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L. Zajíček, Charles University, Sokolovská 83, 186 00 Praha 8, Czechoslovakia

UNPUBLISHED RESULTS OF K. PEKÁR AND H. ZLONICKÁ ON PREPONDERANT DERIVATIVES AND M$_4$ - SETS

These results were obtained in 1981 thesis of K. Pekár [8] and in a 1982 student work of H. Zlonická (now Mrs. H. Palovská) written under the direction of the author of this note. They seem to be of some interest but, for various reasons, were not published.

1 The results of K. Pekár on preponderant derivatives

Pekár in his thesis answered some questions mentioned in [5] and [1]. We shall start with the definitions of notions involving preponderance.

The symbols λ, λ_e and λ_i will stand for the Lebesgue, the outer Lebesgue and the inner Lebesgue measure, respectively.

__Definition 1.__ We shall say that $M \subset R$ is preponderant at $a \in R$ if $\liminf_{h\to 0+} \frac{\lambda_i(M\cap(a,a+h))}{h} > 1/2$ and $\liminf_{h\to 0+} \frac{\lambda_i(M\cap(a-h,a))}{h} > 1/2$. We say that $E \subset R$ is weakly preponderant at a if $\frac{\lambda_i(E\cap I)}{\lambda(I)} > 1/2$ for all sufficiently small intervals I containing a.

__Definition 2.__ Let f be a real function, $a \in R$ and $A \in R \cup \{-\infty, \infty\}$. We say that

(i) (s)-lim $\mathrm{pr}_{x\to a} f(x) = A$ if there is a set E preponderant at a such that $\lim_{x\to a, x\in E} f(x) = A$,

(ii) lim $\mathrm{pr}_{x\to a} f(x) = A$ if $\{x : f(x) > r\}$ ($\{x : f(x) < s\}$) is preponderant at a for each $r < A$ ($s > A$),

(iii) (w)-$\lim \mathrm{pr}_{x \to a} f(x) = A$ if $\{x : f(x) > r\}$ ($\{x : f(x) < s\}$) is weakly preponderant at a for each $r < A$ ($s > A$).

Using these three notions of preponderant limits, three kinds of the preponderant continuity (strong preponderant continuity, preponderant continuity and weak preponderant continuity) and three kinds of the preponderant derivatives ((s)-f'_{pr}, f'_{pr} and (w)-f'_{pr}) are defined in the obvious way.

Note that (w)-f'_{pr} was considered by Denjoy [4], f'_{pr} by Leonard [5] and (s)-f'_{pr} by Bruckner [1].

Pekár proved the following theorem which answers in positive a question posed in [5], p. 763.

Theorem 1. Let f be weakly preponderantly continuous on an interval. Then f is Darboux and in Baire class 1.

Note that O'Malley [7] proved a theorem slightly weaker than Theorem 1, since he used a stronger notion of preponderant continuity. Pekár proved Theorem 1 by a slight modification of O'Malley's proof. The fact that a weakly preponderantly continuous function is in Baire class 1 also easily follows from a general Thomson's theorem ([10], Theorem 33.1.) on the semi-continuity with respect to local systems.

Theorem 1 implies that the Leonard's monotonicity theorem ([5], Theorem 1) remains true if the weak preponderance is considered. Note that the proof of Lemma 3 from [5] is probably incorrect, but [8] contains an alternative proof.

The following theorem is the main result of [8].

Theorem 2. If f has on an interval a preponderant derivative (finite or infinite), then f'_{pr} is in Baire class 1.

Sketch of the proof. Let $a < b$. Choose $a < c < b$. If $f'_{pr}(x) < c$ then we can choose $1 > \delta_x > 0$ and $1/2 > \eta_x > 0$ such that for all $h_1, h_2 \geqq 0$, $h_1 + h_2 > 0$, $h_1 < \delta_x$, $h_2 < \delta_x$ we have

$$\lambda_e(\{z \in (x - h_1, x + h_2) : (f(z) - f(x))(z - x)^{-1} \geqq c\}) < (1/2 - \eta_x)(h_1 + h_2).$$

If we put

$$H = \bigcap_{n=1}^{\infty} \bigcup_{f'_{pr}(x) < c} (x - n^{-1}\eta_x\delta_x, x + n^{-1}\eta_x\delta_x),$$

we see that H is a G_δ-set and $\{x : f'_{pr}(x) \leqq a\} \subset H$. Pekár, using ideas of [9], has proved that $H \subset \{x : f'_{pr}(x) \leqq b\}$. Consequently Theorem 2.12 of [6] implies that f'_{pr} is in Baire class 1.

Note that the fact that the strong preponderant derivative (s)-f'_{pr} is in Baire class 1 is mentioned in [3], p. 113. The same result for a notion of a preponderant derivative (incomparable with f'_{pr}, stronger than (w)-f'_{pr} and weaker than (s)-f'_{pr}) is proved in [6], p. 298. It is probably not known whether the result holds for (w)-f'_{pr}.

Theorem 2 implies that the assumption that f'_{pr} is in Baire class 1 is superfluous in Theorem 10 and Theorem 11 of [5]. Thus the preponderant derivative f'_{pr} of a Darboux function f in Baire class 1 has the Zahorski property M_2 and also the Denjoy property.

On the other hand, Pekár constructed the following easy example.

Example 1. There exists a continuous function f on $(0,1)$ which has a bounded strong preponderant derivative (s)-f'_{pr} and (s)-f'_{pr} has not the Zahorski's M_3-property.

A slightly more difficult construction answers Bruckner's problem (4) of [1], p. 10.

