Michigan Math. J. 54 (2006)

Polynomial Convexity and Rossi's Local Maximum Principle

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Polynomial convexity is an old and fundamental topic in the theory of complex variables. If K is a compact set in $\mathbb{C}^n$ then its polynomial hull, denoted by $\hat{K}$, is the set of $z \in \mathbb{C}^n$ such that $|P(z)| \le \mathrm{Sup}_K |P|$ for every (holomorphic) polynomial P. If $\hat{K} = K$ then K is said to be polynomially convex. It is my opinion that some very basic questions are still worth revisiting. I illustrate this with two examples.

In Part A, I discuss Rossi's local maximum principle. I noticed only recently that this principle becomes a totally trivial exercise if the hull is characterized in terms of plurisubharmonic functions. Rossi's principle then generalizes to almost complex manifolds.

In Part B, which is far less successful, I discuss the old result of polynomial convexity of (smooth enough) arcs. It is a deep result—still with no easy proof—and with an unsatisfactory conclusion, as will be explained later. I would like to see the polynomial convexity of arcs established by some kind of construction similar to the one in Part A (with the soft tool of plurisubharmonic functions). Part B is still far from that goal, but I hope the proof presented there is somewhat more pleasant than previous proofs. Some of its ingredients may be useful.

Part A

1. Introduction

Rossi's maximum principle [15] is an important result in complex analysis. One version reads as follows.

Theorem. *Let K be a compact set in $\mathbb{C}^n$, and let $\hat{K}$ be its polynomial hull. Let $z \in \hat{K} \setminus K$, and let V be a relatively compact neighborhood of z that does not intersect K. Then, for every polynomial P, $|P(z)| \le \mathrm{Sup}_{\hat{K} \cap bV} |P|$ (where bV denotes the boundary of V).*

Thus, although the polynomial hull may not carry any analytic structure [16; 3, Chap. 24], still the maximum principle holds along $\hat{K}$. Following Rossi's original proof [15], proofs such as those for [3, Thm. 9.3] and [8, Thm. 3.2.11] rely on solving $\bar{\partial}$.

Received February 23, 2005. Revision received September 23, 2005.
Partly supported by an NSF grant.

It is a nontrivial but truly fundamental fact that the polynomial hull is the same as the hull with respect to plurisubharmonic functions. This is a transparent consequence of Bremmerman's theorem on plurisubharmonic functions and the Hartogs functions ($\mathrm{Sup}(c_j \log|f_j|)$, $c_j > 0$, f_j holomorphic) (see [4]; see also [9, Thm. 4.2.13] or [7, Sec. Q, Thm. 9]). For an L^2 approach, see Theorem 4.2.7 and Corollary 4.2.8 in [9] (and [8, Thm. 4.3.4]).

More precisely, for $z \in \mathbb{C}^n$, the following are equivalent:

(1) $|P(z)| \le \mathrm{Sup}_K |P|$ for all polynomials P;
(2) $u(z) \le \mathrm{Sup}_K\, u$ for all plurisubharmonic functions u on $\mathbb{C}^n$;
(2′) $u(z) \le \mathrm{Sup}_K\, u$ for all continuous plurisubharmonic functions u on $\mathbb{C}^n$.

I only recently noticed that, by using the equivalence of (1) and (2′), proving the version of Rossi's theorem given here becomes a trivial exercise. This exercise is completed in Section 3. It should be pointed out that this leads, moreover, to a natural generalization of the local maximum principle to arbitrary complex or almost complex manifolds.

The analysis of plurisubharmonic functions on almost complex manifolds seems to be going through some new developments. See in particular [5; 6; 11; 14]. Whether the study of plurisubharmonic hulls in almost complex manifolds will continue to be of interest remains to be seen. In the next section, we state the generalized result.

2. Definitions; Generalized Local Maximum Principle

2.1. REVIEW. Here we summarize material for the convenience of readers who may not be familiar with the theory of plurisubharmonic functions on almost complex manifolds. Elementary proofs of the results discussed in this Section 2.1 may be found in [11].

