ON THE RIEMANN-CESÀRO SUMMABILITY OF SERIES AND INTEGRALS

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1. Introduction and notation. Otto Szász ([5], p. 1139, Theorem 4 and p. 1223, Theorem 1) has proved Tauberian theorems for series, involving the passage from Abel or (A) summability to each of the Riemann summabilities (R, 1) and (R_1), included in the statement:

THEOREM A. *If* $\sum_{k=1}^{\infty} a_k$ *is summable* (A) *to a finite value* l, *and*

$$(T_A) \qquad \text{either } \sum_{k=1}^{n} k|a_k| = O(n), \text{ or } \sum_{k=n}^{2n} (|a_k| - a_k) = O(1), \quad n \to \infty,$$

then $\sum_{k=1}^{\infty} a_k$ *is summable* (R, 1) *to* l *and also summable* (R_1) *to* l.

It is known ([5], p. 1139, Lemma 1) that the second alternative of condition (T_A) along with the summability (A) of $\Sigma\, a_k$ implies the first alternative of (T_A); and so Theorem A need be stated with only the first alternative of (T_A) which Szász uses in his proof of Theorem A without however, explicitly mentioning it as an alternative hypothesis. The main object of this paper is to establish two results: (i) Theorem I′(A) at the end, which is a generalization of Theorem A with the first alternative of hypothesis (T_A), for the Riemann-Cesàro summability (R, p, α) recently defined by Hirokawa ([2], § 1) whose case $p = 1$, $\alpha = -1$ is summability (R, 1) and case $p = 1$, $\alpha = 0$ is summabity (R_1), (ii) an integral analogue of Theorem I′ (A) stated as Theorem I (A) in the last section.[1]

The notation and the definitions used in Theorem I(A) and other integral theorems are as follows. For a real function $a(u)$ bounded and integrable[2]

1) It must be borne in mind that the parallelism between series and integrals is destroyed to some extent by instances of theorems for series, such as the limitation theorem for series summable (C, α), $\alpha > -1$ ([1], Theorem46), which have no integral analogues. Thus one of Hirokawa's general theorems ([2], Theorem 3) has no integral analogue which can be proved by his method since it depends on the limitation theorem referred to. On the other hand, a theorem for integrals, such as Theorem I (A) of this paper, may present additional complications when we try to adapt its proof to obtain its analogue for series. It may be added here that analogous theorems or formulae for integrals and for series, wherever they occur in this paper, bear the same number, unaccented (e.g. I, 1 etc.) or accented (e.g. I′, 1′ etc.), according as the theorems or the formulae are for integrals or for series.

2) As in Hardy [1], integrability is in the Lebesgue sense and every integral $\int_0^\infty$ is defined in the Cauchy-Lebesgue sense as $\lim_{x\to\infty} \int_0^x$.

in every finite interval of $u \geqq 0$, the Cesàro sum of order $\alpha \geqq -1$ and the Cesâro mean of order $\alpha \geqq -1$ are defined by the integrals

$$(1.1) \qquad s_\alpha(u) = \frac{1}{\Gamma(\alpha+1)} \int_0^u (u-x)^\alpha\, a(x)\, dx \text{ if } \alpha > -1, \; s_{-1}(u) = a(u), \; u \geqq 0,$$

nd

$$(1.2) \qquad \sigma_\alpha(u) = \Gamma(\alpha+1) \frac{s_\alpha(u)}{u^\alpha} \quad \text{if } \alpha > -1, \; \sigma_{-1}(u) = u s_{-1}(u), \; u \geqq 0,$$

respectively. The Riemann-Cesàro transform of $a(u)$, of positive integral order p with index $\alpha \geqq -1$, is defined by

$$(1.3) \qquad \rho(p, \alpha, t) = \frac{1}{C_{p,\alpha}} t^{\alpha+1} \int_0^\infty \left(\frac{\sin tu}{tu}\right)^p s_\alpha(u)\, du, \; 0 < t < t_0,$$

where

$$(1.4) \qquad C_{p,\alpha} = \begin{cases} \dfrac{1}{\Gamma(\alpha+1)} \displaystyle\int_0^\infty u^{\alpha-p} \sin^p u\, du & \text{if } \begin{cases} \text{either} & -1 < \alpha < p-1 \\ \text{or} & \alpha = 0, \; p = -1, \end{cases} \\ 1 & \text{if } \; \alpha = -1. \end{cases}$$

and the Laplace-Abel transform of $a(u)$ by

$$(1.5) \qquad j(t) = \int_0^\infty e^{-tu}\, a(u)\, du, \qquad t > 0.$$

The integrals in (1.3) and (1.5) are each supposed to exist for every fixed t in the interval noted against each. There is a similarity between them in that the existence of $j(t)$ for every $t > 0$ enables us to express it as an absolutely convergent integral in the form ([6], Chapter II, proof of Theorem 8.1):

$$(1.6) \qquad j(t) = t^{\alpha+1} \int_0^\infty e^{-tu} s_\alpha(u)\, du, \qquad \alpha \geqq 0.$$

The different kinds of summability of $s(u) = \int_0^u a(x)\, dx$ required in Theorem I(A) and elsewhere are the following, l in each (as everywhere in this paper) denoting a finite number: —

$$(1.7) \begin{cases} \text{summability } (C, \alpha), \; \alpha > -1, \text{ to } l: \lim\limits_{u\to\infty} \sigma_\alpha(u) = l, \\ \qquad\qquad\qquad\qquad \text{written } s(u) \to l \; (C, \alpha), \\ \text{summability (A) to } l: \lim\limits_{t\to+0} j(t) = l, \\ \qquad\qquad\qquad\qquad \text{written } s(u) \to l \, (A), \end{cases}$$

summability (R, p, α) to l : $\lim_{t\to+0} \rho(p, \alpha, t) = l$,

written $s(u) \to l\,(\mathrm{R}, p, \alpha)$.[3]

An additional definition needed in the case of a particular result for $s(u)$, namely, Corollary I(1), is that of (C, ∞) summability which follows.

$$\limsup_{u\to\infty} \sigma_\alpha(u), \qquad \liminf_{u\to\infty} \sigma_\alpha(u)$$

whether finite or not, are defined for every $\alpha \geqq -1$; and they are, for varying $\alpha \geqq 0$, the first a monotonic decreasing function of α and the second a monotonic increasing function of α. Consequently

$$\bar{\sigma}_\infty = \lim_{\alpha\to\infty} \limsup_{u\to\infty} \sigma_\alpha(u), \quad \underline{\sigma}_\infty = \lim_{\alpha\to\infty} \liminf_{u\to\infty} \sigma_\alpha(u)$$

exist, and, whether they are finite or not, $\bar{\sigma}_\infty \geqq \underline{\sigma}_\infty$. We define for $s(u)$

(1.8) summability (C, ∞) to l : $\bar{\sigma}_\infty = \underline{\sigma}_\infty = l$, writing $s(u) \to l\,(\mathrm{C}, \infty)$.

