Some New Directions in Teichmüller Theory

Dissertation Defense of Mark Greenfield
Advisor: Professor Lizhen Ji

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Plan for today

1. History and Background
   - Riemann surfaces, moduli space, and Teichmüller space: a lightning-fast historical introduction for non-mathematicians
2. Teichmüller Theory for Flat Tori
   - Starting from 2-torus, go up in dimension, rather than in genus!
3. Isometric Submersions Between Teichmüller Spaces
   - Rigidity of maps between different Teichmüller spaces: generalizing Royden's theorem

History and Background

Early history: one of Riemann's many contributions

- Dramatic influence over all of mathematics!
- Calculus students learn about Riemann sums
- Einstein's General Relativity is an application of pseudo-Riemannian geometry
- One of the most famous unsolved problems in number theory is the Riemann Hypothesis
- You are about to hear about Riemann surfaces, first studied in Riemann's dissertation (complex analysis/manifolds)
- Sadly, died of tuberculosis at the age of 39

Figure: Bernhard Riemann, 1826-1866 (from Wikipedia)

Introduction to complex numbers

Consider the function $f(x) = \sqrt{x}$.
It "stops" at $x = 0$.
We want to extend it!
(Try solving $x^2 + 1 = 0$)

Introduce an *imaginary number* i with $i^2 = -1$ (so e.g. $f(-4) = 2i$).

Complex numbers extend the real "number line" to a plane (mixing *real* and *imaginary* parts), with a *magnitude* r and an *angle* θ.

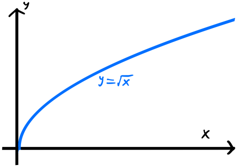


Figure: Plot of $f(x) = \sqrt{x}$ for nonnegative real numbers.

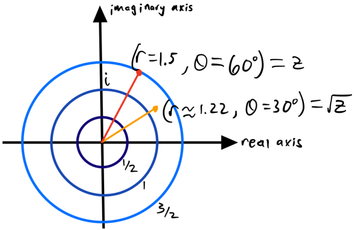
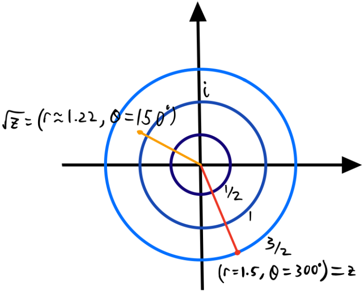


Figure: Square root function for complex numbers: halve the angle, use "ordinary" square root for magnitude.

Problem with this square root function



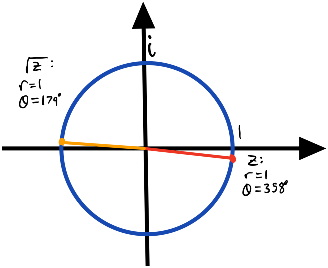


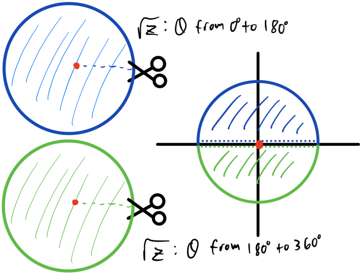
Figure: Halve the angle, square root of magnitude: problem as we come around back to 1 (approaches -1 instead of 1). If we get close to 1, so should the square root, right?

Square root should capture *both* negative and positive square roots for real numbers ($2^2 = (-2)^2 = 4$), and corresponding pairs for complex numbers.

Riemann's idea: introduce a new *domain*.

From complex numbers to Riemann surfaces

Build a new space where part of it yields the "positive" square root and part of it yields the "negative" square root.
Looks like pieces of the complex plane glued together! This is called a *Riemann surface*.



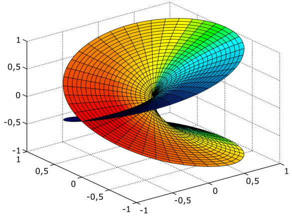


Figure: Glue two pieces on the left to get the surface on the right. Need 4 dimensions to see it properly! Right image from Wikipedia.