An almost complex manifold is a smooth manifold X (manifolds are assumed to be Hausdorff and to have a countable base of topology), of even real dimension $2n$, whose (real) tangent space is equipped at each point p with a complex structure—that is, an endomorphism $J(p)$ satisfying $J(p)^2 = -\mathbf{1}$. For simplicity we assume that the map $p \to J(p)$ is of class $\mathcal{C}^{1,\alpha}$ (first derivatives of Hölder class α) for some $\alpha > 0$. A J-holomorphic disc is a map Φ from the unit disc $\mathbb{D}$ in $\mathbb{C}$ into X whose differential is $(\mathbb{C} - J)$-linear. This means that, for every $z \in \mathbb{D}$ and for every tangent vector (a, b) to $\mathbb{R}^2$ ($\simeq \mathbb{C}$) at z, we have $D\Phi(p)(-b, a) = J(\Phi(p))D\Phi(p)(a, b)$. Long before pseudo-holomorphic curves were made famous by Gromov, it was proved by Nijenhuis and Woolf [13] that, if $p \in X$ and T is a tangent vector to X at p, then there exists a J-holomorphic disc Φ with $\Phi(0) = p$ and $D\Phi(0)(1, 0) = \lambda T$ for some $\lambda > 0$. The J-holomorphic maps are of class $\mathbb{C}^{k+1,\alpha}$ if J is $\mathbb{C}^{k,\alpha}$. An upper semicontinuous function u defined on some open set in X is said to be plurisubharmonic (J-plurisubharmonic if more precision is needed) if its restriction to any J-holomorphic disc (i.e., the function $u \circ \Phi$) is subharmonic on $\mathbb{D}$ in the usual sense.

At least for $\mathcal{C}^2$-functions, there is a characterization of plurisubharmonicity that is similar to the usual characterization in $\mathbb{C}^n$ in terms of the complex Hessian.

One introduces on X the differential operator d^c_J, which is obtained by twisting the de Rham operator d with the complex structure. If f is a function on X then $d^c_J f$ is the 1-form defined as follows. For any tangent vector T at a point $p \in X$, set

$$(d^c_J f)_p(T) = -df_p(J(p)T).$$

For the standard complex structure on $\mathbb{C}^n$ with coordinates $z_j = x_j + iy_j$,

$$d^c f = \sum_j \left(-\frac{\partial f}{\partial y_j} dx_j + \frac{\partial f}{\partial x_j} dy_j \right)$$

and so $d^c = \frac{\partial - \bar{\partial}}{i}$. Plurisubarmonicity of f is equivalent to the positivity condition (for arbitrary tangent vector T at a point p):

$$dd^c_J f_p(T, J(p)T) \geq 0.$$

Observe that we have avoided any complexification of the tangent space. However, by complexifying the tangent space (one more complex structure to deal with!) one can give a characterization in terms of $i\partial_J \bar{\partial}_J$ instead of dd^c_J.

The case of nonsmooth functions is surprisingly much harder to deal with. See [14], where the problem is solved at least for continuous functions.

In real dimension > 2, almost complex manifolds do not carry "J-holomorphic functions" in the absence of an integrability condition. Roughly speaking, complex holomorphic objects can be found only in dimension 1. However, at least locally, plurisubharmonic functions are abundant. It is fairly obvious that the square of the distance to p (in any given Riemannian metric) is plurisubharmonic near p. An interesting nontrivial example with a pole, which has been used in [6] and [11], is due to Chirka. Let J be an almost complex structure defined near 0 in $\mathbb{R}^{2n} \simeq \mathbb{C}^n$ such that $J(0)$ is the standard structure; then, for $A > 0$ large enough, $\log|Z| + A|Z|$ is plurisubharmonic near 0 [11, Lemma 1.4]. Note that $\log|Z|$ need not be J-plurisubharmonic (since it is not strictly plurisubharmonic for the standard structure, a perturbation of the structure allows one to destroy positivity).

Although several results for almost complex manifolds are very similar to the corresponding results for complex manifolds, there are important differences. The question of Levi foliation of a hypersurface $\{\rho = 0\}$ is such an instance. From [11, Sec. 6] we know that the condition $dd^c_J \rho(T, JT) = 0$ for all complex-tangent vectors T is not equivalent to the Frobenius condition $dd^c_J \rho(T, T') = 0$ for all complex-tangent vectors T and T'. In the $\partial\bar{\partial}$ approach followed by Pali and others, we remark that $\partial_J \bar{\partial}_J \rho$ need not be a closed form.

2.2. The Local Maximum Principle on Almost Complex Manifolds. Let X be an almost complex manifold and let K be a compact set in X. We define $\hat{K}$, the *p.s.h. hull* of K, to be the set of $x \in X$ such that $u(x) \leq \operatorname{Sup}_K u$ for every continuous plurisubharmonic function u on X. This is consistent with the same notation already used in $\mathbb{C}^n$.