For a series $\sum_{k=1}^{\infty} a_k$ supposed to be real in this paper, there are the usual analogues of definitions (1.1)—(1.8). For instance,

$$(1.1') \qquad s_n^\alpha = \sum_{k=1}^{n} \binom{n-k+\alpha}{\alpha} a_k \quad \text{for } \alpha > -1, \quad s_n^{-1} = a_n,$$

and

$$(1.2') \qquad \sigma_n^\alpha = \frac{s_n^\alpha}{A_n^\alpha} = s_n^\alpha \Big/ \binom{n+\alpha}{\alpha} \quad \text{for } \alpha > -1, \quad \sigma_n^{-1} = n s_n^{-1}$$ [4]

are the Cesàro sums and the Cesàro means respectively of order $\alpha \geqq -1$ of $\sum a_k$; while the Riemann-Cesàro transform, of positive intgral order p and index $\alpha \geqq -1$, is

$$(1.3') \qquad \rho(p, \alpha, t) = C_{p,\alpha}^{-1}\, t^{\alpha+1} \sum_{k=1}^{\infty} \left(\frac{\sin kt}{kt}\right)^p s_k^\alpha, \quad 0 < t < t_0,$$

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3) *The method of summability* (R, p, α) *for* $s(u)$ *is regular if either* (i) $p > \alpha+1 \geqq 1$, *or* (ii) $p > 1 > \alpha+1 \geqq 0$. For case (i) the proof is that, since $\alpha \geqq 0$, $s(u) \to l\,(u \to \infty)$ implies $\sigma_\alpha(u) \to l$ and the existence of

$$\rho(p, \alpha, t) = \{C_{p,\alpha}\Gamma(\alpha+1)\}^{-1} \int_0^\infty \left(\frac{\sin u}{u}\right)^p u^\alpha \sigma_\alpha(u/t)\, du, \ t > 0,$$

and finally implies $\rho(p, \alpha, t) \to l$ as $t \to +0$ by Lebesgue's theorem of dominated convergence. In case (ii), we argue similarly with

$$-\{C_{p,\alpha}\Gamma(\alpha+2)\}^{-1} \int_0^\infty \frac{d}{du}\left(\frac{\sin u}{u}\right)^p u^{\alpha+1}\, \sigma_{\alpha+1}\,(u/t)\, du, \ t > 0,$$

identifying this integral with $\rho(p, \alpha, t)$ by an integration by parts in which we use $\sigma_{\alpha+1}\,(u) \to l\ (u \to \infty)$.

4) The special definition of σ_n^{-1} is not adopted by Hirokawa and others for the reason that it is inconsistent with the definition of σ_n^α for $\alpha > -1$. However, this definition of σ_n^{-1} enables us to unify in Theorem I′(A) the two cases $\alpha = -1$ and $\alpha > -1$ although the first case often requires more elaborate treatment than the second.

where $C_{p,\alpha}$ is the constant in (1.4). The different kinds of summability of $\sum a_k$ explicitly figuring in this paper are

$$(1.7')^{5)}\quad \begin{cases} \text{summability } (\mathrm{C},\alpha), \alpha > -1 \text{ to } l: \lim_{n\to\infty} \sigma_n^\alpha = l, \\ \qquad \text{written } \sum_1^\infty a_k = l\ (\mathrm{C},\alpha), \\ \text{summability (A) to } l: \lim_{x\to 1-0} \sum_1^\infty a_k x^k = l, \\ \qquad \text{written } \sum_1^\infty a_k = l\ (\mathrm{A}), \\ \text{summability } (\mathrm{R},p,\alpha) \text{ to } l: \lim_{t\to +0} \rho(p,\alpha,t) = l, \\ \qquad \text{written } \sum_1^\infty a_k = l\ (R,p,\alpha). \end{cases}$$

2. Lemmas. In the rest of the paper, the notation and the definitions of the preceding section are used without further explanation. The lemmas which follow are for integrals, and they have series-analogies which are omitted for the reason that there is no special difficulty in either formulating or proving these analogues. In fact, series-analogues of even the subsequent results for integrals are taken for granted unless there is some such difficulty.

The first two lemmas have obvious proofs which are left to the reader.

LEMMA 1. *If $f(u)$ is non-negative and integrable in every finite interval of $u \geqq 0$, $\delta > 0$, $q \neq 0$, then*

$$\int_0^u f(x)\,dx = O(u^{q-\delta}) \textit{ implies } \int_u^c f(x)\,x^{-q}\,dx = O(u^{-\delta}),\ u\to\infty.$$

LEMMA 2. *If $t > 0$, $u > 0$, then*

$$|\phi^*(tu)| \equiv \left|\int_u^\infty \frac{\sin tx}{x}\,dx\right| \leqq \frac{2}{tu}.$$

LEMMA 3. *If $k < 1$, $f(u)$ is continuous in every finite interval of $u \geqq 0$,*

$$F(u) = \int_0^u f(x)\,dx, \textit{ then } f(u) - ku^{-1}F(u) = O(1) \textit{ implies } f(u) = O(1),\ u\to\infty.$$

PROOF. Adapting a method of Hardy's ([1], p. 107), we solve the differential equation for $F(u)$ given by

5) The infinite series involved in the definitions of summability (A) and summability (R,p,α), in (1.7′), are of course supposed to be convergent for the values of x and t, i.e. $|x|<1$ and $0<t<t_0$, which makes the limiting operations of the definitions possible. Another point to be noted is that the method of summability (R,p,α) *for Σa_k is known to be regular for either $p>\alpha+1\geqq 1$ or $p>1>\alpha+1\geqq 0$* ([2], Corollary 1 under Theorem 1) *but not so for $p=1\geqq\alpha+1\geqq 0$* ([2], Theorem 2).

$$F'(u) - ku^{-1}F(u) = g(u)$$

and obtain

$$u^{-k}F(u) = \int^{u} x^{-k}g(x)\,dx + \text{a constant},$$

whence, as a result of our assumption $g(u) = O(1)$, we get successively

$$u^{-1}F(u) = O(1) + O(u^{-1+k}),\ f(u) = ku^{-1}F(u) + O(1) = O(1),$$

LEMMA 4. *If $\alpha > -1$, then*

$$\int_0^u \sigma_\alpha(x)\,dx = O(u) \quad \textit{implies } \sigma_{\alpha+1}(u) = O(1), \qquad u \to \infty.$$

If $\alpha = -1$, the result is still true provided $\sigma_1(u) = O(1)$.

