TOKYO J. MATH.
VOL. 22, NO. 1, 1999

Tunnel Number One Genus One Non-Simple Knots

Hiroshi GODA and Masakazu TERAGAITO

Kobe University and Hiroshima University

(Communicated by K. Kobayasi)

1. Introduction.

A knot in the 3-sphere is said to be *tunnel number one* if there exists an arc attached to the knot at its endpoints so that the complement of a regular neighbourhood of the resulting complex is a genus two handlebody. It is well-known that every torus knot and 2-bridge knot is tunnel number one. Although no composite knot is tunnel number one [7, 8], some non-simple knots are known to be tunnel number one [5]. It seems difficult to characterize tunnel number one knots.

Among prime knots, genus one knots are relatively easy to deal with and possess nice properties. For example, genus one fibred knots are exactly the trefoil knot and the figure-eight knot [1]. Any unknotting number one genus one knot is a double knot [4, 9].

In this note, as the first step for the determination of tunnel number one genus one knots, we shall completely determine all the tunnel number one genus one non-simple knots.

The tunnel number one non-simple knots in S^3 are classified by Morimoto and Sakuma [5]. We review it briefly.

Let K_0 be a non-trivial torus knot $T(p, q)$ of type (p, q) in S^3, and $L=K_1 \cup K_2$ a 2-bridge link $S(\alpha, \beta)$ of type (α, β) in S^3 with $\alpha \geq 4$. Then there is an orientation-preserving homeomorphism $f: E(K_2) \to N(K_0)$ which takes a meridian $m_2 \subset \partial E(K_2)$ of K_2 to a regular fibre $h \subset \partial N(K_0) = \partial E(K_0)$ of the Seifert fibration of $E(K_0)$. Here, for a complex C in S^3, $N(C)$ means the regular neighbourhood of C in S^3, and $E(C)$ means the exterior $S^3 - \operatorname{int} N(C)$. We denote the knot $f(K_1) \subset N(K_0) \subset S^3$ by $K(\alpha, \beta; p, q)$. Then $K(\alpha, \beta; p, q)$ is a tunnel number one non-simple knot, and conversely any tunnel number one non-simple knot is obtained in such a manner.

We will calculate the genera of $K(\alpha, \beta; p, q)$, so that we have the following.

Received June 24, 1997
Revised September 1, 1997
The authors are supported by Grant-in-Aid for Encouragement of Young Scientists, The Ministry of Education, Science, Sports and Culture, Japan.

THEOREM. *Let K be a tunnel number one genus one non-simple knot in S^3. Then $K=K(8m, 4m+1; p, q)$ where $m\neq 0$.*

By the argument in Section 5 in [3], we have the following corollary (cf. Problem 8.3 in [2]).

COROLLARY. *Let K be a tunnel number one genus one non-simple knot in S^3. Then $E(K)$ admits a depth one taut foliation $\mathscr{F}$ such that $\mathscr{F}\cap\partial E(K)$ is a foliation by circles.*

2. Proof of Theorem.

We denote by $g(K)$ the genus of a knot K in S^3. Let K $(=f(K_1))$ be a tunnel number one genus one non-simple knot. By the formula in [10], $g(K)\geq n\cdot g(K_0)+g(K_1')$ where n (≥ 0) is the winding number of K_1 in the solid torus $E(K_2)$, and K_1' is the knot obtained from K_1 by $-1/pq$-surgery on K_2. Thus there are two cases:

Case 1. $n\geq 1$. Since $g(K)=1$ and K_0 is non-trivial, we have $n=1$ and $g(K_0)=1$. Hence K_0 is the trefoil knot. We use the next lemma in [10] (see also [1, Lemma 2.11]).

LEMMA 2.1. *Suppose that $n\neq 0$. Then there is a minimal genus Seifert surface S of K such that $S\cap\partial N(K_0)$ consists of n longitudes of K_0.*

Thus we have a genus one Seifert surface S of K such that $S\cap\partial N(K_0)$ is one longitude. Then $S\cap N(K_0)$ is an annulus, and therefore $K=K_0$ is the trefoil knot. This contradicts that K is non-simple.