We consider only continuous plurisubharmonic functions. On almost complex manifolds, even the local regularization of noncontinuous plurisubharmonic functions seems to raise serious difficulties.

Exactly as in $\mathbb{C}^n$, there are some trivial observations to be made as follows.

LEMMA. *If K is a compact set in an almost complex manifold X, then there exists a continuous plurisubharmonic function ρ on X such that $\rho \equiv 0$ on $\hat{K}$ and $\rho > 0$ off $\hat{K}$.*

Proof. For any $x \in X \setminus \hat{K}$, there exists a continuous plurisubharmonic function v_x on X such that $v_x(x) > \mathrm{Sup}_K\, v_x$. By addition of a constant, we can assume that $v_x(x) > 0 > \mathrm{Sup}_K\, v_x$.

Now $u_x = \mathrm{Max}(v_x, 0)$ satisfies $u_x \equiv 0$ on $\hat{K}$; moreover, $u_x \geq 0$ and $u_x(x) > 0$. Finally one can take $\rho = \sum_{j \in \mathbb{N}} \varepsilon_j u_{x_j}$, where the x_j are chosen so that, at any point of $X \setminus \hat{K}$, there is at least one function $u_{x_j} > 0$ as well as $\varepsilon_j > 0$ for ε_j small enough. □

Next we state the generalized maximum principle.

PROPOSITION. *Let K be a compact set in an almost complex manifold X, and let $\hat{K}$ be its p.s.h. hull. Let $x \in \hat{K} \setminus K$. For any relatively compact neighborhood V of x that does not intersect K and for any continuous plurisubharmonic function u defined on a neighborhood of $\bar{V}$,*

$$u(x) \leq \mathrm{Sup}_{\hat{K} \cap bV}\, u.$$

3. Proofs

In this section we prove the proposition of Section 2 and thus the theorem of Section 1.

In order to reach a contradiction, assume that there exists a continuous plurisubharmonic function u, defined on a neighborhood of $\bar{V}$, such that $u(x) > \mathrm{Sup}_{\hat{K} \cap bV}\, u$. By addition of a constant and multiplication by a positive constant, we can assume that $u(x) = 1$ and $u < 0$ on $\hat{K} \cap bV$. Let ρ be a continuous plurisubharmonic function on X such that $\rho \equiv 0$ on $\hat{K}$ and $\rho > 0$ off $\hat{K}$ (i.e., assume the lemma of Section 2). For $C > 0$ and $z \in V$, set

$$\lambda(z) = \mathrm{Max}(u(z), C\rho(z)).$$

Then $\lambda(x) = 1$ and, if C is large enough, we have $\lambda(z) \equiv C\rho(z)$ near bV. Therefore, λ extends to a continuous global plurisubharmonic function $\tilde{\lambda}$ on X provided we set $\tilde{\lambda}(z) = C\rho(z)$ for $z \notin V$. Hence $\tilde{\lambda}(x) = 1$ but $\tilde{\lambda} = C\rho \equiv 0$ on K, a contradiction.

PART B

4. Preliminaries

Following groundbreaking work of Wermer, Stolzenberg proved the following statement.

THEOREM [17; 20]. *$\mathcal{C}^1$ arcs in $\mathbb{C}^n$ are polynomially convex.*

By "arc" we mean the injective image of $[0,1]$ in $\mathbb{C}^n$. Alexander [1] extended the result to rectifiable arcs, but some smoothness is needed; this is well known [19]. Polynomial convexity of arcs is a deep result with no simple proof known, and the failure of polynomial convexity for (big) totally real discs (see [10, Ex. 6.1, p. 20]) illustrates that point.

A good reference for a proof is [3, Sec. 12]. In my view, this proof is highly nonconstructive. It amounts to a careful study of the structure of a hypothetical hull (it would be a Riemann surface with singularities) that allows global reasoning using a variation of the argument; there is no construction of polynomials or, what should be easier, of appropriate plurisubharmonic functions.