PROOF. For $\alpha > -1$, we have by an integration by parts:

$$\frac{1}{u}\int_0^u \sigma_\alpha(x)\,dx = \frac{1}{\alpha+1}\sigma_{\alpha+1}(u) + \frac{\alpha}{\alpha+1}\frac{1}{u}\int_0^u \sigma_{\alpha+1}(x)\,dx,$$

from which we get the required result by using Lemma 3 with

$$k = -\alpha,\ f(u) = \sigma_{\alpha+1}(u).$$

The case $\alpha = -1$ is obvious since it is simply that $s(u) = O(1)$ when we assume the two conditions:

$$s(u) - \sigma_1(u) = \frac{1}{u}\int_0^u xa(x)\,dx = O(1),\ \ \sigma_1(u) = O(1).$$

LEMMA 5. *If $\alpha \geqq -1$, then*

$$s(u) \to l(\mathrm{A}).\quad \sigma_{\alpha+1}(u) = O_L(1) \ \Rightarrow\ s(u) \to l(\mathrm{C}, \alpha+2)$$

which means that the two hypotheses on the left side, separated by a stop, lead to the conclusion on the right side.

$O_L(1)$ on the left may be replaced by $O_R(1)$, or a fortiori, *by $O(1)$.*

PROOF. The case $\alpha = -1$ is classical ([1], p. 154, integral analogue of Theorem 94). If $\alpha > -1$, one method of proving the result is to note that $s(u) \to l$ (A) implies, by (1.6),

$$\lim_{t\to+0} t^{\alpha+2}\int_0^\infty e^{-tu}\, s_{\alpha+1}(u)\,du = l,$$

and then appeal to a theorem due to Hardy and Littlewood (e.g. [4], pp. 256–7, proof of Theorem t, where r_0 is used instead of $\alpha + 1$).

LEMMA 6. *If $\alpha \geqq -1$, then the two hypotheses*

$$\int_0^u \sigma_\alpha(x)\,dx = O(u), \qquad s(u) \to l\,(\mathrm{A}), \qquad u \to \infty,$$

together imply that

$$\sigma_{\alpha+1}(u) = O(1), \quad \sigma_{\alpha+2}(u) \to l, \qquad u \to \infty.$$

PROOF. If $\alpha > -1$, the conclusion follows at once from Lemmas 4, 5. If $\alpha = -1$, our first hypothesis is that

$$s(u) - \sigma_1(u) = \frac{1}{u} \int_0^u xa(x)\, dx = O(1).$$

Therefore, by a well-known theorem of Szász's ([5], p. 635, Theorem 1; [4], case $r_0 = 0$ of Lemma III), the second hypothesis ensures that $\sigma_1(u) \to l$ and hence also that $s(u) = O(1)$.

LEMMA 7. *If $s(u) \to l$ (C, ∞), then there is an α_0 such that, for $\alpha \geqq \alpha_0$,*

$$\sigma_\alpha(u) = O(1) \qquad \sigma_{\alpha+1}(u) \to l.$$

The lemma is known (e. g. [4], Theorem *t*).

LEMMA 8. *If $\alpha > -1$ and*

$$\sigma_\alpha(u) > -H \quad (H > 0), \quad \sigma_{\alpha+2}(u) = O_R(1)$$

then

$$\sigma_{\alpha+1}(u) = O(1)$$

PROOF. It is obvious from $\sigma_\alpha(u) > -H$ that $\sigma_{\alpha+1}(u) = O_L(1)$. On the other hand, $\sigma_{\alpha+2}(u) = O_R(1)$ implies the existence of a constant $K > 0$ such that, for all large x,

$$Kx^{\alpha+2} > s_{\alpha+2}(x) + H\frac{x^{\alpha+2}}{\Gamma(\alpha+3)} = \int_0^x (x-u)\left[s_\alpha(u) + H\frac{u^\alpha}{\Gamma(\alpha+1)} \right] du$$

$$> \int_0^{x/2} (x-u)\left[s_\alpha(u) + H\frac{u^\alpha}{\Gamma(\alpha+1)} \right] du$$

$$> \frac{x}{2} \int_0^{x/2} \left[s_\alpha(u) + H\frac{u^\alpha}{\Gamma(\alpha+1)} \right] du$$

$$= \frac{x}{2}\left[s_{\alpha+1}\left(\frac{x}{2}\right) + H\frac{(x/2)^{\alpha+1}}{\Gamma(\alpha+2)} \right],$$

i. e. $\quad K > \dfrac{1}{2^{\alpha+2}\,\Gamma(\alpha+2)}[\sigma_{\alpha+1}(x/2) + H]$, or $\sigma_{\alpha+1}(x/2) = O_R(1)$ as $x \to \infty$, so that finally $\sigma_{\alpha+1}(u) = O(1)$ as $u \to \infty$.

LEMMA 9. *If $us(u) > -H(H > 0)$, and either $s(u) \to l$ (C, k), $k > 0$, or $s(u) \to l$ (A),*
then

$$s(u) \to l, \qquad \int_0^u x|a(x)|\, dx = O(u).$$

PROOF. The first conclusion, that $s(u) \to l$, is the result of a classical Tauberian theorem, and it implies

$$\int_0^u xa(x)\,dx = o(u).$$

The second conclusion follows from the above estimate together with the estimate

$$\int_0^u x[\,|a(x)| - a(x)]\,dx < 2\,Hu$$

resulting from the relations

$$x[\,|a(x)| - a(x)] = \begin{cases} 0 & \text{if } a(x) \geqq 0 \\ -2\,xa(x) & \text{if } a(x) < 0 \end{cases} < 2\,H.$$

LEMMA 10. *If $\alpha > -1$ and $s(u)$ is summable $|C, \alpha + 1|$, i.e.*

$$\int_0^\infty |d\sigma_{\alpha+1}(u)| < \infty,$$

then

$$\int_0^u |\sigma_\alpha(x)|\,dx = O(u).$$

If $\alpha = -1$, the result is the familiar one that

$$\int_0^\infty |a(u)|\,du < \infty \quad \textit{implies} \quad \int_0^u x|a(x)|\,dx = o(u).$$

PROOF. For $\alpha > -1$, we have successively

$$(\alpha + 1)^{-1}\,x\frac{d}{dx}\,[\sigma_{\alpha+1}(x)] = \sigma_\alpha(x) - \sigma_{\alpha+1}(x) \quad \text{for almost all } x,$$

$$\int_0^u |\sigma_\alpha(x)|\,dx \leqq \int_0^u |\sigma_{\alpha+1}(x)|\,dx + (\alpha + 1)^{-1}\int_0^u x|\,d\sigma_{\alpha+1}(x)|$$

$$\leqq \int_0^u |\sigma_{\alpha+1}(x)|\,dx + (\alpha + 1)^{-1}\,u\int_0^u |d\sigma_{\alpha+1}(x)| = O(u).$$

3. Tauberian theorems connecting Cesàro summability with Riemann-Cesàro summability. Theorems I, I′ of this section are similar to the following theorem which is mainly due to Obreschkoff and Hirokawa ([2], Theorem 1) and which, in fact, has a case in common with Theorem I′.

