HOMOLOGY AND SEMI-FUNCTORS

BY
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0. Introduction

The general problem of "pasting together" given structures on a class of objects to make a new global structure is well known in all branches of topology, algebra and geometry. We approach this subject by means of a "homology" (the final global structure) in such a way that a "Hurewicz theorem" holds.

In [1] we have constructed corresponding to a given functor $\Phi : \mathfrak{K} \rightarrow \text{Ens}$ and a distinguished subcategory $\mathfrak{L} \subset \mathfrak{K}$ a homotopy functor Φ_π of Φ rel $\mathfrak{L}$. In the present paper this is dualized to the definition of a homology functor H_Φ rel $\mathfrak{L}$. The assertion of Theorem 4.3. which describes H_Φ is completely dual to that of Satz 4 in [1] (which determines Φ_π). One of the advantages of Theorem 4.3. is that it extends immediately to group valued functors. If we start with a functor Φ which is group valued then H_Φ is again a group valued functor. Among the examples for the homology H_Φ we present the singular homology H_* as originating from $\Phi = \pi$ (Theorem 5.1.). This is of course the main justification of the name "homology" for H_Φ. In Section 6 we list more examples for H_Φ including the "globalization" of a given local structure and the completion process of a given semigroup to a group.

The key to all our constructions is again, as in [1] the concept of a semi-functor. Therefore most theorems are concerned with semi-functors rather than with functors. In Theorem 2.2. we give a representation of an arbitrary semi-functor $H = FT^{-1}$ as the inverse of a functor $T : \mathfrak{L} \rightarrow \mathfrak{K}$ followed by a functor $F : \mathfrak{L} \rightarrow \text{Ens}$ (in the sense of Theorem 2.1.). In [1] we proved that every semi-functor H can be converted into a functor $\bar{H}$ in a universal way (see also theorem 3.1.). In Theorem 3.2. and Theorem 3.3. this fact is dualised and extended.

It should be kept in mind that all these theorems are only dual as far as their statements are concerned; the proofs can not be dualised.

As for all theories of this kind one remark must be included: we have to assume throughout this paper that all categories are small. This assures us that everything is in accordance with Bernays-Gödel-von Neumann axiomatics of set theory. However in all our applications (Theorems 5.1–6.5) it can be easily proved directly that all our constructions are legitimate although the different categories $\mathfrak{K}$ are far from being small. The present paper is independent of [1]; only Theorem 3.1. is used without proof.

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1. Semi-functors

We denote as usual by "Ens" the category of sets and by "Ens_0" the category of based sets. A relation $\rho : A \to B$, $A, B \in \text{Ens}$ is a subset $\rho \subset A \times B$. The empty set is not excluded. By $R(A, B)$ we denote the set of all relations $\rho : A \to B$ ("relations from A to B"). If $\rho \in R(A, B)$, $\tau \in R(B, C)$, then $\tau \circ \rho \in R(A, C)$ is defined by

$$\tau \circ \rho = \{ (a, c) \mid \exists\, b \in B,\ (a, b) \in \rho,\ (b, c) \in \tau \}.$$

For any map $f \in \text{Ens}\ (A, B)$ there is a relation $f \in R\ (A, B)$ (denoted by the same letter), the graph of f. In particular $1_A : A \to A$, the graph of the identity, is a member of $R(A, A)$. Relations are partially ordered by inclusion:

$$\rho \leq \tau \Leftrightarrow \rho \subseteq \tau; \quad \rho, \tau \in R(A, B).$$

To each $\rho \in R(A, B)$ there exists the "inverse" relation $f\eth \in R(B, A)$ defined
To each $\rho \in R(A, B)$ there exists the "inverse" relation $d\rho \in R(B, A)$ defined by $d\rho = \{ (b, a) \mid (a, b) \in \rho \}$. The following assertion is immediate:

1.1. (a) $d(\tau\rho) = d\rho\, d\tau$.

(b) $d1_A = 1_A$.

(c) Assume $\rho \in R(A, B)$, $\rho \in R(B, A)$, $\tau\rho = 1_A$, $\rho\tau = 1_B$; then $\tau = d\rho$ and $\rho = d\tau$. Moreover we have $\rho \in \text{Ens}\ (A, B)$ and τ is its inverse.

1.2. Definition. A semi-functor $H : \mathfrak{K} \to \text{Ens}$ where $\mathfrak{K}$ is an arbitrary category is a function which assigns to every $X \in \mathfrak{K}$ a set $H(X) \in \text{Ens}$ and to every $f \in \mathfrak{K}(X, Y)$ a relation $H(f)$ such that the following conditions hold: If

$$X \xrightarrow{\ f\ } Y \xrightarrow{\ g\ } Z$$

are morphisms in $\mathfrak{K}$ one has

$$H(gf) \geq H(g)H(f), \qquad H(1_X) \geq 1_{H(X)}.$$

1.3. Definition: Let $\varphi = \{\varphi_X : H(X) \to H'(X),\ X \in \mathfrak{K}\}$ be a family of transformations in Ens $(H(X), H'(X))$ for two semi-functors H, $H' : \mathfrak{K} \to \text{Ens}$. We call φ a semi-functor transformation if the following naturality condition holds:

If f $\in \mathfrak{K}(X, Y)$, then

$$(a, b) \in H(f) \Rightarrow (\varphi_X(a), \varphi_Y(b)) \in H'(f).$$

1.4 (cf. [1]). (a) If $F : \mathfrak{K} \to \text{Ens}$ is a co (contra-) variant functor, then F is a semi-functor.

(b) If $F, F' : \mathfrak{K} \to \text{Ens}$ are functors and $\varphi : F \to F'$ is a semi-functor transformation then φ is a functor transformation.

(c) Let $H : \mathfrak{K} \to \text{Ens}$ be a semi-functor and select for each $X \in \mathfrak{K}$ a subset

$S(X) \subset H(X)$. For $f \in \mathfrak{K}(X_1, X_2)$ we define

$$(s_1, s_2) \in S(f) \quad \Leftrightarrow \quad (s_1, s_2) \in H(f), \quad s_i \in S(X_i), i = 1, 2.$$

Then S is a semi-functor.

