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ON A NEW RADICAL IN A TOPOLOGICAL RING

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The radical which is referred to in this paper was treated extensively by Wright in the case of topological groups. The present course of attack here is threefold: (1) to show the proximity of large powers of topologically nilpotent elements to the radical in a topological ring, (2) to determine a nilpotence condition on the radical and (3) to characterize the radical of all locally compact simple rings without divisors of zero. For a topological group, the radical possesses little, if any, algebraic structure aside from being a subgroup of the group. Viewed as an additive subgroup of a topological ring R, it is shown that the radical is an ideal of R. Relative to the nilpotence of the radical, the additive group structure of locally compact connected Jacobson semi simple rings is established to within topological isomorphism.

In the final section the theorem on nilpotence is used to characterize the radical of locally compact simple rings having no zero divisors.

1. Preliminaries and definitions. Throughout this paper we adopt the notation and terminology of Wright [8]. All topological groups will be assumed to be Abelian and Hausdorff. For a topological group G, a *maximal* 0-*proper open semigroup* in G is a subset $M \subseteqq G$ satisfying (1) M is a semigroup in G, (2) M is an open set in G, (3) $0 \notin M$, and (4) M is maximal with respect to (1)-(3).

For subset $A \subseteqq G$ we define $s(A) = \{x \in G \mid x + A \subseteqq A\}$ and $b(A) = s(A) \cap s(-A)$. If M is a maximal 0-proper open semigroup in G, $b(M)$ is a closed subgroup of G and G is the disjoint union of $\{M, -M, b(M)\}$. [8; Th. 3.3].

By the *W-radical of G* we mean $\bigcap b(M)$ where the intersection is taken over all maximal 0-proper open semi group M in G. We denote the W-radical by $T(G)$. If G contains no 0-proper open semi groups, (for example if G is finite, or more generally if G is compact) $T(G) = G$ and G is called a *radical group*. If $T(G) = (0)$ we say that G is *W-semi-simple*.

In [8] it is shown that in a locally compact group G, $T(G)$ is a fully warrented hereditary radical [6]. Specifically it is shown that $T(T(G)) = T(G)$, $T(G/T(G)) = (0)$ and if $H \subseteqq G$ is a closed subgroup, $T(H) = H \cap T(G)$.

In general the W-radical of G contains the radical N of Iwasawa which is defined as the maximal compact connected subgroup of a locally compact group G [4]. In the special case that G is connected, both radicals coincide.

2. Locally compact rings. The following result is of major consequence in this paper and establishes a justification for studying the W-radical in topological rings. Furthermore, in view of the remarks in the last section and the fact that $T(R)$ is an ideal of R, the term *ring radical* associated with $T(R)$ is fully justified [6].

THEOREM 2.1. *In any topological ring R, $T(R)$ is an ideal of R. In particular, $T(R)$ is closed under left and right quasi-inverses.*

Proof. We show that if $\phi: R \to R$ is any continuous group homomorphism, then $\phi(T(R)) \subseteqq T(R)$.

Suppose $x \in \phi(T(R))$ so that $x = \phi(t)$ for some $t \in T(R)$. Assume there exists a maximal 0-proper open semigroup M in R with $x \in M$. Then $\phi^{-1}(M)$ is a 0-proper open semigroup in R containing t. An application of Zorn's lemma yields a maximal 0-proper open semigroup $M^* \supseteqq \phi^{-1}(M)$. Thus $t \in M^*$ contradicting that $t \in T(R)$ since R is a disjoint union of $\{M^*, -M^*, b(M^*)\}$. It follows that $x \in T(R)$ and the assertion is established.

To see that $T(R)$ is an ideal of R, let $x \in R$ be fixed, but arbitrary, are consider the continuous group homomorphism $\phi: R \to R: r \to rx$. It follows that $rx \in T(R)$; similarly $xr \in T(R)$ and therefore $T(R)$ is a (two-sided) ideal of R.