THEOREM B. *If $0 < \delta < 1 \leqq p =$ a positive integer,*

$$\sum_{k=1}^\infty a_k = l\,(C, p - \delta),$$

$$(T_B) \qquad \sum_{k=1}^{n} |\sigma_k^{p-\delta-1}| = O(n), \qquad n \to \infty,$$

then, for α *such that* $-1 \leqq \alpha < p - \delta - 1,$

$$\sum_{k=1}^{\infty} a_k = l \ (R, p, \alpha).$$

REMARKS ON THEOREM B. (i) The theorem is due to Obreschkoff in the cases $\alpha = -1$, $p \geqq 1$ ([3], Satz 1), $\alpha = 0$, $p \geqq 2$ ([3], Satz 4) and due to Hirokawa in the remaining cases.

(ii) *The conclusion of Theorem* B *can be proved for* $p = 1$ *and* $\alpha = 0$, *a case ruled out in the enunciation*, by first using Hirokawa's arguments with $p = 1$, $\alpha = -1$, to show that

$$\frac{2}{\pi} \sum_{k=1}^{\infty} a_k \sum_{n=k}^{\infty} \frac{\sin nt}{n} \begin{cases} \text{exists for } t > 0, \\ \text{tends to } l \text{ as } t \to +0, \end{cases}$$

as a result of our two hypotheses and the fact

$$\left| \sum_{n=k}^{\infty} \frac{\sin nt}{n} \right| = t^{-1} O(k^{-1}), \qquad t > 0, \ k \to \infty,$$

which is analogous to Lemma 2 and now serves in lieu of the relation $|\sin kt/kt| = t^{-1}O(k^{-1})$ used by Hirokawa. The proof can then be completed, as pointed out by Szász in a similar context ([5], pp. 1221–2, converse of Lemma 1), by the observation that

$$\frac{2}{\pi} \sum_{k=1}^{\infty} a_k \sum_{n=k}^{\infty} \frac{\sin nt}{n} = \frac{2}{\pi} \sum_{k=1}^{\infty} s_k \frac{\sin kt}{k} \equiv \rho(1, 0, t), \qquad t > 0,$$

because $s_k = O(k^{1-\delta}) = o(k)$, $k \to \infty$, in consequence of the hypothesis $\sigma_k^{1-\delta} \to l$.

(iii) In every case in which Theorem B holds good, it can be restated with only (T_B) changed to

$$\sum_{k=1}^{n} |s_k^{p-\delta-1}| = O(n^{p-\delta}) \qquad n \to \infty,$$

the proof of the changed theorem requiring an obvious modification of but two steps numbered (3. 3) and (3. 4) in Hirokawa's paper [2]. The cases $\alpha = -1$, $p = 1$ and $\alpha = 0, p = 1$ of the theorem thus changed are due to Szász ([5], p. 1159, Theorem A and p. 1228, Theorem 5),

Of our first pair of analogous results for series and for integrals, the one for series, Theorem I′, is stated before its analogue for integrals, Theorem I, but proved after, so as to facilitate the comparison of Theorems B, I′ and present first the simpler of two proofs. Our second pair of analogues, of which the one for integrals is Theorem II and the other for series is taken for granted, can be deduced from the first pair, and the Tauberian condition in them is one-sided.

THEOREM I′. *If* $\alpha + 1 \geqq 0$,

$$\sum_{k=1}^{\infty} a_k = l\ (\mathrm{C}, \alpha+2), \tag{3.1'}$$

$$\sum_{k=1}^{n} |\sigma_k^{\alpha}| = O(n), \qquad n \to \infty, \tag{3.2'}$$

then, for a positive integer $p > \alpha + 1$,

$$\sum_{k=1}^{n} a_k = l\,(\mathrm{R}, p, \alpha).$$

This conclusion is true in the case $\alpha = 0$, $p = 1$ (*ruled out above*) *if* (3.1′) *and* (3.2′) *are assumed for* $\alpha = -1$.

THEOREM I. *If* $\alpha + 1 \geqq 0$,

$$s(u) \to l\,(\mathrm{C}, \ \alpha+2), \tag{3.1}$$

$$\int_0^u |\sigma_\alpha(x)|\, dx = O(u), \qquad u \to \infty, \tag{3.2}$$

then, for a positive integer $p > \alpha + 1$,

$$s(u) \to l\,(\mathrm{R}, p, \alpha).$$

The conclusion remains true for $\alpha = 0$, $p = 1$ *provided that* (3.1), (3.2) *are assumed for* $\alpha = -1$.

PROOF OF THEOREM I. In all cases except the case $\alpha = 0$, $p = 1$, the transform $\rho(p, \alpha, t)$ of (1.3) exists as an absolutely convergent integral for fixed $t > 0$. For, if the integral in (1.3) is taken from a positive U to ∞ and the integrand is replaced by its absolute value, we get

$$\int_U^{\infty} |s_\alpha(u)| \left|\frac{\sin tu}{tu}\right|^p du < \frac{1}{\Gamma(\alpha+1)t^p}\int_U^{\infty} \frac{|\sigma_\alpha(u)|}{u^{p-\alpha}}\, du = O\left(\frac{1}{U^{p-\alpha-1}}\right) \to 0,\ U \to \infty,$$

by (3.2) and an appeal to Lemma 1 with $f(u) = |\sigma_\alpha(u)|$, $q = p - \alpha$, $\delta = p - \alpha - 1$.