Consequently that proof does not allow one to give satisfactory answers to such questions as the following. Consider a smooth arc Γ in $\mathbb{C}^n$ and let d denote the distance function to Γ. Then d^2 is a plurisubharmonic function defined in a neighborhood W of Γ. Choose a neighborhood W_1 of Γ relatively compact in W. Since Γ is polynomially convex, it is easy to construct a plurisubharmonic exhaustion function $\rho \geq 0$, defined on $\mathbb{C}^n$, such that $\rho = 0$ on a neighborhood of Γ and $\rho > 0$ off W_1. Then, for $A > 0$ large enough, the function defined by $\mathrm{Max}(d^2, A\rho)$ on W and coinciding with $A\rho$ off W is a plurisubharmonic function. For $\varepsilon > 0$ small enough, the ε-neighborhood of γ is a sublevel set of that function and hence this ε-neighborhood is polynomially convex. The question is: How small must ε be to ensure that the ε-neighborhood of the arc is polynomially convex? Roughly speaking, if one bounds curvature and avoids near crossing then there is a universal ε, as shown by a normal family argument. Of course, the argument gives absolutely no estimate. We shall restrict our attention to $\mathcal{C}^2$ arcs.

I have been unable to devise a proof that I find truly satisfactory. However, I have come up with a rewriting of the proof of polynomial convexity. I do not know whether readers will agree that this new writing is clearer and easier. The proof unfortunately still uses a study of the structure of a hypothetical hull, but it uses only the first transparent (purely local) step in that study—before things become more complicated to describe. This new proof replaces most of the discussion on Riemann surfaces by "soft" arguments concerning the index of linear maps, and it totally avoids the reasoning based on variation of the argument.

5. Notation

Let Γ be a $\mathcal{C}^2$ arc in $\mathbb{C}^n = \mathbb{C}_z \times \mathbb{C}_w^{n-1}$. The projection of $\mathbb{C}^n$ onto $\mathbb{C}$ will be denoted by Π, $\Pi(z, w) = z$. By abuse of language, Γ will be used to denote the map $\Gamma\colon [0,1] \to \mathbb{C}^n$ as well as just its geometric image. As before, the polynomial hull of Γ will be denoted by $\hat{\Gamma}$; it is the set of points $p \in \mathbb{C}^n$ such that $|P(p)| \leq \mathrm{Sup}_\Gamma |P|$ for any polynomial P. Of course $\hat{\Gamma} \supseteq \Gamma$, but our goal is to prove that $\hat{\Gamma} = \Gamma$. In order to reach a contradiction we shall assume that $\hat{\Gamma} \neq \Gamma$.

Shortening the arc if needed, we can assume that $\Gamma([\varepsilon, 1])$ is polynomially convex for any $\varepsilon > 0$. This will allow us to finish the proof using a purely local argument and without having to consider several cases (this assumption should be useful also in a "semiconstructive" proof when constructing appropriate plurisubharmonic functions). Because the hull of a curve (or, more generally, of a totally real submanifold) cannot stay in a "small" neighborhood of the arc (for a variety of reasons), it is indeed easy to see that the set of ε-values for which the arc $\Gamma([\varepsilon, 1])$ is not polynomially convex is closed.

We denote by γ the projection of Γ on $\mathbb{C}_z$. Given an appropriate (generic) choice of coordinates, we can assume that γ is a curve in $\mathbb{C}$ with only finitely many crossings and that all the crossings are simple. We can also assume that the endpoints of γ are not crossing points. Then $\mathbb{C} \setminus \gamma$ consists of finitely many components $\Omega_0, \Omega_1, \ldots, \Omega_N$, where Ω_1 is the unbounded component and Ω_0 is the component that contains the endpoint $\Pi(\Gamma(0))$. Under our assumptions for a proof by contradiction, it follows that $\Omega_0 \neq \Omega_1$ (but we won't need this); it is precisely the essential difficulty of the problem that one cannot assume $\Pi(\Gamma(0))$ belongs to the unbounded component of $\mathbb{C} \setminus \gamma$. Adjacent components are separated by (relatively open) subarcs that we denote $\gamma_1, \ldots, \gamma_K$. Finally, let γ_0 be the arc $\gamma \cap \Omega_0$, an arc in Ω_0 that does not disconnect Ω_0. We will denote by Γ_j the subarc of Γ whose projection is γ_j. See Figure 1.

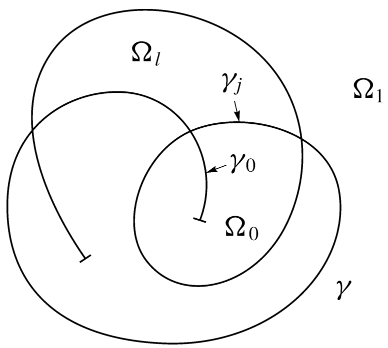

Figure 1

6. Sketch of the Proof

We denote by A the closed subalgebra of $\mathcal{C}(\Gamma)$ generated by polynomials. Once the theorem is proved we shall know that $A = \mathcal{C}(\Gamma)$, by the Oka–Weil theorem, since every continuous function on a $\mathcal{C}^1$ curve can be approximated by holomorphic functions defined on a neighborhood. (Whether there is polynomial approximation on a nonsmooth polynomially convex arc is still an open question.)

