Factorization Homology in 3-Dimensional Topology

Nikita Markarian and Hiro Lee Tanaka

Abstract This Chapter consists of two contributions about the relevance of *factorization homology* (a.k.a. *manifoldic homology* or *topological chiral homology*) in three dimensional topology: 1. Manifoldic Homology and Chern-Simons Formalism, by Nikita Markarian; 2. Factorization Homology and Links Invariants, by Hiro Lee Tanaka.

1 Manifoldic Homology and Chern–Simons Formalism (by Nikita Markarian)

Abstract The aim of this note is to define for any e_n-algebra A and a compact parallelizable n-manifold M without boundary a morphism from the homology of homotopy Lie algebra $A[n-1]$ to the topological chiral homology of M with coefficients in A. This map plays a crucial role in the perturbative Chern-Simons theory.

1.1 Introduction

Manifoldic homology (we suggest this term instead of "topological chiral homology with constant coefficients" from [13]) is a far-reaching generalization of Hochschild homology. In the theory of Hochschild and cyclic homology the additive Dennis

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trace map (e.g. [12, 8.4.16]) plays an important role. Let A be an associative algebra. Denote by $L(A)$ the underlying Lie algebra of A. Then the additive Dennis trace gives a map from $H_*(L(A))$ to $HH_*(A)$. The aim of the present note is to generalize this morphism to any e_n-algebra A.

Let e_n be the operad of rational chains of the operad of little discs and A be an algebra over it. The complex $A[n-1]$ is equipped with homotopy Lie algebra structure, denote it by $L(A)$. Fix a compact oriented n-manifold M without boundary. For simplicity we restrict ourselves to parallelizable manifolds, this restriction may be removed by introducing framed little discs as in [17]. Denote by $HM_*(A)$ the manifoldic homology of A on M introduced in Definition 2, and by $H_*(L(A))$ the Lie algebra homology. In the central Proposition 3 we give a morphism $H_*(L(A)) \to HM_*(A)$ explicitly in terms of the Fulton–MacPherson operad.

For $n = 1$ and $M = S^1$, that is for homotopy associative algebras and Hochschild homology, the above morphism may be presented as the composition of natural morphisms $H_*(L(A)) \to HH_*(U(L(A)) \to HH_*(A)$, where $U(-)$ is the universal enveloping algebra. For $n > 1$ the analogous statement holds, with the universal enveloping algebra replaced by the universal enveloping e_n-algebra. The definition of the latter notion naturally appears in the context of Koszul duality for e_n-algebras, which is still under construction, see nevertheless e.g. [6] and references therein. We need even more, than Koszul duality. The description of application of our construction to manifold invariants requires the Koszul duality for e_n-algebras with curvature. These subjects are briefly discussed in the last section.

Our main construction is exemplary and may be generalized in many ways. For example, one may take some modules over A and put them into some points of M. Then one get a map from homology of $L(A)$ with coefficients in an appropriate module to the manifoldic homology with coefficients in these modules. In particular, if one take copies of A itself as modules, then manifoldic homology with coefficient in them equals to the usual manifoldic homology; thus one get a map from homology of $L(A)$ with coefficients in the tensor product of adjoint modules to $HM_*(A)$. One needs this generalization to build a working theory of invariants of 3-manifolds, I hope to treat this subject elsewhere.

Remark 1 The present note is partially initiated by the work of K. Costello and O. Gwilliam on factorization algebras in perturbative quantum field theory [4], although it is hard to point at exact relations.

1.2 Trees and L_∞

1.2.1 Trees

A tree is an oriented connected graph with three type of vertices: *root* has one incoming edge and no outgoing ones, *leaves* have one outgoing edge and no incoming ones and *internal vertexes* have one outgoing edge and more than one incoming ones.

Edges incident to leaves will be called *inputs*, the edge incident to the root will be called the *output* and all other edges will be called *internal edges*. The degenerate tree has one edge and no internal vertexes. Denote by $T_k(S)$ the set of non-degenerate trees with k internal edges and leaves labeled by a set S.

