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THE HOMOLOGY OF SUBMANIFOLDS OF COMPACT KÄHLER MANIFOLDS

GERALD LEONARD GORDON

1. Introduction

In this article we study certain topological properties of submanifolds of compact Kähler manifolds. Specifically, let $i = X \subset Y$ be the inclusion of a compact manifold X of complex dimension n into a compact Kähler manifold of complex dimension $n + q$. Let $I: H_{p+2q}(Y) \to H_p(X)$ be the map given by transverse intersection, where the coefficients are in K, a fixed field of characteristic zero. Then we ask when do we have the decomposition $H_p(X) = \operatorname{Ker} i_* \oplus I(H_{p+2q}(Y))$ such that if $p = n$, each direct summand is nondegenerate with respect to the intersection pairing. In cohomology this states that $H^p(X) = i^*H^p(Y) \oplus RH^{p+2q-1}(Y - X)$, where R is the Leray-Norguet residue operator. If $n = 1$, then a corollary of this result is that if X_1 and X_2 have this decomposition in Y, i_1 and i_2 are the inclusions, and I_j is the intersections, then the following diagram

$$\begin{array}{ccc} H_{1+2q}(Y) & \xrightarrow{I_1} & H_1(X_1) \\ {\scriptstyle I_2}\downarrow & & \downarrow{\scriptstyle (i_1)_*} \\ H_1(X_2) & \xrightarrow{(i_2)_*} & H_1(Y) \end{array}$$

commutes when restricted to coimage $I_1 \cap$ coimage I_2, i.e., to the set of $\gamma \in H_{1+2q}(Y)$ such that $I_j(\gamma) \neq 0$ for $j = 1$ and 2.

In this article we shall show for $n = 1$ and 2 that this decomposition exists for $p = n$, as well as for submanifolds of complete intersections of CP^N. However, for $p \geq 3$ and any $n \geq 2$, $q \geq 1$ we shall give counterexamples.

This problem arose from questions about the local invariant cycle problem; cf. Griffiths [5, p. 249]. Namely, this decomposition for $n = 1 = p$ is precisely what one needs to prove the problem when one has 2 surfaces intersecting in a double curve [5, p. 292]. In Gordon [4], it is shown that this decomposition for $n = 1 = p$ is essentially what proves the local invariant problem for Kähler surfaces. Furthermore, these counterexamples to the decomposition allows us to construct projective varieties which cannot be embedded in a one-

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dimensional analytic deformation whose generic fibre is a nonsingular compact Kähler manifold; cf. [4]. In § 5 we pose the analogous question about schemes, which should, if true, have applications to studying the monodromy for schemes, over arbitrary algebraically closed fields. The author would like to thank the referee for pointing out a mistake in the original proof of Corollary 3.2.

2. Definition of $A_p(X, Y)$

2.1. In this section Y will always denote a nonsingular connected, compact Kähler manifold and X a nonsingular connected, compact submanifold of complex dimension n, where the complex codimension of X in Y is q. $i\colon X \subset Y$ will denote the inclusion map.

The Poincaré dual class of $0 \neq [X] \in H_{2n}(Y)$ will be denoted by $\Omega_X \in H^{2q}(Y)$, where the coefficients are in K, a fixed field of characteristic zero. Then we have a mapping

$$\wedge \Omega_X\colon H^p(Y) \to H^{p+2q}(Y) .$$

2.1.1. Definition. Let $A_p(X, Y)$ denote the proposition that

$$\operatorname{Ker}\{\wedge \Omega_X\colon H^p(Y) \to H^{p+2q}(Y)\} = \operatorname{Ker}\{i^*\colon H^p(Y) \to H^p(X)\}.$$

Let $I\colon H_{p+2q}(Y) \to H_p(X)$ denote the map given by transverse intersection; it is the vector space dual of the Thom-Gysin map of the normal bundle of X in Y.