Next: What are all the possible ways to build these surfaces?

Fun with gluing pieces of the plane together

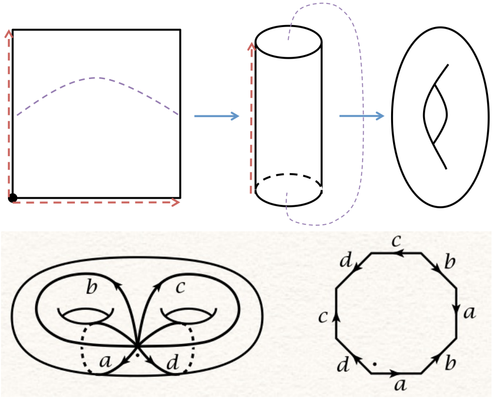


Figure: Different ways of gluing pieces of the complex plane together can yield infinitely many different families of Riemann surfaces. Lower image from *Algebraic Topology*, by A. Hatcher.

The upper image is a *torus*. This and higher-dimensional versions (e.g. gluing opposite faces of a cube) are central to the next section!

Riemann surfaces, constant-curvature metrics, and moduli spaces

- As a consequence of the celebrated Uniformization Theorem, along with some basic facts in geometry and topology:

  Riemann surfaces $\leftrightarrow$ Surfaces with a constant-curvature metric

- By Gauss-Bonnet for closed surfaces, the topology ($\chi(S) = 2 - 2g$) determines whether curvature is 0, 1, or -1 (physicists ask related questions about the shape of the whole 4D universe!)
- Organize each family of Riemann surfaces in a *moduli space*
- Riemann began this with a dimension count: $\dim_{\mathbb{C}} = 3g - 3$ for surfaces of genus (number of handles) $g > 1$.

Teichmüller and his ideas

- An extremely productive mathematician with several highly influential works in a short time
- Worked to understand Riemann's moduli space - led to Teichmüller space
- Laid the foundations for what we now call Teichmüller theory in late 1930s
- Dedicated himself to Nazi party and ideals; killed in action at age 30 after volunteering to join the Nazi army

Figure: Oswald Teichmüller, 1913 - 1943 (from Wikipedia)

Teichmüller spaces

Moduli space of *marked* Riemann surfaces

$$\mathcal{T}(S) = \{[X, f] : X \text{ Riemann sfc, } f : S \to X \text{ o.p. homeo}\} / \sim$$

with $[X, f] \sim [Y, g]$ if there exists a biholomorphism $h \sim f \circ g^{-1}$.

Same geometry, same marking

- Marking keeps track of how pieces of surfaces fit together (and "unfolds corners" in moduli space)
- Equivalent: X is a constant-curvature manifold
- Mapping class group $\text{Mod}(S)$ enables "change-of-marking"
- Mostow Rigidity $\Rightarrow \mathcal{T}$ trivial for hyperbolic n-mfld, $n \geq 3$
- No obstruction for flat manifolds with curvature $\equiv 0 \ldots$

The Teichmüller metric on Teichmüller spaces

- Teichmüller metric d_{Teich}: log of smallest quasiconformal distortion for a marking-respecting map between Riemann surfaces
  - $d_{\text{Teich}}\big((X,h),(Y,j)\big) = \inf_{f \sim j \circ h^{-1}} \log K(f)$

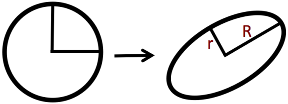


Figure: Locally K-QC. $K = R/r$, related to $f_{\bar{z}}/f_z$.

