Persistent harmonic forms

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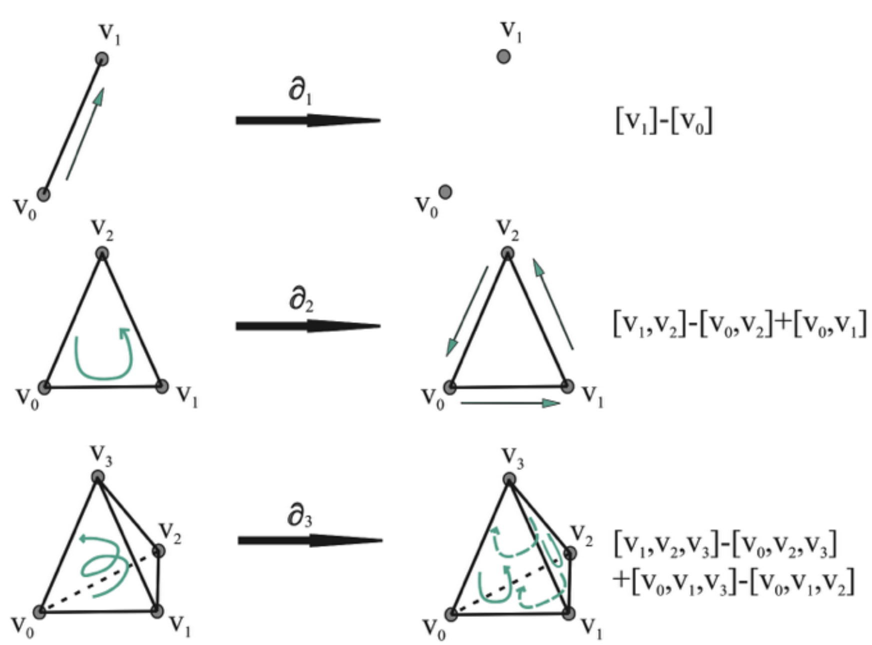
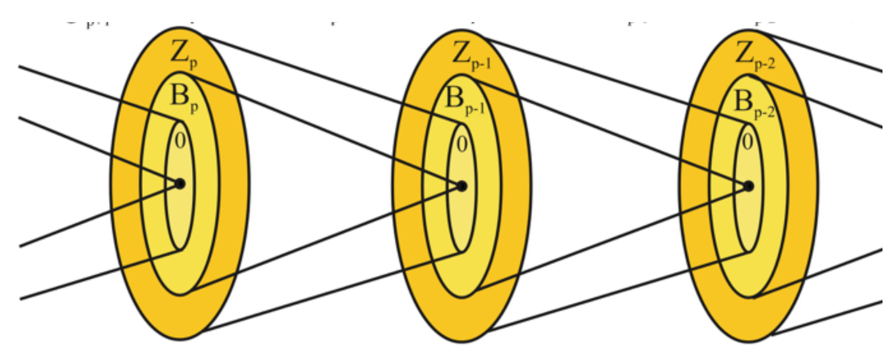
+discussions with: *Dominique Attali* (L^2) + *David Cohen-Steiner* (L^1)

Studying (co-)chains that are L^2 or L^1 minimal in their (persistent) homology class

Homology (simplicial)

support :simplicial complex



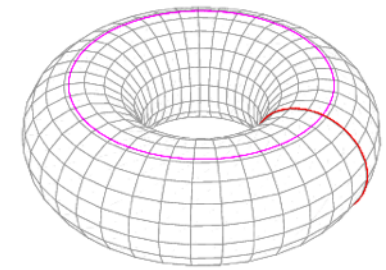


$$H_k = \ker \partial_k / \operatorname{im} \partial_{k-1}$$

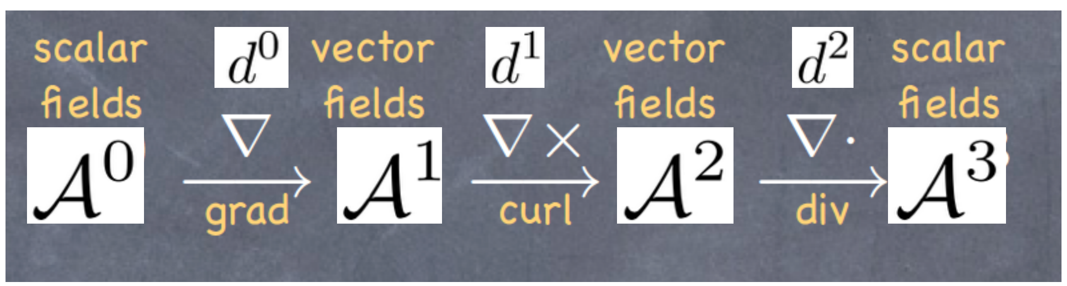
$$\partial_k \colon C_k \to C_{k-1}$$

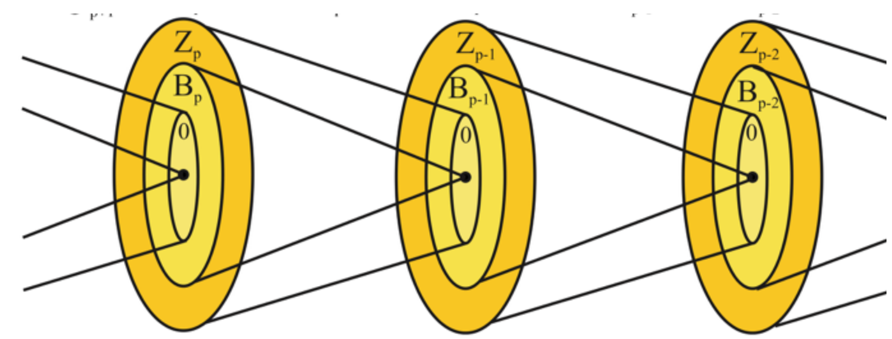
In this talk, the field of coefficients is always $\mathbb{R}$

de Rham Cohomology



support : Riemannian manifold *M*





$$H^n = \ker d^k \,/\, \operatorname{im} d^{k-1}$$

Exteriror derivative:

$$d^k\colon \mathcal{A}^k \to \mathcal{A}^{k+1}$$

$$d^k\left(f dx_{i_1} \wedge \ldots \wedge dx_{i_k}\right) = \sum_j \frac{\partial f}{\partial x_j} dx_j \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k}$$

In this talk, the field of coefficients is always $\mathbb{R}$

Hilbert structure on $C_p(K)$

The vector space of *cochains* $C^p(K)$ is the set of (continuous) linear maps from $C_p(K)$ to $\mathbb{R}$.
We introduce an inner product on $C_p(K)$: $\langle \cdot\,, \cdot \rangle_p : C_p(K) \times C_p(K) \to \mathbb{R}$.

A k-harmonic form is an element ω of $\ker \partial_k$ (or $\ker d^k$, etc...) whose energy is minimal in its (co-)homology class:

$$\partial_k \omega = 0 \text{ and } \forall \alpha \in C_{k+1}, \langle \omega, \omega \rangle \leq \langle \omega + \partial_{k+1}\alpha, \omega + \partial_{k+1}\alpha \rangle$$

This inner product is the « physic » or « geometry » ingredient.
For example DEC is the art of defining it on combinatorial manifolds

For a p-chain (or p-cochain) σ, we call the quantity $\frac{1}{2} \langle \sigma\,, \sigma \rangle_p$ the *energy* of σ.

