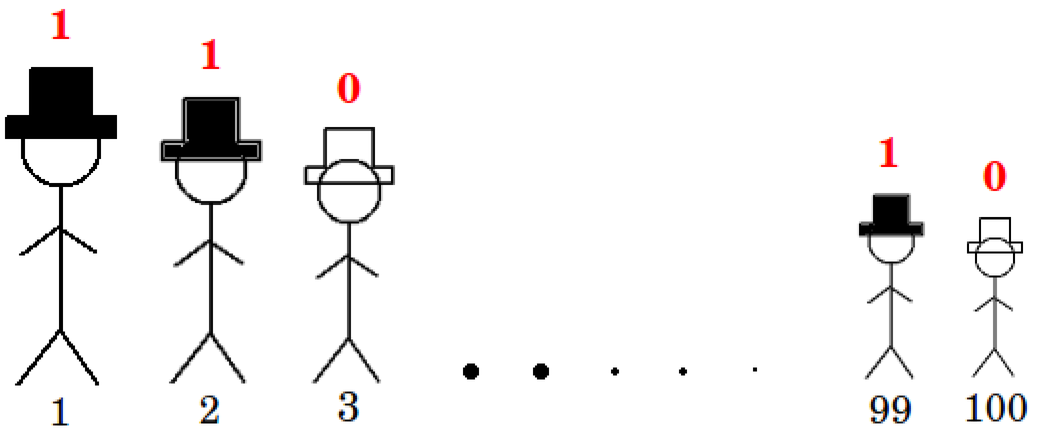
Proof out of scope for the talk!

A popular puzzle :
How many can you save ?



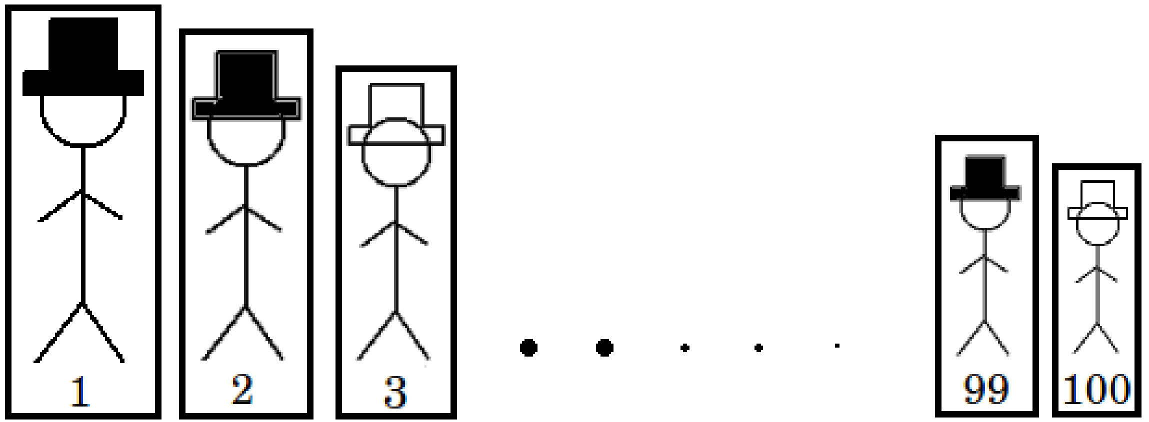
- 100 prisoners in a row
- Each is made to wear a black or white hat
- Each prisoner can look at colours of all hats in front of him, but not his own, nor those behind him.
- Starting from the last, each prisoner is asked to guess the colour of his hat.
- if guessed correctly, the prisoner is released,
  else, the prisoner is executed.

Solution to the puzzle : You can save 99!