If $\alpha = 0$, $p = 1$, the existence of $\rho(p, \alpha, t)$ for fixed $t > 0$ is proved as follows from (3.1) and (3.2) each with $\alpha = -1$. The integral

$$\frac{2}{\pi}\int_0^{\infty} a(x)\phi^*(tx)\, dx \equiv \frac{2}{\pi}\int_0^{\infty} a(x)\, dx \int_x^{\infty} \frac{\sin tu}{u}\, du, \qquad t > 0, \tag{3.3}$$

is absolutely convergent, since

$$\int_U^{\infty} |a(x)|\cdot|\phi^*(tx)|\, dx \leqq \frac{2}{t}\int_U^{\infty} \frac{|a(x)|}{x}\, dx = O\left(\frac{1}{U}\right) \to 0, \quad U \to \infty,$$

by an appeal to Lemma 2, followed by (3.1) with $\alpha = -1$, and an appeal to Lemma 1 with $f(x) = x|a(x)|$, $q = 2$, $\delta = 1$. Since, as a result of (3.1) and (3.2) each with $\alpha = -1$, $s(u) = O(1)$ by the case $\alpha = -1$ of Lemma 4, we get from (3.3), by an integration by parts,

$$(3.4)\qquad \frac{2}{\pi}\int_0^\infty a(x)\phi^*(tx)\,dx = \frac{2}{\pi}\int_0^\infty s(x)\frac{\sin tx}{x}\,dx \equiv \rho(1,0,t), \qquad t>0.$$

We proceed to show that in all the cases considered above, $\rho(p, \alpha, t) \to l$ as $t \to +0$.

Case: $p > \alpha + 1 > 0$. Writing $\phi(u) = (\sin u)^p/u^p$, we get by two integrations by parts, for fixed $t > 0$ and $x > 0$,

$$\rho(p,\alpha,t) = C_{p,\alpha}^{-1}\, t^{\alpha+1}\left(\int_0^{x/t} + \int_{x/t}^\infty\right) s_\alpha(u)\,\phi(tu)\,du$$

$$= C_{p,\alpha}^{-1}\, t^{\alpha+1}\left[s_{\alpha+1}(u)\,\phi(tu) - s_{\alpha+2}(u)\frac{d}{du}\phi(tu)\right]_{u=0}^{u=x/t}$$

$$+ C_{p,\alpha}^{-1}\, t^{\alpha+1}\left[\int_0^{x/t} s_{\alpha+2}(u)\frac{d^2}{du^2}\phi(tu)\,du + \int_{x/t}^\infty s_\alpha(u)\,\phi(tu)\,du\right]$$

$$(3.5)\qquad = C_{p,\alpha}^{-1}(I_1 + I_2 + I_3 + I_4)$$

where

$$I_1 = \frac{\sigma_{\alpha+1}(x/t)}{\Gamma(\alpha+2)}\, x^{\alpha+1}\phi(x), \qquad I_2 = -\frac{\sigma_{\alpha+2}(x/t)}{\Gamma(\alpha+3)}\, x^{\alpha+2}\phi'(x),$$

$$I_3 = \int_0^x \frac{\sigma_{\alpha+2}(u/t)}{\Gamma(\alpha+3)}\, u^{\alpha+2}\phi''(u)\,du, \qquad I_4 = t^{\alpha+1}\int_{x/t}^\infty \frac{\sigma_\alpha(u)}{\Gamma(\alpha+1)}\, u^\alpha\phi(tu)\,du.$$

We shall suppose (as we may) that $t < 1$ and fix x subject to certain conditions to be specified presently. To begin with, (3.2) and Lemma 4 in the case $\alpha > -1$ show that $\sigma_{\alpha+1}(u) = O(1)$ as $u \to \infty$ and that there is a constant M which makes

$$(3.6)\qquad |I_1| < Mx^{\alpha+1}|\phi(x)|$$

for all large x; while (3.2) and Lemma 1 with $f(u) = |\sigma_\alpha(u)|$, $q = p - \alpha$, $\delta = p - \alpha - 1$, show that there is also a constant N making

$$(3.7)\qquad |I_4| < t^{\alpha+1-p}\int_{x/t}^\infty \frac{|\sigma_\alpha(u)|}{\Gamma(\alpha+1)}\frac{du}{u^{p-\alpha}} < t^{\alpha+1-p}N(x/t)^{-p+\alpha+1} = Nx^{-p+\alpha+1}$$

for all large x. On the other hand, (3.1) is $\sigma_{\alpha+2}(u) \to l$ and it gives, when x is fixed and $t \to +0$,

$$I_2 \to -\frac{l}{\Gamma(\alpha+3)}\, x^{\alpha+2}\phi'(x),$$

$$I_3 \to \int_0^x \frac{l}{\Gamma(\alpha+3)}\, u^{\alpha+2}\phi''(u)\,du = \frac{l}{\Gamma(\alpha+3)}\, x^{\alpha+2}\phi'(x) - \frac{l}{\Gamma(\alpha+2)}\, x^{\alpha+1}\phi(x)$$

$$+\int_0^x \frac{l}{\Gamma(\alpha+1)} u^\alpha \phi(u)\,du,$$

so that, by the definition of $C_{p,\alpha}$ in (1.4),

$$(3.8) \quad I_2 + I_3 - C_{p,\alpha}\, l \to -\frac{l}{\Gamma(\alpha+2)} x^{\alpha+1}\phi(x) - \int_x^\infty \frac{l}{\Gamma(\alpha+1)} u^\alpha \phi(u)\,du,\ t\to +0.$$

Given any small $\varepsilon > 0$, x can be chosen so large that the right-hand members of (3.6), (3.7), (3.8) are each less than $C_{p,\alpha}^{-1}\varepsilon/3$ in absolute value. With this choice of x, (3.5) gives when taken along with (3.6), (3.7) and (3.8):

$$\limsup_{t\to+0} |\rho(p,\alpha,t) - l| < \varepsilon, \text{ i.e. } \lim_{t\to+0} \rho(p,\alpha,t) = l.$$

Case: $p \geqq 1$, $\alpha = -1$. (3.5) will now be replaced by

$$(3.5_1) \qquad \rho(p, -1, t) = I_1^{(-1)} + I_2^{(-1)} + I_3^{(-1)} + I_4^{(-1)}$$

where $I_1^{(-1)}$, $I_2^{(-1)}$, $I_3^{(-1)}$, $I_4^{(-1)}$ are I_1, I_2, I_3, I_4 respectively with $\alpha = -1$, so that

$$|I_1^{(-1)}| < M|\phi(x)|,\quad |I_4^{(-1)}| < Nx^{-p},$$

for all large x; and when $t \to +0$, x remaining fixed,

$$I_2^{(-1)} \to -lx\phi'(x),\quad I_3^{(-1)} \to \int_0^x lu\phi''(u)\,du = lx\phi'(x) - l\phi(x) + l\phi(+0).$$

Substitution of these estimates for $I_1^{(-1)}$, $I_2^{(-1)}$, $I_3^{(-1)}$, $I_4^{(-1)}$ in (3.5_1) leads to our conclusion in the penultimate from

$$\limsup_{t\to+0} |\rho(p, -1, t) - l\phi(+0)| < M|\phi(x)| + Nx^{-p} + |l\phi(x)| < \varepsilon$$

since evidently x may be supposed to have been fixed so as to satisfy the last inequality.