If C_k is a Hilbert space it is then isomorphic to its dual

Harmonic forms

A generic definition:

- Orientable Riemannien manifolds (Hodge)
- Discrete Hodge Theories (also DEC)
- Hodge Theory for metric spaces (S.Smale)
- etc…

Define a dot product on C_k (or $\mathcal{A}^k$,etc…):

A k-harmonic form is an element ω of $\ker \partial_k$ (or $\ker d^k$, etc...) whose energy is minimal in its (co-)homology class:

$$\partial_k \omega = 0 \text{ and } \forall \alpha \in C_{k+1}, \langle \omega, \omega \rangle \leq \langle \omega + \partial_{k+1} \alpha, \omega + \partial_{k+1} \alpha \rangle$$

=> Laplacian operator Δ_k , Hodge Theorem, Hodge decomposition theorem follows…

Persistent homology groups

We consider a simplicial complex K and a simplicial subcomplex $L \subset K$.
We denote:

$C_p(K)$: vector spaces of p-chains on K with coefficients in $\mathbb{R}$.

$\partial_p^K : C_p(K) \to C_{p-1}(K)$: the *boundary operator*.

$i : C_p(L) \to C_p(K)$: the inclusion map induced by $L \subset K$

$i^\sharp$: the corresponding map on homology groups.

Recall that the vector space:

$$H_p(K, L) = \frac{\ker(\partial_p^L)}{\mathrm{Im}(\partial_{p+1}^K) \cap C_p(L)} = \mathrm{Im}(i^\sharp)$$

is called the *persistent* p*-homology group of the pair* (K, L).
$\dim(H_p(K, L))$ is the *persistent* p^{th} *Betti number of the pair* (K, L).

Persistent harmonic form (first definition)

From the definition of persistent homology group:

$$H_p(K,L) = \frac{\ker(\partial_p^L)}{\mathrm{Im}(\partial_{p+1}^K) \cap C_p(L)} = \mathrm{Im}(i^\sharp)$$

One get the first definition of persistent harmonic forms:

We can define the *persistent harmonic forms* of the simplicial pair (K, L) as the cycles of $\ker(\partial_p^L)$ with minimal energy in their persistent homology class:

$$\mathcal{H}_p(K,L) = \left\{ \sigma \in \ker(\partial_p^L) \,|\, \forall \beta \in \mathrm{Im}(\partial_{p+1}^K) \cap C_p(L),\ \langle \sigma\,, \sigma \rangle_p \leq \langle \sigma + \beta\,, \sigma + \beta \rangle_p \right\}$$

Adjoint operators (on cochains)

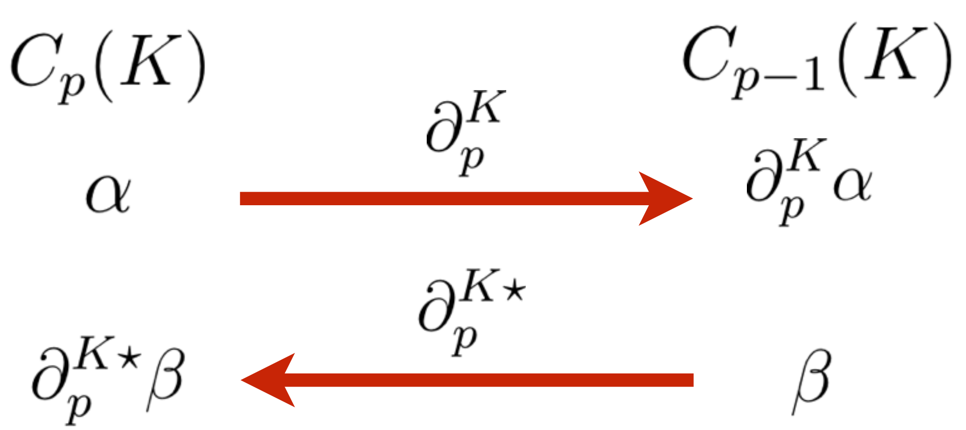
The operators ∂_p^K, ∂_p^L and i induce adjoint operators $\partial_p^{K\star}$, $\partial_p^{L\star}$ and $i^\star$ defined by:

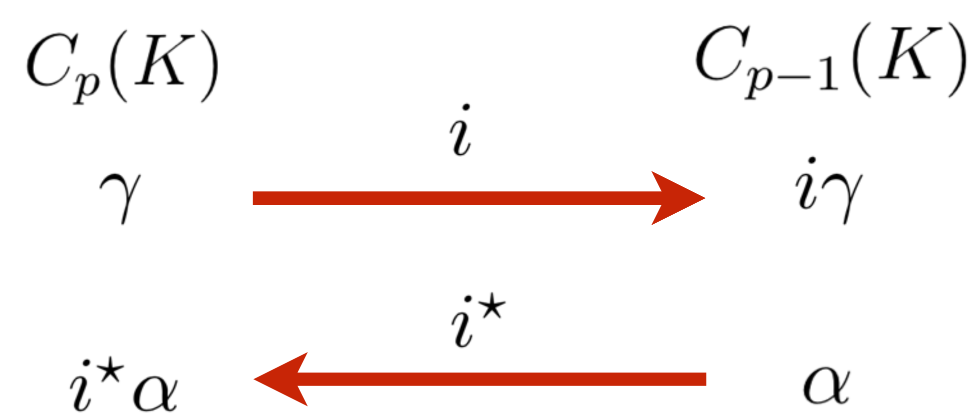
$$\forall \alpha \in C_p(K), \forall \beta \in C_{p-1}(K), \left\langle \alpha \,, \partial_p^{K\star} \beta \right\rangle_p = \left\langle \partial_p^K \alpha \,, \beta \right\rangle_{p-1}$$

and:

$$\forall \alpha \in C_p(K), \forall \gamma \in C_p(L), \langle \alpha \,, i\gamma \rangle_p = \langle i^\star \alpha \,, \gamma \rangle_p$$

$\partial_p^{K\star}$ and $\partial_p^{L\star}$ are called *coboundary* or *exterior derivative* operators. i is the *inclusion* operator, while $i^\star$ is the *restriction* operator.





Trivial reformulation of persistent homology group

Recall:

$$H_p(K,L) = \frac{\ker(\partial_p^L)}{\mathrm{Im}(\partial_{p+1}^K) \cap C_p(L)} = \mathrm{Im}(i^\sharp)$$

$A_p(K,L)$ is the set of p-chain of K whose boundary is in $C_{p-1}(L)$:

$$A_p(K,L) =_{\text{def}} \left\{\alpha \in C_p(K) \,|\, \partial_p^K \alpha \in C_{p-1}(L)\right\} = \left(\partial_p^K\right)^{-1} (C_{p-1}(L))$$

$\overline{\partial}_p$ is simply the restriction of ∂_p^K to $A_p(K,L)$. One has therefore the usual relation $\partial_p^L \overline{\partial}_{p+1} = 0$.