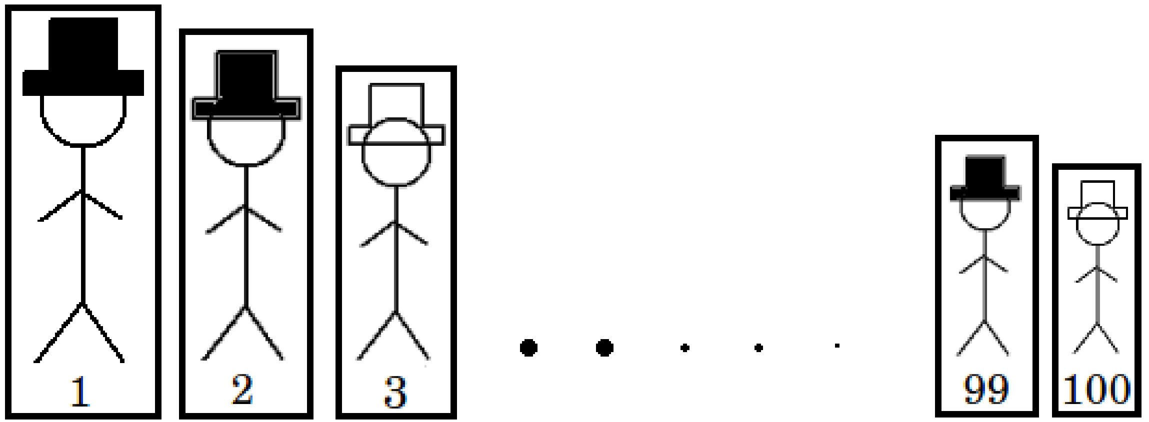
- Represent hat-sequence as a sequence of 1's and 0's.
- Last prisoner adds all the hat numbers in front of him
- guesses ***black*** if answer is *odd,*
  white if answer is *even*.
- Everybody else *comes to know* his hat colour!

Modification of the puzzle : How many can you save ?



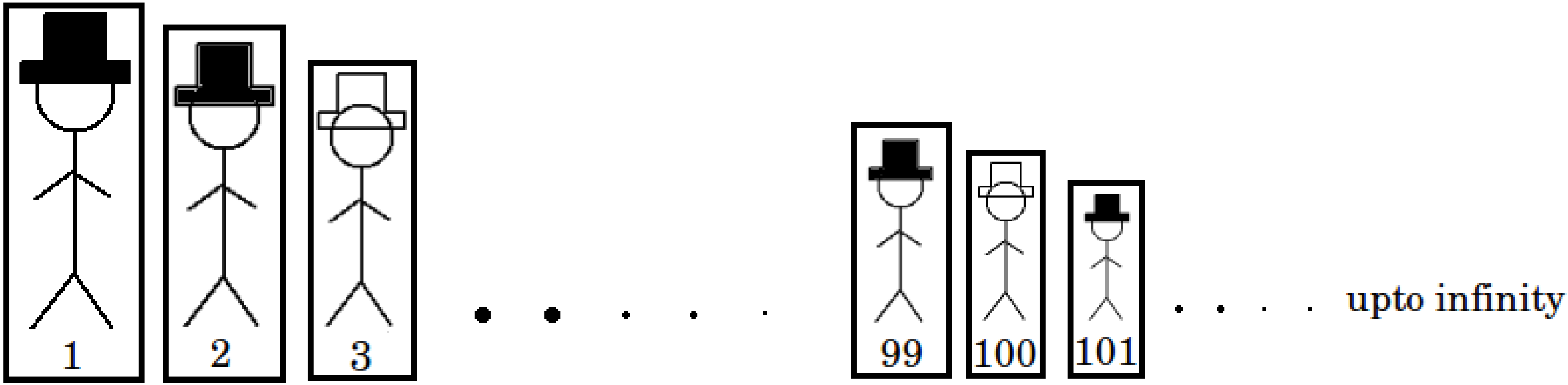
- 100 prisoners in a row, ***standing inside glass boxes.***
- Each prisoner can look at colours of all hats in front of him, but not his own, nor those behind him.
- ***Answer given by a prisoner is not heard by any other prisoner.***

Modification of the puzzle : How many can you save ?