2.2. Proposition.

$$A_p(X, Y) \Rightarrow H_p(X) = \operatorname{Ker} i_* + I(H_{p+2q}(Y)) . \tag{2.2.1}$$

$$A_p(X, Y), A_{2n-p}(X, Y) \Leftrightarrow \begin{cases} H_p(X) = \operatorname{Ker} i_* \oplus I(H_{p+2q}(Y)) , \\ H_{2n-p}(X) = \operatorname{Ker} i_* \oplus I(H_{2n-p+2q}(Y)) . \end{cases} \tag{2.2.2}$$

Proof of (2.2.1). In cohomology, we have the following communative diagram:

$$\begin{tikzcd} H_{2n-p}(X) \arrow[r, "i_*"] & H_{2n-p}(Y) \\ H^p(X) \arrow[u, "D_X", "\wr"'] \arrow[r, "I^*"] & H^{p+2q}(Y) \arrow[u, "D_Y", "\wr"'] \\ H^p(Y) \arrow[u, "i^*"] \arrow[ur, "\wedge \Omega_X"'] & \end{tikzcd}$$

where D_W denotes Poincaré duality in W, and I^* is dual (as vector space) of I. Thus applying Hom to the above diagram, where we identify $H^p(X) \simeq (H_p(X))^* = \operatorname{Hom}_K (H_p(X), K)$ via integration, we get

$$
\begin{tikzcd}
H^{2n-p}(X) \arrow[d, "\wr"', "D_X"] & H^{2n-p}(Y) \arrow[l, "i^*"'] \arrow[d, "\wr", "D_Y"'] \\
H_p(X) \arrow[d] & H_{p+2q}(Y) \arrow[l, "I"'] \arrow[dl, "\cap \Omega_X"] \\
H_p(Y) &
\end{tikzcd}
$$

where $\cap\,\Omega_X$ is cup product. Then $A_p(X, Y)$ implies that $\operatorname{im} i_x = \operatorname{im} \cap\,\Omega_X$.

Thus, if $\alpha \in H_p(X)$ with $i_*\alpha \neq 0$, then $i_*\alpha = \cap\,\Omega_X(\beta)$. But by the communative diagram, $i_*I(\beta) = i_*\alpha$, hence $I(\beta) - \alpha \in \ker i_*$, i.e., $\alpha = \gamma + I(\beta)$ for some $\gamma \in \operatorname{Ker} i_*$.

Proof of (2.2.2). Suppose we have $A_p(X, Y)$ and $A_{2n-p}(X, Y)$. If $\gamma_p \in H_p(X)$ with $\gamma_p = I(\gamma_{p+2q})$, then by the above diagram $i^*(D_Y(\gamma_{p+2q})) = D_X(\gamma_p) \in H^{2n-p}(X)$, and $A_{2n-p}(X, Y)$ implies that $\wedge\Omega_X(D_Y(\gamma_{p+2q})) \neq 0$. But by the first communative diagram in the proof of (2.2.1), we have $\wedge\Omega_X(D_Y(\gamma_{p+2q})) \neq 0 \Leftrightarrow i_*\gamma_p \neq 0$. Hence $H_p(X) = \operatorname{Ker} i_* \oplus I(H_{p+2q}(Y))$. By duality we have the direct summand decomposition for $H_{2n-p}(X)$.

The converse of (2.2.2) is clear.

2.3. Proposition. *If X is a positive hypersurface of Y, then $A_p(X, Y)$ is true for all p.*

This is an immediate consequence of the hard Lefschetz theorem.

2.3.1. The difficulty is that when one wants to work with problems as the local invariant cycle problem, one wants to apply $A_p(X, Y)$ when X is a hypersurface which comes from a monoidal transform, and hence is very negative.

3. Study of $A_p(X, Y)$

3.1. Proposition. *Let P^p_W denote he primitive cohomology of $H^p(W; C)$ for any compact Kähler manifold W. For $i\colon X \subset Y$, if $i^*P^q_Y \subset P^q_X$ for all $q \leq p \leq n$, then $A_p(X, Y)$ is true.*