For two trees $t_1 \in T_{k_1}(S_1)$ and $t_2 \in T_{k_2}(S_2)$ and an element $s \in S_1$ the composition of trees $t_1 \circ_s t_2 \in T_{k_1+k_2+1}$ is obtained by identification of the input of t_1 corresponding to s and the output of t_2. Composition of trees is associative and the degenerate tree is the unit. The set of trees with respect to the composition forms an operad.

Call the tree with only one internal vertex the *star*. Any non-degenerate tree with k internal edges may be uniquely presented as a composition of $k+1$ stars.

The operation of *edge splitting* is the following: take a non-degenerate tree, present it as a composition of stars and replace one star with a tree that is a product of two stars and has the same set of inputs . The operation of an edge splitting depends on a internal vertex and a subset of more than one incoming edges.

1.2.2 L_∞

For a non-degenerate tree t denote by $\mathrm{Det}(t)$ the one-dimensional $\mathbb{Q}$-vector space that is the determinant of the vector space generated by internal edges. For $s > 1$ consider the complex

$$L(s)\colon \bigoplus_{t\in T_0([\mathbf{s}])} \mathrm{Det}(t) \to \bigoplus_{t\in T_1([\mathbf{s}])} \mathrm{Det}(t) \to \bigoplus_{t\in T_2([\mathbf{s}])} \mathrm{Det}(t) \to \cdots, \tag{1}$$

where $[\mathbf{s}]$ is the set of s elements, the cohomological degree of a tree $t \in T_k([\mathbf{s}])$ is $2-s+k$ and the differential is given by all possible splitting of an edge (see e.g. [9]). The composition of trees equips the sequence $L(i)\otimes \mathrm{sgn}$ with the structure of a dg-operad, here sgn is the sign representation of the symmetric group.

This operad is called L_∞ *operad*. Denote by $L_\infty[n]$ the dg-operad given by the complex $L(s)[n(s-1)]\otimes(\mathrm{sgn})^n$ and refer to it as n-shifted L_∞ operad.

1.3 Fulton–MacPherson Operad

1.3.1 Fulton–MacPherson Compactification

The Fulton–MacPherson compactification is introduced in [8, 14], see also [1, 17]. We cite here its properties that are essential for our purposes.

For a finite set S denote by $(\mathbb{R}^n)^S$ the set of ordered S-tuples in $\mathbb{R}^n$. For a finite set S denote by $\Delta_S\colon \mathbb{R}^n \to (\mathbb{R}^n)^S$ the diagonal embedding. We will denote by $[\mathbf{n}]$ the set of n elements.

Let $\mathscr{C}^0_S(\mathbb{R}^n) \subset (\mathbb{R}^n)^S$ be the space of ordered pairwise distinct points in $\mathbb{R}^n$ labeled by S. The *Fulton–MacPherson compactification* $\mathscr{C}_S(\mathbb{R}^n)$ is a manifold with corners with interior $\mathscr{C}^0_S(\mathbb{R}^n)$. The projection $\mathscr{C}_S(\mathbb{R}^n) \xrightarrow{\pi} (\mathbb{R}^n)^S$ is defined, which is an isomorphism on $\mathscr{C}^0_S(\mathbb{R}^n) \subset \mathscr{C}_S(\mathbb{R}^n)$. Moreover there is a sequence of manifolds with corners $F_n(S)$ labeled by finite sets and maps $\phi_{S_1,\dots,S_k}$ that fit in the diagram

$$\begin{array}{ccc} F_n(S_1)\times\cdots\times F_n(S_k)\times\mathscr{C}_{[\mathbf{k}]}(\mathbb{R}^n) & \xrightarrow{\phi_{S_1,\dots,S_k}} & \mathscr{C}_{(S_1\cup\cdots\cup S_k)}(\mathbb{R}^n) \\ \downarrow{\pi} & & \downarrow{\pi} \\ (\mathbb{R}^n)^k & \xrightarrow{\Delta_{S_1}\times\cdots\times\Delta_{S_k}} & (\mathbb{R}^n)^{(S_1\cup\cdots\cup S_k)} \end{array}$$