(a) For $a \in \mathbb{C} \setminus \gamma$, define the index of a to be the codimension of $(z-a)A$ in A. Note that $(z-a)A$ is obviously a closed subspace of A. Totally soft arguments show that the index is constant on each connected component of $\mathbb{C} \setminus \gamma$. One can therefore speak about the index of each component Ω_j; this index is the essential tool of the proof.

If finite, the index is the maximum number of functions in A that are linearly independent in $A/(z-a)A$. By approximation for any dense subalgebra B of A, the index is also the maximum number of elements of B that are linearly independent in $A/(z-a)A$. Several times we will take B, the algebra of polynomials, but in Section 8.3 we take the algebra of functions that are holomorphic on a fixed neighborhood of $\hat{\Gamma}$.

(b) One needs to understand how the index may change when crossing an arc γ_j. If the arc is real-analytic (much less is needed, as will be seen), then a soft argument still shows that when crossing the arc the index may not change or may change by ± 1 (and so is always finite). See Section 7.

(c) To finish with real-analytic arcs, one may show (using our minimality assumption) that when crossing γ_0 the index should jump by ± 1 but when crossing γ_0 one stays in the same connected component Ω_0 on which the index is constant—a contradiction. For this part of the proof (see Section 8), I have not been able to avoid the beginning (but just the easy beginning) of a study of a hypothetical hull.

(d) The case of a non–real-analytic arc can be somewhat reduced to the real-analytic case by making a distinction between the subarcs γ_j. Soft arguments show that, in some sense, some of these arcs are not essential (e.g., because they would not contain any peak point). The other arcs that are essential, and γ_0 would be one of them, can (partly) be replaced by real-analytic arcs. Here the proof again uses the previous easy partial study of the structure of the hull. Although nothing really difficult is done in (d), I cannot say that I am very pleased with it.

Section 7 contains the soft arguments that lead to (a) and (b). Section 8 deals with the well-known first step in the study of the polynomial hull of Γ and concludes the proof for the real-analytic case. The case of non–real-analytic arcs is treated in Section 9.

We retain the notation used so far. All steps in the proof, except the final one, are valid for closed curves as well. Some more general statements could obviously be given, but we avoided going in that direction.

7. Soft Arguments

7.1. Proof of (a). First assume that, for some point $a \in \mathbb{C} \setminus \gamma$, $(z-a)A$ has finite codimension equal to k in A. Let $f_1, \ldots, f_k \in A$ be such that the map $(g, \lambda_1, \ldots, \lambda_k) \mapsto (z-a)g + \lambda_1 f_1 + \cdots + \lambda_k f_k$ is a bijective map from $A \times \mathbb{C}^k$ onto A. For b close to a, the map $(g, \lambda_1, \ldots, \lambda_k) \mapsto (z-b)g + \lambda_1 f_1 + \cdots + \lambda_k f_k$ is still an isomorphism and so the set of points $a \in \mathbb{C} \setminus \gamma$ with index k is open. Since we have not yet ruled out that the index may be infinite, note that the exact same reasoning with injective instead of bijective maps (taking into account that $(z-a)A$ is closed in A) shows that the set of all a with index $> k$ is also open in $\mathbb{C} \setminus \gamma$. Therefore the set of all a having a given index k is both open and closed in $\mathbb{C} \setminus \gamma$.

7.2. Proof of (b). Consider the case when Γ_j has a real-analytic subarc. Let then $\varphi = (\alpha, \beta)$ be a nonconstant map from the unit disc $\mathbb{D}$ (with variable ζ) into $\mathbb{C}^n = \mathbb{C} \times \mathbb{C}^{n-1}$ such that $\varphi((-1, +1)) \subset \Gamma_j$ and $\alpha(\zeta) \notin \gamma$ if $\zeta \notin \mathbb{R}$. Assume that, for some $\zeta_0 \notin \mathbb{R}$, $(z - \alpha(\zeta_0))A$ has codimension $\geq (k+1) \geq 1$ in A. Then, by approximation, there exist $k+1$ polynomials in (z,w) that are linearly independent in $A/(z-\alpha(\zeta_0))A$. Hence there exist at least k polynomials $P_1,\dots,P_k$ that are linearly independent in $A/(z-\alpha(\zeta_0))A$ and that satisfy $P_l(\varphi(\zeta_0))=0$. As a result, the functions $F_l(z,w)=P_l(z,w)/(z-\alpha(\zeta_0))$ defined on Γ are linearly independent in $\mathcal{C}(\Gamma)/A$ and so there exist k continuous linear forms on $\mathcal{C}(X)$, $\psi_1,\dots,\psi_k$, that are vanishing on A but such that the determinant of the matrix $(\psi_m(F_l))_{(m,l)}\neq 0$.