The notion of topological nilpotence has a natural generalization to its algebraic analogue. In a sequel to this paper we shall see how topological nilpotence of elements plays a vital role in the characterization of the prime radical of $T(R)$ in a certain subclass of locally compact commutative rings. Here we develop some elementary properties relating topological nilpotence to the W-radical of a locally compact ring.

DEFINITION 2.2. [5; p. 162]. Let R be a topological ring. An element $x \in R$ is called topologically nilpotent in case $\lim x^n = 0$.

Our first result along these lines shows that in locally compact rings with compact component of 0, powers of topologically nilpotent elements are eventually in the W-radical. We begin by establishing the following lemma:

LEMMA 2.3. *Let R be locally compact totally disconnected ring. If $x \in R$ is topologically nilpotent, there exists $N \in \boldsymbol{Z}^+$ such that $n \geqq N$ implies $x^n \in T(R)$.*

Proof. Assume $\lim x^n = 0$. Since R is locally compact and totally disconnected, find a compact open neighborhood U of 0. Choose $N \in \boldsymbol{Z}^+$ so that $n \geqq N$ implies $n^n \in U$. (For $y \in R$ denote by

$\langle y \rangle$ the additive subgroup generated by y.) Now $\langle x^n \rangle \subseteqq U$ and hence $\mathrm{Cl}\,\langle x^n \rangle \subseteqq U$. The compactness of U implies that $\mathrm{Cl}\,\langle x^n \rangle$ is compact, whence $\mathrm{Cl}\,\langle x^n \rangle \subseteqq T(R)$ [8; Th. 8.10]. We conclude that $x^n \in T(R)$.

Throughout the remainder of this paper the symbol K will be used to denote the connected component of 0.

THEOREM 2.4. *Let R be a locally compact ring with K compact. If $x \in R$ is topologically nilpotent, $x^n \in T(R)$ for all n sufficiently large.*

Proof. Since K is closed, R/K is locally compact and totally disconnected. If $\lim x^n = 0$ for $x \in R$ it is easy to see that $\lim (x+K)^n = K$. Applying 2.3 we have that $(x + K)^n = x^n + K \in T(R/K)$ for all n sufficiently large. Since K is compact, $K \subseteqq T(R)$. From [8; Th. 4.7] we have $T(R/K) = T(R)/K$ whence, $(x + K)^n \in T(R)/K$. This shows that $x^n \in T(R)$.

We remark that the condition $K \subseteqq T(R)$, which is insured by compactness of K, is emphatic in 2.4. The real numbers as a topological ring shows that the compactness of K cannot be dropped from our assumptions. (For example, 1/2 is topologically nilpotent, but $(1/2)^n \neq 0$ for any $n \geqq 1$.) Locally compact W semi-simple rings with compact K are necessarily totally disconnected. In such rings topological nilpotence is reduced (in fact is equivalent) to algebraic nilpotence. This comment is generalized in § 3.

For the next few results we digress slightly to the study of the radical structure of locally compact abelian groups.

Wright [8] shows that in every topological abelian group G there is a unique (closed) maximal radical subgroup which we denote by H. The symbol $\bigoplus \sum_{\alpha \in \mathscr{A}} G_\alpha$ will denote the weak direct product of groups G_α, $\alpha \in \mathscr{A}$, where $\mathscr{A}$ is an arbitrary index set.

LEMMA 2.5. *Let $G = \bigoplus \sum_{\alpha \in \mathscr{A}} G_\alpha$ where each G_α is a topological abelian group, and let H_α be the maximal radical subgroup of G_α. Then $\bigoplus \sum_{\alpha \in \mathscr{A}} H_\alpha$ is the maximal radical subgroup of G.*