Remark : The persistent homology group of the pair(K,L) can equivalently be defined as:

$$H_p(K,L) = \frac{\ker(\partial_p^L)}{\mathrm{Im}(\overline{\partial}_{p+1})}$$

(persistent) Laplacian operator

Recall:

$$\mathcal{H}_p(K,L) = \left\{\sigma \in \ker(\partial_p^L) \,|\, \forall \beta \in \mathrm{Im}(\partial_{p+1}^K) \cap C_p(L),\ \langle \sigma\,,\sigma \rangle_p \leq \langle \sigma+\beta\,,\sigma+\beta \rangle_p \right\}$$

Even if this is not immediately obvious from this definition, next lemma shows that $\mathcal{H}_p(K,L)$ is a linear space.

Let us define the persistent laplacian operator $\Delta_p : C_p(L) \to C_p(L)$ by:

$$\Delta_p = \partial_p^{L\star}\,\partial_p^L + \overline{\partial}_{p+1}\,\overline{\partial}_{p+1}^{\star}$$

Lemma 1. *One has :*

$$\mathcal{H}_p(K,L) = \ker(\partial_p^L) \cap \ker(\overline{\partial}_{p+1}^{\star}) = \ker \Delta_p$$

(persistent)-Hodge Theorems

(persistent) Hodge Theorem

Lemma 2. *$\mathcal{H}_p(K, L)$ is isomorphic to the persistent homology group $H_p(K, L)$*

$$\mathcal{H}_p(K, L) \xrightarrow{\text{isomorphism}} H_p(K, L)$$

(persistent) Hodge decomposition Theorem

Lemma 3. *We have the following decomposition as a direct sum:*

$$C_p(L) = \mathrm{Im}(\overline{\partial}_{p+1}) \oplus \mathcal{H}_p(K, L) \oplus \mathrm{Im}(\partial_p^{L\star})$$

Moreover these three subspaces are pairwise orthogonal.

Canonical orthogonal basis of persistent harmonic forms

Lemma 4. *If $M \subset L \subset K$ are finite simplicial complexes, then:*

$$\mathcal{H}_p(K, M) \subset \mathcal{H}_p(L, M)$$

The inclusion in the lemma is trivially injective. A consequence of this lemma is that, given a filtration of finite simplicial complexes $K_0 \subset K_1 \subset \ldots \subset K_n$, such that K_n has trivial homology (for example it could be contractible), then we have the sequence of inclusion:

$$\{0\} = \mathcal{H}_p(K_n, K_0) \subset \mathcal{H}_p(K_{n-1}, K_0) \subset \ldots \mathcal{H}_p(K_1, K_0) \subset \mathcal{H}_p(K_0, K_0)$$

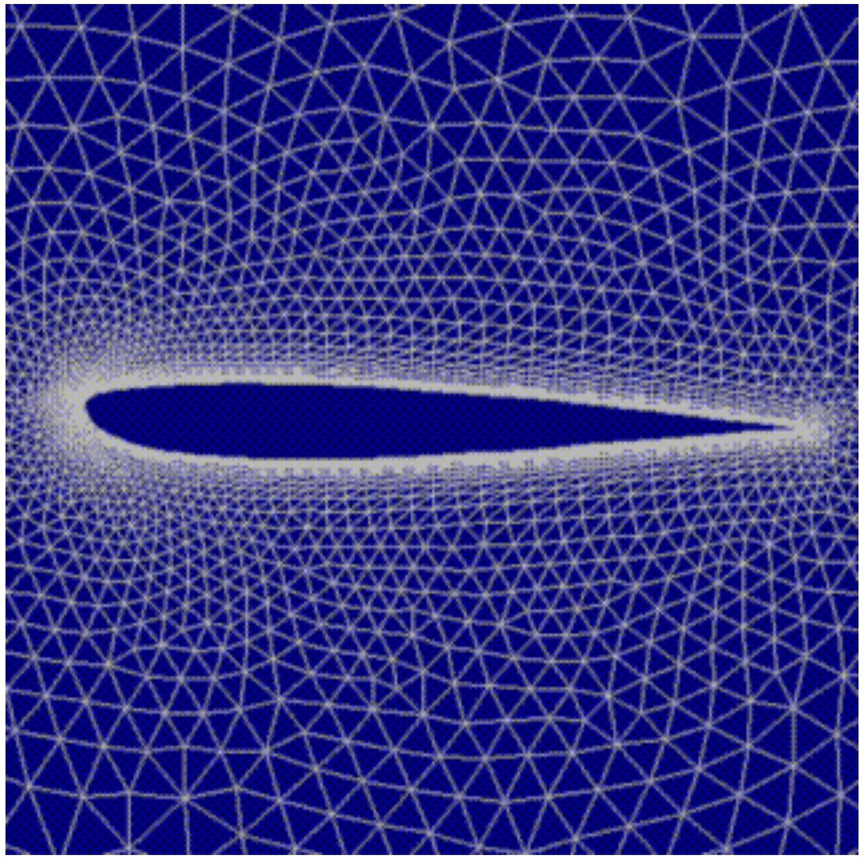
If the filtration is fine enough, the dimension will never increase by more than 1 along this sequence. This allows to build a canonical orthonormal basis.

Application to linear PDEs ?

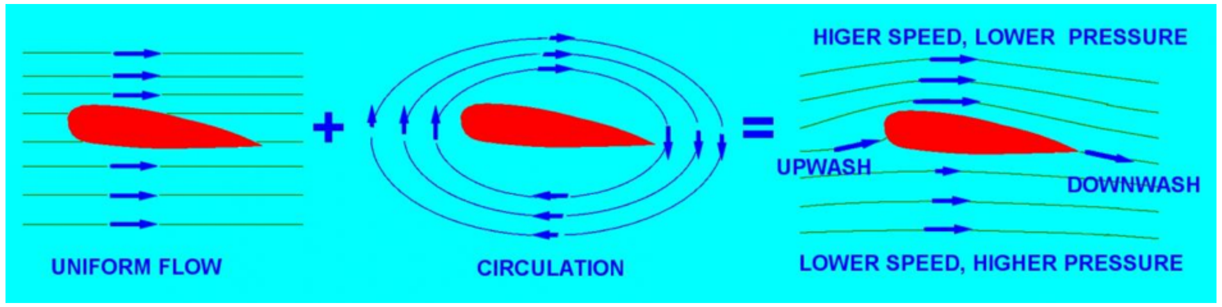
References DEC: (Desbrun,…

Incompressible fluids:

K is the meshed domain and $L \subset K$ the meshed domain boundary. $C_1(K, L) = C_1(K)/C_1(L)$ and $C_0(K, L) = C_0(K)/C_0(L)$ the relative chains and $\partial_{K,L} : C_1(K, L) \to C_0(K, L)$ the corresponding boundary operator.