- There does not seem to be any good strategy. :(
- On an *average*, you will save **50**. :(
- In the *worst case*, you will save **0**. :(

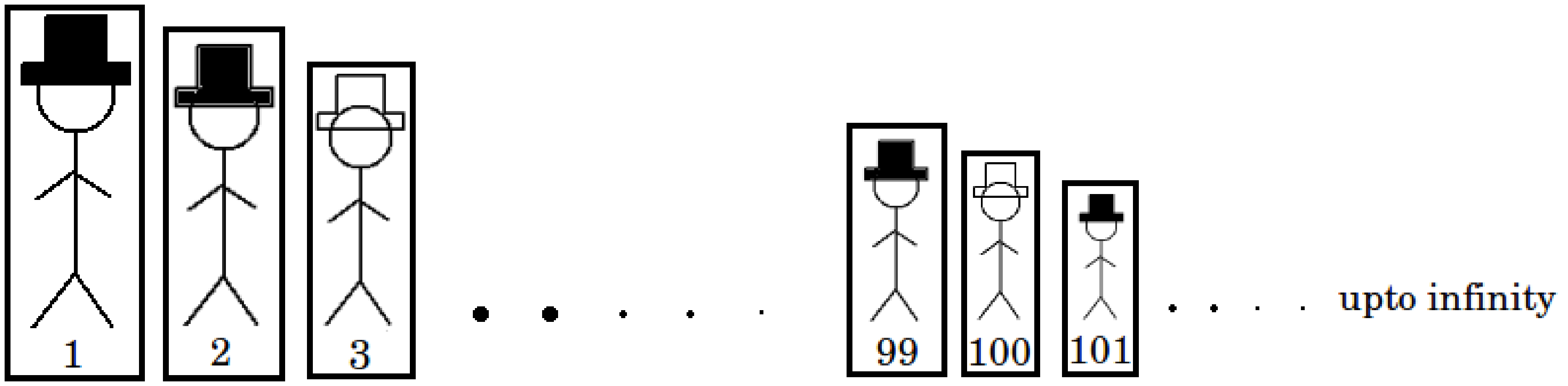
Another modification : How many can you save ?



- ***Infinite*** prisoners in a row, standing inside glass boxes.
- Remaining rules as before!

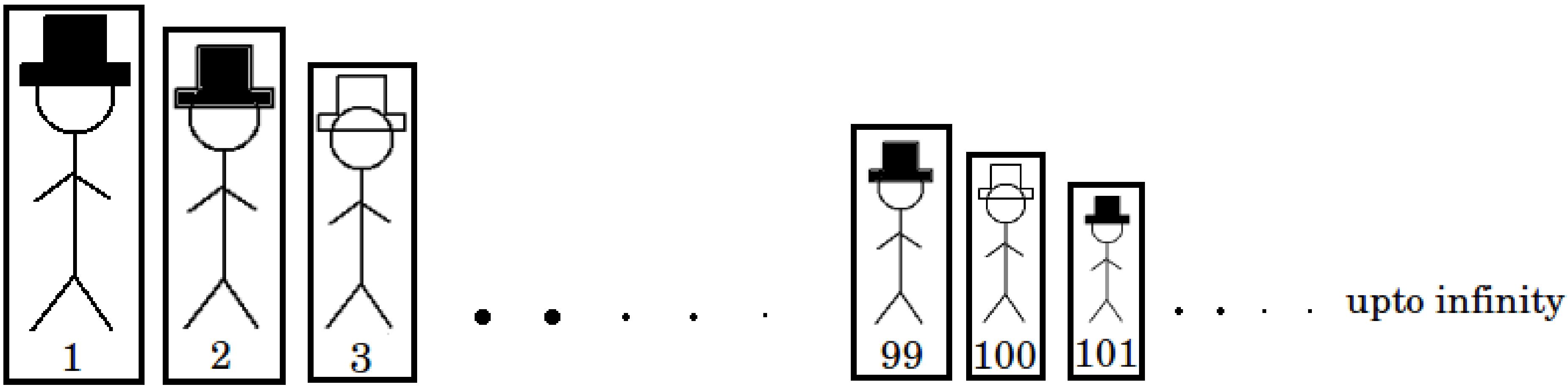
(taken from : http://cornellmath.wordpress.com/2007/09/13/the-axiom-of-choice-is-wrong/)

Another modification :
How many can you save ?