- Idea: quantitative measure of how much a surface must be distorted (using the marking to compare surfaces)

Post-WWII Teichmüller theory

- L. Ahlfors and L. Bers took up the task of understanding, formalizing, and extending Teichmüller's work starting in the 1950s and 1960s
- Among others, their students C. Earle, F. Gardiner, and I. Kra, continued studying Teichmüller spaces
- Many extensions of earlier work to punctured surfaces and surfaces of infinite type

Figure: Top row: Lars Ahlfors, Lipman Bers. Bottom row: Clifford Earle, Frederick Gardiner, Irwin Kra (from Wikipedia, cuny.edu, and ams.org)

William Thurston and the hyperbolic perspective

- Considered one of the greatest figures in geometry and topology
- Numerous beautiful contributions across geometry, topology, (complex) dynamics,...
- Two most important for us today:
  - Defined and studied the *Thurston metric*
  - Constructed a *compactification* of Teichmüller space using foliations on surfaces to understand how surfaces can degenerate
- Thurston's asymmetric metric d_{Th}: log of smallest Lipschitz constant for a map between surfaces

Figure: William Thurston, 1946 - 2012 (from cornell.edu)

A digression on Symmetric Spaces

- First studied rigorously by Élie Cartan in early 20th century
- Cartan developed much of the theory of representations of Lie groups
- We will use some basics of this theory to understand compactifications of certain Teichmüller spaces
- Comparisons to symmetric spaces yield many interesting questions to ask about Teichmüller space
- H. L. Royden showed Teichmüller spaces of hyperbolic surfaces have no symmetric points

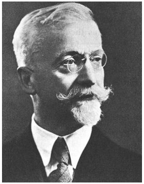


Figure: Élie Cartan, 1869 - 1951 (from Wikipedia)

Figure: Halsey Royden, 1928 - 1993 (from stanford.edu)

Symmetric spaces

- Riemannian manifold with geodesic-reversing isometries at every point
- Examples: $\mathbb{R}^n$, S^n, $\mathbb{H}^n$ (constant-curvature, simply-connected)
- Non-example: $\mathbb{R}^+$ (positive numbers). One direction is infinite, the other is finite!
- Our main example: $\mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)$, identified with space of lattices in $\mathbb{R}^n$, or with positive-definite symmetric matrices
- When $n = 2$: upper half-plane $\mathbb{H}^2 = \mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2)$

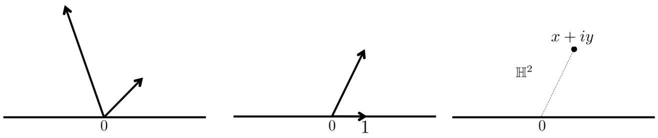


Figure: Identification of $\mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2)$ with $\mathbb{H}^2$.

How to think about Teichmüller space

- Each point of Teichmüller space is a *marked Riemann surface*
- The marking keeps track of which way the surface is “facing” (so e.g. we know which curves to compare to each other)
- Distances between points are based on how “badly” we must distort the first surface to transform it into the other surface

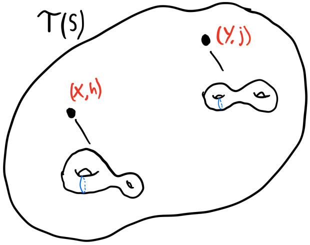


Figure: Teichmüller space is a space whose points are spaces!

Teichmüller theory today

- Spaces of other kinds of structures on surfaces
- Ways to continuously deform surfaces: Teichmüller dynamics
- Special spaces of representations: generalizing an alternative perspective on Teichmüller theory
- Geometric properties of surfaces: e.g. how many loops can be drawn of certain lengths
- Using Teichmüller space: spaces of punctured spheres representing ways nice maps can deform $\mathbb{C}$

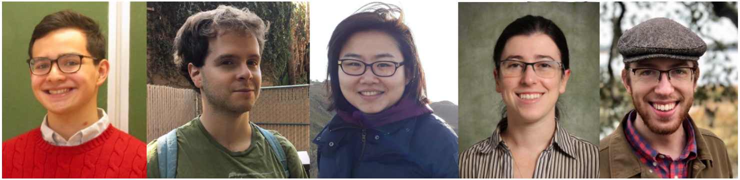


Figure: Some of my friends who have worked on Teichmüller theory and related subjects. Left to right: A. Calderon, D. Gekhtman, M. He, J. Sapir, and B. Zykoski.