Proof. Define $\mathscr{S} = \{S \subseteqq G \mid S \subseteqq \bigoplus \sum_{\alpha \in \mathscr{A}} H_\alpha$ and S is a radical subgroup$\}$. Partially order $\mathscr{S}$ by inclusion. If $\{S_\beta\}$ is a chain in $\mathscr{S}$, by [8; p. 483] it can be shown that $\bigcup S_\beta \in S$, whence $\{S_\beta\}$ is bounded above. Let $M \in \mathscr{S}$ be a maximal element. We show that $M = \bigoplus \sum_{\alpha \in \mathscr{A}} H_\alpha$. To see this, let $H = \bigoplus \sum_{\alpha \in \mathscr{A}} H_\alpha$ and assume $M \neq H$. Then there exists $\alpha \in A$ with $M \neq M + H_\alpha$. Now $M + H_\alpha$ is W-radical since both M and H_α are so; thus we have $M \neq M + H_\alpha \subseteqq \bigoplus \sum_{\alpha \in \mathscr{A}} H_\alpha$ contradicting the maximality of M. Therefore $M = H$.

In every locally compact group the W-radical is the maximal radical subgroup of G [8; Th. 8.10]. As a useful consequence to 2.5 we recite the following result whose proof is immediate.

THEOREM 2.6. *Let $G = \bigoplus\sum_{\alpha\in\mathscr{A}} G_\alpha$ be a locally compact abelian group. Then $T(G) = \bigoplus\sum_{\alpha\in\mathscr{A}} T(G_\alpha)$.*

We are now in a position to prove the most general result involving topologically nilpotent elements in a topological ring R and the W-radical of R. In particular this result shows that in W-semisimple rings, topologically nilpotent elements approach their limit via points in Euclidean n-space. We designate by $\|\ \|$ the Euclidean norm in Euclidean n-space.

THEOREM 2.7. *Let R be a locally compact ring and assume $x \in R$ is topologically nilpotent. Then given $M \in Z^+$ there is an $n \in Z^+$ such that for $K \geqq n$, $x^k = e_k + g_k$ where $g_k \in T(R)$ and $\| e_k \| < 1/M$.*

Proof. As topological groups, $R = E^n \oplus G$ where G is a group having compact component of 0 [3; Th. 24.30]. Assume $\lim x^k = \lim (e_k + g_k) = 0$, where $e_k \in E^n$ and $g_k \in G$ for $k \geqq 1$. It follows that $\lim e_e = \lim g_k = 0$. Abiding by the argument of 2.3 and 2.4 it is plain that $g_k \in T(G)$ and $\| e_k \| < 1/M$ for large k. By 2.6 $T(R) = T(E^n) \oplus T(G)$, and since $T(E^n) = (0)$, $T(G)$ is topologically isomorphic to $T(R)$ in which case the theorem obtains.

3. Locally compact rings with connected W-radical. This section constitutes the central theme of this paper. Here we formulate a sufficient condition to insure the algebraic nilpotence of the W-radical. Then by example we show the condition is not necessary. In conjunction with [8; Th. 8.3] our main theorem establishes up to topological isomorphism the group structure of connected locally compact Jacobson semisimple rings. This consequence may be compared to the following two results, the first of which follows trivially from 3.2 and [8; Th. 8.10]; the second of which is proved in [5; Th. 16].

Every compact connected Jacobson semisimple ring is (0).

A compact Jacobson semi simple ring in homeomorphic to a direct sum of finite simple rings.

We begin by reciting a well known result whose proof follows directly from the definition of a topological ring and the notion of a compact set.

LEMMA 3.1. *Let R be a topological ring, and let $C \subseteqq R$ be a* *compact subset. Then given an open neighborhood U of 0 there exists an open neighborhood V of 0 such that $V \cdot C \subseteqq U$.*

THEOREM 3.2. *Let R be a locally compact ring with $T(R)$ connected. Then $T(R)$ is a nilpotent ideal of index at most two. In particular $T(R) \subseteqq J(R)$.*

Proof. By [3] it will suffice to show that if $T(R)$ is viewed as a topological abelian group, and if f is an arbitrary character on $T(R)$, $f(T^2(R)) = 0$.

