Information content and computational complexity of recursive sets *

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An honest function is, roughly speaking, a unary, recursive, and strictly increasing function with a very simple graph. Thus if f is an honest function, then the growth of f reflects the computational complexity of f. The honest elementary degrees are the degree structure induced on the honest functions by the reducibility relation "being (Kalmár) elementary in". (Other subrecursive reducibility relations will also work, for example "being primitive recursive in", but not "being polynomial time computable in the *length of input*". "Being polynomial time computable in the *input*" might work, at least in some respects.) A recursive function turns out to be total iff it is elementary in some honest function. Thus, since the set of functions elementary in a particular honest function constitutes a complexity class, the structure of honest elementary degrees will provide a measure for the computational complexity of any total recursive function f. If f is not elementary in a honest function of degree $\mathbf{a}$, it is because f is too hard to compute, i.e. it requires more resources to compute f than the honest degree $\mathbf{a}$ allows.

The structure of subrecursive honest degrees is studied, explicitly or implicitly, in Meyer and Ritchie [11], Basu [2], Machtey [8] [9] [10], Simmons [16], and Kristiansen [5] [6]. Machtey shows that the structure of elementary honest degrees is a lattice with strong density properties, for instance between any degrees $\mathbf{a}, \mathbf{b}$ such that $\mathbf{a} < \mathbf{b}$ there are two incomparable degrees. Kristiansen studies a jump operator on the structure. Among other results he shows that it is possible to invert the jump; there exist low degrees; there exist degrees which are neither high nor low; every situation compatible with $\mathbf{a}' \cup \mathbf{b}' \leq (\mathbf{a} \cup \mathbf{b})'$ is realized in the structure; every situation compatible $\mathbf{a} \leq \mathbf{b} \Rightarrow \mathbf{a}' \leq \mathbf{b}'$ is realized in the structure, e.g. we have incomparable degrees $\mathbf{a}, \mathbf{b}$ such that $\mathbf{a}' < \mathbf{b}'$ and incomparable degrees $\mathbf{a}, \mathbf{b}$ such that $\mathbf{a}' = \mathbf{b}'$ etcetera. Moreover there is a close relationship between the elementary honest degrees and the subrecursive hierarchies described in the book of Rose [15]. Let $\mathbf{0}$ be the degree of the elementary functions, let $\cdot'$ be the jump operator from [6], and let $\mathcal{E}^0, \mathcal{E}^1, \mathcal{E}^2, \ldots$ denote the classes in the Grzegorczyk hierarchy. Then the class $\mathcal{E}^3$ is exactly the functions elementary in an honest function of degree $\mathbf{0}$, the class $\mathcal{E}^4$ is exactly the functions elementary in an honest function of degree $\mathbf{0}'$, the class $\mathcal{E}^5$ corresponds to the degree $\mathbf{0}''$ and so on. By introducing an ω-jump in an obvious way, we will be able to

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climb beyond the Grzegorczyk classes and generate elementary honest degrees that correspond to higher levels in the transfinite hierarchies. Heaton and Wainer [4] study the relationship between subrecursive hierarchies and a jump operator similar to $\cdot'$.

The most popular current subrecursive degree theory is *not* a theory of honest degrees, but a theory that is concerned with reducibility between recursive sets. Due to its importance in computer science, "being polynomial time computable in" is the most popular reducibility relation between the sets. In our discussion it is convenient to use the reducibility relation "being (Kalmár) elementary in", and we use $\leq_E$ to denote this relation. Thus the degrees in the set-degree theory are the equivalence classes induced on the recursive sets by the $\leq_E$-relation. (For the time being it is not important whether we are talking about m-degrees or T-degrees.) This set-degree theory is in many respects different from the honest degree theory induced by the same reducibility relation. The proof methods are different, the degree structures are different, and it seems that a theory of set-degrees does not admit a jump operator.

In a classical paper Ladner [7] proves that neither the polynomial m- nor T-degrees of recursive sets are a lattice. (They are just upper semi-lattices.) He also shows density results and minimal pair results for the same structures. Ladner's proof methods are based on traditional recursion theoretic constructions, and his methods and results generalize to a wide variety of subrecursive reducibilities, e.g. to "being elementary in" and "being primitive recursive in". After Ladner researchers have used refinements of his methods in further studies of the structure of polynomial time degrees of recursive sets. See Ambos-Spies [1] for an overview and further references. I believe the techniques developed in the area can be transferred to all other reasonable subrecursive reducibility relations between sets, and as far as I know all the techniques involve some kind of constructions. In the study of the honest degrees it is possible to obtain a lot of results without doing any constructions at all. Instead in our proofs we exploit that there exists a bound on the growth of the functions in a honest degree. For instance all the results in Kristiansen [6] are achieved by such means. Exactly how far we can get in the study of the honest degrees without constructions, is an interesting question.