Consider the holomorphic function h defined on $\mathbb{D}\setminus\mathbb{R}$ by

$$h(\zeta)=\det\left(\psi_m\left(\frac{P_l(z,w)-P_l(\varphi(\zeta))}{z-\alpha(\zeta)}\right)\right).$$

Thus $h(\zeta_0)\neq 0$. Next observe that h is holomorphic on $\mathbb{D}\setminus\mathbb{R}$ and extends holomorphically across $\mathbb{R}$. This is why we dropped from $k+1$ to k in order to have the needed vanishing of the numerator in $(P_j(z,w)-P_j(\varphi(\zeta)))/(z-\alpha(\zeta))$ if $z=\alpha(\zeta)$ and $(z,w)\in\Gamma$ (so $(z,w)=\varphi(\zeta)$). Therefore, $h\neq 0$ except at isolated zeros. But if $\zeta\notin\mathbb{R}$ and $h(\zeta)\neq 0$, then the functions $(P_j(z,w)-P_j(\varphi(\zeta)))/(z-\alpha(\zeta))$ $(j=1,\dots,k)$ are k functions on Γ that are linearly independent in $\mathcal{C}(\Gamma)/A$. Hence the polynomials $Q_j(z,w)=P_j(z,w)-P_j(\varphi(\zeta))$ are linearly independent in $A/(z-\alpha(\zeta))A$. For such a point, the codimension of $(z-\alpha(\zeta))A$ in A is thus not less than k. In view of (a), (b) follows trivially.

COMMENT. More is accomplished by these soft arguments than may appear to be the case. For example, if the hull is trivial on one side of the curve then our argument shows that, on the other side, the hull is trivial or parameterized by the map φ. If the hull is not trivial then it shows $P(\varphi(\zeta))=0$ provided that $P/(z-\alpha(\zeta))$ is the uniform limit of a polynomial on Γ. Treated directly (e.g., via a Cousin problem with bounds, since local approximation on $\hat{\Gamma}$ is then clear), this is not such an obvious fact! Neither is it obvious from an abstract point of view, since there exist nonlocal uniform algebras [12; 18, Sec. 8.4].

8. Next Steps and the Real-Analytic Case

In proving (c), we are unable to avoid the beginning of a study of a hypothetical hull. The reader should note that, unless stated otherwise, it is not necessary to assume real analyticity in 8.1–8.3.

8.1. We first discuss the hull near a point in Γ whose projection (on γ) is on (the regular part of) the boundary of Ω_1 (the unbounded component of $\mathbb{C}\setminus\gamma$) or, more generally, on the boundary of any component Ω_{j_1} whose index is 0.

Assume that Ω is one of the bounded components of $\mathbb{C}\setminus\gamma$ that has a common boundary γ_r (for some $r\in\{1,\dots,K\}$) with Ω_{j_1}. It follows from the local maximum principle that $\Pi^{-1}(\gamma_r)\cap\hat{\Gamma}=\Gamma_r$; that is, the fiber of $\hat{\Gamma}$ over $z\in\gamma_r$ is trivial. Then, either $\hat{\Gamma}\cap\Pi^{-1}\Omega$ is empty or there exists a holomorphic map Ψ from Ω into $\mathbb{C}^{n-1}$, extending continuously to γ_r, such that $\hat{\Gamma}\cap\Pi^{-1}\Omega$ is the graph of Ψ

and such that the restriction of Ψ to γ_r gives a parameterization of Γ_r. Although this has been shown by other techniques in the early works of Wermer (and in the real-analytic case it follows rather easily from the fact that, on Ω, the index is at most 1), I believe the Alexander–Wermer characterization [2] of hulls of sets fibered over the circle with convex fibers provides the most transparent explanation. By the local maximum principle, $\hat{\Gamma} \cap \Pi^{-1}(\Omega)$ is included in the polynomial hull of the set fibered over the boundary of Ω whose fiber over z is reduced to the point in Γ over z if $z \in \gamma_r$; otherwise, whose fiber is, say, a large ball in $\mathbb{C}^{n-1}$. Clearly, there can be only one Alexander–Wermer disc in the present situation.