...and the rest of today's discussion!

Teichmüller Theory for Flat Tori

Teichmüller spaces of flat n-tori

Denote the Teichmüller space of marked flat volume 1 n-tori by $\mathcal{T}(n)$. What kind of space is this?

- Torus geometry determined by a lattice in $\mathbb{R}^n$, up to $\mathrm{SO}(n)$
- Marking determined by specifying a basis of the lattice
- Bijective correspondence:

$$\mathcal{T}(n) \leftrightarrow \mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)$$

- Flat 2-torus: $\mathcal{T}(2) \cong \mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2) \cong \mathbb{H}^2$ (classical)

This identifies the points of $\mathcal{T}(n)$. Let's build some tools to understand the intrinsic geometry.

Extremal maps between flat tori

Proposition (G-Ji)

The map $\psi : S \to S'$ which lifts to the unique **affine map** $\tilde{\psi} : \mathbb{R}^n \to \mathbb{R}^n$ *realizes both the minimal Lipschitz stretching and minimal quasiconformal distortion between marked flat tori.*

Proof Idea.

Let $g : S \to S'$ be a K-Lipschitz or K-quasiconformal candidate, and lift it to $G : \mathbb{R}^n \to \mathbb{R}^n$. Consider $G_k : \mathbb{R}^n \to \mathbb{R}^n$ defined by:

$$G_k(\mathbf{x}) = \frac{G(k\mathbf{x})}{k}, \ k = 1, 2, \ldots.$$

This uniformly converges to the affine map ("averaging out" the "wobbles") which must also be K-Lipschitz or K-quasiconformal! $\square$

Metrics on $\mathcal{T}(n)$

Let $G = \mathrm{SL}(n, \mathbb{R})$, $K = \mathrm{SO}(n)$. Preceding proposition enables quick computation of d_{Th} and d_{Teich} ($g, h \in G$):

$$d_{Th}(gK, hK) = \log ||hg^{-1}||_{op}$$

with $|| \cdot ||_{op}$ the operator norm (max stretch of a vector).

$$d_{Teich}(gK, hK) = \max\big(d_{Th}(gK, hK), d_{Th}(hK, gK)\big).$$

From the perspective of symmetric spaces, we have:

$$d_{Sym}(gK, hK) = \left(\sum_i (\log a_i)^2\right)^{1/2}$$

where a_i are singular values of $g^{-1}h$. d_{Sym} is induced by the G-invariant Riemannian metric on $\mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)$.

Comparisons of the metrics on $\mathcal{T}(n)$

After computing and comparing the metrics, we have the following:

Theorem (G-Ji)

The three metrics d_{Th}, d_{Teich}, and d_{Sym} define Finsler structures on $\mathcal{T}(n)$ and can be computed explicitly. Moreover:

- *d_{Sym} matches the Weil-Petersson metric (based on distances between metrics in classical Teichmüller theory)*
- *d_{Teich} is a symmetrization of d_{Th}*
- *They are mutually distinct when $n \geq 3$*
- *When $n = 2$, all these metrics coincide with the hyperbolic metric on $\mathbb{H}^2 \cong SL(2, \mathbb{R})/SO(2)$*

Theory of compactifications

Definition

A *compactification* of a locally compact space X is a dense topological embedding

$$i : X \hookrightarrow C$$

into a compact space C. If G acts on X and $i : X \hookrightarrow C$ is G-equivariant, it is said to be a *G-compactification*.

Many examples and applications!

- Affix "endpoints" to $\mathbb{R}$ for a compact number line $[-\infty, \infty]$
- $\mathbb{C} \hookrightarrow \hat{\mathbb{C}}$ the one-point compactification
- $\mathbb{H}^2 \hookrightarrow \overline{\mathbb{D}^2}$ is a $\mathrm{SL}(2, \mathbb{R})$-compactification

We consider compactifications of $X = \mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)$ from Teichmüller theory and symmetric space perspectives.