Case: $p = 1$, $\alpha = 0$. We see from (3.4) that the treatment of this case is like that of the case $p = 1$, $\alpha = -1$ with the difference that we have now $2\pi^{-1}\phi^*(u)$ instead of $\phi(u)$. Hence our conclusion will now assume the penultimate form

$$\limsup_{t\to+0} \left| \rho(1,0,t) - \frac{}{\pi}\phi^*(0)\right| < M|\phi^*(x)| + Nx^{-1} + \left|\frac{2l}{\pi}\phi^*(x)\right| < \varepsilon,$$

a choice of x satisfying the last inequality being possible by Lemma 2.

PROOF OF THEOREM I′. The *case* $\alpha + 3 = p > \alpha + 1 > 0$ is reducible to Theorem B with its two hypotheses replaced by the single stronger hypothesis $\sigma_n^{p-\delta-1} \to l$ as $n \to \infty$. For, (3.2′) implies, by the series-analogue of Lemma 4, $\sigma_n^{\alpha+1} \equiv \sigma_n^{p-2} = O(1)$ which together with (3.1′) leads to $\sigma_n^{p-\delta-1} \to l$ for every positive $\delta < 1$ by a well-known theorem ([1], p. 127, Theorem 70). The *case* $p = 2, \alpha = -1$ follows from the case $p = 1$, $\alpha = -1$, since we have, assuming the result for $p = 1, \alpha = -1$,

$$\sum_{k=1}^{\infty} a_k = l\,(\mathrm{C},1), \quad \sum_{k=1}^{n} k|a_k| = O(n) \;\Rightarrow\; \sum_{k=1}^{\infty} a_k = l\,(\mathrm{R},1) \;\Rightarrow\; \sum_{k=1}^{\infty} a_k = l\,(\mathrm{R},2),$$

it being well-known that $(\mathrm{R},1)\Rightarrow(\mathrm{R},2)$ ([1], Appendix III). Hence *we may suppose that* $\alpha + 3 \neq p > \alpha + 1 \geqq 0$.

Case: $\alpha + 3 \neq p > \alpha + 1 > 0$. After establishing the existence of the $\rho(p, \alpha, t)$ in (1.3′) as in the proof of Theorem I, we proceed to the step corresponding to (3.5) there, which is now obtained by two Abel or partial-summation transformations, in the form

$$\begin{aligned}\rho(p,\alpha,t) &= C_{p,\alpha}^{-1}\Big[t^{\alpha+1} s_m^{\alpha+1}\,\phi(mt) + t^{\alpha+1} s_{m-1}^{\alpha+2}\,\Delta\phi(\overline{m-1}t)\\ &\qquad + t^{\alpha+1}\sum_{k=1}^{m-2} s_k^{\alpha+2}\,\Delta^2\phi(kt) + t^{\alpha+1}\sum_{k=m+1}^{\infty} s_k^{\alpha}\phi(kt)\Big]\\ &= C_{p,\alpha}^{-1}(I_1' + I_2' + I_3' + I_4'),\ \text{say},\end{aligned}$$

where $m =$ the integral part of x/t, $x > 0$, $t > 0$. By (3.2′) and the series-analogue of Lemma 4, $\sigma_k^{\alpha+1} = O(1)$ as $k\to\infty$, so that, as in the proof of Theorem I, there is a constant M such that

$$|I_1'| < Mx^{\alpha+1}|\phi(x)| \tag{3.6′}$$

for all large x. By (3.2′) and the series-analogue of Lemma 1, there is also a constant N such that

$$|I_4'| > Nx^{\alpha+1-p} \tag{3.7′}$$

for all large x. If x is fixed, using the hypothesis (3.1′) that $\sigma_k^{\alpha+2}\to l$ as $k\to\infty$, we can show that

$$\begin{aligned}&\limsup_{t\to+0} |I_2' + I_3' - C_{p,\alpha} l|\\ &\leqq \frac{|l|}{\Gamma(\alpha+2)}\,x^{\alpha+1}|\phi(x)| + \left|\int_x^{\infty} \frac{l}{\Gamma(\alpha+1)}\,u^{\alpha}\phi(u)\,du\right| + F(x)\end{aligned} \tag{3.8′}$$

where $F(x)$ is a certain positive function of x tending to 0 as $x\to\infty$. After this we can complete the proof like that of Theorem I, postulating that x has been chosen so as to make right-hand members of (3.6′), (3.7′) and (3.8′) each less than $C_{p,\alpha}^{-1}\,\varepsilon/3$.

It remains to show how (3.8′) is reached. $F(x)$ having been defined as in (3.18′) below and $x > 1$ having been chosen as indicated above, we can state the hypothesis (3.1′) in the form

$$s_k^{\alpha+2} = l\,A_k^{\alpha+2} + \varepsilon_k\, l\,A_k^{\alpha+2}, \quad \varepsilon_k l A_k^{\alpha+2}/k^{\alpha+2}\to 0 \text{ as } k\to\infty,$$

and find m_0 so that, for $k \geqq m_0 + 1$,

$$s_k^{\alpha+2} = l\,A_k^{\alpha+2} + \varepsilon_k\, l\,A_k^{\alpha+2}, \quad |\varepsilon_k\, l\,A_k^{\alpha+2}| < x^{-2|\alpha+3-p|}\,k^{\alpha+2}. \tag{3.9′}$$

With this choice of m_0 and with $m =$ the integral part of $x/t > m_0 + 2$ in (3.5′)[6],

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6) Since finally $t\to+0$, the supposition regarding m, viz., that $m \geqq m_0+3$, is valid.

we can write

$$I_2' + I_3' = t^{\alpha+1}\, l\, A_{m-1}^{\alpha+2}\, \Delta\phi(\overline{m-1}t) + t^{\alpha+1}\, l\, \varepsilon_{m-1}\, A_{m-1}^{\alpha+2}\, \Delta\phi(\overline{m-1}t)$$

$$+ t^{\alpha+1}\sum_{k=1}^{m-2} l\, A_k^{\alpha+2}\, \Delta^2\phi(kt) + t^{\alpha+1}\Big(\sum_{k=1}^{m_0} + \sum_{k=m_0+1}^{m-2}\Big) l\, \varepsilon_k A_k^{\alpha+2}\, \Delta^2\phi(kt)$$