Proof. We first prove $A_p(X, Y)$ for complex coefficients. Consider the following diagram:

$$
\begin{tikzcd}
H^{2n-p}(X; C) \arrow[dr, phantom, "\square"] & H^{2n-p}(Y; C) \arrow[l, "i^*"'] \arrow[d, "\wr", "\mathrm{Hom}_Y"'] \arrow[dd, bend left=60, "*_Y"] \\
H_{2n-p}(X; C) \arrow[u, "\wr"', "\mathrm{Hom}_X"] \arrow[r, "i^*"] \arrow[dr, phantom, "\square"] & H_{2n-p}(Y; C) \arrow[d, "\wr", "D_Y"'] \\
H^p(X; C) \arrow[u, "\wr"', "D_X"] \arrow[uu, bend left=60, "*_X"] & H^{p+2q}(Y; C) \\
H^p(Y; C) \arrow[u, "i^*"'] \arrow[ur, "\wedge\Omega_X"'] &
\end{tikzcd}
$$

where Hom is vector space duality via integration, and $*_W$ is the usual real star operator on forms on a manifold W, which induces an isomorphism on harmonic forms; $\bar{*}_W$ is complex conjugation followed by $*_W$. The $\square$ means the diagram commutes where $\mathrm{Hom} \circ W_W = *_W$ for W compact follows from the definition of $*_W$ and the fact that

$$\int_W \alpha \wedge \beta = \int_{P_W(\beta)} \alpha \ .$$

Furthermore Hom_X is natural in the sense that if $0 \neq \omega \in H^{2n-p}(X) \cap \mathrm{Im}\, i^*$, then $i_*(\mathrm{Hom}_X)^{-1}(\omega) \neq 0$. To see this, let $\omega = i^*\omega'$ and $\alpha = (\mathrm{Hom}_X)^{-1}(\omega)$. Then

$$\int_{i_*(\alpha)} \omega' = \int_\alpha i^*(\omega') = \int_\alpha \omega = (\mathrm{Hom}_X)^{-1}(\omega)(\alpha) = 1 \ .$$

Hence $i_*(\alpha) \neq 0$, as it has nonzero periods.

The converse is also true in the sense that if $\alpha \in H_{2n-p}(X)$ with $i_*\alpha \neq 0$, then the projection of $\mathrm{Hom}_X(\alpha)$ onto the subspace $\mathrm{Im}\, i^*$ is nonzero.

Thus by the commutative diagram to show $A_p(X, Y)$ it suffices to show that if $i^*\omega \neq 0$, then $\Omega_X(\omega) \neq 0$. Since i^* respects (r, s) type, it suffices to consider forms of pure type.

Suppose $i^*\omega^{s,p-s} \neq 0$, for $\omega^{s,p-s} \in H^{s,p-s}(Y; C)$. We must show that $i_* D_X i^* \omega^{s,p-s} \neq 0$. By the above remark, this follows if we show that

$$\mathrm{Hom}_X \circ D_X i^* \omega^{s,p-s} \in \mathrm{Image}\, i^*, \text{ i.e., } \bar{*}_X i^* \omega^{s,p-s} \in \mathrm{Image}\, i^* \ .$$

By the Hodge decomposition theorem, we can write $\omega^{s,p-s} = \sum_r L_Y^r \omega^{s-r,p-s-r}$ where the $\omega^{s-r,p-s-r} \in P_Y^{p-2r}$. Then, since $p - 2r \leq p \leq n$, $i^*\omega^{s-r,p-s-r} \in P_X^{p-2r}$ by hypothesis. Hence, by a standard identity for compact Kähler manifold (cf. Weil [8, p. 23]),

$$\begin{aligned}
\bar{*}_X i^* \omega &= \sum_r \bar{*}_X L_X^r i^* \omega^{s-r,p-s-r} \\
&= \sum_r (-1)^{1/2(p-2r)(p-2r+1)} \frac{r!}{(n-p+r)!} L_X^{n-p+r} \bar{C} i^* \bar{\omega}^{s-r,p-s-r} \\
&= i^*\left(L_Y^{n-p} \sum_r (-1)^{1/2(p-2r)(p-2r+1)} \frac{r!}{(n-p+r)!} (-1)^p (\sqrt{-1})^s L_Y^r \bar{\omega}^{s-r,p-s-r}\right) ,
\end{aligned}$$

where we have used the identity $L_X i^* = i^* L_Y$ and the fact that L_X and i^* are real operators, and where C is the Weil operator. But complex conjugation sends $H^{s-r,p-s-r}(Y; C)$ isomorphically onto $H^{p-s-r,s-r}(Y; C)$ and Λ_Y, the adjoint of L_Y, is a real operator, hence $\bar{\omega}^{p-r,s-p-r} \in P_Y^{p-2r}$.