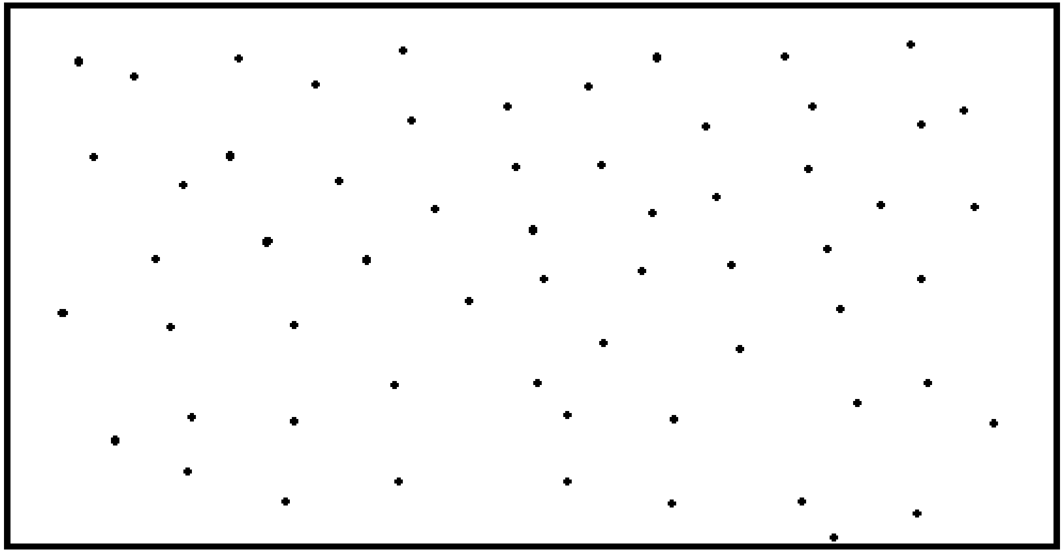
- There doesn't seem to be any strategy! :(
- *Average Case* : you lose **infinite** people! :(
- *Worst Case* : you **lose everybody**... :'(

Another modification : How many can you save ?



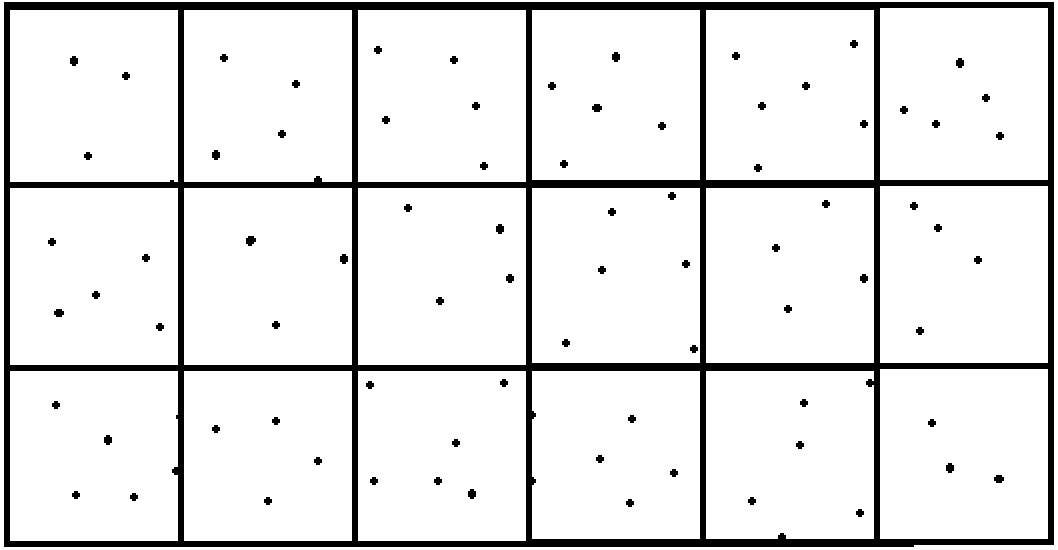
- ***Axiom of Choice*** comes to the rescue! :)
- *Worst Case* : you lose **atmost finite**ly many prisoners! \m/

Alternate view of the Axiom of Choice



- Given :
  - A *Set*

Alternate view of the Axiom of Choice



- Given :
  - A *Set*
  - A *Partition* over the set (each partition is non-empty).
- it is possible to ***choose*** an element from each *partition.*

How to partition a set?