Two general types of compactifications

- Horofunction (Gromov) compactification: based on embedding $i : X \hookrightarrow C(X, \mathbb{R})$, depends on metric
  - $x \mapsto d(\cdot, x) - d(x, p)$ ($p \in X$ fixed)
  - Get $\overline{X}$ by taking closure of $i(X) \subseteq C(X, \mathbb{R})$
  - Often used for CAT(0) spaces: generalization of geodesic (visual) boundary
- Satake compactification: comes from representation $G \to \mathrm{PSL}(m, \mathbb{C})$ using $X = G/K$
  - Representation induces isometric embedding $X \to \mathrm{PSL}(m, \mathbb{C})/\mathrm{PSU}(m)$
  - Take closure again! Only finitely many up to isomorphism (combinatorics of root system...)

Geometric compactification for $\mathcal{T}(n)$

Geometric/Analytic viewpoint: Given $A \cdot \mathrm{SO}(n) \in \mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)$, $A^T A$ is positive-definite form giving (flat) metric on $\mathbb{T}^n$.

- Compactify $\mathcal{T}(n)$ by including positive-semidefinite forms
- Geometric interpretation for $\overline{\mathcal{T}(n)} = \mathcal{T}(n) \cup \partial\mathcal{T}(n)$?

Let $Q \in \partial\mathcal{T}(n)$. Foliate $\mathbb{T}^n$ by the kernel. Intuitive idea: "collapsing" torus along this foliation! Transverse directions have "relative sizes" (determined by the projective measure).

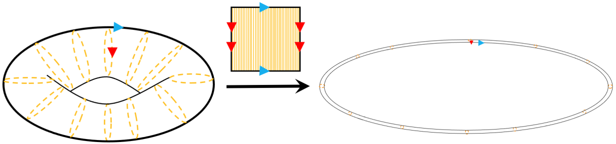


Figure: Foliation of $\mathbb{R}^2$ by lines parallel to the kernel. Q is nonzero on transverse directions.

Compare to Thurston's compactification of Teichmüller space: "collapse" along a foliation, measure transverse to leaves.

Summary of results on compactifications of $\mathcal{T}(n)$

Theorem (G-Ji)

The following are $SL(n, \mathbb{R})$-equivariantly isomorphic:

1. *Thurston compactification via measured foliations on n-tori*
2. *Horofunction compactification w.r.t. Thurston metric*
3. *Satake compactification w.r.t. to standard rep. of $SL(n, \mathbb{R})$.*

Proof ideas:

- Certain measured foliations $\leftrightarrow$ degenerate quadratic forms
- Natural topology and $\mathrm{SL}(n, \mathbb{R})$ action yield homeomorphic and equivariant compactifications
- Compute unit ball for metrics and use result of Haettel-Schilling-Walsh-Wienhard showing unit ball determines Satake compactification for symmetric spaces

Isometric submersions between Teichmüller spaces

Maps between Teichmüller spaces

Theme: given $F : \mathcal{T}_{g,n} \to \mathcal{T}_{k,m}$ preserving analytic/geometric structure, show F is induced by some map $f : S_{g,n} \leftrightarrow S_{k,m}$ preserving topological structure.

Example of f inducing F: if $\varphi \in \mathrm{Mod}(S)$ then $(X, h) \mapsto (X, h \circ \varphi^{-1})$ is a map $\mathcal{T}_{g,n} \to \mathcal{T}_{g,n}$

Beautiful central idea: geometry (metric, $\mathbb{C}$-structure, etc.) of $\mathcal{T}_{g,n}$ reflects topology of $S_{g,n}$

Maps between Teichmüller spaces: past results

Theorem (Royden, '71)

If $F : \mathcal{T}_g \to \mathcal{T}_g$ is a biholomorphism then F is induced by some $\varphi \in Mod(S_g)$ (up to hyperelliptic invol. for $g = 2$)

Generalized several times (Earle, Kra, Gardiner, Lakic 70s - 90s), finally by Markovic (2003) to include all ∞-type surfaces of non-exceptional type.