(3.10′) $$= I_{21}' + I_{22}' + I_{31}' + I_{32}' + I_{33}', \text{ say.}$$

Since there is a constant K such that $|\Delta^2\phi(kt)| < Kt^{2-p}k^{-p}$ for all $t > 0$ and $k \geqq 1$ ([3], p. 443), we first get, using (3.9′) in I_{33}'

$$|I_{33}'| < K\, t^{\alpha+3-p}\, x^{-2|\alpha+3-p|} \sum_{k=m_0+1}^{m-2} k^{\alpha+2-p}.$$

Since there is evidently also a constant K_1 such that $\sum_{k=1}^{m} k^{\alpha+2-p} < K_1 m^{\alpha+3-p}$, the above step gives us

$$|I_{33}'| < KK_1 (tm)^{\alpha+3-p}\, x^{-2|\alpha+3-p|}$$

where $tm \to x$ as $t \to +0$. Thus

(3.11′) $$\limsup_{t\to+0} |I_{33}'| \leqq KK_1\, x^{\alpha+3-p-2|\alpha+3-p|} \leqq KK_1\, x^{-|\alpha+3-p|}.$$

Secondly, there is a constant L such that $|\Delta\phi(kt)| < Lt^{1-p}k^{-p}$ for all $t > 0$ and $k \geqq 1$ and so we find, recalling that

$$|l\varepsilon_{m-1} A_{m-1}^{\alpha+2}| < x^{-2|\alpha+3-p|}(m-1)^{\alpha+2},$$

$$|I_{22}'| < L(t\overline{m-1})^{\alpha+2-p}\, x^{-2|\alpha+3-p|}$$

whence follows, exactly like (3.11′),

(3.12′) $$\limsup_{t\to+0} |I_{22}'| \leqq L\, x^{\alpha+2-p-2|\alpha+3-p|} \leqq L\, x^{-1-|\alpha+3-p|}.$$

Thirdly, $|\Delta^2\phi(kt)| = O(t^2)$ as $t \to +0$ where the constant implied by $O(t^2)$ can be chosen to be the same for $k = 1, 2, 3, \ldots, m_0$; while $l\varepsilon_k A_k^{\alpha+2}/k^{\alpha+2}$ is bounded for $k \geqq 1$. Hence there is a constant L_1 such that, for all sufficiently small t,

(3.13′) $$|I_{32}'| < L_1\, t^{\alpha+3}\sum_{k=1}^{m_0} k^{\alpha+2}, \text{ i.e. } \lim_{t\to+0} |I_{32}'| = 0.$$

Lastly, we have, by two Abel or partial-summation transformations,

(3.14′) $$I_{21}' + I_{31}' = t^{\alpha+1}\sum_{k=1}^{m} l\, A_k^{\alpha}\phi(kt) - t^{\alpha+1}\, l\, A_m^{\alpha+1}\, \phi(mt)$$

where, since $mt \to x$ as $t \to +0$,

(3.15′) $$t^{\alpha+1} A_m^{\alpha}\phi(mt) \sim \frac{(mt)^{\alpha+1}}{\Gamma(\alpha+2)}\, \phi(mt) \to \frac{x^{\alpha+1}\phi(x)}{\Gamma(\alpha+2)};$$

and further, as shown by an argument elsewhere ([3], proof of Lemma 3),

(3.16′) $$t^{\alpha+1}\sum_{k=1}^{m} A_k^{\alpha}\, \phi(kt) \sim \frac{1}{\Gamma(\alpha+1)}\int_0^x u^{\alpha-p}(\sin u)^p\, du, \quad t \to +0.$$

From (3.14′), (3.15′), (3.16′) it follows that

$$\lim_{t\to+0}(I'_{21}+I'_{31})=\frac{l}{\Gamma(\alpha+1)}\int_0^x u^{\alpha-p}(\sin u)^p\,du-\frac{l\,x^{\alpha+1}\phi(x)}{\Gamma(\alpha+2)}$$

$$(3.17')\qquad =lC_{p,\alpha}-\frac{l}{\Gamma(\alpha+1)}\int_x^\infty u^{\alpha-p}(\sin u)^p\,du-\frac{l\,x^{\alpha+1}\phi(x)}{\Gamma(\alpha+2)}.$$

Using (3.11′), (3.12′), (3.13′), (3.17′) in (3.10′), we finally obtain (3.8′) with

$$(3.18)\qquad F(x)=KK_1\,x^{-|\alpha+3-p|}+Lx^{-1-|\alpha+3-p|}$$

where K, K_1, L are absolute constants. This completes the proof as already explained.

Case: $\alpha+1=0,\ 2\neq p\geqq 1$. We merely put $\alpha+1=0$ in all the steps of the preceding proof except (3.8′), (3.14′)——(3.17′). The excepted steps, after the changes obviously necssary, culminate in the following steps instead of (3.17′) and (3.8′):

$$\lim_{t\to+0}(I'_{21}+I'_{31})=l-l\phi(x),\quad \limsup_{t\to+0}|I'_2+I'_3-l|\leqq|l\phi(x)|+F(x).$$

Otherwise the proof is as before.

Case: $\alpha=0,\ p=1$. The passage to this case from the case $\alpha=-1$, $p=1$ is as in the proof of Theorem I with the necessary modifications for series indicated in Remark (ii) following Theorem B.

DEDUCTIONS FROM THEOREM I.

COROLLARY I(1). *If $s(u)\to l$ (C, ∞), then, for all large α, say $\alpha\geqq\alpha_0$, $s(u)\to l\,(R,p,\alpha)$ if p is a positive integer chosen corresponding to each α, so that $p>\alpha+1$.*

PROOF. By Lemma 7, the hypotheses (3.2), (3.1) of Theorem I now obtain in the stronger form

$$\sigma_\alpha(u)=O(1),\quad \sigma_{\alpha+1}(u)\to l,$$

Corollary I(1) invites comparison with the known result $\alpha\geqq\alpha_0$. ([4], Lemma II) that, if $s(u)\to l$ (C, ∞) then, for all $\alpha\geqq\alpha_0$,

$$t^{\alpha+1}\int_0^\infty e^{-tu}s_\alpha(u)\,du\to l,\qquad t\to+0.$$

COROLLARY I(2). *If $\alpha\geqq -1$, $s(u)$ is summable $|C,\alpha+1|$ to l, then $s(u)$ is summable (C, p,α) to l, if p is a positive integer such that $p>\alpha+1$.*

The conclusion is true for $p=1$, $\alpha=0$ on the hypothesis that $\int_0^\infty a(u)\,du$ is absolutely convergent.

