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THE CONTRIBUTIONS OF ANTONI ZYGMUND TO REAL ANALYSIS

At the beginning of Zygmund's career he wrote a few papers on real analysis which were not directly related to trignometric series. It is interesting to note, however, that these papers and the few such that he wrote later in life were always joint, the problems presumably arising from a confluence of interests of the parties.

Typical of these are the paper of 1923 with Sierpinski showing that there exist functions discontinuous on every set having the power of the continuum and another with Saks in 1924 on the sheaf of tangent lines from a point to a planar curve[1] Another such result appears in Saks's *Theory of the Integral,* noted (in the original French version) as Zygmund's Lemma, and used in the proof of the Fundamental Theorem of the Calculus for the Perron integral.

Most of Zygmund's contributions are subsumed under the rubric of "differentiation theory", which is a natural consequence of his primary interests in trigonometric series. The sequence of partial sums $\{s_n(x)\}$ of the trigonometric Fourier series of a function f is transformed by a summability method into a sequence $\{\sigma_n(x)\}$ and the difference $\sigma_n(x) - f(x)$ is represented by the convolution of a symmetric difference and a summability kernel. The simplest example is the case where the method is convergence itself and $s_n(x) - f(x)$ is essentially the convolution of $f(x+t)+f(x-t)-2f(x)$ and $(\sin nt)/t$. Similarly, his work in 1934 and his joint paper of 1935 (with Jessen and Marcinkiewicz) on the differentiability of multiple integrals arose from problems in the summability of double Fourier series.

In the study of trigonometric series, the conjugate series is of great importance; in the case of the Fourier series of a function, the associated conjugate function was the center of much of Zygmund's work. In 1929 he extended the result of M. Riesz on the integrability of the conjugate function from the class L^p, $p > 1$, to the class $L \log L$, which proved to be an essential ingredient of much of his later work. In the same paper he also introduced the

[1]We will refer to papers only by their dates of publication. The exact references can be found by consulting the bibliography of Zygmund published in his *Selected Papers.*

spaces L_φ, which are now called Orlicz spaces although the paper of Orlicz appeared in 1931. He explored the relationships between many derivatives; the Peano, Schwarz, Borel, Riemannian and approximate derivatives were all considered. In a particularly deep paper of 1936 with Marcinkiewicz, although most of the paper dealt with differentiation, he defined the T integral of Perron type. With the T integral he solved a problem of Denjoy by showing that if a trigonometric series converges everywhere to a function f, then the series is the T-Fourier series of f.

The notion of smoothness which appears in 1945 is is a generalization of symmetric continuity and Λ_* is a generalization of the the class of integrals of $Lip1$ functions. In this paper he also studies fractional integrals and derivatives. Many of these results were generalized to n-dimensions with Calderon in 1954, and the notion of smoothness is applied to explain a result of Salem in a paper with Mary Weiss in 1959. He returned to the basic questions in the 1945 paper in a series of deep papers with E. M. Stein in 1961, 1964 and 1965.

In the 1952 paper with Calderon on the existence of singular integrals, a covering lemma is of critical importance. This lemma provides the basis for the decomposition of a function into its "good" and "bad" parts. These techniques are featured in several papers published with Calderon in the 50's and early 60's. The most notable may be the 1961 paper on local properties of solutions of elliptic PDE's.