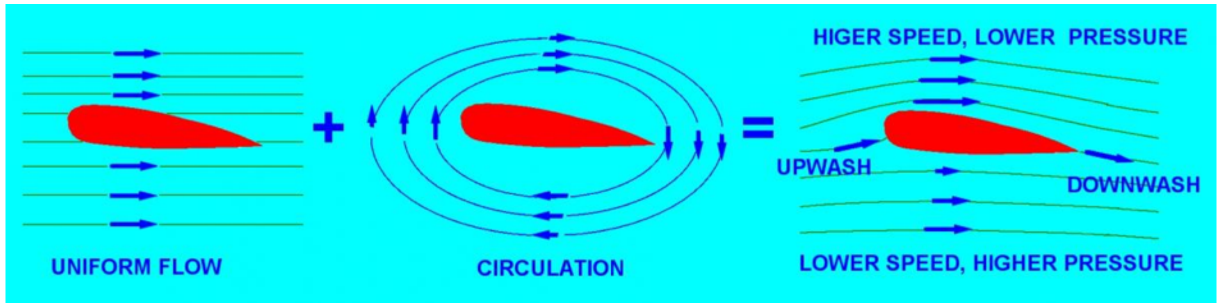
Taking the right inner product (finite elements, DEC) and minimizing the energy of 1-forms in their relative (co-)homology class we get a set of irrotational flow (incompressible Euler) as «relative harmonic form». The circulation around the wing gives the lift.

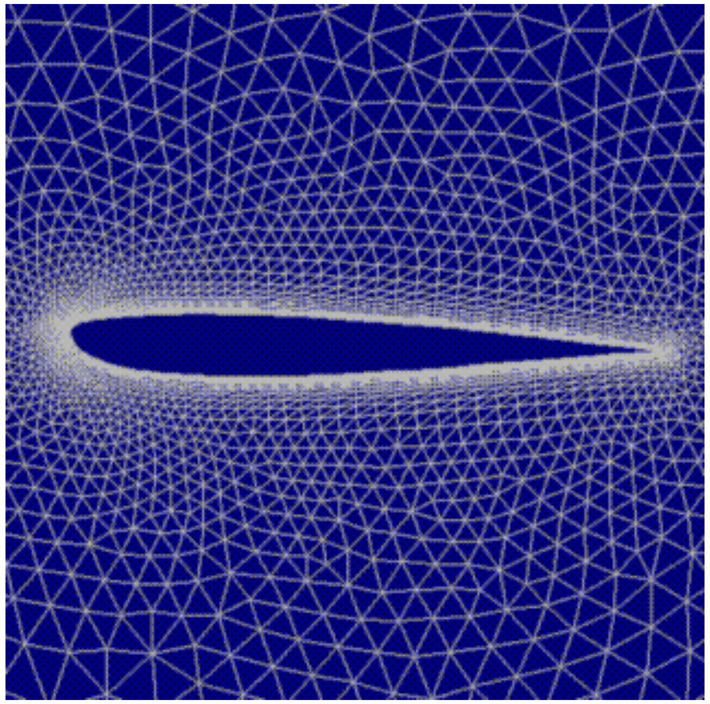


Application to linear PDEs ?

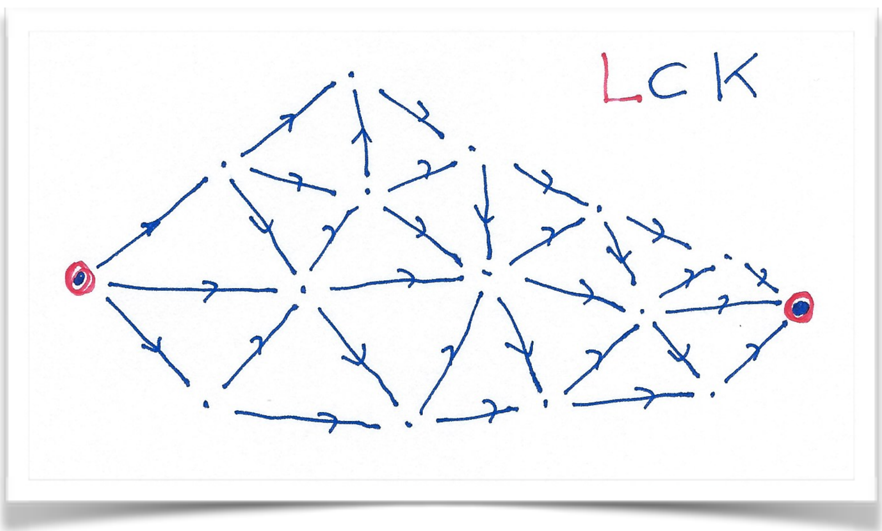
In the previous example, the homology of the domain is crucial.

- What if the domain of the PDE is known through approximations (point sample) ?
- If we are able to build a pair of complex capturing the homology of the domain, could we approximate the solution of the PDE by « relative persistent harmonic forms » ?
- The inner product can be inherited from DEC if the complexes are embedded (alpha-complex). But is it possible to design a reasonable inner product if the complexes are not embedded ? (Cech, Rips) ?





L^2 minima are not sparse

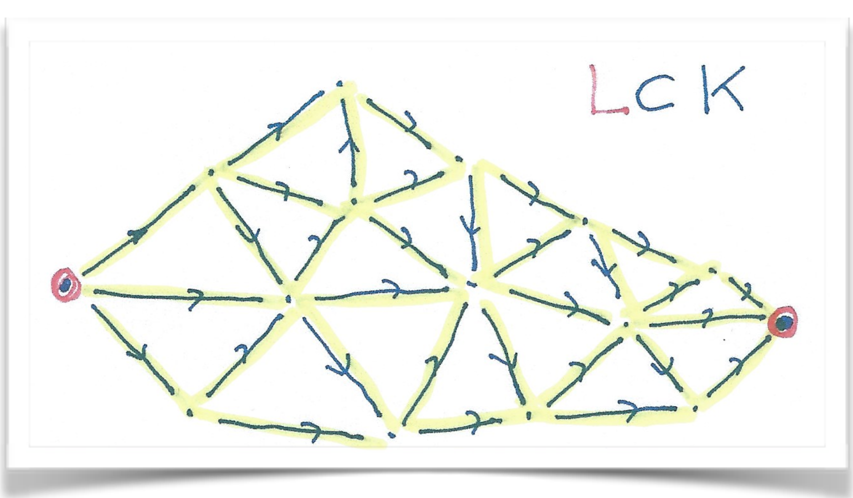


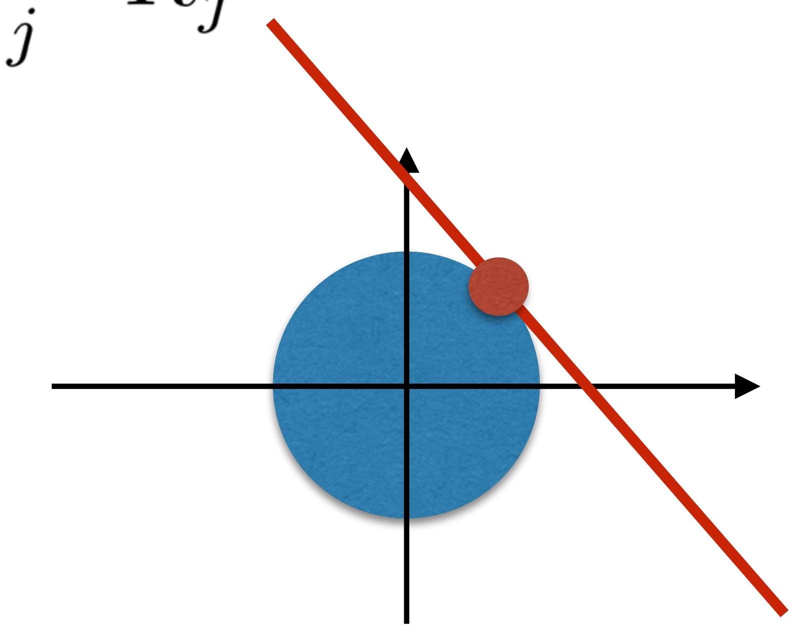
K is a 2-dimensional simplicial complex,
L is a 0-dimensional simplicial complex
The relative Homology group $H_1(K,L)$ has dimension 1.