Assume f is a character on $T(R)$. For each $x \in T(R)$ define $T_x = \{t \in T(R) \mid f(t \cdot \mathrm{Cl}\,(x)) = 0\}$, where $\mathrm{Cl}\,(x)$ denotes the closure of the additive subgroup generated by x. Let $T' = \{a \in T(R) \mid f(a \cdot T(R)) = 0\}$. We show $\bigcap_{x \in T(R)} T_x = T'$. If $y \in T'$ and $x \in T(R)$ is arbitrarily chosen, then $f(y \cdot \mathrm{Cl}\,(x)) = 0$; hence $y \in T_x$ and therefore $y \in \bigcap T_x$ where $x \in T(R)$. Conversely, if $y \in \bigcap T_x$ with $x \in T(R)$, let $t \in T(R)$ be arbitrary. Then $f(y \cdot \mathrm{Cl}\,(t)) = 0$ since, in particular, $y \in T_t$; whence $f(yt) = 0$. Thus $y \in T'$. This establishes the claim. Next we show that for each $x \in T(R)$, T_x is an open subgroup of $T(R)$. It is plain that T_x is a subgroup; we concentrate on the openness of T_x. It suffices to exhibit an open neighborhood of $T(R)$ which is contained in T_x. Since $f(0) = Z$ (Z is viewed as a point in R/Z) and since f is continuous find an open neighborhood U of 0 (open in $T(R)$) such that $f(U) \subseteqq (-1/4, 1/4) + Z$. Since $x \in T(R)$, $\mathrm{Cl}\,(x)$ is compact by [8; Th. 8.10]. By 3.1 there is an open neighborhood V in $T(R)$ such that $V \cdot \mathrm{Cl}\,(x) \subseteqq U$. Now let $y \in V$ be arbitrary. Now since $nz \in \mathrm{Cl}\,(x)$ whenever $z \in \mathrm{Cl}\,(x)$, it follows that if $n \in Z^+$ then $nyz \in U$.

Now suppose that for some $y \in V$ and some $z \in \mathrm{Cl}\,(x)$, $f(yz) = r + Z$ with $r \in (-1/4, 1/4)$. Then $f(nyz) = nf(yz) = nr + Z$; hence $nr \in (-1/4, 1/4)$ for all $n \in Z^+$. It follows that $r = 0$ and so $f(yz) = Z$. This argument shows that $V \subseteqq T_x$ and thus T_x is open in $T(R)$; since T_x is a subgroup of $T(R)$, T_x is also closed. By assumption $T(R)$ is connected and so $T_x = T(R)$ for each $x \in T(R)$. It now follows that $f(T^2(R)) = 0$ since $T' = T(R) = \bigcap T_x$ where $x \in T(R)$. By Pontryagin duality, $T^2(R) = (0)$. Finally since $J(R)$ contains all nilpotent ideals, our theorem is established.

The next example shows that in locally compact rings, the connectedness of the W-radical is mandatory.

EXAMPLE. Let $Z[x]$ be the ring of polynomials over the integers and let $R = Z[x]/(2x)$. Topologize R with the discrete topology. Since $x + x \equiv 0 \pmod{2x}$, x has torsion. Hence $T(R) \neq (0)$. Moreover, $T^2(R) \neq (0)$ because $x^2 \not\equiv 0 \pmod{2x}$. (Recall that $T(R)$ contains the additive torsion subgroup of R.)

Nonconnected locally compact rings R having the property that $T(R) \neq (0)$ and $T^2(R) = (0)$ are plentiful and easy to construct.

The following two results whose contents are undeniably known follow as corollaries to 3.2. We present them here since they are derived intrinsically from the W-radical.

COROLLARY. [Cf. 2.] *Every nontrivial compact ring R containing no divisors of zero is totally disconnected.*

Proof. We need only remark that R is either connected or totally disconnected [5; p. 169]. The remainder of the proof follows directly from 3.2.

COROLLARY. *If R is any topological ring with K compact, then $K^2 = (0)$.*

Proof. Since K is compact, $T(K) = K$; by 3.2 $T^2(K) = K^2 = (0)$.

The next result reveals an interesting interplay between algebraic nilpotence and K.

