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Continuous Restrictions of Marczewski Measurable Functions

The proofs of the new results announced here will appear in [1]. We study theorems about functions from the unit interval I = [0,1] into the reals, R. c denotes the cardinality of the continuum, and CH refers to the Continuum Hypothesis.

The measurable functions we will be interested in are defined in terms of the following σ-algebras of subsets of a complete metric space X which has no isolate points:

B_w: Baire property in the wide sense [14],

B_r: Baire property in the restricted sense [14],

L: Lebesgue measurable sets (assuming X is the reals),

U: Universally measurable sets (a set M is universally measurable if it is measurable with respect to the completion of every Borel measure on X),

(s): Marczewski measurable sets (a set M is Marczewski measurable provided that for every perfect subset P of X, there exists a perfect subset Q of P which either misses M or is a subset of M), and

B: Borel measurable.

The Marczewski measurable sets are most easily visualized as follows. Let the statement that a set M is "Bernstein dense" in a set P mean that M intersects every perfect subset of P. Then, a set M is Marzcewski measurable or (s)-measurable provided there is no perfect set P in which both M and its complement are Bernstein dense (we would say that $M \cap P$ is one half of a

"Bernstein subdivision of P" if both M and its complement were Bernstein dense in P). Property (s) for sets was defined by Marczewski in [16], where he established their basic properties and showed that the (s)-measurable functions were the same as the class of functions (studied by Sierpinski in [19]) f which are such that for every perfect set P, there exists a perfect set $Q \subset P$ such that $f|Q$ is continuous.

It is well known that these measurablilty properties are related to each other according to the following diagram of implications:

$$
\begin{tikzcd}
 & & L \\
 & U \arrow[ur] \arrow[dr] & \\
B \arrow[ur] \arrow[dr] & & (s) \\
 & B_r \arrow[ur] \arrow[dr] & \\
 & & B_w
\end{tikzcd}
\tag{I}
$$

We will have occasion to refer to the σ-ideals associated with these σ-algebras:

FC: first category sets,

AFC: always first category sets [14],

L_0: Lebesgue measure zero sets,

U_0: universal null sets (a set M is a universal null set provided it has measure zero relative to the completion of every continuous Borel measure on X),

(s^0): Marczewski null sets (a set M is a Marcewski null set provided it is true that for every perfect subset P of X, there exists a perfect subset Q of P which misses M), and count: countable sets.

These singularity properties can be thought of as **hereditary** B_w, B_r, U, L, (s), and B, respectively, and therefore fit into the following diagram of implications:

(II)
$$
\begin{tikzcd}
 & & L_0 & \\
 & U_0 \arrow[ur] \arrow[dr] & & \\
\text{count} \arrow[ur, dashed] \arrow[r] \arrow[dr] & (c-) \arrow[r] & (s^0) \arrow[r] & TI \\
 & AFC \arrow[ur] \arrow[dr] & & \\
 & & FC &
\end{tikzcd}
$$

(c-) means of cardinality less than c, and TI or "totally imperfect" means that the set contains no perfect subset.

If P is one of the singularity properties in the above diagram, we will say that a set D is "non-P dense in an interval J" if every subinterval of J intersects D in a non-P set. However, we will say that D is "uncountably dense in J" rather than "non-count dense in J", that D is "categorically dense in J" rather than "non-FC dense in J", that D is "c-dense in J" rather than "non-(c-) dense in J", and that D is "perfectly dense in J" rather than "non-TI dense in J", since this latter phenomenon occurs only if every subinterval of J intersects D in a set which contains a perfect set.

We are interested in the following categoric notions of "bigness" of subsets D of I:

(1) D residual in I (i.e. I-D is FC),

(2) D categorically dense in I,

(3) D non-FC and uncountably dense in I,

(2') D perfectly dense in I,

(3') D (non-s^0)-dense in I,

(4+) D c-dense in I,

(4) D uncountably dense in I,

(5+) D of cardinality c and dense in I.

(5) D uncountable and dense in I, and

(6) D dense in I.

We know these notions are related as follows,

(III)
$$\begin{tikzcd}
(1) \arrow[r, dashed] \arrow[dr] & (2') \arrow[r] & (3') \arrow[r] & (4+) \arrow[r] \arrow[dr] & (5+) \arrow[dr] & & \\
& (2) \arrow[r] & (3) \arrow[rr] & & (4) \arrow[r] & (5) \arrow[r] & (6)
\end{tikzcd}$$

We are also interested in the following measure theoretic notions of "bigness" of subsets D of I:

(7) D of outer measure 1,

(8) D non-L_0 dense in I,

(9) D non-L_0 dense in some subinterval of I,

(10) D non-L_0, and

(7+), (8+), (9+), and (10+), are the same as (7) through (10), respectively, except that D is assumed to be measurable.