Minimizing L^2 norm
=> relative harmonic form:

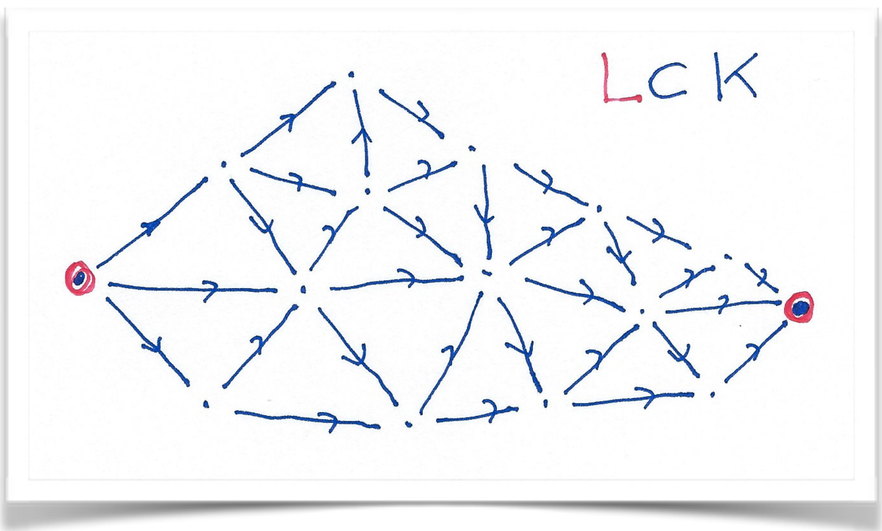
$$\sum_j \frac{1}{R_j} I_j^2$$

(Electric power)





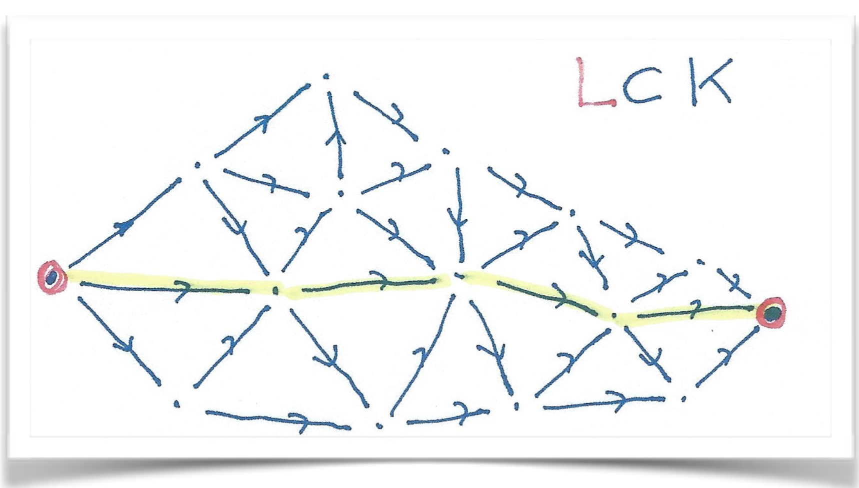
L^1 minima are sparse



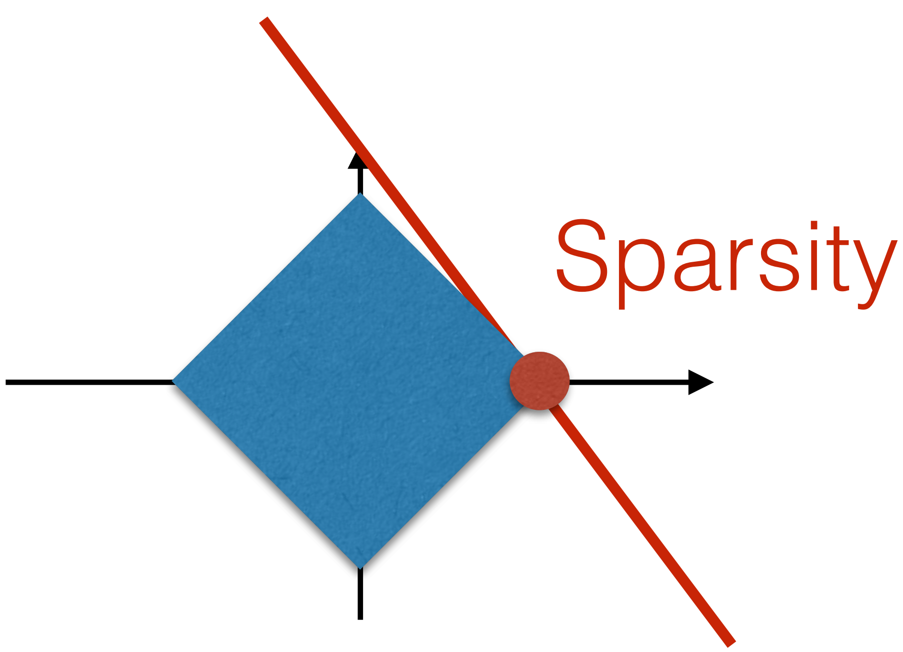
K is a 2-dimensional simplicial complex,
L is a 0-dimensional simplicial complex
The relative Homology group $H_1(K,L)$ has dimension 1.

Minimizing L^1 norm :
=> shortest path

$$\sum_j l_j |I_j|$$ (length)



generically 1-manifold ??



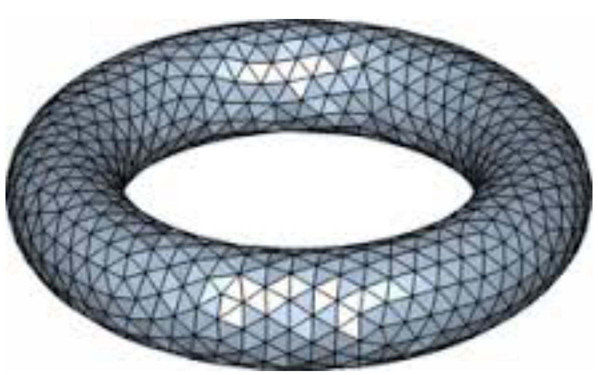
Convergence of L[1] minimal cycles

M is a connected orientable compact embedded k-manifold such that:
$\forall t \in [0, 4\epsilon]$, the inclusion $M \to M^t$ is a homotopy equivalence.