Example 2. There exists a Lipschitz monotone function f on $(0,1)$ which has a strong preponderant derivative (s)-f'_{pr} at each point and the approximate derivative exists at no point of the Cantor ternary set C.

Construction. Let $I_n = (a_n, b_n)$, $n = 1, 2, \ldots$, be intervals contiguous to C. Put $c_n = (b_n - a_n)/20$ and define a function g such that

(i) if $x \in C$ or $x \in (a_n, a_n + 8c_n] \cup [a_n + 12c_n, b_n)$, then $g(x) = 0$,

(ii) if $x \in [a_n + 9c_n, a_n + 11c_n]$, then $g(x) = (b_n - a_n)/2$,

(iii) g has a continuous derivative with $|g'| \leqq 20$ on each (a_n, b_n).

It is easy to prove that $f(x) = g(x) + 30x$ has all the desired properties.

2 The result of H. Zlonická on M_4-sets.

This result which gives a relatively simple characterization of Zahorski's M_4-sets is based on a theorem of Bruckner and Thorne [2].

The density (right density) of a set $M \subset R$ at $x \in R$ will be denoted by $d(M,x)(d_+(M,x))$. Now recall the definition of Zahorski's M_4-sets.

Definition 3. Let E be an F_σ-set. We say E belongs to class M_4 if there exists a sequence of closed sets $\{K_n\}$ and a sequence of positive numbers $\{\eta_n\}$ such that $E = \cup K_n$ and for each $x \in K_n$ and each $c > 0$ there exists a number $\varepsilon(x,c) > 0$ such that if h and h_1 satisfy $hh_1 > 0$, $h/h_1 < c$ and $|h + h_1| < \varepsilon(x,c)$, then $\frac{\lambda(E \cap (x+h, x+h+h_1))}{|h_1|} > \eta_n$.

The following definition is new.

Definition 4. We say that an F_σ-set M is a D-set if there exists $d > 0$ such that for each $x \in M$ there exists a set $M_x \subset M$ such that $d(M_x, x) = d$.

The result obtained by H. Zlonická is the following.

Theorem 3. A set $X \subset R$ is an M_4-set iff it is a countable union of D-sets.

Sketch of the proof. By Theorem 1 of [2], for a measurable set $A \subset R$ and $0 \leqq r \leqq 1$, the following conditions are equivalent:

(1) there exists a set $C \supset A$ such that $d_+(C,0) = r$,

(2) $\limsup_{b \to 0+} \frac{\lambda(A \cap (b(k-1)/k, b))}{b/k} \leqq r$ for every natural k.

Using this result, it is easy to see that for an $x \in R$ the following conditions are equivalent:

(3) there exists a set $D \supset A$ such that $d(D,x) = r$,

(4) for each $c > 0$ there exists $\varepsilon > 0$ such that if h and h_1 satisfy $hh_1 > 0$, $h/h_1 < c$ and $|h + h_1| < \varepsilon$, then $\frac{\lambda(A \cap (x+h, x+h+h_1))}{|h_1|} \leqq r$.

Using this fact for the complement of a measurable set $B \subset R$, we obtain that for each $0 \leqq s \leqq 1$ and $x \in R$ the following conditions are equivalent:

(5) there exists a set $E \subset B$ such that $d(E,x) = s$,

(6) for each $c > 0$ there exists $\varepsilon > 0$ such that if h and h_1 satisfy $hh_1 > 0$, $h/h_1 < c$ and $|h + h_1| < \varepsilon$, then $\frac{\lambda(B \cap (x+h, x+h+h_1))}{|h_1|} \geqq s$.

Therefore it is clear that each D_σ-set is an M_4-set. Now, if E is an M_4-set and K_n are the sets from Definition 3, we can find an F_σ-set $T \subset E$ such that $\lambda(E - T) = 0$ and T has the density 1 at each of its points, and put $D_n = K_n \cup T$. Then $E = \cup_{n=1}^{\infty} D_n$ and, since (5) is equivalent to (6), all D_n are D-sets.

References

[1] A.M. Bruckner, Some observations about Denjoy's preponderant derivative, Bull. Math. Soc. Sci. Math. R. S. Roumanie 21 (69) (1977), 1-10.

[2] A.M. Bruckner and Ch.L. Thorne, Density covers, Ann. Scuola Norm. Sup. Pisa 27 1973, 97-104.

[3] A.M. Bruckner, R.J. O'Malley and B.S. Thomson, Path derivatives: A unified view of certain generalized derivatives, Trans. Amer. Math. Soc. 283 (1984), 97-126.

[4] A. Denjoy, Mémoire sur la totalisation des nombres dérivées non-sommables, Ann. École Norm. Sup. 33 (1916), 127-236; 34 (1917), 181-238.

[5] J.L. Leonard, Some conditions implying the monotonicity of a real function, Rev. Roumaine Math. Pures Appl. 17 (1972), 757-780.

[6] J. Lukeš, J. Malý and L. Zajíček, Fine topology methods in real analysis and potential theory, Lecture Notes in Mathematics 1189, Springer 1986.

[7] R.J. O'Malley, Note about preponderantly continuous functions, Rev. Roumaine Math. Pures Appl. 21 (1976), 335-336.

[8] K. Pekár, Preponderant continuity and preponderant derivatives, Thesis, Charles University, Prague, 1981.

[9] D. Preiss, Approximate derivatives and Baire classes, Czechoslovak Math. J. 21 (96) (1971), 373-382.

[10] B.S. Thomson, Real Functions, Lecture Notes in Mathematics 1170, Springer 1985.

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