Maps between different Teichmüller spaces are relatively unexplored!

More Background: Quadratic differentials

- Holomorphic quadratic differential q on a Riemann surface X is locally $q(z)dz^2$ with $q(z)$ holomorphic
- q is integrable if the 1-norm is finite:

$$||q|| := \int_X |q| < \infty$$

- Define $Q(X) = \{\text{integrable holomorphic q.d.s on } X\}$; $\dim_{\mathbb{C}} Q(X) = 3g - 3 + n$.
- For X punctured, write $\hat{X}$ for filled-in surface. Fact: $Q(X) = Q(\hat{X}) \cup \{\text{q.d.s w/ simple poles at punctures}\}$

More Background: Local structure of Teichmüller space

Some classical facts:

There is a dual pairing between the holomorphic tangent space and the space of quadratic differentials for each $X \in \mathcal{T}_{g,n}$:

$$T_X \mathcal{T}_{g,n} \leftrightarrow Q^*(X)$$

The *dual norm* for the infinitesimal Teichmüller metric $||\cdot||_T$ is given by the 1-norm on the space of quadratic differentials:

$$\sup_{v \in T_X \mathcal{T}_{g,n}} \frac{||\phi \cdot v||_T}{||v||_T} = ||\phi|| = \int_X |\phi| \text{ for } \phi \in Q(X)$$

Understanding a map's induced behavior on quadratic differentials gives valuable information about its behavior on Teichmüller space.

Isometric submersions

Definition

An **isometric submersion** $F : M \to N$ is a C^1 *submersion* such that for all $x \in M$,

$$dF_x(\text{unit ball in } T_x M) = \text{unit ball in } T_{F(x)} N$$

"Metric form" of projection

Equivalent: the *co-derivative*

$$dF_x^* : T_{F(x)}^* N \to T_x^* M$$

is an *isometric embedding* with respect to dual norms on cotangent spaces.

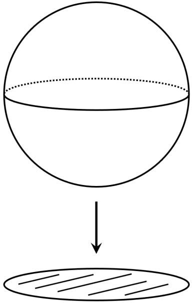


Figure: Infinitesimal isometric submersion

Key example of isometric submersion: the forgetful map

$$F : \mathcal{T}_{g,n} \to \mathcal{T}_{g,m}, \text{ where } m < n$$

Coderivative is inclusion:

$$dF_X^* : Q(\hat{X}) \hookrightarrow Q(X)$$

$$\{\text{q.d.s holo on } \hat{X}\} \hookrightarrow \{\text{q.d.s holo on } X\}$$

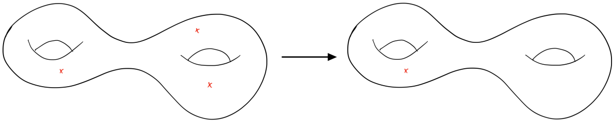


Figure: Forgetful map for $\mathcal{T}_{2,3} \to \mathcal{T}_{2,1}$.

Our question

What are the possible holomorphic isometric submersions between Teichmüller spaces?

The result

Theorem (Gekhtman - G)

Let $F : \mathcal{T}_{g,n} \to \mathcal{T}_{k,m}$ *be a holomorphic isometric submersion with* $k \geq 1$ *and* $2k + m > 4$.
Then $g = k$, $n \geq m$, *and* F *is a forgetful map.*

Remark

1. *We do not assume same genus.*
2. *Conjecture: For* $\mathcal{T}_{2,0}$, $\mathcal{T}_{1,2}$, *the only exceptions involve* $\mathcal{T}_{2,0} \cong \mathcal{T}_{0,6}$ *and* $\mathcal{T}_{1,2} \cong \mathcal{T}_{0,5}$.