If we denote by $\mathfrak{H} = \mathfrak{H}(\mathfrak{K})$ the category of all semi-functors with transformations as in 1.3., and by $\mathfrak{F} \subset \mathfrak{H}$ the subcategory of functors, then 1.4. (b) can be rephrased as "$\mathfrak{F}$ is a full subcategory of $\mathfrak{H}$".

Similarly one can define $\mathfrak{H}_0(\mathfrak{K})$, the category of all semi-functors $H : \mathfrak{K} \to \text{Ens}_0$. In [4] Mac Lane defines additive relations for an abelian category. However this notion can easily be extended to the category of groups. If one replaces relations $\rho : A \to B$ by additive relations in this sense we obtain the category $\mathfrak{H}_G(\mathfrak{K})$ where now $H : \mathfrak{K} \to \text{Grp}$ (= category of groups) is a group-valued semi-functor. As long as we work with a fixed $\mathfrak{K}$, we will simply write $\mathfrak{H}$ $(\mathfrak{F}, \mathfrak{H}_G, \cdots)$ instead of $\mathfrak{H}(\mathfrak{K})$ $(\mathfrak{F}(\mathfrak{K}), \mathfrak{H}_G(\mathfrak{K}), \cdots)$. By a functor we mean throughout this paper a covariant functor.

It should be mentioned that a semi-functor H is a bifunctor from the trivial bicategory $\mathfrak{K}$ to the bicategory of sets and relations. Thus the concept of a semi-functor is not new.

2. Some constructions in $\mathfrak{H}$

The aim of this section is to construct new objects in $\mathfrak{H}$ from given ones. Let $T : \mathfrak{L} \to \mathfrak{K}$ be a covariant functor for two arbitrary categories $\mathfrak{L}$ and $\mathfrak{K}$ and $F : \mathfrak{L} \to \text{Ens}$ a given semifunctor. We are going to construct a semi-functor $H = FT^{-1} : \mathfrak{K} \to \text{Ens}$ and a transformation $\eta : F \to HT$, $\eta \in \mathfrak{H}(F, HT)$ such that the pair (H, η) is universal in the following sense:

For any other pair (H', η'), $H' : \mathfrak{K} \to \text{Ens}$, $\eta' \in \mathfrak{H}(F, H'T)$ there exists a unique transformation $\varphi \in \mathfrak{H}(H, H')$ such that for the corresponding transformation $\varphi_T : HT \to H'T$ one has $\varphi_T \eta = \eta'$. Using this definition, the following existence theorem holds:

2.1 THEOREM. *Let $F \in \mathfrak{H}(\mathfrak{L})$ and $T : \mathfrak{L} \to \mathfrak{K}$ be given. Then there exists a universal pair (H, η).*

Proof. We will assume that for a semi-functor H and any two objects $L_1 \neq L_2$ one has $H(L_1) \cap H(L_2) = \emptyset$. Different objects can be mapped into equivalent but never into equal objects. Because the category Ens is large enough this assumption does not cause any difficulty.

We define for given $X \in \mathfrak{K}$,

$$H(X) = \bigcup F(L), \quad T(L) = X$$

and for given $f \in \mathfrak{K}(X, Y)$, $\zeta \in H(X)$, $\zeta' \in H(Y)$, $(\zeta, \zeta') \in H(f) \Leftrightarrow \exists\, f_{\mathfrak{L}} : L_X \to L_Y$ in $\mathfrak{L}$ such that

$$T(f_{\mathfrak{L}}) = f, \quad \zeta \in F(L_X), \quad \zeta' \in F(L_Y), \quad (\zeta, \zeta') \in F(f_{\mathfrak{L}}).$$

If there is no L with $T(L) = X$, we set $H(X) = \emptyset$. In this way we have constructed a semi-functor; since for fixed $L \in \mathfrak{L}$, $F(L) \subset HT(L)$, we also have a transformation $\eta : F \rightarrow HT$. Now let $\eta' : F \rightarrow H'T$, $H' \in \mathfrak{H}(\mathfrak{L})$ such that $T(L) = X$ and $\zeta \in F(L)$. Hence we can define

$$\varphi(\zeta) = \eta'(\zeta) \in H'T(L) = H'(X).$$

By construction $\varphi \in \mathfrak{H}(H, H')$ and $\varphi_T \eta = \eta'$. Now take any $\bar{\varphi} \in \mathfrak{H}(H, H')$ with the property $\bar{\varphi}_T \eta = \eta'$ and let $\zeta \in H(X)$ be given. The object L with $T(L) = X$ and $\zeta \in F(L)$ is unique. Hence $\bar{\varphi}(\zeta) = \eta'(\zeta)$ and therefore $\bar{\varphi} = \varphi$.

This completes the proof of Theorem 2.1.

Clearly H is nothing other than a left Kan extension of a semifunctor along a functor.

There is an alternative formulation of Theorem 2.1. which for some purposes seems to be more suitable:

Associated to the functor $T : \mathfrak{L} \rightarrow \mathfrak{K}$ there is a functor

$$\mathfrak{H}(T) : \mathfrak{H}(\mathfrak{K}) \rightarrow \mathfrak{H}(\mathfrak{L})$$

defined by $\mathfrak{H}(T)F = FT$, $F \in \mathfrak{H}(\mathfrak{K})$, The relation between F and (H, η) gives rise to a functor $R : \mathfrak{H}(\mathfrak{L}) \rightarrow \mathfrak{H}(\mathfrak{K})$ which in fact turns out to be the left adjoint of $\mathfrak{H}(T)$. By Kan's definition of left adjointness [4] there has to be an isomorphism

$$\lambda : \mathfrak{H}(\mathfrak{K})(R(F), F') \approx \mathfrak{H}(\mathfrak{L})(F, \mathfrak{H}(T)F')$$

which is natural in both variables. The result can be established by standard arguments: Let $\varphi \in \mathfrak{H}(\mathfrak{K})(R(F), F')$. Then there exists a transformation

$$\varphi_T : HT \; (= \mathfrak{H}(T)R(F)) \rightarrow F'T.$$

We set $\lambda(\rho) = \rho_T \eta$ which on the other hand uniquely determines a transformation φ (by universality).