Case 2. $n=0$. In this case we will show that the 2-bridge link $L=K_1\cup K_2$ is $S(8m, 3m+1)$.

LEMMA 2.2. *There is a genus one Seifert surface S of K_1 such that $S\cap K_2=\emptyset$.*

PROOF. Let R be a genus one Seifert surface of K. Suppose that $R\cap\partial N(K_0)\neq\emptyset$. By a suitable isotopy, we can assume that $R\cap\partial N(K_0)$ contains no inessential circle on $\partial N(K_0)$. Then neither $E(K_0)\cap R$ nor $N(K_0)\cap R$ contains disk components. If the component of $N(K_0)\cap R$ meeting K is an annulus, we have a contradiction, since $n=0$. Therefore $N(K_0)\cap R$ consists of some annuli and a twice punctured disk P whose boundary contains K. Consider an annulus $Q\subset\partial N(K_0)$ bounded by two components of ∂P. We note that the two components of ∂P are not homologous on $\partial N(K_0)$ when their orientations are induced by that of P, since $n=0$. Then $S=P\cup Q$ is orientable, and therefore it gives a genus one Seifert surface of K contained in $N(K_0)$. Hence $f^{-1}(S)$ is a desired Seifert surface of K_1 disjoint from K_2. □

LEMMA 2.3. *There is an annulus A embedded in the solid torus $E(K_2)$ such that $A\cap K_1=\emptyset$ and that one boundary component of A is lying in S and the other is a meridian of $\partial E(K_2)$.*

PROOF. Let $(B^i, K_1^i \cup K_2^i)$ $(i=1, 2)$ be trivial 2-string tangles corresponding to a 2-bridge representation of $L=K_1 \cup K_2$. Here K_j^i $(i=1, 2)$ are subarcs of K_j $(j=1, 2)$. Let τ be a simple arc corresponding to a tunnel of the link L as illustrated in Figure 1. By an isotopy, we can suppose that $B^1=N(K_1^1 \cup K_2^1 \cup \tau; S^3)$, and therefore we can suppose that $B^1 \cap S$ consists of disks as in Figure 1. Assume that the number of components of $B^1 \cap S$ is minimal. Then $B^2 \cap S$ is incompressible in $B^2-(K_1^2 \cup K_2^2)$, since S is incompressible in $E(L)$.

Let D be a disk properly embedded in B^2 which separates K_1^2 and K_2^2. Then, since $B^2 \cap S$ is a torus with holes and is incompressible in $B^2-(K_1^2 \cup K_2^2)$, it cannot be contained in a component of B^2-D. Hence $(B^2 \cap S) \cap D \neq \varnothing$. By the incompressiblity of $B^2 \cap S$, we may assume that there is no circle component of $(B^2 \cap S) \cap D$. Let α be an outermost arc component of $(B^2 \cap S) \cap D$ in D. Boundary-compression of $B^2 \cap S$ along α gives a band connecting one component of $B^1 \cap S$ with itself as illustrated in Figure 2 by the minimality of the number of components of $B^1 \cap S$. Therefore, we can find an annulus stated in Lemma 2.3. □