The three main steps in the proof

Given $F : \mathcal{T}_{g,n} \to \mathcal{T}_{k,m}$, pick $X \in \mathcal{T}_{g,n}$, then $dF^*_{F(X)} : Q(F(X)) \to Q(X)$ is an isometric embedding.

1. Generalize methods of Markovic and Earle-Markovic (2003) to find a holomorphic map $h : X \to F(X)$ inducing $dF^*_{F(X)}$
2. Riemann-Hurwitz and dimn count $\Rightarrow$ h is forgetful
3. Show h varies continuously across $\mathcal{T}_{g,n}$, so the same forgetful map works everywhere

Topological answer, to a geometric question... via analytic methods!

A quick look at the case of infinite punctures: Challenges

Much of the classical theory has been extended to infinite-dimensional Teichmüller spaces (especially by Earle, Gardiner, and Kra).

- General case: $T_\tau \mathcal{T}(S) = Q^*(X)$
- Finite dimensions: same as saying $T_\tau^* \mathcal{T}(S) = Q(X)$
- Infinite dimensions: vector spaces are irreflexive - cannot simply take coderivative of a map $\mathcal{T}(X) \to \mathcal{T}(Y)$ to get a map $Q(Y) \to Q(X)$
- Need new tools in order to convert results about maps $Q(Y) \to Q(X)$ to results about maps $\mathcal{T}(X) \to \mathcal{T}(Y)$

A quick look at the case of infinite punctures: Results

- Earle-Gardiner Adjointness Theorem: If $T : Q^*(X) \to Q^*(Y)$ is a linear isometry, then there is a linear isometry $S : Q(Y) \to Q(X)$ with $S^* = T$. (Can find a pre-dual!)
- Does not generalize to isometric *submersions*. Sufficient technical condition: T is weak*-sequentially continuous

We can show the following, assuming X and Y are finite-genus:

Theorem (G)

If $F : \mathcal{T}(X) \to \mathcal{T}(Y)$ is a holomorphic isometric submersion whose derivatives are weak*-sequentially continuous, then

1. *X and Y have the same genus.*
2. *If Y has finite punctures, then at each point dF_τ is induced by an inclusion.*

Part (2) uses similar approach to finite case. By postcomposing F with a forgetful map, Part (1) follows from Part (2).

Conclusion

Some research directions about maps between Teichmüller spaces

- Completing the generalization to infinite punctures, expanding to infinite genus (arbitrary Teichmüller spaces)
- Finite type: the remaining cases of $\mathcal{T}_{1,2} \cong \mathcal{T}_{0,5}$ and $\mathcal{T}_{2,0} \cong \mathcal{T}_{0,6}$ with surfaces of exceptional type
- Can we get rid of "holomorphic" or "isometric" from the conditions "holomorphic isometric submersion"?
  - Not both, since a topological submersion would only remember that $\mathcal{T}_{g,n} \cong \mathbb{R}^{6g-6+2n}$
  - Not "submersion" since then we are back to Royden's theorem

Some research directions for $\mathcal{T}(n)$

- Complex tori (i.e. $\mathbb{C}^n/\Lambda$) - maybe can recover some of the complex-analytic theory of Teichmüller space
- Study maps between these Teichmüller spaces: modular interpretation of maps $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n) \to \mathrm{SL}(m,\mathbb{R})/\mathrm{SO}(m)$
- Action of mapping class group and moduli space: metrics and compactifications of $\mathrm{SL}(n,\mathbb{Z})\backslash\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$, the moduli space of unmarked flat tori

Accepted works on which this thesis is based

- Gekhtman, D., Greenfield, M. *Isometric submersions of Teichmüller spaces are forgetful*. To appear in Israel Journal of Mathematics. Available at arXiv:1901.02586.
- Greenfield, M., Ji, L. *Metrics and compactifications of Teichmüller spaces of flat tori*. To appear in Asian Journal of Mathematics. Available at arXiv:1903.10655.

Thank you for Zooming in!

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I have many other marvelous acknowledgments, but this margin is too narrow to contain them