- One way : Manually *group objects into partitions*.

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- Alternate way :

  - Relate different objects, and group them together.

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  - Relation should be such that :

    if A is related to B
    and B is related to C, (Transitivity)
    then A is related to C.

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  - Relate different objects, and group them together.

  - Relation should be such that :
    if A is related to B
    and B is related to C, (Transitivity)
    then A is related to C.

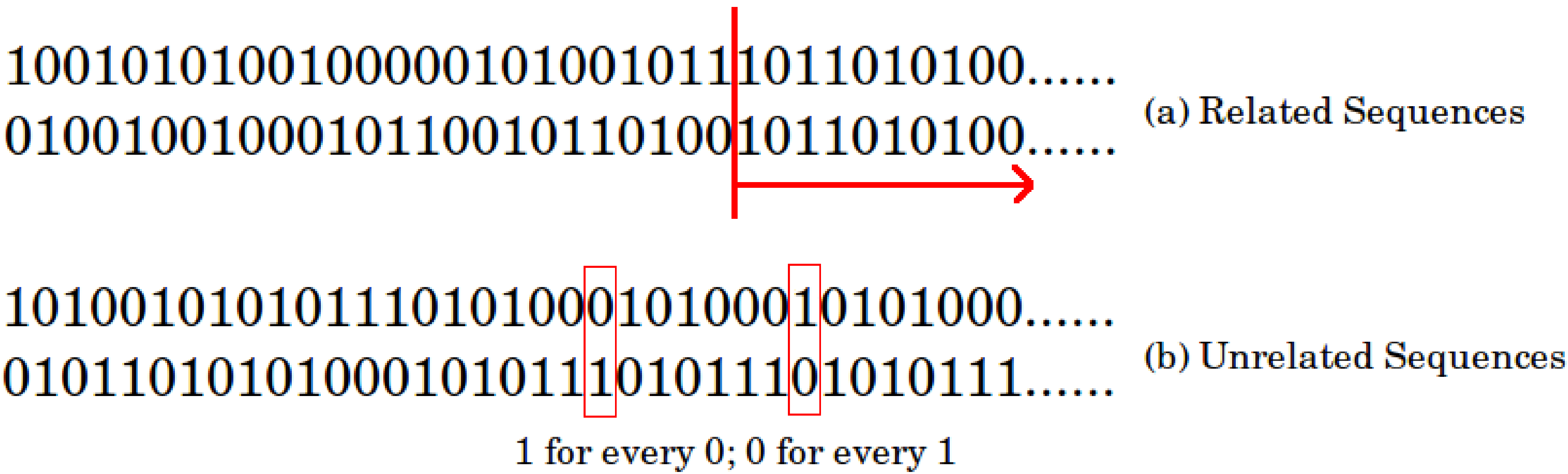
  - Group related objects into one *partition*.

  - This naturally induces a *partition* over the set.

Solution to the *Infinite Prisoners* Problem.