Furthermore $p \leq n$, so that $0 \leq n - p < n + s - p = \dim_C Y - p$: hence

$$L_Y^{n-p}\left(\sum_r (-1)^{1/2(p-2r)(p-2r+1)} \frac{r!}{(n-p+1)!} (-1)^p (\sqrt{-1})^s L_Y^r \bar{\omega}^{s-r,p-s-r}\right) \neq 0$$

by the hard Lefschetz theorem and the uniqueness of the Hodge decomposition; cf. Weil [8, p. 75].

But if K is any subfield of C, then $H_*(X; C) = H_*(H; K) \otimes_K C$, while the operators i^* and $\wedge \Omega_X$ are integral operators, hence are defined on any subfield of C. Thus $\operatorname{Ker} i^* | H^p(Y; C) = \operatorname{Ker} \wedge \Omega_X | H^p(Y; C)$ implies that $\operatorname{Ker} (i^* | H^p(Y; K)) = \operatorname{Ker} \wedge \Omega_X | H^p(Y; K)$.

3.2. Corollary. *For all X and Y, $A_p(X, Y)$ is true for $p \leq 2$.*

Proof. For $p = 0$, $P_Y^0 = P_X^0 = 0$, while for $p = 1$, $P_X^1 = H^1(X; C)$.

Similarly $H^{2,0}(X; C) \cap P_X^2 = H^{2,0}(X; C)$ and the same for $H^{0,2}(X; C)$. Hence to prove the corollary, it suffices to consider $P_Y^{1,1}$.

3.2.1. Lemma. *$\omega^{1,1} \in P_Y^{1,1} \Rightarrow$ either $i^*\omega^{1,1} \in P_X^{1,1}$ or $i^*\omega^{1,1} = \alpha L_X(\mathbf{1})$ for $\alpha \in C$ where $\mathbf{1}$ is a generator of $H^0(X; C)$.*

Note. This says that the restriction of a primitive 2-form does not split up into two nontrivial components in the Hodge decomposition of the subspace.

Proof. If it is not so, then we have $i^*\omega^{1,1} = \alpha L_X(\mathbf{1}) + \omega_X$ where $0 \neq \omega_X \in P_X^{1,1}$ and $0 \neq \alpha \in C$. Then $\alpha L_X(\mathbf{1}) = i^*(\alpha L_Y(\mathbf{1}))$, hence $0 \neq i^*(\omega^{1,1} - \alpha L_Y(\mathbf{1})) \in P_X^{1,1}$. But by the uniqueness of the Hodge decomposition, $(\omega^{1,1} - \alpha L_Y(\mathbf{1})) \notin P_Y^{1,1}$. Thus we have a nonprimitive form on Y whose restriction to X is primitive, which is impossible. q.e.d. for Lemma 3.2.1.

Suppose we have $\omega \in H^{1,1}(Y; C)$ with $\omega = \omega^{1,1} + \beta L_Y(\mathbf{1})$ for $\beta \in C$, $\mathbf{1}$ a generator of $H^0(Y; C)$, and $\omega^{1,1} \in P_Y^{1,1}$. Then if $i^*\omega \neq 0$, we must show $\wedge \Omega_X(\omega) \neq 0$; by the diagram in (3.1) and the remarks after the diagram, it suffices to show $*_X i^*(\omega) \in \operatorname{Im} i^*$.

If $i^*\omega^{1,1} \in P_X^{1,1}$, then by Proposition 3.1 we are done. Hence by Lemma 3.2.1 it suffices to assume that $i^*\omega^{1,1} = \alpha L_X(\mathbf{1})$. But then $i^*\omega = \alpha L_X(\mathbf{1}) + i^*\beta L_Y(\mathbf{1}) = (\alpha + \beta) L_X(\mathbf{1})$, and $\alpha + \beta \neq 0$ by hypothesis that $i^*\omega \neq 0$. Hence $*_X i^*\omega = i^*(-(\alpha + \beta) L_Y(\mathbf{1}))$. q.e.d. for Corollary 3.2.