PROOF. The two hypotheses (3.1), (3.2) of Theorem I are ensured, the first one obviously and the second one by Lemma 10; and so every case of Corollary I(2) is reducible to the corresponding case of Theorem I.

THEOREM II. *If* $\alpha \geqq -1$, $H > 0$, *then, for a positive integer* $p > \alpha + 1$,

$$s(u) \to l\,(\mathrm{C}, \alpha + 2). \quad \sigma_\alpha(u) > -H \Rightarrow s(u) \to l\,(\mathrm{R}, p, \alpha).$$

The above result is additionally true with $\alpha = -1$ *in the hypothesis (left side) and* $\alpha = 0$, $p = 1$ *in the conclusion (right side).*

PROOF. If $\alpha > -1$, Lemma 8 shows that $\sigma_{\alpha+1}(u) = O(1)$ and that consequently

$$\frac{1}{u}\int_0^u |\sigma_\alpha(x) + H|\,dx = \frac{1}{\alpha+1}[\sigma_{\alpha+1}(u) + H] + \frac{\alpha}{\alpha+1}\frac{1}{u}\int_0^u [\sigma_{\alpha+1}(x) + H]dx = O(1).$$

Thus Theorem II is simply Theorem I with $\sigma_\alpha(x) + H$ instead of $\sigma_\alpha(x)$ in (3.1) and corresponding changes elsewhere, in particular, with the conclusion, for $p > \alpha + 1 > 0$,

$$C_{p,\alpha}^{-1}\, t^{\alpha+1}\int_0^\infty \left[s_\alpha(u) + H\frac{u_\alpha}{\Gamma(\alpha+1)}\right]\left(\frac{\sin tu}{tu}\right)^p du \begin{cases} \text{exists for } t > 0, \\ \text{tends to } l + H \text{ as } t \to +0. \end{cases}$$

This is the conclusion sought since the terms in H can be removed from both sides.

If $\alpha = -1$, Lemma 9 gives

$$\int_0^x x|a(x)|\,dx = O(u)$$

which together with the hypothesis $s(u) \to l\,(\mathrm{C}, 1)$ shows that the case $p > \alpha + 1 = 0$ of Theorem II is included in the same case of Theorem I.

Finally, the additional result follows from the case $\alpha = 0$, $p = 1$ of Theorem I.

COROLLARY II. *For* $s(u) > -H$, *summabilities* $(\mathrm{C}, 2)$ *and* (R_2) *are equivalent.*

PROOF. For the class of $s(u)$ in question, we have, taking $\alpha = 0$, $p = 2$ in Theorem I,

$$(\mathrm{C}, 2) \Rightarrow (\mathrm{R}_2)$$

while, by a known theorem ([1], p. 305, Theorem 237),

$$(\mathrm{R}_2) \Rightarrow (C, 1) \Rightarrow (C, 2).$$

4. Tauberian theorems connecting Abel summability with Riemann-Cesàro summability. The theorems of this section are derived from those of the last section.

THEOREM I(A). *Theorem I can be restated with the hypothesis* $s(u) \to l$ $(\mathrm{C}, \alpha + 2)$ *of* (3.1) *changed to* $s(u) \to l\,(\mathrm{A})$ *and no other change.*

PROOF. From (3.2) and $s(u) \to l(\mathrm{A})$, it follows, by an appeal to Lemma 6, that $s(u) \to l\,(\mathrm{C}, \alpha+2)$. Theorem I(A) is thus reduced to Theorem I.

The integral analogue of Szász's Theorem A at the outset is generalized in the following corollary obtained by taking $\alpha = -1$ and $\alpha = 0$ in Theorem I(A).

COROLLARY I(A).

$$s(u) \to l\,(\mathrm{A}). \begin{cases} \textit{either} \displaystyle\int_0^u x|a(x)|\,dx = O(u)^{7)} \\ \textit{or} \displaystyle\int_u^{\lambda u} [|a(x)| - a(x)]\,dx = O(1), \lambda > 1 \end{cases} \Rightarrow \begin{cases} s(u) \to l(\mathrm{R}, p) \\ s(u) \to l(\mathrm{R}_1). \end{cases}$$

THEOREM II(A). *Theorem* II *is true with* $s(u) \to l(\mathrm{C}, \alpha+2)$ *replaced by* $s(u) \to l(\mathrm{A})$.

PROOF. For $\alpha \geqq -1$, the hypothesis $\sigma_\alpha(u) > -H$ gives $\sigma_{\alpha+1}(u) = O_L(1)$ which, in conjunction with the hypothesis $s(u) \to l\,(A)$ leads to $s(u) \to l(\mathrm{C}, \alpha+2)$ by Lemma 5. And so Theorem II(A)is reducible to Theorem II.

Whether for series or for integrals, the known theorems ([1], Appendix III) connecting each of the summabilities (R, 1), (R, 2), (R_2) with summability (A) are that

$$(\mathrm{R}, 1) \to\to (\mathrm{R}, 2) \to\to (A), \quad (\mathrm{R}_2) \to\to (A).$$

There are non-trivial converses of these theorems for integrals, under a one-sided Tauberian condition, contained in Theorem II(A) and explicitly stated below.

COROLLARY II(A).

$$s(u) \to l(A).\ \ ua(u) > -H \to\to \begin{cases} s(u) \to l\,(\mathrm{R}, 1) \\ s(u) \to l\,(\mathrm{R}_1), \end{cases}$$

$$s(u) \to l(A).\ \ s(u) > -H \to\to \quad s(u) \to l\,(\mathrm{R}_2).$$

The series-analogues of all the results in this section can be stated and proved on the same lines as these results. In particular, Theorem I(A) has the series-analogue which is given below and deduced from Theorem I′ exactly as Theorem I(A) from Theorem I.

THEOREM I′(A). *In Theorem* I′, *the hypothesis* $\sum a_k = l(\mathrm{C}, \alpha+2)$ *of* (3.1′) *can be changed to* $\sum a_k = l(\mathrm{A})$ *without any other change.*

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7) On the hypotheses of Corollary I(A), $s(u)$ is summable (C, δ) for every $\delta > 0$ (cf. [5], p. 1141, Lemma 3), but not necessarily convergent. The negative part of this statement has been demonstrated by Szász by means of an example for the analogous case of series ([5], p. 1145). In this analogous case, there is also an additional conclusion proved by Hirokawa ([2], Theorem 4), namely, that the series considered is summable $(\mathrm{R}, 1, \alpha)$, $-1 < \alpha < 0$.

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