8.2. From 8.1 we may derive two easy generalizations as follows.

8.2.1. Consider the hull near a (peak) point $p \in \Gamma$ such that, for some polynomial P, $P(p) = 1$ but $|P| < 1$ on $\Gamma \setminus \{p\}$ and p is not a critical point for P on Γ. Then, map Γ into an arc $\tilde{\Gamma}$ in $\mathbb{C}^{n+1}$ by the map $(z, w) \mapsto (z, w, t) = (z, w, P(z, w)) \in \mathbb{C} \times \mathbb{C}^n \times \mathbb{C}$. Because P is a polynomial, it maps $\hat{\Gamma}$ to the hull of $\tilde{\Gamma}$. Now, instead of using the projection of the first factor $\mathbb{C}_z$, use the projection on the last factor $\mathbb{C}_t$. Then the point $(p, P(p))$ has its projection at the boundary of the unbounded component, as in 8.1.

We conclude that if p is as just described and if, near p, $\hat{\Gamma}$ is not reduced to Γ then, near p, $\hat{\Gamma}$ is given by a holomorphic disc (on one side of γ) attached to Γ (continuously along part of its boundary).

Let us state this conclusion more precisely for the case when arc Γ is real-analytic near p. There exists a holomorphic map $\varphi = (\alpha, \beta)$ from $\mathbb{D}$ into $\mathbb{C}^n$ as in 7.2 and so (using φ), near p, the arc γ_j is parameterized by $[-1, +1]$ and $\hat{\Gamma}$ is parameterized by the upper half-disc.

8.2.2. Let P be a polynomial. For simplicity we shall assume that the image of Γ under P is a curve with only simple crossing (a genericity hypothesis). Let $\Lambda = \{(z, w) \in \Gamma;\ |P(z, w)| > 1\}$. Then, by 8.1 and the reasoning in 8.2.1, either the hull of Γ is trivially reduced to Γ near any point of Λ or there is a point in Λ near which the hull is given by a holomorphic disc attached (on one side) to Γ.

8.3. Jump of the Index. We now consider the case of the subarcs γ_j that contain a p such that, near p, Γ is real-analytic. We assume that, near p, $\hat{\Gamma}$ is given by a holomorphic disc (on one side of γ) attached to Γ. (The disc, but not the hull, extends across Γ.) We know that the index changes by at most 1 when crossing γ_j. We shall prove that if the index is finite (as we already know it is for real-analytic arcs) then the index cannot stay constant: it must change by ± 1.

For this, we go back to 7.2 but now substituting, for the algebra of polynomials, the algebra B of holomorphic functions on a neighborhood W of $\hat{\Gamma}$. We take a connected Runge neighborhood W of $\hat{\Gamma}$ such that, for any (z, w) in some neighborhood of p, the connected component of $\{z\} \times \mathbb{C}^{n-1} \cap W$ that contains (z, w) does not contain any point of $\hat{\Gamma}$ *except* the point on the half-disc parameterized by φ when z is on the right side of γ.

The argument used in 7.2 shows that, if we look at the maximum number of P (with $P \in B$ and $P(\varphi(\zeta)) = 0$) that are linearly independent in $A/(z-\alpha(\zeta))A$, then that number does not change when $\alpha(\zeta)$ crosses γ_j (with possible isolated exceptions). We restrict the domain of ζ so that $\varphi(\zeta) \in W$. Hence the whole question is whether there exist any $P \in B$, with $P(\varphi(\zeta)) \neq 0$, that belong to $(z-\alpha(\zeta))A$. The answer is simple: if $\varphi(\zeta) \in \hat{\Gamma}$, it is obviously impossible. On the other hand, if $\varphi(\zeta) \notin \hat{\Gamma}$ and if $\zeta \simeq 0$ then there exists a $P \in B$ such that, as a function of w, $P(\alpha(\zeta), w) \equiv 0$ on a neighborhood of $\hat{\Gamma}$ yet $P(\varphi(\zeta)) \neq 0$. By the Oka–Weil theorem, such a P belongs to $(z-\alpha(\zeta))A$ because $P/(z-\alpha(\zeta))$ defines a holomorphic function on a neighborhood of $\hat{K}$. This creates the jump of ± 1.