2.2. THEOREM. *The functor $\mathfrak{H}(T)$ is provided with a left adjoint R.*

In case F is a functor, $H = FT^{-1}$ can be considered as the composition of the "inverse" of a functor T and a given functor. Even under these circumstances H need not be a functor but only a semi-functor. It is interesting to observe that there is a converse to this statement:

2.3. THEOREM. *Every $H \in \mathfrak{H}$ is isomorphic to a FT^{-1} where $T : \mathfrak{L} \rightarrow \mathfrak{K}$ and $F : \mathfrak{L} \rightarrow$ Ens are suitable covariant functors.*

This is an immediate generalization of a well-known fact about Kan extensions and functors.

Proof. Take for $\mathfrak{L}$ the category whose objects are the elements $\zeta \in H(X)$ for some $X \in \mathfrak{K}$ and whose maps are triples (ζ, ζ', f), where $f \in \mathfrak{K}(X, Y)$ and

$(\zeta, \zeta') \in H(f)$. The composition of two morphisms

$$(\zeta, \zeta', f) \in \mathfrak{L}(\zeta, \zeta'), \qquad (\zeta', \zeta'', g) \in \mathfrak{L}(\zeta', \zeta'')$$

is defined by $(\zeta, \zeta'', gf) \in \mathfrak{L}(\zeta, \zeta'')$. The distinguished morphism $(\zeta, \zeta, 1) : \zeta \to \zeta$ serves obviously as the identity. Under these circumstances $\mathfrak{L}$ is a category and $T : \mathfrak{L} \to \mathfrak{K}$

$$T(\zeta) = X, \zeta \in H(X), \qquad T(\zeta, \zeta', f) = f,$$

is a covariant functor. The functor $F : \mathfrak{L} \to \text{Ens}$ is trivial: assign to each $\zeta \in \mathfrak{L}$ the set $F(\zeta)$ which consists of one element (namely ζ itself) and to each $(\zeta, \zeta', f) \in \mathfrak{L}(\zeta, \zeta')$ the trivial map in Ens between the one-point-sets $F(\zeta)$ and $F(\zeta')$. We are now ready to compute $FT^{-1}(X)$ for given $X \in \mathfrak{K}$. By definition

$$FT^{-1}(X) = \bigcup F(\zeta), T(\zeta) = X;$$

thus

$$FT^{-1}(X) = \bigcup_{\zeta \in H(X)} \{\zeta\} = H(X).$$

Furthermore for a given $f \in \mathfrak{K}(X, Y)$ it is a simple matter to prove that

$$(\zeta, \zeta') \in FT^{-1}(f) \Leftrightarrow (\zeta, \zeta') \in H(f).$$

The only properties of the category Ens which were used in the proofs of the results in this section are (1) the existence of "relations" in Ens and (2) some cocompleteness properties (existence of sums). This indicates a proof of the following result:

2.4. COROLLARY. *The assertions of Theorems* 2.1–2.3 *are still true if* Ens *is replaced by* Ens_0 *or by* Grp.

3. Relations between $\mathfrak{H}$ and $\mathfrak{F}$

In [1] we proved the following theorem:

3.1 THEOREM. *For any $H \in \mathfrak{H}$, there exists a pair $(\bar{H}, \eta)$ such that:*

(a) $\eta \in \mathfrak{H}(H, \bar{H}), \bar{H} \in \mathfrak{F}$.

(b) *If (F, σ) is another pair for which* (a) *is true, then there exists a unique $\varphi \in \mathfrak{H}(\bar{H}, F)$ such that $\varphi\eta = \sigma$.*

This theorem allows a dualization:

3.2. THEOREM. *For any $H \in \mathfrak{H}$ there exists a pair $(\bar{H}, \eta)$ such that:*

(a) $\eta \in \mathfrak{H}(\bar{H}, H), \bar{H} \in \mathfrak{F}$.

(b) *If (F, σ) is another pair for which* a) *is true, then there exists a unique $\varphi \in \mathfrak{H}(F, \bar{H})$ such that $\eta\varphi = \sigma$.*

We will prove the following generalization of Theorem 3.2:

3.3. THEOREM. *Let $\mathfrak{L} \subset \mathfrak{K}$ be a subcategory and $H \in \mathfrak{H}(\mathfrak{L})$. Then there exists a pair $(\bar{H}, \eta)$ such that:*

(a) $\eta \in \mathfrak{H}(\mathfrak{L})(\bar{H}i, H), \bar{H} \in \mathfrak{F}(\mathfrak{K})$ *($i : \mathfrak{L} \subset \mathfrak{K}$, the inclusion).*

(b) *If* (F, σ) *is another pair with property* (a) *then there exists a unique* $\varphi \in \mathfrak{H}(\mathfrak{K})(F, \bar{H})$ *such that* $\eta(\varphi/\mathfrak{L}) = \sigma$ *on* $\mathfrak{L}$.

Proof. A "net" $N = N_X$ for given $X \in \mathfrak{K}$ is a function which assigns to each $f \in \mathfrak{K}(X, Y) Y \in \mathfrak{L}$ a $\zeta_f = N(f) \in H(Y)$ such that the following condition holds:

(N) If for $f_i \in \mathfrak{K}(X, Y_i)$, $i = 1, 2$ and $r \in \mathfrak{L}(Y_1, Y_2)$ $rf_1 = f_2$, then

$$(\zeta_{f_1}, \zeta_{f_2}) \in H(r).$$

Let $g \in \mathfrak{K}(X, X')$ be an arbitrary map and $N = N_X$ a given net, then $N' = g_* N$ is defined by

$$N'(f') = N(f'g), \quad f' \in \mathfrak{K}(X', Y), \; Y \in \mathfrak{L}.$$

The following proposition is immediate:

3.4. N' is a net; if

$$X \xrightarrow{g_1} X' \xrightarrow{g_2} X''$$

are given maps then $(g_2 g_1)_* N = g_{2*}(g_{1*} N)$; $(1_X)_* N = N$.