I have argued that the honest degrees are degrees of computational complexity, and I am about to argue that the set-degrees are not. Let B be a recursive non-elementary set, and let $\mathcal{B} = \{A \mid A \leq_E B \text{ and } A \text{ is a set}\}$. Then $\mathcal{B}$ is not a complexity class in the sense that every set computable within a certain amount of resources belongs to $\mathcal{B}$. No, $A \leq_E B$ iff B contains enough information to decide membership in A within elementary time. So $\mathcal{B}$ is the class of sets that is computable in elementary time if we have access to the information in B, i.e. if we do not have to compute B. It is reasonable to view the set-degrees as degrees of information content and not as degrees of computational complexity. If a set A is not subrecursively reducible to a set in the set-degree $\mathbf{a}$, it is because A contains too much information, i.e. more information than the degree $\mathbf{a}$ permits.

The objective of this paper is to obtain some relationships between the computational complexity and the information content of sets. Let the set A be of a certain computational complexity, i.e. A is of a certain honest degree. How much information can A possibly contain, i.e. which sets are subrecursively reducible to A? This paper gives answers to such questions, and hopefully this paper contributes to bridge a gap between the two different approaches to subrecursive degree theory.

1. General preliminaries and definitions

I assume the reader is familiar with the most basic concepts of classical recursion theory. An introduction and survey can be found in the books [12] and [14]. I also assume acquaintance with subrecursion and, in particular, with the elementary functions. An introduction to this subject can be found in [13] or [15]. Here I just state some important basic facts and definitions. See [13] and [15] for proofs.

The initial elementary functions are the successor ($\mathcal{S}$), projections ($\mathcal{I}_i^n$), zero (0), addition (+), and modified subtraction ($\dot{-}$) functions. *The elementary schemes* are *composition*, i.e. $f(\mathbf{x}) = h(g_1(\mathbf{x}), \ldots, g_m(\mathbf{x}))$ and bounded *sum* and *product*, i.e. $f(\mathbf{x}, y) = \sum_{i<y} g(\mathbf{x}, i)$ and $f(\mathbf{x}, y) = \prod_{i<y} g(\mathbf{x}, i)$. *The class of elementary functions* is the least class which contains the initial elementary functions and is closed under the elementary schemes. A *relation* or a *predicate* $R(\mathbf{x})$ *is elementary* when there exists an elementary function f with range $\{0, 1\}$ such that $f(\mathbf{x}) = 0$ iff $R(\mathbf{x})$ holds. That a function f has an *elementary graph* means that the relation $f(\mathbf{x}) = y$ is elementary. If we can define a function g from the function f plus the initial elementary functions by the elementary schemes, we say that *g is elementary in f*.

The definition scheme $(\mu z < x)[\ldots]$ is called the *bounded μ-operator*, and $(\mu z < y)[R(\mathbf{x}, z)]$ denotes the least $z < y$ such that the relation $R(\mathbf{x}, z)$ holds. Let $(\mu z < y)[R(\mathbf{x}, z)] = 0$ if no such z exists. The elementary functions are closed under the bounded μ-operator. If f is defined by a primitive recursion over g and h and $f(\mathbf{x}, y) \leq j(\mathbf{x}, y)$, we say that f is a *limited recursion* over g, h and j. (It is convenient to think about limited recursion as a scheme with g, h and j as parameters, although the j is actually not used to generate f.) The elementary functions are closed under limited recursion, but not under primitive recursion. Moreover, the elementary relations are closed under the operations of the propositional calculus and under bounded quantification, i.e. $(\forall x < y)[R(x)]$ and $(\exists x < y)[R(x)]$.

The class of elementary functions is the closure of $\{0, \mathcal{S}, \mathcal{I}_i^n, 2^x, \max\}$ under composition and limited recursion. A proof of this characterization of the elementary functions can be found in [15] or [3]. They prove that the elementary functions equal the third Grzegorczyk class $\mathcal{E}^3$, and $\mathcal{E}^3$ is defined to be the class $\{0, \mathcal{S}, \mathcal{I}_i^n, E_2, \max\}$ closed under composition and limited recursion. It is easy to see that the class $\mathcal{E}^3$ remains the same if we use 2^x in place of E_2 in the definition. ($E_2(x) = E_1^x(2)$ where $E_1(x) = x^2 + 2$.) Thus it follows that the class of functions elementary in f is the closure of $\{0, \mathcal{S}, \mathcal{I}_i^n, 2^x, \max, f\}$ under composition and limited recursion.

All the closure properties of the elementary functions can be proved by using Gödel numbering and coding techniques. Uniform systems for coding the finite sequences of natural numbers are available inside the class of elementary functions. Let $F_f(x)$ be the code for the sequence $\langle f(0), f(1), \ldots f(x)\rangle$. Then F_f belongs to the elementary functions if f does. We are quite informal and indicate the use of coding functions with the notations $\langle \ldots \rangle$. Our coding system is monotone, i.e. $\langle x_0, \ldots, x_n\rangle < \langle x_0, \ldots, x_n, y\rangle$ for every value of y, and $\langle x_0, \ldots, x_i, \ldots, x_n\rangle < \langle x_0, \ldots, x_i + 1, \ldots, x_n\rangle$.

A relation $\sim$ between two functions holds *almost everywhere* (a.e.) iff there is a number k such that for all $x > k$ we have $g(x) \sim f(x)$. Notation: $g(x) \overset{\text{(a.e.)}}{\sim} f(x)$. The function f^k is the k'th iterate of the unary function f, i.e. $f^0(x) = x$ and $f^{k+1}(x) = ff^k(x)$.

