THE MARCH MEETING OF THE CHICAGO SECTION.

THE eleventh regular meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY was held on Saturday, March 29, 1902, at the University of Chicago. The following fifteen members were present:

Professor Oskar Bolza, Professor D. F. Campbell, Professor L. E. Dickson, Professor Thomas F. Holgate, Dr. H. G. Keppel, Professor Kurt Laves, Professor H. Maschke, Professor E. H. Moore, Dr. F. R. Moulton, Professor S. W. Shattuck, Professor H. E. Slaught, Professor E. J. Townsend, Professor L. G. Weld, Professor H. S. White, Professor J. W. A. Young.

A morning and an afternoon session were held, Professor Townsend performing the duties of chairman at the first and Professor Moore, President of the Society, at the second.

At the last Christmas meeting of the Section a committee, composed of Professors Waldo, Bolza and Townsend, was appointed to consider and report a scheme of uniform requirements for the master's degree for candidates who make mathematics their major subject. This committee presented a preliminary report, which was discussed and ordered to be manifolded for the use of members. It is expected that the report will come up for further discussion at the next meeting of the Section.

The following papers were presented at this meeting:

(1) Professor O. STOLZ: "Nachtrag zum Artikel: 'Zur Erklärung der Bogenlänge und des Inhaltes einer krummen Fläche.'"

(2) Mr. A. T. BELL: "The mutual independence of Hilbert's axioms within the various groups."

(3) Professor H. MASCHKE: "On the superosculation of surfaces."

(4) Professor L. G. WELD: "A certain conic connected with the isotomic relation."

(5) Professor OSKAR BOLZA: "Proof of the sufficiency of Jacobi's condition for a permanent sign of the second variation in the so-called isoperimetric problems."

(6) Professor OSKAR BOLZA: "Concerning the isoperimetric problem on a given surface."

(7) Professor L. E. DICKSON: "On the group defined for any given field by the multiplication table of any given finite group."

(8) Professor KURT LAVES: "Some remarkable cases of libration among the small planets of the Hilda type."

(9) Professor E. J. TOWNSEND: "On the interchange of the order of differentiation."

(10) Professor L. E. DICKSON: "Theorems on the residues of multinomial coefficients with respect to a prime modulus"

Mr. Bell was introduced to the Society by Professor Townsend. The paper by Professor Stolz was presented and read by Professor Moore. The following abstracts will serve to characterize the papers:

Professor Stolz gives a comparison of his exposition of the theory of rectification of curves (*Transactions*, volume 3, pages 23–37) with that of C. Jordan (Cours d'Analyse, 2d edition, volume 1, nos. 105–111).

Hilbert in his "Grundlagen der Geometrie" considers the mutual independence of the five groups into which he divides his system of axioms. He also calls attention to the fact that the axioms within the various groups are mutually independent, but merely refers the reader to his lectures on Euclidean geometry (1898–99) where he shows the independence of *particular* axioms from others of the same group. It is the purpose of Mr. Bell's paper to complete this work by constructing the necessary geometry in each case.

Professor Maschke deduces the conditions which a point P of a surface S has to satisfy in order that at P a contact of the third order between S and a general surface of the second order be possible. These conditions, which for every given case reduce to two, are in general given by the vanishing of a differential covariant of the third order. It follows by a simple proof that the developable surfaces and the surfaces of the second order are the only ones for every point of which a surface of second order exists having a contact of the third order at this point. The paper will be published in the *Transactions*.

If L, M, N are the midpoints of the sides a, b, c of a triangle ABC, and if we lay off on a and b, respectively, $LL_1 = \sigma$ and $MM_1 = r\sigma$; also $LL' = -\sigma$ and $MM' = -r\sigma$, then P', determined by AL', BM' is the *isotomic conjugate* of P_1, determined by AL_1, BM_1. Assuming $r =$ const., Professor Weld shows that the locus of P_1, or of P', is a conic

φ, through A, B, the centroid G of the triangle, and the ex-centroid G_c. This conic reduces to the "chord of contact" form when $r = b/a$; to the C-medial and the side c if $r = -b/a$; to the A-medial and the exterior B-medial, *i. e.*, the line through B parallel to b, if $r = \pm \infty$; to the B-medial and the exterior A-medial if $r = 0$. Chords through isotomic conjugates are concurrent in a point R, lying upon the exterior C-medial; the correspondence between r and R being one to one. The conic φ has the same relation to the triangle ABR as to the triangle ABC when r is any one of the roots of the cubic

$$r^3 - 5\frac{b}{a}r^2 - 5\frac{b^2}{a^2}r + \frac{b^3}{a^3} = 0;$$

i. e., for the three values of r thus determined, ABR and ABC may be appropriately called isotomic reciprocal triangles with respect to φ. One of these values is $-b/a$, for which R coincides with C. The two conjugate values, $(3 \pm 2\sqrt{2})\, b/a$, determine R_a''' and R_b''' upon the exterior C-medial, which points may be called isotomic foci of the given triangle. There are two other pairs of foci, viz., R_b', R_c' upon the exterior A-medial, and R_c'', R_a'' upon the exterior B-medial. These six foci are con-conic.