We know these notions are related as follows:

(IV)
$$\begin{tikzcd}
(7+) \arrow[rr] \arrow[dr] & & (8+) \arrow[rr] \arrow[dr] & & (9+) \arrow[rr] \arrow[dr] & & (10+) \arrow[dr] & \\
& (7) \arrow[rr] & & (8) \arrow[rr] & & (9) \arrow[rr] & & (10)
\end{tikzcd}$$

Lusin proved the following theorem in 1912 [15]:

Theorem 1: For every L-measurable $f : I \to R$, there exists $D \subset I$, D of positive measure (10+), such that $f|D$ is continuous.

Nikodym proved the following category version of Lusin's theorem in 1929 [18]:

Theorem 2: For every B_w-measurable $f : I \dashrightarrow R$, there exists $D \subset I$, D residual in I (1), such that $f|D$ is continuous.

Kuratowski [13] extended Theorem 2 to the metric case, where the range space is separable, and the case where the range space is non-separable has recently been established in [4] and [5].

It is well-known that you can make the set D of Lusin's Theorem have as large a positive measure less than 1 as desired, but Theorems 1 and 2 are best possible in terms of having the set D satisfy one of the properties of (III) or (IV).

For example, it is well known that you cannot make the set D of Lusin's theorem satisfy (9) or simultaneously satisfy (10) and (6), even for B-measurble f.

Nor can you make the set D of the Nikodym-Kuratowski theorem satisfy (10+) or (under CH) (10).

Ceder [3] recently gave an example which showed that you cannot make the set D of Lusin's Theorem satisfy (5+) or (under CH) (5).

Ceder asked in [3], "Are there 'nice' kinds of functions, f, not having the property of Baire, for which there exists a dense subset D of I with D uncountable such that f|D is continuous?" We show that the Marczewski measurable or (s)-measurable functions form such a class and that an even stronger result holds.

Theorem 3: For every (s)-measurable $f : I \dashrightarrow R$, there exists $D \subset I$, D perfectly dense in I (2'), such that $f|D$ is continuous.

Assuming CH, you cannot make the set D of Theorem 3 satisfy

(10+) or (1).

It is still not known if Theorem 3 is best possible in terms of having D satisfy one of the properties of (III) or (IV).

Problem 1: Can you show (even under CH) that the set D of Theorem 3 cannot be made to satisfy (2)? Note that if you could make D satisfy (3) or just be non-FC, then repeated applications of the Banach Category Theorem would show that it could be made categorically dense in I (2).

Problem 2: Can you show (even under CH) that the set D of Theorem 3 cannot be made to be non-L_0 (10)?

It follows as a corollary to Theorem 3 that the following strengthened version of Lusin's Theorem holds for U-measurable f:

Theorem 4: For every U-measurable f : I --> R, there exists $D_1 \subset I$ and $D_2 \subset I$, D_1 perfectly dense in I (2') and D_2 of positive measure (10+), such that $f|D_1$ and $f|D_2$ are continuous.

You cannot find a single set D which will accomplish both jobs simultaneously in Theorem 4, even for B-measurable f. Under CH, it also follows that you cannot make the set D of Theorem 4 satisfy (1).

Problem 3: Can the set D for U-measurable functions be made to be non-FC? A negative solution to this problem also solves Problem 1, and the author conjectures that this is the way things will turn out.

The conclusion to Theorem 2 obviously holds for B_r-measurable functions.

Problem 4: For B_r-measurable f, does there necessarily exist a non-L_0 set D such that $f|D$ is continuous? A negative answer to this question also solves Problem 2, and the author conjectures that this is the way things will turn out.

Problem 5: Several of the examples announced above rely on the use of Lusin or Sierpiniski sets, whose existence depends on the CH assumption. It would be preferable to obtain the examples without assuming CH. This might be a rather tall order. The existence of B_r sets which are not L sets, or U sets which are not B_w sets, is not known in ZFC, as far as this author knows. Indeed, the related examples given in the 1976 edition of Kuratowski's and Mostowski's book [14] still rely on CH. Grzegorek and Ryll-Nardzewski [5] - [12] have recently made remarkable progress in obtaing related results in ZFC which were previously only known under CH. See [2] and [17] for expository treatments of this subject.

NOTE: Problems 1 - 4 were solved during the Symposium by Karel Prikry and the author.

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