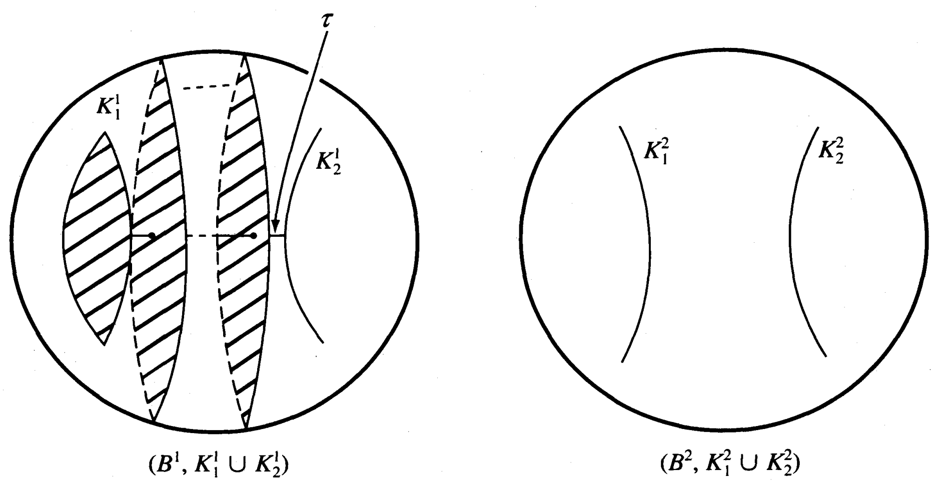

FIGURE 1

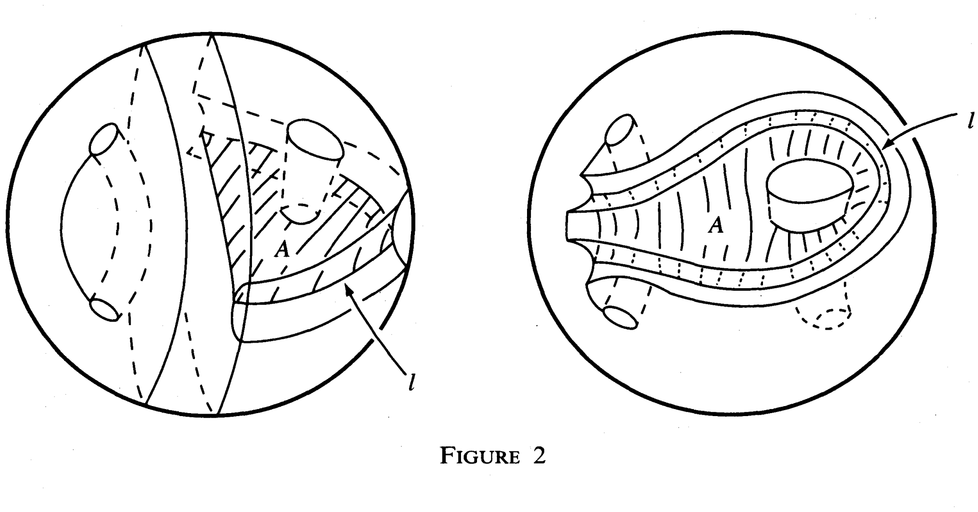

FIGURE 2

Let l be the component of ∂A lying in S. If l is separating in S, then l is parallel to $K_1(=\partial S)$. However this implies that K_1 is a meridian of K_2, which is impossible. Therefore l is non-separating in S. Let D be the disk in S^3 obtained from A by capping one boundary component of A in $\partial N(K_2)$ off by a meridian disk of $N(K_2)$. Compressing S along D gives a disk S' such that $\partial S' = K_1$ and S' meets K_2 in two points of opposite sign.

Let B_1 be a thin regular neighbourhood of S' in S^3. Then $(B_1, B_1 \cap L)$ gives the tangle (the 2-string Hopf tangle) as shown in Figure 3.

We note that K_1 is contained in B_1. Let $B_2 = \mathrm{cl}(S^3 - B_1)$. Then $(B_2, B_2 \cap L) = (B_2, B_2 \cap K_2)$ is a 2-string tangle. The tangle $(B_1, B_1 \cap L)$ is a prime tangle [6]. Since any 2-bridge link cannot have a decomposition into two prime tangles [6], the 2-string tangle $(B_2, B_2 \cap K_2)$ is a trivial tangle. Since L has the form as shown in Figure 4, L is a Montesinos link. However L is 2-bridge, therefore $(B_2, B_2 \cap L)$ is an integral tangle as shown in Figure 5.