3.3. Corollary. *If Y is a complete intersection in CP^N, then $A_p(X, Y)$ is true for $p < \dim_C Y$ for any submanifold X in Y.*

This follows because $P_Y^p = 0$ for $p < \dim_C Y$.

3.4. Proposition. *Let $n \geq 2$, and let p and q be fixed such that $3 \leq p \leq 2n - 1$ and $q \geq 1$. Then there exist projective algebraic manifolds X and Y such that $A_p(X, Y)$ is false.*

Proof. The first case to consider is $p = 3$, $n = 2$ and $q = 1$. Let $T \subset CP^3$ be the nonsingular elliptic curve of degree 3, and let $\Pi: Y \to CP^3$ be the monoidal transform with center T. Let $X = \Pi^{-1}(T)$ and $i: X \subset Y$ be the inclusion, where Y is projective algebraic.

Then by Séminaire Géométrie Algébrique 5, vii $i_*: H_3(X) \simeq H_3(Y) \simeq H_1(T) \oplus H_3(CP^3) = K \oplus K$ and $H_1(Y) = 0$. Then by Poincaré duality, $H^5(Y) = 0$. But then $i^*: H^3(Y) \simeq H^3(X)$ and $\wedge \Omega_X: H^3(Y) \to (H^5(Y) = 0)$ is the zero map, hence $A_3(X, Y)$ is false.

Next consider $p = 3$, any n, and $q > 1$. All we need to do is to take $X \times CP^{n-2}$ and $Y \times CP^{n-2} \times CP^{q-1}$. Then by the Kunneth formula, i^* is still an isomorphism for $p = 3$, but $\dim_K (\operatorname{Ker} \wedge \Omega_X) = 2$ for $p = 3$.

For $p = 4$ and $n \geq 3$ and any q, we need only consider $X \times CP^{n-3} \times T$ and $Y \times CP^{n-3} \times T \times CP^{q-1}$. Let $0 \neq \omega \in H^3(Y)$, $0 \neq \gamma \in H^1(T)$. Then $i^*(\omega, 0, \gamma, 0) \neq 0$, while $\wedge \Omega_X(\omega, 0, \gamma, 0) = 0$.

In general, if $p = 2k + 3$, for any n and q, take $X \times CP^{n-2}$ and $Y \times CP^{n-2} \times CP^{q-1}$ and consider $(\omega, \gamma_k, 0)$ for $0 \neq \omega \in H^3(Y)$ and $0 \neq \gamma_k \in H^{2k}(CP^{n-2})$ to get a counterexample to $A_p(X \times CP^{n-2}, Y \times CP^{n-2} \times CP^{q-1})$.

Finally, if $p = 2k + 4$, for any n and q, take $X \times CP^{n-3} \times T$ and $Y \times CP^{n-3} \times T \times CP^{q-1}$ and $(\omega, \gamma_k, \gamma, 0)$ will give the counterexample.

3.4.1. The counterexamples for $p > n$ arise from the fact that one has an $\omega^p \in H^p(Y)$ with $i^*\omega^p = L_X^{p-n}\omega_X^{2n-p}$ and $*_X i^*\omega^p = \alpha\omega_X^{2n-p}$ for $0 \neq \alpha \in C$. But there is nothing to guarantee that $\omega_X^{2n-p} \in \operatorname{Image} i^*$, e.g., one could have $H^{2n-p}(Y; C) = 0$. The basic reason for this is that $i^*\Lambda_Y \neq \Lambda_X i^*$.