Now we define $\bar{H}(X)$ to be the set of all nets $N = N_X$ and $\bar{H}(g)$ (for $g \in \mathfrak{K}(X, X')$) to be g_*. Then $\bar{H}$ is in $\mathfrak{F}$ and by

$$\eta_X(N) = N(1_X), \quad X \in \mathfrak{L},$$

we obtain a morphism in $\mathfrak{H}(\mathfrak{L})(\bar{H}i, H)$. In fact, let $g \in \mathfrak{L}(X, X')$ be a map and $N' = g_* N$, then $N'(1_{X'}) = N(g)$ and by (N) one has $(N(1_X), N(g)) \in \bar{H}(g)$.

Let (F, σ) be a second pair and $\zeta_1 \in F(X)$ $f : X \to Y \in \mathfrak{L}$ a morphism; then $N(f) = \sigma(F(f)\zeta_1)$ will be a well-defined net in $H(X)$ and $\varphi(\zeta_1) = N$ defines a morphism $\varphi \in \mathfrak{H}(\mathfrak{K})(F, \bar{H})$. The naturality of φ is trivial. Let $\bar{\varphi} \in \mathfrak{H}(F, \bar{H})$ be a second morphism with $\eta(\bar{\varphi}|\mathfrak{L}) = \sigma$ and for given $\zeta_1 \in F(X)$ set $N_1 = \bar{\varphi}(\zeta_1)$. It is immediate that $N_1(f) = \sigma(F(f)\zeta_1)$, whence $N_1 = N$ and $\varphi = \bar{\varphi}$ follows.

This completes the proof of the theorem.

Theorem 3.1. can be generalized in the same way. One simply has to replace a given semi-functor $H \in \mathfrak{H}(\mathfrak{L})$ by Hi^{-1} where $i : \mathfrak{L} \subset \mathfrak{K}$ is again the inclusion. Now the generalized form of Theorem 3.1. follows immediately.

When we start with the category Grp instead of Ens the assertion of Theorem 3.3. still holds. The reason is that the elements of $\bar{H}(X)$ in Theorem 3.3. are nets and nets can be multiplied: $N_1 \circ N_2(f) = N_1(f) \circ N_2(f)$. In the same way one has $N^{-1}(f) = N(f)^{-1}$; hence $\bar{H}(X)$ inherits a group structure. If $g \in \mathfrak{K}(X, X')$ and $N_1, N_2 \in \bar{H}(X)$ then

$$g_*(N_1 \circ N_2)(f') = N_1 \circ N_2(f'g) = N_1(f'g) \circ N_2(f'g) = g_* N_1 \circ g_* N_2 .$$

In the same way one proves that $\varphi : F \to \bar{H}$ is in $\mathfrak{H}_G(F, \bar{H})$, provided F and σ are in $\mathfrak{H}_G$.

We summarize this by stating the following:

3.5. COROLLARY. *In Theorem* 3.3 *one can replace the category* Ens *by* Ens_0 *or by* Grp.

The statement about Ens_0 is trivial.

Denote by $\bar{\bar{\mathfrak{H}}}(\mathfrak{K}) = \bar{\bar{\mathfrak{H}}}$ the category of all semi-functors H which are defined on some subcategory $\mathfrak{L} \subset \mathfrak{K}$. Let $H_1 \in \mathfrak{H}(\mathfrak{L}_1)$, $H_2 \in \mathfrak{H}(\mathfrak{L}_2)$ be given, $i : \mathfrak{L}_1 \subset \mathfrak{L}_2$. Then a morphism $\varphi \in \bar{\bar{\mathfrak{H}}}(H_2, H_1)$ (resp. $\psi \in \bar{\bar{\mathfrak{H}}}(H_1, H_2)$) is simply a transformation

$$\varphi \in \mathfrak{H}(\mathfrak{L}_1)(H_2\, i, H_1) \quad (\text{resp. } \psi \in \mathfrak{H}(\mathfrak{L}_1)(H_1, H_2\, i))$$

This defines a category. There is an inclusion $I : \mathfrak{F}(\mathfrak{K}) \subset \bar{\bar{\mathfrak{H}}}(\mathfrak{K})$. Now Theorem 3.1. (in the generalized form) and Theorem 3.3. are equivalent to the statements that I is provided with a left (resp. right) adjoint.

There is a generalization of Theorem 3.2. which will be useful in all applications:

One can replace the single semi-functor H by a directed diagram $\mathcal{S} \subset \bar{\bar{\mathfrak{H}}}$ (i.e. a subcategory of $\bar{\bar{\mathfrak{H}}}$ with certain well-known properties). One can immediately extend Theorem 3.3. to this case:

3.6. THEOREM. *There exists an* $\bar{\mathcal{S}} \in \mathfrak{F}$ *and to each* $S \in \mathcal{S}$ *a transformation* $\eta_S : \bar{\mathcal{S}} \to S$ *in* $\bar{\bar{\mathfrak{H}}}$ *such that*:

(a) *For each* $\sigma \in \mathcal{S}(S, S')$ *one has* $\sigma\eta_S = \rho_{S'}$.

(b) *If* $(F, \{\rho_S\})$ *is any family for which* (a) *holds, then there exists a unique* $\varphi : F \to \bar{\mathcal{S}}$ *in* $\mathfrak{F}(\mathfrak{K})$ *such that* $\eta_S\, \varphi = \tau_S$ *in* $\bar{\bar{\mathfrak{H}}}$ *for all* $S \in \mathcal{S}$.