The details may be painful but the outline is clear. The presence of a disc in the hull on one side of the curve has created one additional condition for the approximation of $P/(z-a)$ by imposing the vanishing of P at the appropriate point.

8.4. CONCLUSION OF THE REAL-ANALYTIC CASE. There could be points in $p \in \Gamma_0$ in the neighborhood of which the hull of Γ is trivial (reduced to Γ). We denote the set of such points by I (for inessential), which is a relatively open subset of Γ. If $p \in I$ then, by Rossi's local maximum principle, there exists a polynomial P such that $|P(p)| > \operatorname{Sup}_{\hat{\Gamma}\setminus V}|P|$. Of course, we can impose the condition that p not be a critical point of $P|\Gamma$. By the elementary theory of single complex variables, any continuous function f on $P(\hat{\Gamma})$ that is 0 on a neighborhood of the closure of $P(\hat{\Gamma} \setminus \Gamma)$ (so its support is made of images of arcs) can be approximated by holomorphic functions defined on a neighborhood of $P(\hat{\Gamma})$. Hence $f \circ P \in A$. Therefore, any continuous function on Γ coincides in a sufficiently small neighborhood of p with a function belonging to A.

It follows that the hull of Γ is simply the union of I and of the hull of $\Gamma \setminus I$. Various reasons can be given. One of them is that any Jensen representing measure whose support contains a point p (as described here) must be a point mass.

Because the arc Γ has been chosen to be minimal, $\Gamma \setminus \Gamma_0$ is polynomially convex. We know that $\Gamma_0 \setminus I$ cannot be empty and must contain peak points for A (a general fact of function algebras). Hence there exists a polynomial P (as in 8.2.2) such that $\operatorname{Sup}_{\Gamma_0\setminus I}|P| > 1 > \operatorname{Sup}_{\Gamma\setminus\Gamma_0}|P|$. By 8.2.2 there is a point $p \in \Gamma_0$ near which the hull is given by a holomorphic disc (on one side of γ_0) attached to Γ. By 8.3, the index must change by ± 1 when crossing γ_0, leading to a contradiction as already explained in (c).

9. The Smooth Case

We should first eliminate the subarcs μ of Γ that, roughly speaking, do not contribute to the existence of a nontrivial hull. First, there are those arcs μ in the neighborhood of which the hull of Γ is reduced to Γ; they have been studied in the previous section. Other nonessential subarcs are those arcs μ that do not contain any peak point for A. Then convergence of a sequence of polynomials on $\Gamma \setminus \mu$ implies convergence on Γ. It is clear that, when crossing one of the arcs γ_j that contains a (nonempty!) subarc of one of these two kinds, the index cannot change.

The index can change only when crossing an arc that contains a peak point. Then, as shown in Section 8, a nontrivial hull near some point p will be described by a holomorphic disc attached on one side. We can then deform the arc Γ near p by pushing it and pushing the point p in this analytic disc, making a portion of the new Γ real-analytic near the new point p. By Rossi's local maximum principle, this will not destroy the supposed nontriviality of $\hat{\Gamma}$, and the set of inessential points can only increase. Creating such a deformation along each subarc where this is needed, we now have that the index changes by at most 1 when crossing any of the arcs γ_j. Therefore, the index is finite. When crossing γ_0 it should jump by 1, given the reasoning of 8.4 (after a local real-analytic modification of the arc). This leads to the same contradiction.

Added in proof. Our goal has been to obtain a clearer proof of the polynomial convexity of arcs, not to obtain the sharpest possible results. We restricted our attention to $\mathcal{C}^2$ arcs so that, after a simple linear change of coordinates, one can assume that Π (the projection of $\mathbb{C} \times \mathbb{C}^{n-1}$ on $\mathbb{C}$) defines a global immersion of Γ into $\mathbb{C}$.

By an automorphism of $\mathbb{C}^n$, any smooth arc in $\mathbb{C}^n$ $(n > 1)$ can be approximately straightened to a line segment. (See J.-P. Rosay, *Straightening of arcs,* Astérisque 217 (1993), 217–225; see also F. Forstneric and J.-P. Rosay, *Approximation of biholomorphic mappings of* $\mathbf{C}^n$, Invent. Math. 112 (1993), 323–349 [and erratum in Invent. Math. 118 (1994), pp. 573–574].) After straightening, polynomial convexity of the arc is totally obvious, but polynomial convexity is the hard result needed for the proof of straightening.

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