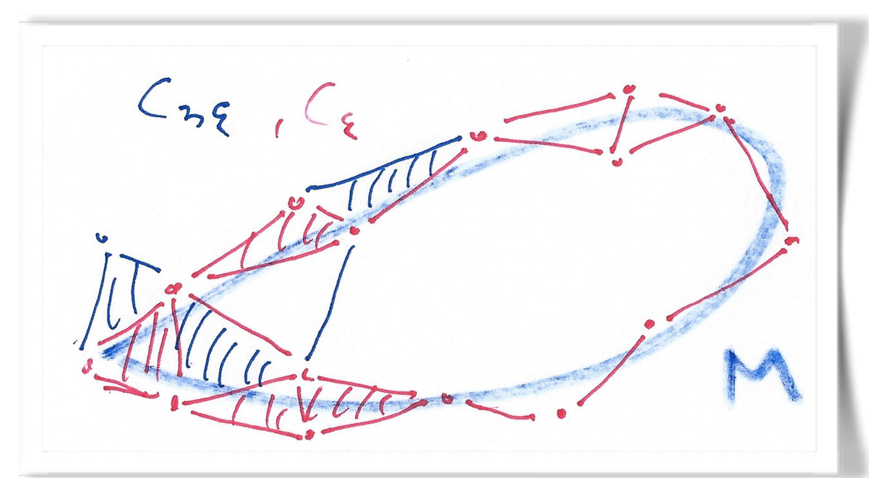
For a cloud P such that $d_H(P, M) < \eta$, we denote:

$$(K, L) = (\mathrm{Cech}(P, 3\epsilon_0), \mathrm{Cech}(P, \epsilon_0))$$

If $\eta \leq \epsilon_0$, the simplicial pair (K, L) captures the homology of M, in particular:
$H_k(K, L)$ captures the fundamental class of M and we have $\dim(H_k(K, L)) = 1$



Fundamental class of a torus



Approximating pair (alpha-shape)

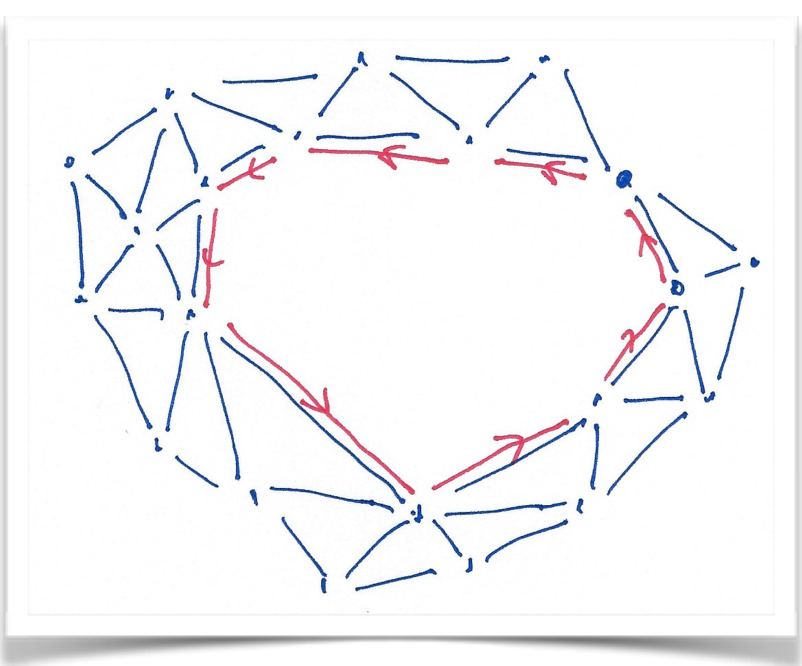
Convergence of L^1 minimal cycles

If α is a k-chain on a Cech (or Rips) complex, we define the L^1 norm $\| \alpha\|_1$ as:

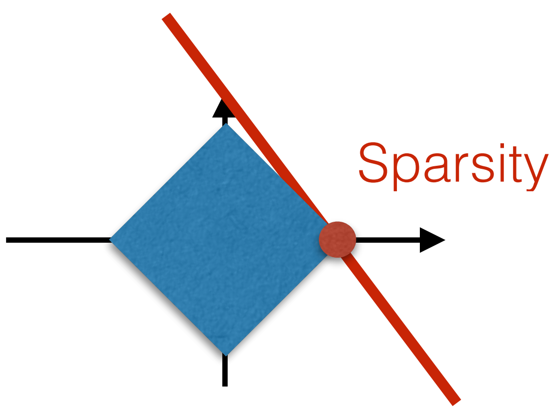
$$\| \alpha\|_1 = \sum_{\sigma \in k\text{-simplex}} \mathrm{Vol}(\sigma)|\alpha(\sigma)|$$

Example: k=1, M is a circle $\qquad \dim(H_k(K, L)) = 1$

If k=1 : we get non trivial cycle of minimal length,.



If k=1, this cycle generically induces an homeomorphic manifold



Convergence of L¹ minimal cycles

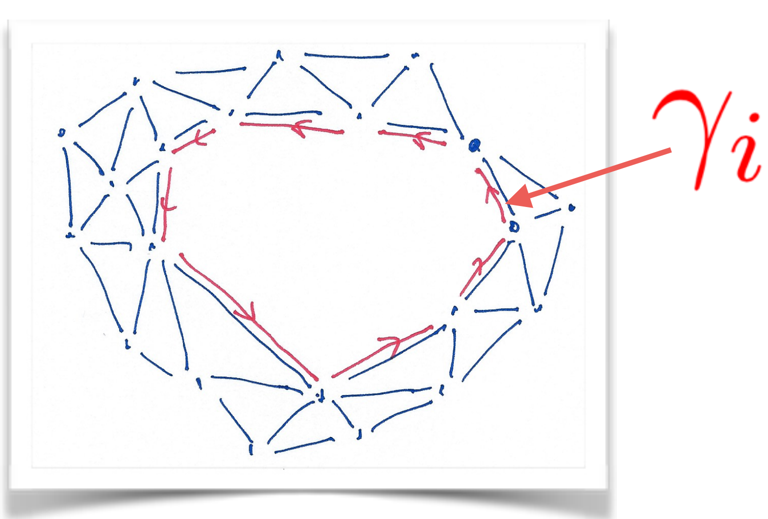
We consider a sequence of point clouds $P_i, i \in \mathbb{N}$, such that $d_H(P_i, M) < \eta_i$, and the k-persistent homology group of the pair $(C(P_i, 3\epsilon_0), C(P_i, \epsilon_i))$, with :

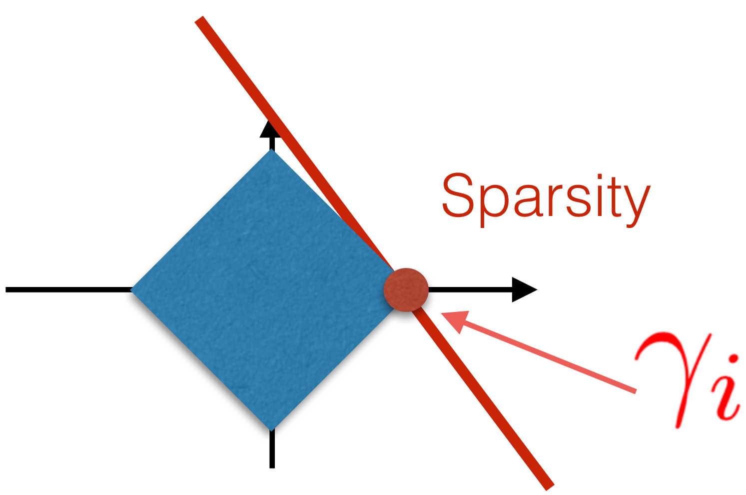
$$\lim_{i\to\infty} \epsilon_i = 0 \quad \text{and} \quad \lim_{i\to\infty} \frac{\eta_i}{\epsilon_i} = 0$$

Let γ_i be a k-cycle of $C_k(C(P_i, \epsilon_i))$ L^1 minimal in its persistent homology class.