2. Theorems on total recursive functions

Definition 2.1. *A function f is* honest *iff (i) f is unary, (ii) $f(x) \geq 2^x$, (iii) f is monotone (nondecreasing), and (iv) f has an elementary graph.*

It is clause (iv) in the definition which is essential. We require that an honest function has an elementary graph because we want no "hidden complexity" in the function. We want the growth of the function to mirror the computational complexity of the function. The structure of honest degrees would be the same if an honest function were not required to satisfy (i), (ii), and (iii), but those requirements are needed for other purposes. Meyer and Ritchie [11] give some characterizations of the honest functions.

We define *recursive function*, *recursive index*, *computation tree*, and other well-known concepts in the usual way. When e is a recursive index for the function f, we adopt the traditional abuse of notation and write $\{e\}(\mathbf{x})$ both for (i) the computation of $f(\mathbf{x})$ associated with e and for (ii) the eventual result of the computation. Let $\mathcal{U}$ be the function such that $\mathcal{U}(\langle x_1, \ldots, x_m\rangle) = x_m$, i.e. a function that gives the last coordinate of a sequence number. When y codes the empty sequence, or when y is not a sequence number at all, let $\mathcal{U}(y) = 0$. Let $\mathcal{T}_n$ be the Kleene predicate for $n = 0, 1, 2, \ldots$, i.e. the predicate $\mathcal{T}_n(e, x_1, \ldots, x_n, t)$ holds iff t is a computation tree for $\{e\}(x_1, \ldots, x_n)$. The relation $\mathcal{T}_n$ is elementary. (According to Rose [15], this was one of the main motivations for introducing the elementary functions in the first place.) The function $\mathcal{U}$ is also elementary, and for each total recursive f we have

$f(x_1,\ldots,x_n) = \{e\}(x_1,\ldots,x_n) = \mathcal{U}(\mu z[\mathcal{T}_n(e,x_1,\ldots,x_n,z)])$ when e is a recursive index for f. We will now state refined versions of the Kleene Normal Form Theorem and the Second Recursion Theorem. The proofs are close to the usual proofs of the original theorems, just a little bit of additional book-keeping is required.

Theorem 2.1 (The Normal Form Theorem). *An n-ary function g is elementary in an honest function f iff there exist a recursive index e for g and a fixed number k such that*

$$\{e\}(x_1,\ldots,x_n) \;=\; \mathcal{U}(\mu y < f^k(\max(x_1,\ldots,x_n))[\mathcal{T}_n(e,x_1,\ldots,x_n,y)]) \;.$$

Proof. Suppose $g(\mathbf{x}) = \{e\}(\mathbf{x}) = \mathcal{U}(\mu y \leq f^k(\max(\mathbf{x}))[\mathcal{T}_n(e,\mathbf{x},y)])$. Then it is trivial that g is elementary in f; the Kleene predicate $\mathcal{T}_n$ is elementary, the functions $\mathcal{U}$ and max are elementary, and the elementary functions are closed under composition and the bounded μ-operator. To prove the other direction of the equivalence, assume that g is elementary in the honest function f. Then g can be generated from the functions $0, \mathcal{S}, \mathcal{I}_i^n$, max and f by composition and limited recursion. Complete the proof of the theorem by induction on such a generation of g.

Theorem 2.2 (The Recursion Theorem). *Let g be an $n{+}1$-ary function elementary in an honest function f. Let $\mathbf{x} = x_1,\ldots,x_n$. Then there exists a recursive index e and a fixed number k such that*

$$g(e,\mathbf{x}) \;=\; \{e\}(\mathbf{x}) \;=\; \mathcal{U}(\mu z < f^k(\max(\mathbf{x}))[\mathcal{T}_n(e,\mathbf{x},z)]) \;.$$

Proof. Prove a refined version of the S_n^m-theorem. Such proof of has a structure similar to the proof of the ordinary S_n^m-theorem. Then carry out a proof that is similar to Kleenes proof of the original Second Recursion Theorem. No surprises come up, and we leave the details.

Definition 2.2. $\{e\}^f(\mathbf{x}) \stackrel{\text{def}}{=} \mathcal{U}(\mu t < f(\max(\mathbf{x}))[\mathcal{T}_n(e,\mathbf{x},t)])$.

Under this notation the Normal Form Theorem says that a function g is elementary in an honest function f iff there exists a recursive index e (for g) and a fixed number k such that $g(\mathbf{x}) = \{e\}^{f^k}(\mathbf{x}) = \{e\}(\mathbf{x})$. The Recursion Theorem says that if g is elementary in f, then there exist numbers e,k such that $g(e,\mathbf{x}) = \{e\}^{f^k}(\mathbf{x})$. The following implication is trivial:

$$g(\max(\mathbf{x})) \geq f(\max(\mathbf{x})) \;\wedge\; \{e\}(\mathbf{x}) = \{e\}^f(\mathbf{x}) \quad\Rightarrow\quad \{e\}(\mathbf{x}) = \{e\}^g(\mathbf{x}) \;.$$