For $p \leq n$, the problem arises because we no longer have Proposition 3.1., i.e., the restriction of primitive forms need not be primitive for $p > 2$, e.g., in our example for $p = 3 = n$ and $q = 1$ we have $X \times CP^1 \subset Y \times CP^1$. Then $H^1(Y \times CP^1; C) = 0$, so that $H^3(Y \times CP^1; C) \simeq C \oplus C$ is all contained in the primitive cohomology. But $b_1(X \times CP^1) = 2$, $b_3(X \times CP^1) = 4$ and the map

$$
\begin{aligned}
L_{X \times CP^1} \colon H^1(X \times CP^1; C) &\to H^3(X \times CP^1; C) \\
&\cong H^3(X; C) \oplus (H^1(X; C) \otimes H^2(CP^1; C))
\end{aligned}
$$

has $L_{X \times CP^1}(\alpha) = (L_X\alpha, 0, \alpha, 0)$, $L_{X \times CP^1}(\beta) = (0, L_X\beta, 0, \beta)$ for α, β generators of $H^1(X; C)$.

Thus, if we take $\omega^{2,1} \in H^{2,1}(Y \times CP^1; C)$, then $i^*\omega^{2,1} = \alpha L_X\omega^{1,0} + \eta^{2,1}$ where $0 \neq \alpha \in C$, $\eta^{2,1} \in P^{2,1}_{X \times CP^1}$ and $\omega^{1,0} \in P^{1,0}_X$. Then $*_X i^*\omega^{2,1} = \sqrt{-1}(\alpha L_X\omega^{1,0} - \eta^{2,1})$, which is not in the image of i^* because of the change of sign before $\eta^{2,1}$. In homology this states that we have a finite 3-cycle γ_1 and a nonfinite 3-cycle γ_2 in a subspace which are homologous when injected into the ambient space.

4. Some consequences of $A_n(X, Y)$

4.1. Corollary. *Suppose $A_n(X, Y)$ is true. In particular, if $n = 1$ or 2 or if Y is a complete intersection, then*

$$
H_n(X) = \operatorname{Ker} i_* \oplus I(H_{n+2q}(Y)) , \qquad H^n(X) = i^*H^n(Y) \oplus RH^{n+2q-1}(Y - X) ,
$$

Furthermore, the restriction of the intersection pairing to each of the summands is nondegenerate (equivalently, the restriction of cup product on $H^n(X)$ is nondegenerate on each of the summands).

Proof. The decomposition for homology follows from Proposition 2.2 and

Corollary 3.2. The Thom-Gysin sequence in homology for $X \subset Y$ can be written

$$\begin{tikzcd} H_{n+2q}(Y) \arrow[r, "I"] \arrow[d] & H_n(X) \arrow[r, "\tau"] \arrow[d] & H^c_{n+2q-1}(Y - X) \arrow[r] \arrow[d] & H_{n+2q-1}(Y) \arrow[d] \\ H^{n+2q}(Y) & H^n(X) \arrow[l] & H_F^{n+2q-1}(Y - X) \arrow[l, "R"'] & H^{n+2q-1}(Y) \arrow[l] \end{tikzcd}$$

where we take vector space duality via integration to get the vertical isomorphisms, c denotes compact support, F denotes closed support and R is the Leray-Norguet-Poincaré residue.

The duality via integration between homology with compact support and forms with closed support was proven for $q = 1$ by Leray [6]. For $q > 1$, this was done by Norguet. For an exposition of the dualities between homology with compact support and cohomology with closed support, the reader is referred to Fotiadi, et al. [1, part III].

It can be shown, cf., e.g., Poly [3], that every cohomology class α of $H_F^{n+2q-1}(Y - X)$ can be represented by a closed C^∞ form of the type $\theta \wedge K_X + \eta$ where θ and η are C^∞ forms with singularities on X. Furthermore $R(\alpha) = [\theta | X]$ where $\theta | X$ is closed. Hence Image $I \simeq \operatorname{Ker} \tau \simeq$ Image R, so that the decomposition in cohomology follows.

The cup product pairing is nondegenerate on each summand because in the proof of Proposition 3.1, we showed if $\omega \in i^*H^n(Y)$, then $\operatorname{Hom}_X \circ D_X(\omega) \in i^*H^n(Y)$, but $\int_X \omega \wedge \operatorname{Hom}_X \circ D_X(\omega) > 0$. Also, if $D_X^* = \operatorname{Hom}_X \circ D_X$, then $(D_X^*)^2 = (-1)^n \operatorname{Id}$, where Id is the identity on $H^n(X)$, hence this gives the nondegeneracy on $RH^{n+2q-1}(Y - X)$.