where the left arrow is the projection to the point on the first factors and π on the last one. Restrictions of $\phi_{S_1,\dots,S_k}$ to $F_n(S_1)\times\cdots\times F_n(S_k)\times\mathscr{C}^0_{[\mathbf{k}]}(\mathbb{R}^n)$ are isomorphisms onto the image. It follows that $F_n(S) = \pi^{-1}\mathbf{0}$, where $\mathbf{0} \in (\mathbb{R}^n)^S$ is S-tuple sitting at the origin. Being restricted on $F_n(S) \subset \mathscr{C}_S(\mathbb{R}^n)$, maps ϕ equip $F_n(S)$ with an operad structure:

$$\phi_{[\mathbf{s_1}],\dots,[\mathbf{s_k}]} \colon F_n([\mathbf{s_1}])\times\cdots\times F_n([\mathbf{s_k}])\times F_n([\mathbf{k}]) \to F_n([\mathbf{s_1}+\cdots+\mathbf{s_k}]).$$

Manifolds $\mathscr{C}_{[\mathbf{k}]}(\mathbb{R}^n)$ and $F_n([\mathbf{k}])$ are equipped with a k-th symmetric group action consistent with its natural action on $\mathscr{C}^0_{[\mathbf{k}]}(\mathbb{R}^n)$ and all maps are compatible with this action.

Definition 1 [8, 14, 17] The sequence of spaces $F_n([\mathbf{k}])$ with the symmetric group action and composition morphisms as above is called the *Fulton–MacPherson operad.*

1.3.2 Strata, Trees and L_∞

There is a map of sets $F_n(S) \xrightarrow{\mu} T(S)$ that subdivides $F_n(S)$ into smooth strata. This map is totally defined by the following properties. Firstly, μ is consistent with the operad structure in the sense that the preimage of a composition is the composition of preimages. Secondly, the zero codimension stratum corresponding to a star tree is the intersection of $\pi^{-1}\mathbf{0}$ and the stratum of $\mathscr{C}_S(\mathbb{R}^n)$ that is the blow-up of the small diagonal minus pull backs of other diagonals. These latter strata freely generate the Fulton–MacPherson operad as a set.

Denote by $C_*(F_n)$ the dg-operad of rational chains of the Fulton–MacPherson operad. For a tree $t \in T(S)$ let $[\mu^{-1}(t)] \in C_*(F_n(S))$ be the chain presented by its preimage under μ.

Proposition 1 *Map $[\mu^{-1}(\cdot)]$ gives a morphism from shifted L_∞ operad $L(s)[s(1-n)]$ to the dg-operad $C_*(F_n([\mathbf{s}]))$ of rational chains of the Fulton–MacPherson operad.*

Proof To see that the map commutes with the differential note, that two strata given by μ with dimensions differing by 1 are incident if and only if one of the corresponding trees is obtained from another by edge splitting. In this way we get a basis in the conormal bundle to a stratum labeled by the internal edges. It follows the consistency of the map from the statement with signs.

It follows that there is a morphism of dg-operads

$$L_\infty[1-n] \to C_*(F_n) \tag{2}$$

Let e_n be the dg-operad of rational chains of the operad of little n-discs.

Proposition 2 *Operad $C_*(F_n)$ is weakly homotopy equivalent to e_n.*

Proof See [17, Proposition 3.9].

Thus there is a homotopy morphism of operads $L_\infty[1-n] \to e_n$.

1.4 Manifoldic and Lie Algebra Homology

1.4.1 Manifoldic Homology

Let M be an n-dimensional parallelized compact manifold without boundary. In the same way as for $\mathbb{R}^n$ there is the Fulton-MacPherson compactification $\mathscr{C}_S(M)$ of the space $\mathscr{C}_S^0(M)$ of ordered pairwise distinct points in M labeled by S; inclusion $\mathscr{C}_S^0(M) \hookrightarrow \mathscr{C}_S(M)$ is a homotopy equivalence. There is a projection $\mathscr{C}_S(M) \xrightarrow{\pi} M^S$ and maps $\phi_{S_1,\dots,S_k}$ that fit in the diagram