3. The honest functions and the elementary degrees

Definition 3.1. *Let $f \leq_E g$ denote that f is elementary in g, let $f <_E g$ denote that $f \leq_E g$ and $g \not\leq_E f$, and let $f \equiv_E g$ denote that $f \leq_E g$ and*

$g \leq_E f$. *The equivalence classes induced by* $\equiv_E$ *are* the elementary degrees. *We let* $deg(f) \stackrel{\text{def}}{=} \{g \mid g \equiv_E f\}$ *and refer to* $deg(f)$ *as* the degree of f. *We use* $<, \leq$ *for the ordering induced on the degrees by* $<_E, \leq_E$. *An elementary degree* $\mathbf{a}$ *is* honest *iff* $\mathbf{a} = deg(f)$ *for some honest* f. *We will use small bold-faced letters early in the Latin alphabet, i.e.* $\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots$, *to denote honest elementary degrees. If* $\mathbf{a} \leq \mathbf{b} \leq \mathbf{c}$ *then* $\mathbf{a}$ *is a degree* below $\mathbf{b}$, *and* $\mathbf{b}$ *is a degree* above $\mathbf{a}$, *and* $\mathbf{b}$ *is a degree* between $\mathbf{a}$ *and* $\mathbf{c}$. *Every degree in this paper is an* honest elementary degree. *If we just say* degree *or* honest degree, *we do really mean* honest elementary degree. *From now on we reserve the letters* f *and* g *to denote honest functions only.*

The next theorem is important. It enables us to avoid the usual recursion theoretic constructions when we are proving results on honest degrees and functions.

Theorem 3.1 (The Growth Theorem). *Let* f *and* g *be honest functions. Then*

$$g \leq_E f \;\Leftrightarrow\; g(x) < f^k(x) \textit{ for some fixed } k\,.$$

Proof. The left-right direction of the equivalence follows trivially from the Normal Form Theorem. Now suppose that $g(x) < f^k(x)$. Since g is honest, the relation $g(x) = y$ is elementary. We have $g(x) = (\mu y < f^k(x))[g(x) = y]$. Hence $g \leq_E f$ since the elementary functions are closed under composition and the bounded μ-operator.

Definition 3.2. *Let* f *be an honest function. We define the function* f' *by* $f'(x) \stackrel{\text{def}}{=} f^x(x)$. *We call* $\cdot'$ *the* jump operator. *Let* $\mathbf{a}$ *be an honest elementary degree. Then* $\mathbf{a}' \stackrel{\text{def}}{=} deg(f')$ *where* f *is some honest function in* $\mathbf{a}$. *(The degree* $\mathbf{a}'$ *does not depend on the choice of* f *in* $\mathbf{a}$. *This follows from the Growth Theorem.) We let* $\mathbf{0}$ *denote the honest degree* $deg(2^x)$, *i.e.* $\mathbf{0}$ *is the class of elementary functions. Further we let*

$$[\langle d,k\rangle]^f(\mathrm{x}) \stackrel{\text{def}}{=} \{d\}^{f^k}(\mathrm{x}) \stackrel{\text{def}}{=} \mathcal{U}(\mu t < f^k(\max(\mathrm{x}))[\mathcal{T}_n(d, x_1, \mathrm{x}, t)])\,.$$

We say that e *is an* f-elementary index for ψ *whenever* $\psi(\mathrm{x}) = [e]^f(\mathrm{x})$ *and* f *is some honest function.*

We may also define a meet and a join operator on the honest elementary degrees since the structure is a lattice, but we do not need such operators in this paper. Anyway, the function $\max(f(x), g(x))$ is the l.u.b. of the honest functions f and g, and the function $\min(f(x), g(x))$ is the g.l.b. of the honest functions f and g. See [6] and [9].

The jump operator seems a bit arbitrary, but it is very natural. The next few lemmas tell us that it is indeed an analogue to the jump operator on the Turing degrees.

Lemma 3.1. *Let* f *be an honest function. Then* $\{[e]^f\}_{e\in\omega}$ *is an enumeration of the functions elementary in* f.

Proof. Let e be an arbitrary natural number. Let $\psi = [e]^f$ and let $e = \langle d, k\rangle$. Then, straightaway from the definitions, we have

$$\psi(\mathbf{x}) \;=\; \{d\}^{f^k}(\mathbf{x}) \;=\; \mathcal{U}(\mu t < f^k(\max(\mathbf{x}))[\mathcal{T}_n(d, \mathbf{x}, t)])\;.$$

Thus ψ is elementary in f since $\mathcal{U}, \mathcal{T}_n$ etc. are elementary. Further, suppose ψ is elementary in f. By the Normal Form Theorem there exists a recursive index d for ψ and a fixed number k such that $\psi(\mathbf{x}) = \{d\}^{f^k}(\mathbf{x})$. Let $e = \langle d, k\rangle$. Then we have $\psi = [e]^f$.