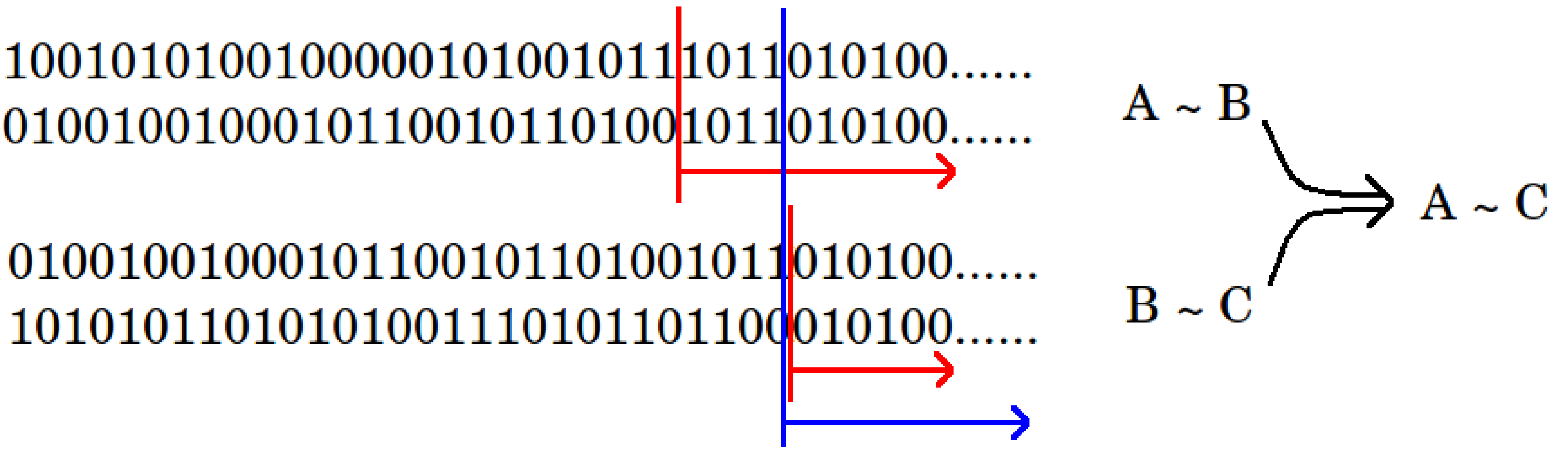
- Strategy :
  - Let S be the collection of all *binary* sequences.
  - Call two sequences as *related* if they are equal after a certain position.

- For example,



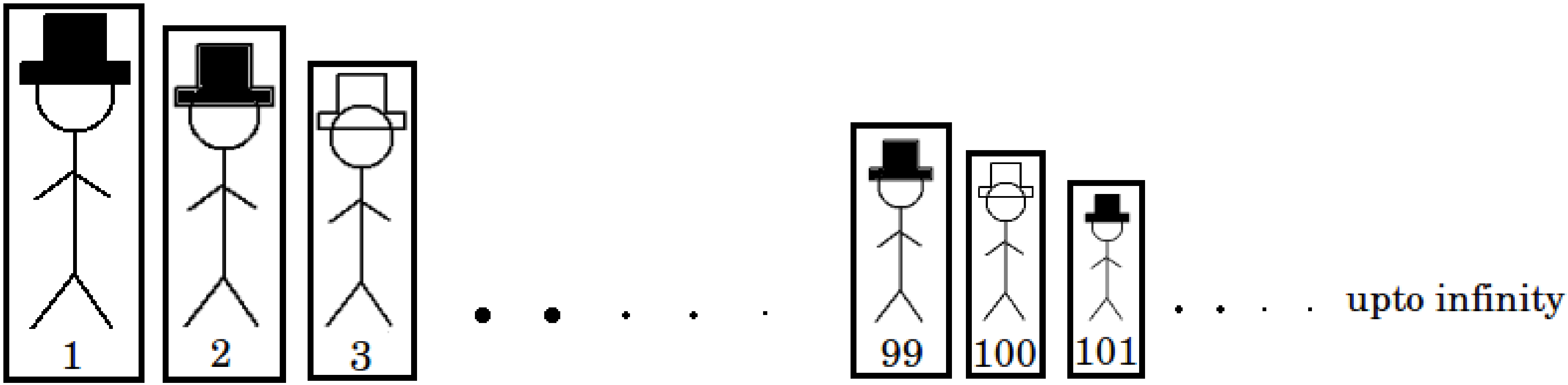
Solution to the *Infinite Prisoners* Problem.

- This *relation* satisfies *transitivity* :
  if   A is related to B
  and B is related to C,
  then A is related to C.



- This induces *partition* on the set of binary sequences.