Proof. Construct for each $S \in \mathcal{S}$ the corresponding $(\bar{S}, \zeta_S)$ as in Theorem 3.3. To a given $\sigma \in \mathcal{S}(S, S')$ there corresponds a transformation $\bar{\sigma} \in \mathfrak{F}(\bar{S}, \bar{S}')$ with $\zeta_{S'}\, \bar{\sigma} = \sigma\zeta_S$. Now we define:

$$\bar{\mathcal{S}}(X) = \text{inv lim}_{\Sigma}\, S(X)$$

where $\Sigma = \{\bar{\sigma}\}$ is a directed family and obtain a functor $\bar{\mathcal{S}} \in \mathfrak{F}$ as well as a transformation $\eta_S : \bar{\mathcal{S}} \to S$ for given $S \in \mathcal{S}$.

The verification of (a) and (b) is straightforward and left to the reader.

If we take for $\mathcal{S}$ the trivial directed diagram (consisting of one single object in $\bar{\bar{\mathfrak{H}}}$) we obviously get back Theorem 3.3.

Let $\bar{\bar{\mathfrak{H}}}_D$ be the category whose objects are directed diagrams $\mathcal{S}$ in $\bar{\bar{\mathfrak{H}}}$ and with inclusions $\mathcal{S}_1 \subset \mathcal{S}_2$ of the corresponding subcategories as morphisms. Then one has a functor $J : \mathfrak{F} \to \bar{\bar{\mathfrak{H}}}_D$ (the usual inclusion). Theorem 3.6. simply means that there exists a right adjoint to J.

4. Hurewicz's theorem

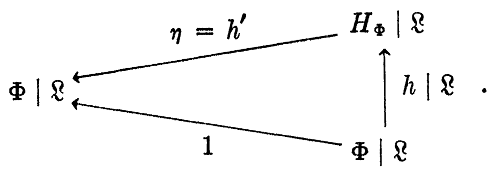
We start again with a category $\mathfrak{K}$, a given subcategory $\mathfrak{L}$ and a functor $\Phi : \mathfrak{K} \to \text{Ens}$. In [1] we constructed a functor Φ_π as well as two morphisms

$$h : \Phi_\pi \to \Phi, \qquad h' : \Phi \to \Phi_\pi$$

such that $hh' = 1$. The last relation is called the "Hurewicz-theorem" while h is the "Hurewicz-homomorphism". Furthermore the triple (Φ_π, h, h') could be characterized by a universality condition (Satz 4 in [1]).

From the point of view of this paper, all this is the "coextension-case" of a Hurewicz-theory. The "extension-case" is still missing and will be settled in the course of this section.

Let $\Phi \in \mathfrak{F}_0(\mathfrak{K})$ be any functor, $\mathfrak{L} \subset \mathfrak{K}$ a subcategory. We set $H = \Phi_i$ ($i : \mathfrak{L} \subset \mathfrak{K}$ the inclusion functor), and by applying Theorem 3.3. we get a functor $\bar{H} = H_\Phi$ as well as a transformation $\eta : H_\Phi\, i \to \Phi i$ on $\mathfrak{L}$. As is traditional we denote η by h'. By using property (b) in Theorem 3.3, we obtain a transformation $h : \Phi \to H_\Phi$ of functors such that on $\mathfrak{L}$ the following triangle commutes:



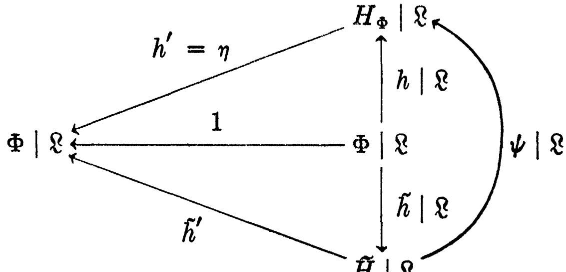
4.1. Definition. We call H_Φ the homology of the functor rel $\mathfrak{L}$ and $h : \Phi \to H_\Phi$ the Hurewicz-homomorphism.

4.2. Theorem. *For an arbitrary $\Phi \in \mathfrak{F}(\mathfrak{K})$ and subcategory $\mathfrak{L} \subset \mathfrak{K}$ there exists a homology functor H_Φ, a Hurewicz homomorphism h and a morphism $h' : H_\Phi \to \Phi$ on $\mathfrak{L}$, such that*

$$h'h = 1.$$

4.3. Theorem. *Let $\Phi \in \mathfrak{F}(\mathfrak{K})$ be a given functor and $(\tilde{H}, \tilde{h}, \tilde{h}')$ a triple, $\tilde{H} \in \mathfrak{F}(\mathfrak{K})$, $\tilde{h} \in \mathfrak{F}(\Phi, \tilde{H})$, $\tilde{h}' \in \mathfrak{F}(\tilde{H}/\mathfrak{L}, \Phi/\mathfrak{L})$ such that $\tilde{h}'\tilde{h} = 1$. Then there exists a unique $\psi : \tilde{H} \to H_\Phi$ such that $\psi\tilde{h} = h$ and $h'\psi = \tilde{h}'$ on $\mathfrak{L}$.*

Proof. By property (b) in Theorem 3.3. there exists a transformation $\psi : \tilde{H} \to H_\Phi$ which is uniquely determined by the fact that $h'(\psi/\mathfrak{L}) = \tilde{h}'$. To complete the proof of Theorem 4.3. one merely has to prove the relation $\psi\tilde{h} = h$. To this end consider the following commutative diagram:



The transformation $h : \Phi \to H_\Phi$ was uniquely determined by the fact that $h'h = 1$ on $\mathfrak{L}$. However $\psi\tilde{h}$ is also a transformation with the same property. Thus one has $\psi\tilde{h} = h$.

4.4. COROLLARY. *In Theorems* 4.2, 4.3. *one can replace the category* Ens *by the category of groups.*

This follows immediately from Corollary 3.5.

As in Theorem 3.6, everything carries over to the case of a directed system $\tilde{\mathfrak{L}}$ of subcategories of $\mathfrak{K}$ instead of a single category $\mathfrak{L} \subset \mathfrak{K}$. The precise formulation of the related theorems (4.2., 4.3.) is immediate and left to the reader.