Lemma 3.2. *Let $\mathcal{J}(f)(\langle x_1, x_2\rangle) = [x_1]^f(x_2)$. Then $f' \equiv_E \mathcal{J}(f)$ whenever f is an honest function.*

Proof. First we prove that f' is elementary in $\mathcal{J}(f)$. The k'th iterate of f, i.e. f^k, is elementary in f for all $k \in \omega$. There exists an elementary function ψ such that $f^k(x) = [\psi(k)]^f(x)$. Thus we have $f'(x) = \mathcal{J}(f)(\langle\psi(x), x\rangle)$ and thereby $f' \leq_E \mathcal{J}(f)$. Next we prove that $\mathcal{J}(f)$ is elementary in f'. Let $a\langle x_1, x_2\rangle = x_1$ and $b\langle x_1, x_2\rangle = x_2$. Then

$$\mathcal{J}(f)(x) \;=\; [ax]^f(bx) \;=\; \{aax\}^{f^{bax}}(bx) \;=\; \mathcal{U}(\mu t < f^{bax}(bx)[\mathcal{T}_1(aax, bx, t)]\;.$$

(The first equality holds by the definition of $\mathcal{J}(f)$, the second by the definition of $[\cdot]^f$, and the third by the definition of $\{\cdot\}^{f^k}$.) It is trivial that the function $f^{bax}(bx)$ is elementary in f'. Thus $\mathcal{J}(f) \leq_E f'$ since $\mathcal{U}, \mathcal{T}_1$ etc. all are elementary functions.

Definition 3.3. *A binary function ρ is a* universal function *for an honest degree $\mathbf{a} = deg(f)$ iff for all unary $\xi \leq_E f$ there exists an n such that $\xi(x) = \rho(n, x)$. Let f and g be honest functions. We write $f \ll g$ when there is a universal function ρ for the degree $deg(f)$ such that $\rho \leq_E g$. We also write $\ll$ for the corresponding relation on the degrees.*

That the relation $\mathbf{a} \ll \mathbf{b}$ holds means that in some sense $\mathbf{b}$ lies far above $\mathbf{a}$. The situation $\mathbf{a} \ll \mathbf{b}$ implies that $\mathbf{a} < \mathbf{b}$, but there exist degrees $\mathbf{a}, \mathbf{b}$ such that $\mathbf{a} < \mathbf{b}$ and $\mathbf{a} \not\ll \mathbf{b}$. The next theorem gives a characterization of the $\ll$-relation. Meyer and Ritchie [11] prove related results.

Theorem 3.2. *Let g and f be honest functions. Then (1) $g \ll f$, (2) $(\exists m)(\forall k)[g^k(x) \overset{\text{(a.e.)}}{<} f^m(x)]$, and (3) $(\exists\psi \leq_E f)(\forall\phi \leq_E g)[\phi(x) \overset{\text{(a.e.)}}{<} \psi(x)]$ are equivalent.*

Proof. (2) $\Rightarrow$ (3): Assume (2). Then we can pick a number m such that $g^k(x) \overset{\text{(a.e.)}}{<} f^m(x)$ for all k. By the Growth Theorem we have that $f^m \leq_E f$, and that every function elementary in g is bounded by g^k for some fixed k. Thus (3) follows. (3) $\Rightarrow$ (1): Assume (3). Then there exists a function ψ elementary in f which majorizes (a.e.) the function g^m, i.e. $g^m(x) \overset{\text{(a.e.)}}{<} \psi(x)$ for every fixed m. This implies that for every m there exists an n such that

$g^m(x) < n + \psi(x)$ (*). Let ξ be any unary function elementary in g. By the Normal Form Theorem we have a recursive index d for ξ and a fixed number m such that

$$\begin{aligned} \xi(x) \;=\; \{d\}^{g^m}(x) &= \mathcal{U}((\mu t < g^m(x))[\mathcal{T}_1(d,x,t)]) \\ &\overset{(*)}{=} \mathcal{U}((\mu t < n + \psi(x))[\mathcal{T}_1(d,x,t)]) \;. \end{aligned}$$

Let $\rho(\langle d,n\rangle,x) \overset{\text{def}}{=} \mathcal{U}((\mu t < n+\psi(x))[\mathcal{T}_1(d,x,t)])$. Then $\rho \leq_E f$, and for every unary function $\xi \in \deg(g)$ there exists a number k such that $\xi(x) = \rho(k,x)$. Thus (1) holds. (1) $\Rightarrow$ (2): Let $\psi(x) = (\max_{i\leq x} \max_{j\leq x} \rho(i,j)) + 1$, where ρ is a universal function for $\deg(g)$ and $\rho \leq_E f$. Now $\psi \leq_E f$, so the Growth Theorem yields a fixed m such that $\psi(x) < f^m(x)$. It is easy to verify that ψ majorizes (a.e.) every function which is elementary in g. Since $g^k \leq_E g$ for every fixed k, we have $g^k(x) \overset{\text{(a.e.)}}{<} \psi(x) < f^m(x)$ for every fixed k. Thus (2) holds.