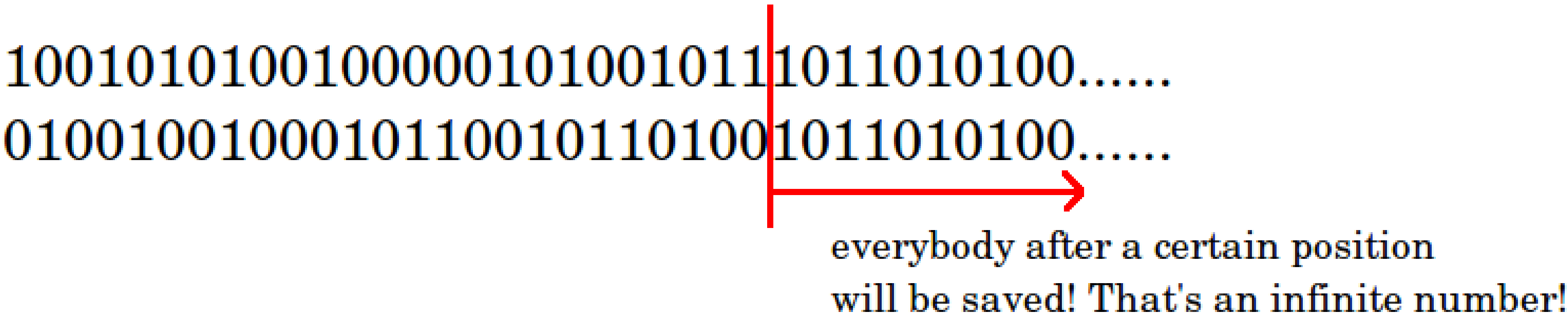
THE SOLUTION to the *Infinite Prisoners* Problem.



- All prisoners meet and decide upon a *choice* from each partition.
- Each prisoner now looking at the sequence in front of him can decide which *partition* this sequence lies in.
- Every prisoner now guesses that colour which he was wearing in the *chosen sequence* from that partition.

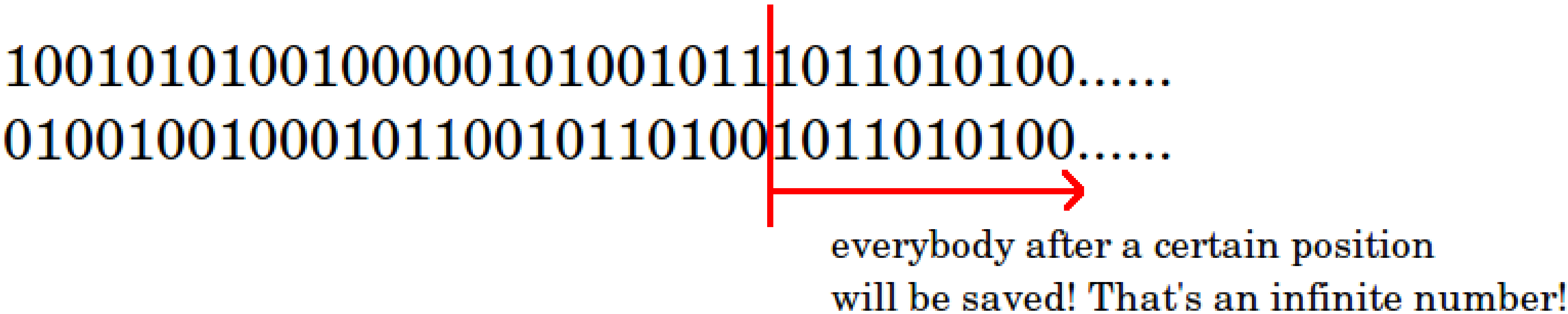
So why does this work?

- Since the *chosen* sequence and the *given* sequence match after a certain position, all the prisoners after that certain position will be saved!



So why does this work?

- Even if the person setting up the hats knows the prisoners' strategy, he cannot make the number of people executed, greater than a finite number!



Another quote on the Axiom of Choice

- "The *Axiom of Choice* is obviously true, the *Well-Ordering Principle* obviously false, and who can tell about *Zorn's lemma*?"

— Jerry Bona

Questions??