Generically, γ_i is unique, up to a multiplicative constant (=>need for normalization).

In which sense could we say (or hope) that γ_i converges toward M as $i \to \infty$?



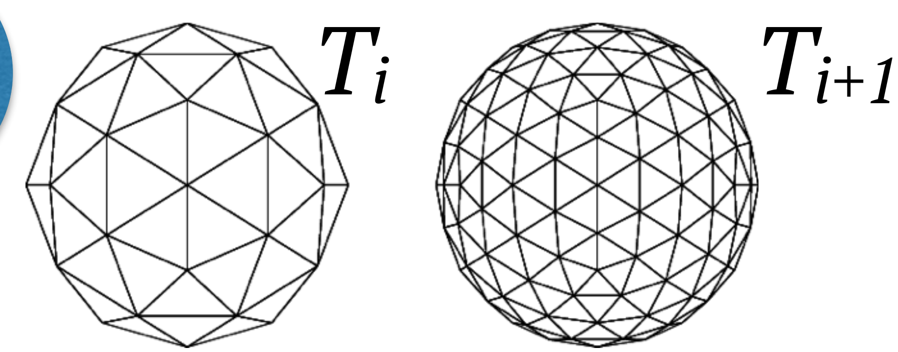


Convergence of L^1 minimal cycles

Rectifiability assumption on the manifold M:

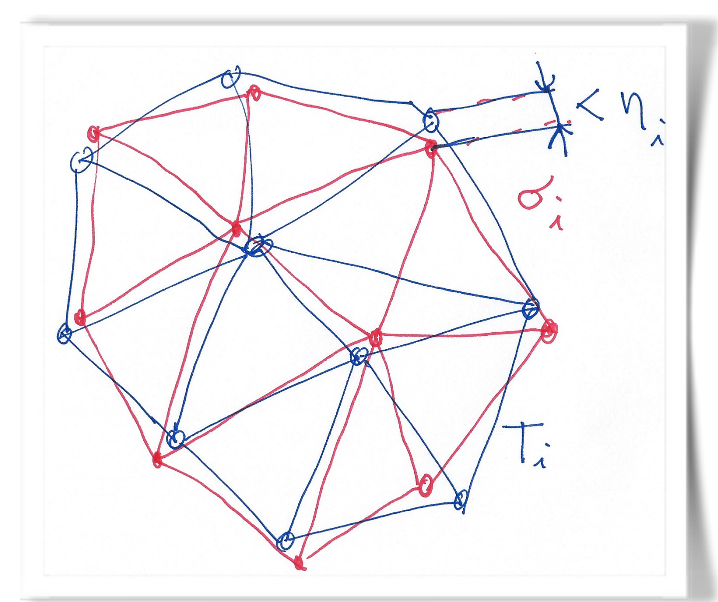
There is a constant $C > 0$ and a sequence of triangulations T_i of M such that:

- the radius of any simplex of T_i is less than $\frac{1}{2}\epsilon_i$,
- maximal simplices of T_i contain a ball of radius $C\epsilon_i$
- $\lim_{i\to\infty} \mathrm{Vol}(T_i) = \mathrm{Vol}(M)$



Since $\lim_{i\to\infty} \frac{\eta_i}{\epsilon_i} = 0$, for each triangulation T_i, there is a cycle $\sigma_i \in C_k(C(P_i, \epsilon_i))$ "close enough" to T_i with:

$$\lim_{i\to\infty} \frac{|\,\mathrm{Vol}(T_i) - \|\sigma_i\|_1|}{\mathrm{Vol}(T_i)} = 0$$

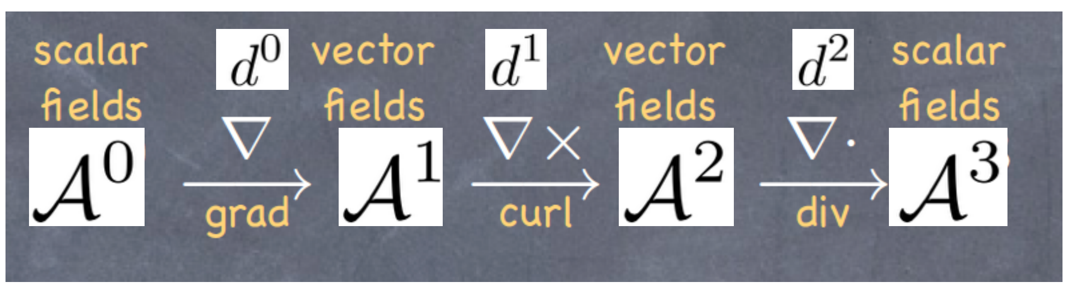


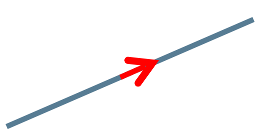
Which gives:

$$\limsup \|\gamma_i\|_1 \leq \mathrm{Vol}(M)$$

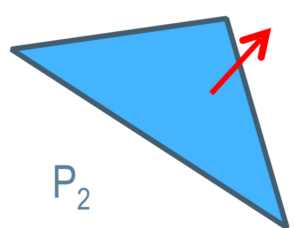
Embedded polyhedral chains and embedded oriented surfaces as Whitney chains and currents

There is an obvious way to integrate a k-differential form of the euclidean ambient space $\mathbb{R}^n$ on an oriented k-simplex.
By additivity, a cycle $\gamma_i \in C_k(C(P_i, \epsilon_i))$ is continuous linear form on the set of differential forms, in other words, a k-current denoted Γ_i with mass $M(\Gamma_i) = \|\gamma_i\|_1$.
Similarly, it is possible to integrate k-differential forms on the oriented embedded manifold M which therefore define a k-current denoted Γ_M with mass $M(\Gamma_M) = \mathrm{Vol}(M)$.