4.1.1. For $n = 1 = q$, the nondegenerate decomposition also can be proven by the Poincaré complete reducibility theorem: the map $I^* : H^1(X) \to H^3(Y)$ is derived from the map of Albanese varieties with the nondegenerate cup product structure, hence the Poincaré complete reducibility theorem states that the image has a direct summand which respects the nondegenerate structure.

4.2. Corollary. *Let X_j, $j = 1, \cdots, k$, be nonsingular submanifolds of complex dimension* 1 *in Y, a compact Kähler manifold of complex dimension* $1 + q$. *Let $i: \cup X_j \subset Y$ and $i_j : X_j \subset Y$ be the inclusions. If $\gamma_{1+2q} \in H_{1+2q}(Y)$ is such that $0 \neq \gamma_{1+2q} \cap X_j = \gamma_{1,j} \in H_1(X_j)$ for $j = 1, \cdots, k$, then $(i_{j_1})_* \gamma_{1,j_1} = (i_{j_2})_* \gamma_{1,j_2}$ for $1 \leq j_1 < j_2 \leq k$.*

Proof. It suffices to assume $k = 2$ by looking at the X_j two by two.

Let $X = \bigcup_{j=1}^2 X_j$, which is a subvariety of Y and $i: X \subset Y$ the inclusion.

In Gordon [1, Chapter 4] it is shown that one has the diagram of exact rows

$$\begin{tikzcd}
H_{1+2q}(Y) \arrow[r, "I"] \arrow[d] \arrow[dr, phantom, "\square"] & H_1(X)_\Delta \arrow[r, "\tau"] \arrow[d] \arrow[dr, phantom, "\square"] & H^c_{1+2q-1}(Y - X) \arrow[d] \\
H^1(Y) \arrow[r, "i^*"] \arrow[d] & H^1(X) \arrow[r, "\partial^*"] \arrow[d] & H^{2+1}(Y, X) \arrow[d] \\
H_1(Y) & H_1(X) \arrow[l, "i^*"'] & H_{2+1}(Y, X) \arrow[l, "\partial^*"']
\end{tikzcd}$$

where the first row is isomorphic to the second row by either Poincaré-Lefschetz duality or by the duality theorem proven in Gordon [3], where a definition of $H_1(X)_\Delta$ is also given. Basically, $H_1(X)_\Delta$ are those cycles in X over which one can construct "tubes" in $Y - X$. Thus they are the cycles which lie in the non-singular part of X or intersect transversally the singular locus of X. The second is isomorphic to the third row by vector space duality. $\square$ means the diagram commutes. $H_1(X) \subset \oplus_j H_1(X_j) \oplus H_0(X_{12})$ by the Maier-Vietoris sequence for $X_1 \cup X_2$, where $X_{12} = X_1 \cap X_2$. By Gordon [1, Corollary 4.13] $H_1(X)_\Delta \simeq \oplus_j H_1(X_j)_\Delta \oplus \tau X_{12}$. Also $H_1(X_j)_\Delta \subset H_1(X_j)$ and τX_{12} is generated by tubes over classes in X_{12}, i.e., if $0 \neq I(\gamma_{1+2q})$ has a representative which is homologous to zero in X, then this representative can be chosen so that it is a tube over a lower dimensional cycle in X_{12}. Furthermore, under the isomorphism $H_1(X) \simeq H_1(X)_\Delta$, $H_1(X) \cap H_0(X_{12}) \simeq \tau X_{12}$.