The proof announced in Professor Bolza's title is based upon an extension of the lemma on linear differential expressions by which Jacobi proves the analogous theorem in the unconditioned problem. It will be published in the *Transactions*.

Professor Bolza's second paper gives a simple proof of the well known theorem that the extremals for the isoperimetric problem on a given surface are curves of constant geodesic curvature. This paper will be published in the *Mathematische Annalen*.

In his first paper Professor Dickson develops for an arbitrary field a theory which generalizes the results of Burnside announced in a paper "On the continuous group defined by any given group of finite order" (*Proceedings of the London Mathematical Society*, volume 29, 1898). His proofs by means of the Lie theory are replaced by rational operations valid for any field F. In particular, by taking for F the various Galois fields, we obtain a doubly infinite system of finite groups corresponding to each given finite group. The results find application in the problem to represent a given finite group as a linear group in a given field upon the smallest number of variables.

Laplace has shown that libration in longitude will occur under the following condition: If θ be an angle of the character of the longitude in the orbit, it is defined by the differential equation

$$\frac{d^2\theta}{dt^2} = -\frac{m'}{2} h^2 \sin\theta, \tag{1}$$

where h^2 is a function of certain elements of the perturbed and perturbing planets, m' the mass of the perturbing planet. Calling c the constant of integration, we obtain from (1)

$$h\sqrt{m'}dt = \frac{d\theta}{\sqrt{\cos\theta + c}}. \tag{2}$$

The condition for libration is then $c^2 < 1$; θ will then oscillate between two finite values. Determining the value of c from the osculating elements of a planet, it can be shown, that this condition may be written as follows:

$$[in_1 - (i+1)n'] < h \cdot \cos\frac{\theta_1}{2}\sqrt{2m'},$$

for all planets which are of such a type that their mean daily motion is to that of Jupiter as $i+1 : i$, where i is an integer. In the foregoing formula n_1 represents an osculating value of the mean daily motion of the perturbed planet, n' the mean daily motion of Jupiter; θ_1 is a special value of θ. Tisserand, in searching for an explanation of his "lacunes" in the ring of planetoids seized upon the possibility of Jupiter's effecting librations among the planetoids at the places where these "lacunes" take place. He was not able to discover librational effects in these cases. It is a curious result that appears from Professor Laves's paper that libration takes place at those very places $n = 450''$ and $n = 400''$ where *no* "lacunes" occur.

Professor Townsend discussed the conditions under which the relation

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

is valid. It is usual to show that this holds at any point in whose neighborhood

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x \partial y} \left(\text{or } \frac{\partial^2 f}{\partial y \partial x} \right)$$

exist and are continuous in the two variables x, y together. In this paper it is shown that a narrower condition is sufficient, namely when

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \text{ and } \frac{\partial^2 f}{\partial x \partial y} \left(\text{or } \frac{\partial^2 f}{\partial y \partial x} \right)$$

exist and are continuous in x alone and in y alone. When these conditions exist, the points are everywhere dense upon any line $x = \text{constant}$, or $y = \text{constant}$, in which $\frac{\partial^2 f}{\partial x \partial y}$ is a continuous function of both variables together. It follows that at these points $\frac{\partial^2 f}{\partial y \partial x}$ also exists and equals $\frac{\partial^2 f}{\partial x \partial y}$. A necessary and sufficient condition is then developed for the interchange of the order of limits in general. By the application of this theorem, the existence and continuity of

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x \partial y} \left(\text{or } \frac{\partial^2 f}{\partial y \partial x} \right)$$

in each variable separately is shown to be a sufficient condition for the interchange of the order of differentiation.

Professor Dickson shows in his second paper that the residue, modulo p, of the sum of certain general sets of multinomial coefficients is either zero or a single reduced multinomial coefficient. This result is a generalization of a theorem on binomial coefficients due to Glaisher (*Quarterly Journal of Mathematics*, volume 30 (1899), pages 361–383). The proof depends on two theorems announced by the author in the *Annals of Mathematics*, 1st series, volume 11 (1897), pages 75–76.

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