$$\begin{array}{ccc} F_n(S_1)\times\cdots\times F_n(S_k)\times\mathscr{C}_{[\mathbf{k}]}(M) & \xrightarrow{\phi_{S_1,\dots,S_k}} & \mathscr{C}_{(S_1\cup\cdots\cup S_k)}(M) \\ \downarrow \pi & & \downarrow \pi \\ M^k & \xrightarrow{\Delta_{S_1}\times\cdots\times\Delta_{S_k}} & M^{(S_1\cup\cdots\cup S_k)} \end{array}$$

and are isomorphisms on $F_n(S_1)\times\cdots\times F_n(S_k)\times\mathscr{C}^0_{[\mathbf{k}]}(M)$, where $\Delta_S\colon M\to M^S$ are the diagonal maps. It follows that spaces $\mathscr{C}_*(M)$ form a right module over the PROP generated by the Fulton-MacPherson operad

$$P(F_n)(m,l) = \bigcup_{\sum m_i = m} F_n(m_1)\times\cdots\times F_n(m_l).$$

This module as a set is freely generated by $\mathscr{C}_*^0(M)$. The stratification on F_n defines a stratification on $\mathscr{C}_*(M)$.

Denote by $C_*(\mathscr{C}_{[\mathbf{k}]}(M))$ the complex of rational chains of the Fulton-MacPherson compactification.

Definition 2 For a $C_*(F_n)$-algebra A and a compact parallelized n-manifold without boundary M call the complex $CM_*(A) = C_*(\mathscr{C}_*(M)) \otimes_{C_*(P(F_n))} A$ the *manifoldic* *chain complex* of A on M. Call the homology of the manifoldic chain complex the *manifoldic homology* of A on M.

This definition is based on Definition 4.14 from [17]. By Proposition 2 one may pass from a $C_*(F_n)$-algebra to an e_n-algebra. As it is shown in [13], the manifoldic homology is the same as the topological chiral homology with constant coefficients introduced in *loc. cit* of this e_n-algebra.

1.4.2 Morphism

Let $(\mathfrak{g}, d)$ be a L_∞-algebra. Let $l_{i>1}\colon \Lambda^i\mathfrak{g}[i-2] \to \mathfrak{g}$ be its higher brackets, that is, the operations in complex (1) corresponding to the star trees. The structure of L_∞-algebra may be encoded in a derivation $D = D_1 + D_2 + \cdots$ on the free super-commutative algebra generated by $\mathfrak{g}^\vee[1]$, where D_1 is dual to d and D_i is dual to l_i on generators and are continued on the whole algebra by the Leibniz rule. The *Chevalley–Eilenberg chain complex* $CE_*(\mathfrak{g})$ is the super-symmetric power $S^*(\mathfrak{g}[-1])$ with the differential $\mathrm{d_{tot}} = d + \theta_2 + \theta_3 + \cdots$, where θ_i is dual to D_i.

Denote by $[\mathscr{C}^0_{[\mathbf{k}]}] \in C_*(\mathscr{C}_{[\mathbf{k}]}(M))$ the chain given by the submanifold $\mathscr{C}^0_{[\mathbf{k}]}(M)$ in $\mathscr{C}_{[\mathbf{k}]}(M)$. For a $C_*(F_n)$-algebra A and a cycle $c \in C_*(\mathscr{C}_{[\mathbf{k}]}(M))$ denote by $(a_1 \otimes \cdots \otimes a_k) \otimes_{\Sigma_k} c \in CM_*(A)$ the chain given by the tensor product over the symmetric group. Recall that by (2) for any $C_*(F_n)$-algebra A the complex $A[n-1]$ is equipped with a L_∞ structure. Denote this L_∞-algebra by $L(A)$. Denote by $\mathrm{Alt}(a_1 \otimes \cdots \otimes a_k)$ the sum $\sum_\sigma \pm a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(k)}$ by all permutations, where signs are sign given by the sign of the permutation and the Koszul sign rule.