If $I(\gamma_{1+2q}) \neq 0$, then $I(\gamma_{1+2q}) \in \oplus H_1(X_i)_\Delta$ or $I(\gamma_{1+2q}) \in \tau X_{12}$. For if not, this would give nontrivial relations among the $H_1(X_i)_\Delta$ and τX_{12} in $H^c_{1+2q-1}(Y - X)$. But Gordon [2, Corollary 4.19] has shown that if one looks at the Leray spectral sequence of the inclusion map $j \colon Y - X \subset Y$, then

$$E_2^{r,s} \Rightarrow (\tau H_{r+s-2q+1}(X)_\Delta \subset H^c_{r+s}(Y - X)) ,$$

and in particular,

$$E_2^{1,2q-1} \Rightarrow \left(\bigoplus_i H_1(X_j)_\Delta \subset \bigoplus_j H_1(X_j)\right) ,$$

while

$$E_2^{1+2q-1-s,s} \Rightarrow \tau X_{12} \qquad \text{for } s > 2q - 1 .$$

But since we are working over a field, there can be no nontrivial relations between $E_\infty^{1,2q-1}$ and $E_\infty^{1+2q-1-s,s}$ for $s > 2q - 1$. Hence

$$(\text{Image } I) \cap \bigoplus_j H_1(X_j) \simeq (\text{coimage } i^*) \cap \bigoplus_j H_1(X_j)$$

by the exactness of the sequences and duality of vector spaces. Moreover, the isomorphism is given by $\text{Hom} \circ D_j$, where D_j is Poincaré duality on X_j. Let $D_j^* = \text{Hom} \circ D_j$. Then by Corollary 4.1, if

$$\gamma_{1+2q} \cap X_j = \gamma_{1,j} \neq 0 ,$$

there is a

$$\gamma'_{1+2q} \in H_{1+2q}(Y)$$

with

$$\gamma'_{1+2q} \cap X_j = D_j^* \gamma_{1,j} .$$

Hence

$$D_1^*\gamma_{1,1} - D_2^*\gamma_{1,2} \notin \operatorname{Ker} \tau, \text{ i.e.,} \quad \tau(D_1^*\gamma_{1,1} - D_2^*\gamma_{1,2}) = 2\tau D_1^*\gamma_{1,1} \neq 0 .$$

Thus

$$(D_1^* \circ D_1^*)\gamma_{1,1} - (D_2^* \circ D_2^*)\gamma_{1,2} \notin \operatorname{coker} \partial_*, \text{ i.e.,} \quad (D_1^*)^2\gamma_{1,1} - (D_2^*)^2\gamma_{1,2} \in \operatorname{Image} \partial_* .$$

But $(D_i^*)^2 = -\mathrm{Id}$, where Id is the identity map on $H_1(X_i)$.

5. A question on schemes

5.1. Suppose that Y is an integral algebraic k-scheme, where k is an arbitrary fixed algebraically closed field of any characteristic. We assume that Y is a smooth subscheme of projective space $P_N(k)$, and dimension of Y is $n + q$. Suppose furthur that X_1 and X_2 are smooth subschemes of Y of dimension q, and $i_j : X_j \subset Y$ is the inclusions. Consider the following diagram

$$\begin{array}{ccc} H^n(Y) & \xrightarrow{i_1^*} & H^n(X_1) \\ {\scriptstyle i_2^*}\downarrow & & \downarrow{\scriptstyle G_1} \\ H^n(X_2) & \xrightarrow{G_2} & H^{n+2q}(Y) \end{array}$$

where the G_i are the Gysin maps, where we are facing the l-adic cohomology, for l prime to the characteristic of k.

5.1.1. Question. When does the diagram commute with respect to coim i_1^* $\cap$ coim i_2^*, i.e., if $i_1^*\gamma$, $i_2^*\gamma \neq 0$, does $G_1 i_1^*\gamma = G_2 i_2^*\gamma$ for $n = 1$?

Over the complex numbers, this is the dual statement in cohomology to Corollary 4.2. The reason one believes it might be true for $n = 1$, is that one needs essentially only the strong Lefschetz theorem to prove Corollary 4.2., but the analogue of the strong Lefschetz theorem is true in étale-cohomology. However, the Kähler identities do not have an immediate analogue.

If the answer to question 5.1.1 is true for $n = 1$, one could probably prove the local invarient cycle problem for deformations of smooth schemes of dimension 2, using the analogues of the geometric constructions in [4].

Added in Proof. Some of the results in this paper have been generalized; see the author's paper, *On the primitive cohomology of submanifolds*, to appear in Illinois J. Math.

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UNIVERSITY OF ILLINOIS, CHICAGO