Thus the 2-bridge link L corresponds to the continued fraction

$$2 + \cfrac{1}{-2m - \cfrac{1}{2}} = \frac{8m}{4m+1}.$$

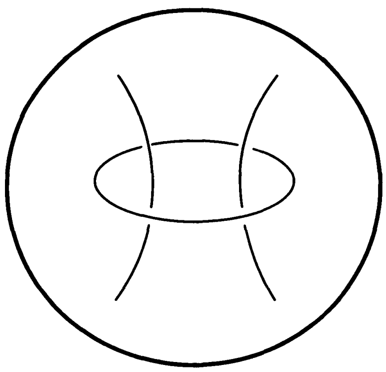

FIGURE 3

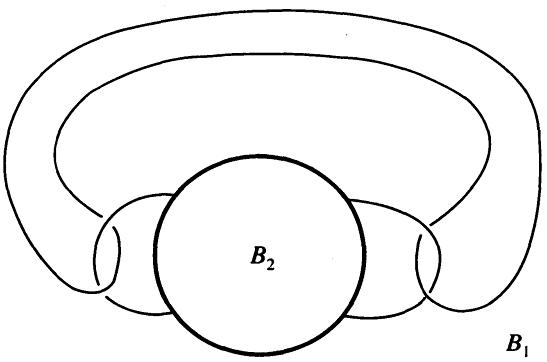

FIGURE 4

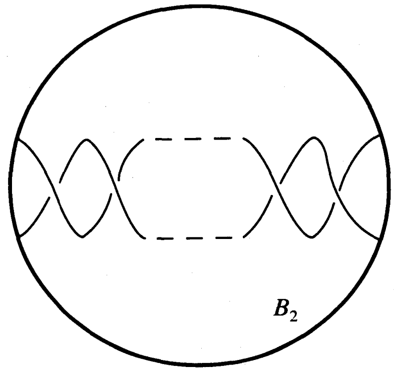

FIGURE 5

This completes the proof of Theorem.

References

[1] G. BURDE and H. ZIESCHANG, *Knots*, Walter de Gruyter Studies in Math. **5** (1985).
[2] D. GABAI, Problems in foliations and laminations, *Geometric Topology* (W. H. Kazez, ed.), AMS/IP Studies in Adv. Math. (1997), Part 2, 1–33.
[3] H. GODA, Depth of foliations on tunnel number one genus one knot complements, *Proc. VII Internat. Colloq. Differential Geometry, Analysis and Geometry in Foliated Manifolds* (X. Masa, ed.), World Scientific (1995), 39–53.
[4] T. KOBAYASHI, Minimal genus Seifert surfaces for unknotting number one knots, Kobe J. Math. **6** (1989), 53–62.
[5] K. MORIMOTO and M. SAKUMA, On unknotting tunnels for knots, Math. Ann. **289** (1991), 143–167.
[6] Y. NAKANISHI, Prime and simple links, Math. Sem. Notes Kobe **11** (1983), 249–256.
[7] F. H. NORWOOD, Every two generator knot is prime, Proc. Amer. Math. Soc. **86** (1982), 143–147.
[8] M. SCHARLEMANN, Tunnel number one knots satisfy the Poenaru conjecture, Topology Appl. **18** (1984), 235–258.
[9] M. SCHARLEMANN and A. THOMPSON, Link genus and the Conway moves, Comment. Math. Helv. **64** (1989), 527–535.
[10] H. SCHUBERT, Knoten und Vollringe, Acta Math. **90** (1953), 131–286.

Present Addresses:
HIROSHI GODA
THE GRADUATE SCHOOL OF SCIENCE AND TECHNOLOGY, KOBE UNIVERSITY,
ROKKODAI, NADA-KU, KOBE, 657–8501 JAPAN.

MASAKAZU TERAGAITO
DEPARTMENT OF MATHEMATICS EDUCATION, FACULTY OF SCHOOL EDUCATION,
HIROSHIMA UNIVERSITY, KAGAMIYAMA, HIGASHI-HIROSHIMA, 739–8524 JAPAN.