THEOREM 3.3. *Let R be a locally compact ring with $T(R)$ connected. If K is compact, then topological nilpotence is equivalent to algebraic nilpotence.* [Cf. *to the remark following* 2.4.]

Proof. Let $x \in R$ be topologically nilpotent so that $\lim x^n = 0$. By 2.4 $x^n \in T(R)$ for all n sufficiently large. By 3.2 $(x^n)^2 = x^{2n} = 0$; hence, x is algebraically nilpotent. The converse implication is trivial.

The pathological behavior relative to the W-radical and the Jacobson radical is in general somewhat surprising. We conclude this section by citing three examples illustrating this peculiar behavior.

EXAMPLE. A topological ring R in which $T(R) = (0)$ and $J(R) = R$.

Let R' be any ring with torsion subgroup (0), and let $R = J(R')$. Give R the discrete topology. Then $J(R) = R$ and $T(R) = (0)$ [8; Th. 4.1].

EXAMPLE. A topological ring R in which $T(R) = R$ and $J(R) = (0)$.

Let R be any finite field with the discrete topology. Since R is compact, $T(R) = R$; since R is a field $J(R) = (0)$.

EXAMPLE. A topological ring R in which $T(R) \neq J(R) \neq R$.

Let R be the ring of formal power series over the real numbers $\boldsymbol{R}$. Consider R as a discrete topological ring. It is well known that $J(R) = (x)$, the set of elements having constant term equal to zero. Now define a family of continuous real valued group homomorphisms $f_n: R \to \boldsymbol{R}$ by $f_n(\sum a_i x^i) = a_n$. If $p(x) = \sum b_i x^i \in T(R)$ and $n \geqq 0$ is arbitrary, it follows by [9] that $f_n(p(x)) = b_n = 0$. Let $\{f_\alpha\}$ be the collection of all real valued continuous group homomorphisms on R with $K_\alpha = \ker f_\alpha$ for each α. Then $T(R) = \bigcap K_\alpha \subseteqq \bigcap_{n=1}^{\infty} K_n$ where $K_n = \ker f_n$ for each $n \geqq 1$. Hence, $p(x) = 0$ and therefore,

$$T(R) = (0) \subsetneqq (x) = J(R) \subsetneqq R .$$

4. **Locally compact simple rings.** Locally compact non-discrete fields are completely characterized in the literature [7]. Using such a characterization one is easily able to classify those fields which are W-radical and those which are W-radical free. In this section we generalize this method of classification to locally compact simple rings containing no divisors of zero; we do this intrinsically without appealing to the inherent structure of the ring. Our main instrument here is Theorem 3.2. We begin by eliminating the discrete case.

THEOREM 4.1. *For a discrete simple ring R without divisors of zero, $T(R) = (0)$ if and only if* Char $R = 0$.

Proof. Assume $T(R) \neq (0)$ and pick a nonzero $x \in T(R)$. Since R is simple, $T(R) = R$. By [8; Th. 4.1] R is an additive torsion group and therefore has nonzero characteristic. Conversely, if $T(R) = (0)$ and if $x \in R$ is nonzero, then $nx \neq 0$ for all $n \in Z^+$ since $T(R)$ contains the torsion subgroup of R. This shows that Char $R = 0$.

Every locally compact ring without divisors of zero is either connected or totally disconnected [5; p. 169]. In view of this result and Theorem 4.1, our next theorem completely characterizes the W-radical of the rings in question.

THEOREM 4.2. *Let R be a nondiscrete locally compact simple ring without divisors of zero. Then*

(1) *If R is connected, $T(R) = (0)$.*

(2) *If R is totally disconnected, $T(R) = R$.*

Proof. (1) Since R is connected, so is $T(R)$; by 3.2 $T^2(R) = (0)$. Since R has no zero divisors, $T(R) = (0)$ obtains.

(2) If R is totally disconnected, then R contains a compact open subgroup U. Now since U is compact, $U \subseteqq T(R)$ ($T(R)$ contains all compact subgroups). Finally, R is nondiscrete implies $U \neq (0)$ whence, $T(R) = R$.

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