4. Main results

Definition 4.1. *A* set *is a unary function with range* $\{0,1\}$. *We will use the first few capital letters in the Latin alphabet to denote sets, and we will write* $x \in A$ *instead of* $A(x) = 0$ *etcetera. We denote the sets in the* lower cone *of the honest degree* $\mathbf{a}$ *by* $\leq_{\scriptscriptstyle E}(\mathbf{a})_*$, *i.e.*

$$\leq_{\scriptscriptstyle E}(\mathbf{a})_* \;\overset{\text{def}}{=}\; \{A \mid A \textit{ is a set and } A \leq_E f \textit{ for some } f \textit{ such that } \deg(f) = \mathbf{a}.\}$$

A set A *is (elementarily)* m-reducible *to a set* B *iff there exists an elementary function* ψ *such that* $x \in A \Leftrightarrow \psi(x) \in B$. *A set* B *is* $\leq_{\scriptscriptstyle E}(\mathbf{a})_*$-hard *iff every set in* $\leq_{\scriptscriptstyle E}(\mathbf{a})_*$ *is* m*-reducible to* B.

Fix an honest function f *such that* $\deg(f) = \mathbf{a}$. *A set* B *is* effectively $\leq_{\scriptscriptstyle E}(\mathbf{a})_*$-hard *iff there exists an elementary function* ψ *such that for every* f*-elementary index* e *for a set* $A \in \leq_{\scriptscriptstyle E}(\mathbf{a})_*$ *we have* $x \in A \Leftrightarrow \psi(e,x) \in B$.

Theorem 4.1. *The cone* $\leq_{\scriptscriptstyle E}(\mathbf{b})_*$ *contains an effectively* $\leq_{\scriptscriptstyle E}(\mathbf{a})_*$*-hard set iff* $\mathbf{a}' \leq \mathbf{b}$.

Proof. Let $B \overset{\text{def}}{=} \{\langle e,x\rangle \mid [e]^g(x) = 0\}$ where g is an honest function. We have $B \leq_E g'$ by Lemma 3.2. Let $A \leq_E g$, and let e be a g-relative index for A. Then $x \in A \Leftrightarrow \langle e,x\rangle \in B$. Thus, whenever $\mathbf{b}$ lies above $\mathbf{a}'$, the cone $\leq_{\scriptscriptstyle E}(\mathbf{b})_*$ contains an effectively $\leq_{\scriptscriptstyle E}(\mathbf{a})_*$-hard set.

Now assume that $\mathbf{a}' \not\leq \mathbf{b}$ and that $B \in \leq_{\scriptscriptstyle E}(\mathbf{b})_*$ is an effectively $\leq_{\scriptscriptstyle E}(\mathbf{a})_*$-hard set. We shall derive a contradiction from these assumptions. Let $\mathbf{a} = \deg(g)$ and $\mathbf{b} = \deg(f)$. Further let $A_i(\langle e,x\rangle) \overset{\text{def}}{=} 1 \dot{-} \{e\}^{g^i}(x)$ if $\{e\}$ is a unary function, and let $A_i(\langle e,x\rangle) \overset{\text{def}}{=} 1$ if $\{e\}$ is not unary. Then $A_i \leq_E g$ for every $i \in \omega$. Thus there exists an elementary function ξ such that for all fixed i

we have $x \in A_i \Leftrightarrow \xi(e,x) \in B$ whenever e is a g-elementary index for A_i. Let $\psi(i)$ give a g-elementary index for A_i. Note that ψ is elementary. Let $\phi(y,x) \stackrel{\text{def}}{=} B(\xi(\psi(\langle y,x\rangle),\langle y,x\rangle))$. Then we have $\phi(y,x) = A_{\langle y,x\rangle}(\langle y,x\rangle)$ (i). We also have $\phi \leq_E f$ because $B \leq_E f$. Thus the Recursion Theorem yields a recursive index e and a fixed k such that $\phi(e,x) = \{e\}^{f^k}(x) = \{e\}(x)$ (ii). Since $\deg(g') = \mathbf{a}' \not\leq \mathbf{b} = \deg(f)$, the Growth Theorem says that for every fixed k there exist infinitely many x such that $g^x(x) = g'(x) \geq f^k(x)$ (iii). When we put (i), (ii) and (iii) together we get

$$
\begin{array}{rcll}
\phi(e,x) & \stackrel{\text{(ii)}}{=} & \{e\}^{f^k}(x) & \\
& = & \{e\}^{g^x}(x) & \text{(iii) for some large } x \\
& = & \{e\}^{g^{\langle e,x\rangle}}(x) & \langle\cdot,\cdot\rangle \text{ is monotone} \\
& \neq & 1 \dot{-} \{e\}^{g^{\langle e,x\rangle}}(x) & \\
& = & A_{\langle e,x\rangle}(\langle e,x\rangle) & \text{def. of } A_i \\
& = & \phi(e,x)\ . & \text{(i)}
\end{array}
$$

So there exist e,x such that $\phi(e,x) \neq \phi(e,x)$. Contradiction!