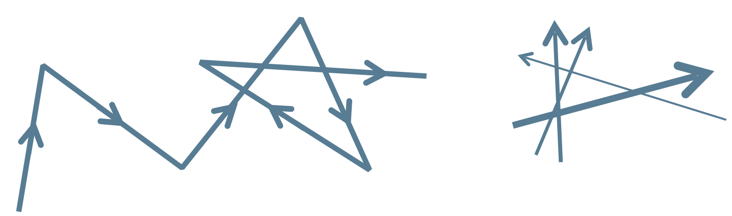


P_1



P_2

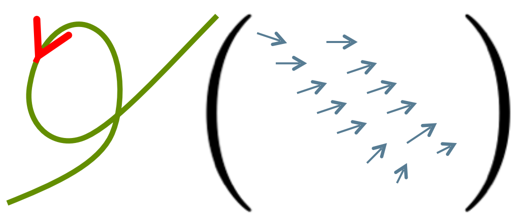
Sums (more generally linear combinations) of such elementary currents are called a Polyhedral chains :



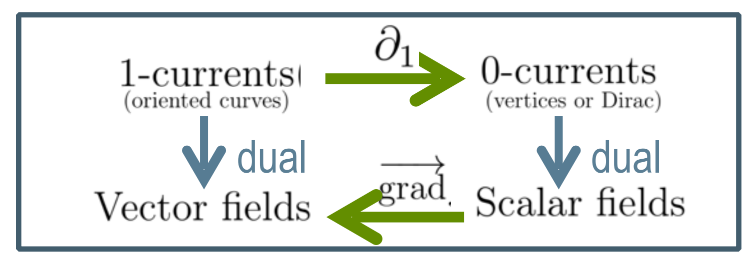
Currents, Mass, flat norm,..

A k-current is something on which one can integrate differential forms. Formally it is the dual of the space of k-differential forms.
Boundary is defined as the adjoint of exterior derivative operator (think of Stokes or Archimedes Theorems).

Boundary:



$$= \int_0^1 \vec{V}\,(\mathcal{C}(t)) \cdot \frac{\overrightarrow{d\mathcal{C}}}{dt}(t)\, dt$$



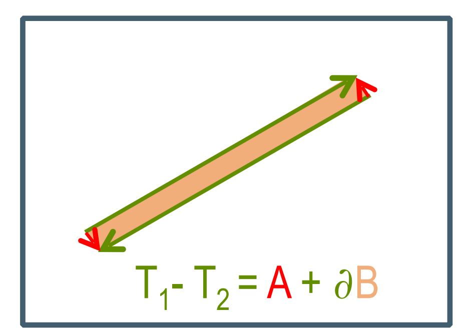
$$\int_x \partial^\star \alpha = \int_{\partial x} \alpha$$

$$(\partial^\star \alpha) x \quad = \quad \alpha(\partial x)$$

Mass and flat norms

$$M(T) = \sup\{T(\phi),\ \|\phi\|_\infty \leq 1\}$$

$$\begin{aligned} F(T) &= \sup\{T(\phi),\ \|\phi\|_\infty \leq 1 \text{ and } \|d\phi\|_\infty \leq 1\} \\ &= \min\{M(A) + M(B),\ T = A + \partial B\} \end{aligned}$$

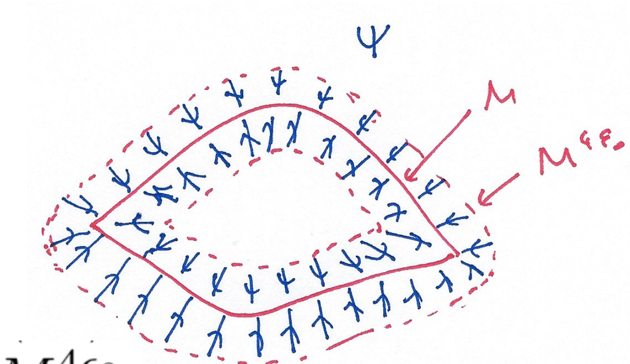


Convergence of L¹ minimal cycles

Lipchitz deformation retract assumption on the manifold M:

There are constants C_1, C_2 and a deformation retract $\psi : M^{4\epsilon_0} \to M$ such that:

- $\forall x_1, x_2 \in M^{4\epsilon_0},\ d(\psi(x_1), \psi(x_2)) \leq C_1 d(x_1, x_2)$
- $\forall x \in M^{4\epsilon_0},\ d(x, \psi(x)) \leq C_2 d(x, M)$

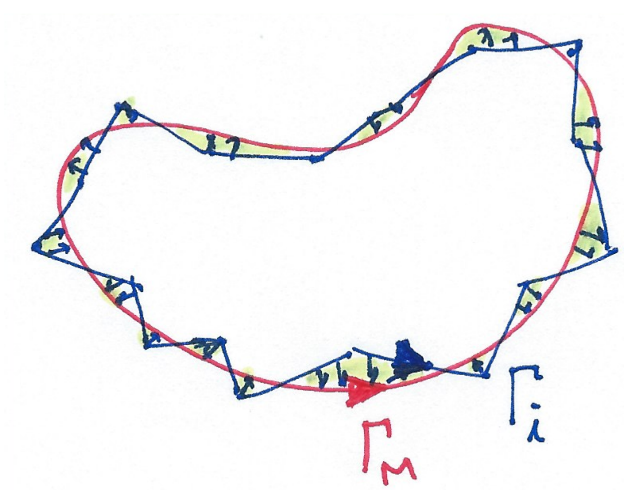


With the assumption above, if we consider the homotopy $h : [0,1] \times M^{4\epsilon_0} \to M^{4\epsilon_0}$ defined by $h(t,x) = (1-t)x + t\psi(x)$ we get a $(k+1)$-current $H_i = h_\sharp([0,1] \times \Gamma_i)$ that"span the space between" Γ_i and Γ_M, formally:

$$\partial H_i = \Gamma_M - \Gamma_i$$

Since $\limsup M(\Gamma_i) \leq \mathrm{Vol}(M)$ and from the properties of ψ we get that:

$$\lim_{i\to\infty} M(H_i) = 0, \text{ in other words: } \lim_{i\to\infty} F(\Gamma_M - \Gamma_i) = 0$$



Convergence of L^1 minimal cycles

One has (flat norm convergence entails weak convergence):

$$\lim_{i\to\infty} F(\Gamma_M - \Gamma_i) = 0 \Rightarrow \lim \inf_{i\to\infty} M(\Gamma_i) \geq M(\Gamma_M) = \mathrm{Vol}(M)$$

This together with $\lim\sup_{i\to\infty} M(\Gamma_i) \leq M(\Gamma_M)$ gives us:

$$\lim_{i\to\infty} \|\gamma_i\|_1 = \lim_{i\to\infty} M(\Gamma_i) = M(\Gamma_M) = \mathrm{Vol}(M)$$

Convergence of L^1 minimal cycles

Perspectives:

- Understanding first rectifiability condition (the existence f regular triangulations) beyond piecewise smooth manifolds.
- Understanding second rectifiability condition (Lipchitz deformation retract) beyond pieces smooth and positive mu-reach.
- Normalization in practice ?
- Sparcity => Homeomorphic manifold in the limit :

$\exists i, \forall j > i, \sigma_j$ defines a manifold homeomorphic to$M$?

By "defines" one means that the coefficients of k-simplices of $C(P_j, \epsilon_j)$ for σ_j are in $\{0, 1\}$ (or in $\{0, \lambda\}$, with $\lambda \neq 0$ if the normalization is not done) and selecting the set of simplifies with non-zero values in σ_j gives the manifold.

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Application to Smale L^2 Hodge theory ?

Dimension of $\ker \Delta_p$ is the k^{th} persistent Betti number.
It is finite under mild conditions.
Are they mild conditions for the laplacian operator to be closed ?
Why not revisit Smale Hodge Theory in the context of persistence?