At the end of this section it seems to be helpful to indicate a direct construction of the functor H_Φ and of the transformations h, h': let $X \in \mathfrak{K}$ be any object; an element $N \in H_\Phi(X)$ is a function which assigns to each $f : X \to L \in \mathfrak{L}$ an element $N(f) \in \Phi(L)$ in such a way that for any $l \in \mathfrak{L}(L_1, L_2)$ and $f_i : X \to L_i$ in $\mathfrak{K}$ with $1 f_1 = f_2$, one has

$$\Phi(l)N(f_1) = N(f_2). \tag{N}$$

Let $g : X \to Y$ be a morphism in $\mathfrak{K}$. Then $H_\Phi(g) : H_\Phi(X) \to H_\Phi(Y)$ is defined by

$$(H_\Phi(g)N)(f) = N(fg), \quad f \in \mathfrak{K}(X, L), L \in \mathfrak{L}.$$

Let $a \in \Phi(X)$ be given. Then we define a $N_a \in H_\Phi(X)$ by $N_a(f) = \Phi(f)a$. The transformation $h : \Phi \to H_\Phi$ is defined by $h(a) = N_a$.

If $L \in \mathfrak{L}$ is a given object and $N \in H_\Phi(L)$ then $N(1_X) \in \Phi(L)$ is defined. For obvious reasons N is uniquely determined by $N(1_X) = a$. We set $h'(N) = a$. The relation $h'h = 1$ is immediate.

Remark. This theory works only if Φ is in fact a functor rather than merely a semi-functor. The definition of h depends strongly upon this assumption. However there seems to be no reason not to use from now on the name "homology" for any $\overline{\Phi i}$ in the sense of Theorem 3.3. (and denote it also by H_Φ) where $\Phi \in \mathfrak{H}(\mathfrak{K})$ is now any semi-functor which is defined on $\mathfrak{K}$.

5. Homology

We denote by $\mathfrak{T}$ the homotopy category of based topological spaces which are simply connected and of the homotopy type of a CW-complex. Many of our assumptions are perhaps unnecessary although they are very convenient. The functor Φ with which we start is the homotopy functor $\pi : \mathfrak{T} \to \{\text{graded groups}\}$. The distinguished subcategory $\mathfrak{L}$ is the category whose objects are Eilenberg-MacLane spaces $K(G, n)$ $(n > 1)$ and a map $f : K(G, n) \to K(G', n')$ is contained in $\mathfrak{L}$ if $n = n'$.

In Section 4 we constructed the homology functor H_Φ of Φ rel $\mathfrak{L}$.

We denote by $H_* : \mathfrak{T} \to \{\text{graded groups}\}$ the singular homology functor, and by $h_\pi : \pi \to H_*$ the ordinary Hurewicz homomorphism which splits on $\mathfrak{L}$ by a map $h_\pi' : H_* \to \pi$.

5.1. THEOREM. *There exists an isomorphism $\varphi : H_* \to H_\pi$ such that $\varphi h_\pi = h$ and $h'\varphi = h'_\pi$ on $\mathfrak{L}$.*

The rest of this section is devoted to a proof of this theorem. We state the following proposition which, according to Theorem 4.3, is equivalent to Theorem 5.1:

5.2. Let $\tilde{H} : \mathfrak{T} \to \{\text{graded groups}\}$ be any functor together with transformations $\tilde{h} : \pi \to \tilde{H}$, and $\tilde{h}' : \tilde{H} \to \pi$ on $\mathfrak{L}$, such that $\tilde{h}'\tilde{h} = 1$ on $\mathfrak{L}$. Then there exists a unique $\psi : \tilde{H} \to H_*$ such that $\psi\tilde{h} = h_\pi$ and $h'_\pi\, \psi = \tilde{h}'$ on $\mathfrak{L}$.

Let $x_0 \in X \in \mathfrak{T}$ be a point. Then we can replace $\tilde{H}_n(X)$ by the cokernel of $\tilde{H}_n(i)$ where $i : \{x_0\} \subset X$ is the inclusion. Since X is simply connected, this is independent of the choice of x_0 and so we can assume without loss of generality that

5.3. $\tilde{H}_n(x_0) = \{0\}$ for all n.

Thus we will use reduced homology. We divide the construction of ψ into three parts.

Case 1. $X = L \in \mathfrak{L}$. We define $\psi = \psi_{X,n} : \tilde{H}_n(X) \to H_n(X)$ as hh'.

Case 2. X is $(n - 1)$-connected. We kill the higher homotopy groups of X and obtain an inclusion $j : X \subset K(G, n) = L \in \mathfrak{L}$ where $H_n(j)$ is an isomorphism. Now we can define

$$\psi_{X,n} = H_n(j)^{-1}\psi_{L,n}.$$

Case 3. X an arbitrary object in $\mathfrak{T}$. Let us start with some preliminary considerations.

5.4. Let $X \in \mathfrak{T}$ be an arbitrary object in $\mathfrak{T}$. Without loss of generality we assume that X is a CW-complex; by attaching m-cells $(2 \leq m \leq n - 1)$ to X successively we get a space $X' = X_n$ and an inclusion $i : X \subset X'$ with the following properties:

$$\pi_k(X') = \{0\},\ k < n,$$

$$\pi_n(X') \approx H_n(X'),$$

$$H_n(i) \text{ is a monomorphism.}$$

5.5. The cokernel K of $H_n(i)$ is a free group with generators $\{\zeta\}$. To each ζ there is related a $\xi \in H_{n-1}(X_{n-1})$ (X_{n-1} is the space obtained after the application of the first $n - 3$ steps of our killing process) of order $\xi = p_\zeta$.

5.6. If we set $H_n(X') = G$ and if $\zeta \in K$ is given then

$$G = C \oplus C',$$

where C is the smallest subgroup of G which contains ζ. Furthermore C is finitely generated (and clearly free abelian).

5.7. In X' we can kill the higher homotopy groups $\pi_{n+1}(X')$, $\cdots$ and obtain a space $K(G, n)$ as well as an inclusion $j : X \subset K(G, n)$, and $H_n(j)$ is again a monomorphism.