Lemma 4.1. *Assume that f,g are honest functions and that there exist infinitely many x such that $g^i(x) > f^{k+1}(x)$. Then, for every number m there exist infinitely many x such that $g^i(\langle m,x\rangle) > f^k(\langle m,x\rangle)$.*

Proof. The pairing function $\langle\cdot,\cdot\rangle$ is polynomial and monotone in both arguments. Let m be any fixed number. Then we have $\langle m,x\rangle \stackrel{\text{(a.e.)}}{<} f(x)$ since f is an honest function. (We have $f(x) \geq 2^x$ for any honest f.) Therefore it is possible to pick an arbitrarily large x such that

$$f^k(\langle m,x\rangle) \ \leq\ f^{k+1}(x) \ <\ g^i(x) \ \leq\ g^i(\langle m,x\rangle)\ .$$

Theorem 4.2. *Assume $\mathbf{0} \ll \mathbf{b}$. The cone $\leq_E(\mathbf{b})_*$ contains a $\leq_E(\mathbf{a})_*$-hard set iff $\mathbf{a} \ll \mathbf{b}$.*

Proof. Let g and f be honest functions such that $\deg(g) = \mathbf{a} \ll \mathbf{b} = \deg(f)$. By Theorem 3.2 there exists a universal function ρ for the degree $\mathbf{a}$ such that $\rho \leq_E f$. Let $B \stackrel{\text{def}}{=} \{\langle y,x\rangle \mid \rho(y,x) = 0\}$. Then $B \leq_E f$ and thus $B \in \leq_E(\mathbf{b})_*$. Let A be any set such that $A \leq_E g$. We show that A is m-reducible to B. Since ρ is an $\mathbf{a}$-universal function there exists n such that $A(x) = \rho(n,x)$. Fix such an n and let $\xi(x) = \langle n,x\rangle$. Then ξ is elementary and $x \in A \Leftrightarrow \xi(x) \in B$. Thus A is m-reducible to B. That was the proof of the if-direction.

In order to prove the only-if-direction assume that $B \in \leq_E(\mathbf{b})_*$ is $\leq_E(\mathbf{a})_*$-hard, that $\mathbf{a} \not\ll \mathbf{b}$, and that $\mathbf{0} \ll \mathbf{b}$. We will derive a contradiction from these assumptions. Let g and f be honest functions such that $\deg(g) = \mathbf{a}$ and $\deg(f) = \mathbf{b}$. By Theorem 3.2 we have

$$(\forall k)(\exists i)\ [\ g^i(x) > f^k(x) \text{ for infinitely many } x\ ] \qquad \text{(I)}$$

since $\mathbf{a} \not\ll \mathbf{b}$. Let

$$A_{i,k}(\langle y,x\rangle) \stackrel{\text{def}}{=} \begin{cases} 1 \dot{-} \{y\}^{f^k}(\langle y,x\rangle) & \text{if } \{y\} \text{ is unary} \\ & \text{and } f^k(\langle y,x\rangle) < g^i(\langle y,x\rangle) \\ 0 & \text{otherwise} \end{cases}$$

For every fixed i and k the set $A_{i,k}$ is elementary in g. (Here is an argument to support the this claim: The graph of f is elementary since f is honest. By induction on k it is easy to show that the graph of f^k is elementary. Thus the function $\xi(x) \stackrel{\text{def}}{=} (\mu z < g^i(x))[f^k(x) = z]$ is elementary in g because the relation $f^k(x) = z$ is elementary. Moreover, if $\xi(x) \neq 0$ then $f^k(x) < g^i(x)$, and if $\xi(x) = 0$ then $f^k(x) \not< g^i(x)$. Therefore it is possible to decide elementarily in g whether $f^k(x) < g^i(x)$ for fixed i and k. Now it is easy to see that it is also possible to decide elementarily in g whether $x \in A_{i,k}$. So $A_{i,k} \leq_E g$.) The set B is elementary in f. The Normal Form Theorem says that there exists a recursive index e_0 for B and a fixed number j such that $B(x) = \{e_0\}^{f^j}(x)$. By assumption B is $\leq_E(\mathbf{a})_*$-hard. Thus, for all fixed i and k there exists an elementary ψ such that $x \in A_{i,k} \Leftrightarrow \psi(x) \in B$. We have also assumed $\mathbf{0} \ll \mathbf{b} = \deg(f)$. This assumption implies that for every elementary function ψ there exists a recursive index e_1 such that $\psi(x) \stackrel{\text{(a.e.)}}{=} \{e_1\}^f(x)$. We have $A_{i,k}(x) = B(\psi(x))$ for some elementary ψ. Thus for all fixed i,k there exists a recursive index e_1 such that $A_{i,k}(x) \stackrel{\text{(a.e.)}}{=} \{e_0\}^{f^j}(\{e_1\}^f(x))$, and $\{e_0\}^{f^j}(\{e_1\}^f(x)) = \{e\}^{f^m}(x)$ for some recursive index e and fixed number m. So it is possible to pick a fixed number m such that for every i,k there exists a recursive index e such that

$$A_{i,k}(x) \stackrel{\text{(a.e.)}}{=} \{e\}^{f^m}(x). \tag{II}$$

Note that e depends on i and k, but m does not.

Now fix m such that (II) holds. Fix k such that $k > m$. Fix i such that $g^i(x) > f^{k+1}(x)$ for infinitely many x. Such an i exists by (I). Then fix a d such that $A_{i,k}(x) \stackrel{\text{(a.e.)}}{=} \{d\}^{f^m}(x)$. Such a d exists by (II). Now we have $A_{i,k}(x) \stackrel{\text{(a.e.)}}{=} \{d\}^{f^m}(x) = \{d\}^{f^k}(x)$. Thus, when we substitute $\langle d,x\rangle$ for x in the last formula, we get

$$A_{i,k}(\langle d,x\rangle) = \{d\}^{f^k}(\langle d,x\rangle) \text{ for every sufficiently large } x. \tag{III}$$

We have chosen i such that $g^i(x) > f^{k+1}(x)$ holds for infinitely many x. Hence, by Lemma 4.1, there exists an arbitrarily large number n such that $g^i(\langle d,n\rangle) > f^k(\langle d,n\rangle)$. Fix a sufficiently large such n. Now we have $A_{i,k}(\langle d,n\rangle) = 1 \dot{-} \{d\}^{f^k}(\langle d,n\rangle)$ by the definition of $A_{i,k}$, but we also have $A_{i,k}(\langle d,n\rangle) = \{d\}^{f^k}(\langle d,n\rangle)$ by (III). So we have numbers i,k,d,n such that $A_{i,k}(\langle d,n\rangle) \neq A_{i,k}(\langle d,n\rangle)$, i.e. we have a contradiction.

Let **a** be any honest degree such that $\mathbf{0} \ll \mathbf{a}$. The structure of the sets in the lower cone of **a** is obviously an ideal (with respect to the partial ordering $\leq_E$). By the previous theorem we may infer that such a structure is never a principal ideal since $\mathbf{b} \not\ll \mathbf{b}$ for all honest degrees **b**.

We know that there exists a whole $\ll$-dense set of honest degrees between **a** and $\mathbf{a}'$. (See Meyer and Ritchie [11], Simmons [16] and Kristiansen [5].) Unfortunately it is not known whether the elementary honest degrees are $\ll$-dense. (This is stated as an open problem in Meyer and Ritchie [11].) Anyway, let us suppose that the answer to this open question is positive. Suppose also that $\mathbf{0} \ll \mathbf{a}$. Then there isn't any least degree **b** such that $\leq_E(\mathbf{b})_*$ contains a $\leq_E(\mathbf{a})_*$-hard set. This is a consequence of the previous theorem. In contrast Theorem 4.1 says that there is a least degree **b** such that $\leq_E(\mathbf{b})_*$ contains an *effectively* $\leq_E(\mathbf{a})_*$-hard set, namely $\mathbf{b} = \mathbf{a}'$. No degree **b** strictly below or incomparable to $\mathbf{a}'$ is such that $\leq_E(\mathbf{b})_*$ contains an effectively **a**-hard set. (It is a pity that we need the condition $\mathbf{0} \ll \mathbf{b}$ in the previous theorem. Is it possible that $\leq_E(\mathbf{a})_*$ possesses a $\leq_E(\mathbf{0})_*$-hard set when **a** is some honest degree very close to $\mathbf{0}$?)

Let **b** be an arbitrary set which lies strictly above **a**, i.e. $\mathbf{a} < \mathbf{b}$. The previous theorem says that $\leq_E(\mathbf{b})_*$ does not necessarily possess a $\leq_E(\mathbf{a})_*$-hard set. (We may have $\mathbf{a} < \mathbf{b}$, but not $\mathbf{a} \ll \mathbf{b}$.) This leads us to ask if it is necessarily the case that $\leq_E(\mathbf{b})_*$ contains any set at all which isn't also contained in $\leq_E(\mathbf{a})_*$; or if **a** and **b** are incomparable, is it necessarily the case that $\leq_E(\mathbf{a})_*$ and $\leq_E(\mathbf{b})_*$ are incompatible? The next theorem answers these questions.

Theorem 4.3. *We have the equivalence* $\leq_E(\mathbf{a})_* \subseteq \leq_E(\mathbf{b})_* \;\Leftrightarrow\; \mathbf{a} \leq \mathbf{b}$.

Proof. If $\mathbf{a} \leq \mathbf{b}$, then $\leq_E(\mathbf{a})_* \subseteq \leq_E(\mathbf{b})_*$ follows trivially. Assume $\mathbf{a} \not\leq \mathbf{b}$ and that $\mathbf{a} = \deg(f)$ and $\mathbf{b} = \deg(g)$. Let $A(\langle y, x\rangle) \overset{\text{def}}{=} 1 \dot{-} \{y\}^f(x)$ whenever y is an index for a unary function. (Let $A(\langle y, x\rangle) \overset{\text{def}}{=} 1$ if $\{y\}$ is not a unary function.) It is obvious that A is elementary in f, i.e. $A \in \leq_E(\mathbf{a})_*$. We will now derive a contradiction from the assumption that $A \in \leq_E(\mathbf{b})_*$. So assume $A \leq_E g$. Then the Recursion Theorem proclaims the existence of e and k such that $A(\langle e, x\rangle) = \{e\}^{g^k}(x)$ (*). But since $f \not\leq_E g$ we may use the Growth Theorem and pick an m such that $f(m) \geq g^k(m)$ (**). Now the following contradiction emerges:

$$A(\langle e,m\rangle) \overset{(*)}{=} \{e\}^{g^k}(m) \overset{(**)}{=} \{e\}^f(m) \neq 1 \dot{-} \{e\}^f(m) = A(\langle e,m\rangle)\,.$$

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