Statements 5.4. and 5.7. are immediate consequences of well-known theorems on the killing of homotopy (resp. homology) groups. In the course of our killing process (from X to X_{n-1} and finally in a last step from X_{n-1} to X') the homology $H_n(X)$ will be affected for the first time during the last step. By attaching cells of a dimension less than n to the space X one does not change H_n. Observe that by Hurewicz's theorem $H_{n-1}(X_{n-1})$ is isomorphic to $\pi_{n-1}(X_{n-1})$. The elements of cokernel $H_n(i)$ correspond to elements $\xi \in H_{n-1}(X_{n-1})$ which are of finite order $p_\xi = p$. To each such ζ there corresponds a unique n-cell τ_s^n in X' (which kills ξ). Since there do not exist m-cells σ^m in X', $m > n$, such that a τ_s^n lies on their boundaries, assertion 5.5. follows.

Now let $\zeta \in K$, $\zeta = \Sigma m_i \zeta_i$, where $\{\zeta_i\}$ is a subset of a basis of K. Then C is the group generated by the $\zeta_1, \cdots, \zeta_k$. The direct sum decomposition $G = C \oplus C'$ follows from the construction of C as a subgroup of K (which is in fact a direct summand of G).

We will now continue with the proof of 5.2.:

Let $\tilde{a} \in \tilde{H}_n(X)$ be any element and $\tilde{a}_1 = \tilde{H}_n(i)\tilde{a}$. We know already how to define $\psi(\tilde{a}_1) \in \tilde{H}_n(X')$. We claim that $\psi(\tilde{a}_1) \in \operatorname{im} H_n(i)$. Because $H_n(i)$ is a monomorphism this would give us the right to define:

$$\psi_{X,n}(\tilde{a}) = H_n(i)^{-1}\psi(\tilde{a}_1).$$

Therefore the construction of ψ will be complete as soon as we know the following fact:

5.8. $\psi(\tilde{a}_1) \in \operatorname{im} H_n(i)$.

Proof. We have the map $j' : X' \subset K(G, n)$. Let us assume that $\psi(\tilde{a}_1) \notin \operatorname{im} H_n(i)$.

From the direct sum decomposition of G in 5.6 we obtain a group C, $c \in C$, $\psi(\tilde{a}_1) = c + c'$, a projection $\eta : G \to C$ and a map

$$\eta_* : K(G, n) \to K(C, n).$$

Since C is finitely generated by elements $\zeta_1, \cdots, \zeta_k$, we can define $p = p_1 \cdots p_k$ ($p_i = p_{\zeta_i}$) and consider the map $p_* : K(C, n) \to K(C, n)$ which corresponds to the multiplication by p in C. Now it is easy to see that $p_* \eta_* j'i = 0$: due to a well-known argument (concerning the relation of homology to cohomology) $\operatorname{im} H^n(\eta_* j'i; C)$ is a subgroup of $H^n(X; C)$ which is isomorphic to a C-module generated by the elements $\xi_1, \cdots, \xi_k$ where ξ_i is of order p_i. Thus all elements of this group are of an order which divides p; consequently $p \cdot \operatorname{im} H^n(\eta_* j'i; C) = 0$. On the other hand one has

$$p \cdot \operatorname{im} H^n(\eta_* j'i; C) = \operatorname{im} H^n(p_* \eta_* j'i; C).$$

This proves $p_* \eta_* j'i = 0$.

We know by construction that $0 \neq H_n(\eta_* j')\psi(\tilde{a}_1) = c \in C$. Because C is a free group we have

$$H_n(p_* \eta_* j')\psi(\tilde{a}_1) = \psi H_n(p_* \eta_* j')\tilde{a}_1 = \psi H(p_* \eta_* j'i)(\tilde{a}) \neq 0.$$

However this is impossible because $p_* \eta_* j'i = 0$ and (by 5.3), $\tilde{H}_n(0)\tilde{a} = 0$. This defines the transformation completely. The verification of its properties ($\psi\tilde{h} = h_\pi$, $\tilde{h}'\psi = h'_\pi$ on $\mathfrak{L}$, as well as the uniqueness of ψ) is immediate and left to the reader. This completes the proof of Theorem 5.1.

6. Examples

In this section we give some examples which are typical for the results which we have achieved in the preceding paragraphs. Proofs are mostly immediate and therefore omitted.

Our definition of homology covers all conceivable kinds of "globalization" of a given local structure.

To give an idea how this works, let us go back to the explicit construction of H_Φ at the end of section 4. Let X be a topological space, $\mathfrak{U} = \{U\}$ an open covering (which contains together with $U_1, \cdots, U_n$ also their intersection $\bigcap_{i=1}^n U_i$) and ζ a vector bundle over X which is trivial on each $U \in \mathfrak{U}$. Let $\zeta_U = \zeta/U$ be the trivial bundle on U and $i_U : U \subset X$ the inclusion. The system $\zeta = \{\zeta_U\}$ behaves like a "net" N if one defines $N_\zeta(i_U) = \zeta_U$. The condition (N) in Section 4 now simply states that ζ_U and $\zeta_{U'}$ fit together on their intersection $U \cap U'$.

In this way we can interpret $\mathfrak{U}$ as a category $\mathfrak{L}$ with inclusions as mappings (in fact we have to go over to the dual category to remain within the frame of our theory). The category $\mathfrak{K} \supset \mathfrak{L}$ is the category whose objects are all unions of objects in $\mathfrak{L}$ (e.g. X itself is an object in $\mathfrak{K}$). The trivial vector bundles on each $U \in \mathfrak{U}$ define in an obvious way a semi-functor on $\mathfrak{L}$. The functor H_Φ gives us the desired global bundle structure on X (resp. each subspace of X in $\mathfrak{K}$).

This would indicate what we mean by a "globalization of a local structure".

Let X be a topological space and $X_\#$ the category of all open subspaces with inclusions as morphisms. By $\tilde{\mathfrak{L}} = \{\mathfrak{L}\}$ we denote the directed diagram of all full subcategories of $\mathfrak{K} = X_\#^*$ (the dual of $X_\#$) with the following property:

(*) To each $x \in X$ there exists an object $U \in \mathfrak{L}$, $x \in U$.

Let $\Phi(Y)$ be the class of all trivial n-dimensional vector bundles over $Y \in \mathfrak{K}$. Clearly $\Phi : \mathfrak{K} \to \mathrm{Ens}_0$ becomes a semifunctor, where the product bundle serves as a base point. One has:

6.1. Theorem. *$H_\Phi(X)$ is the family of all n-dimensional (locally trivial) vector bundles over X.*

Remark. In this example $\Phi(X)$ is surely *not* a set. However all constructions which lead to H_Φ go through immediately (although $H_\Phi(X)$ is again only a class). On the other hand the reader may be inclined to bring the preceding construction in accordance with all set-theoretical requirements by working with a suitable "universe" which contains all vector bundles in $\Phi(X)$ (and consequently in $H_\Phi(X)$). This example breaks down as soon as one uses isomorphism classes of bundles instead of bundles, because all trivial bundles are isomorphic.

As a second example in this direction we consider differentiable structures: Let M^n be a given (topological) manifold and $\mathfrak{K}^*$ the category of all open submanifolds $A \subset M^n$ with inclusions as mappings. Let $\mathfrak{L}^* \subset \mathfrak{K}^*$ be the full subcategory whose objects are those $A \in \mathfrak{K}^*$ which are homeomorphic to R^n. Let $H(L)$, $L \in \mathfrak{L}^*$, be the set of all charts on L. A chart is an equivalence class $\{\varphi_L\}$ of homeomorphisms $\varphi_L : L \approx R^n$; two φ_L, φ'_L are equivalent if $\varphi_L \varphi'^{-1}_L$ is differentiable. If $i : A \subset B$ is a map in $\mathfrak{L}^*$, then $H(i)\{\varphi_B\} = \{\varphi_B/A\}$. This determines a semi-functor $H : \mathfrak{K} \to \text{Ens}$ ($\mathfrak{K}$ is dual to $\mathfrak{K}^*$), where we set $H(A) = \emptyset$ if $A \notin \mathfrak{L}$ (the dual to $\mathfrak{L}^*$).

By applying Theorem 3.1 we obtain a functor $\bar{H} : \mathfrak{K} \to \text{Ens}$ such that $\bar{H} \mid \mathfrak{L} = H \mid \mathfrak{L}$. By adding to $H(A)$ a new abstract element ζ_A we get a functor $\phi : \mathfrak{K} \to \text{Ens}_0$. We are now in a position to compute H_Φ :

6.2. THEOREM. *$H_\Phi(M^n)$ consists of the basepoint ζ_{M^n} and the set of all differentiable structures on M^n.*

Concerning all set-theoretical difficulties we refer to the preceding remark.

6.3. Let $\Phi : \mathfrak{K} \to \text{Ens}$ be any functor such that for every object $L \in \mathfrak{L}$, $\Phi(L)$ carries a group structure which is natural with respect to the morphisms in $\mathfrak{L}$. Then $H_\Phi(X)$ also carries a natural group structure (now for any $X \in \mathfrak{K}$ and with respect to arbitrary maps in $\mathfrak{K}$) which converts H_Φ into a functor which maps into the category of groups.

Proof. The values of a net are in $\Phi(L)$ for given $L \in \mathfrak{L}$. Thus, nets can be multiplied and $H_\Phi(X)$ inherits a group-structure which is easily seen to be natural.

By Sgrp we denote the category of semigroups with unit.

6.4. THEOREM. *Let Φ : Sgrp $\to$ Sgrp be the identity functor and $\mathfrak{L} \subset$ Sgrp the subcategory of groups. Then H_Φ : Sgrp $\to$ Grp is the "completion functor" in the sense of* [3, p. 103].

A similar assertion holds for semi-rings.

Let $\mathfrak{K}$ be a category with the notion of a homotopy and $T : \mathfrak{K} \to \mathfrak{K}_H$ the projection [2]. If $F : \mathfrak{K}_d \to \text{Ens}$ is any semifunctor, we observe, that $(FT^{-1})^-$ is a functor from $\mathfrak{K}_H$ into Ens (the bar means the functor which corresponds to FT^{-1} in the sense of Theorem 3.1).

6.5. THEOREM. *Let $\mathfrak{K} = \mathfrak{S}_E$ be the category of Kan-complexes and $F = Z_n(\ ;G)$ the functor which assigns to each object the group of cycles (with coefficients in G). Then one has*

$$(FT^{-1})^{-} \approx H_n(\ ;G).$$

Let $\mathfrak{B}_0$ be the homotopy category of based topological spaces (with homotopy classes of base point preserving continuous maps as morphisms) and $\mathfrak{B}_S$ the S-category in the sense of Spanier and Whitehead (see [5] or [6]). Then there exists a functor $\alpha : \mathfrak{B}_0 \to \mathfrak{B}_S$ which assigns to each $X \in \mathfrak{B}_0$ the pair $(X, 0) \in \mathfrak{B}_S$. Let

$$\pi : \mathfrak{B}_0 \to \{\text{graded groups}\}$$

be the functor of the Hurewicz homotopy groups.

6.6. THEOREM. *The functor $\pi_S = (\pi\alpha^{-1})^{-}$ (bar in the sense of Theorem 3.1.) is the functor of the stable homotopy groups.*

REFERENCES

1. F. W. BAUER, *Halbfunktoren*, Math. Nachr., vol. 47 (1970), pp. 155–159.
2. F. W. BAUER AND J. DUGUNDJI, *Categorical homotopy and fibrations*, Trans. Amer. Math. Soc., vol. 140 (1969), p. 239–256.
3. D. HUSEMOLLER, *Fibre bundles*, MacGraw-Hill, New York, 1966.
4. S. MACLANE, *Homology*, Springer, Berlin, 1963.
5. D. PUPPE, *Stabile Homotopietheorie I*, Math. Ann., vol. 169 (1967), pp. 243–274.
6. M. TIERNEY, *Categorical constructions in stable homotopy theory*, Lecture notes in mathematics, vol. 87, Springer Verlag, Berlin, 1969.

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