Proposition 3 *For a $C_*(F_n)$-algebra A and a parallelized compact manifold without boundary M the map* $\mathrm{T}\colon a_1 \wedge \cdots \wedge a_k \mapsto \mathrm{Alt}(a_1 \otimes \cdots \otimes a_k) \otimes_{\Sigma_k} [\mathscr{C}^0_{[\mathbf{k}]}]$ *defines a morphism from Chevalley–Eilenberg complex $CE_*(L(A))$ to the manifoldic chain complex $CM_*(A)$.*

Proof Denote the total differentials on both complexes $CE_*(L(A))$ and $CM_*(A)$ by $\mathrm{d_{tot}}$. One needs to show that $\mathrm{d_{tot}} \circ \mathrm{T} = \mathrm{T} \circ \mathrm{d_{tot}}$.

The boundary of $[\mathscr{C}^0_{[\mathbf{k}]}]$ in $C_*(\mathscr{C}_{[\mathbf{k}]}(M))$ is the sum of all codimension one strata: $\partial[\mathscr{C}^0_{[\mathbf{k}]}] = \sum_i \theta_i[\mathscr{C}^0_{[\mathbf{k-i+1}]}]$. Here θ_i is the symmetrization in $C_*(P(F_n))$ of the operation in $C_*(F_n)$ that corresponds by Proposition 1 to the star with i inputs. This means that

$$\mathrm{d_{tot}} \circ \mathrm{T}\,(a_1 \wedge \cdots \wedge a_k) = (d\,\mathrm{Alt}(a_1 \otimes \cdots \otimes a_k)) \underset{\Sigma_k}{\otimes} [\mathscr{C}^0_{[\mathbf{k}]}] + \mathrm{Alt}(a_1 \otimes \cdots \otimes a_k) \underset{\Sigma_k}{\otimes} \sum_{i>1} \theta_i [\mathscr{C}^0_{[\mathbf{k-i+1}]}]$$

One may carry θ's from one factor of $\otimes_{\Sigma_k}$ to another by the very definition of the tensor product over $C_*(P(F_n))$. And the action of θ's on the alternating sum again by definition is given by the higher brackets of the L_∞-algebra. After summing with d it gives the differential on the Chevalley–Eilenberg complex. It follows that $\mathrm{d_{tot}} \circ \mathrm{T} = \mathrm{T} \circ \mathrm{d_{tot}}$.

1.5 Sketch: Invariants of a Parallelized Manifold and Koszul Duality

1.5.1 Invariant of a Parallelized Manifold

The idea how to apply manifoldic homology to manifolds invariant is the following. Below (Definition 3) we sketch a construction of a e_n-algebra $\mathfrak{D}^n(V)$ such that for any n-dimensional parallelized compact manifold without boundary M manifoldic homology $HM_*(\mathfrak{D}^n(V))$ is one-dimensional (Proposition 4). Then,

$$H_*(L(\mathfrak{D}^n(V))) \to HM_*(\mathfrak{D}^n(V)) \tag{3}$$

given by Proposition 3 supplies us with a cocycle in the Lie algebra cohomology of $L(\mathfrak{D}^n(V))$.

In this way we obtain an invariant that is conjecturally related to the universal Chern–Simons invariant (see [1, 3]) which takes value in "graph cohomology", with the Chevalley–Eilenberg cochain complex of $L(\mathfrak{D}^n(V))$ representing the "graph complex".

1.5.2 Koszul Duality

Quillen duality [11, 16] gives an equivalence between homotopy categories of Lie algebras Lie and connected cocommutative coalgebras $coCom$. Koszul duality [6, 13] is an analogous equivalence between the categories of augmented e_n- algebras and coaugmented e_n-coalgebras satisfying certain conditions analogous to connectedness. I hope to elaborate on these conditions elsewhere. Denote the above mentioned categories by $e_n - alg$ and $e_n - coalg$. The relationship between Quillen and Koszul dualities is displayed in the diagram

$$\begin{array}{ccccc}
L: & e_n - alg & \rightleftarrows & Lie & : U^n \\
\text{Koszul duality} & \updownarrow & & \updownarrow & \text{Quillen duality} \\
\mathrm{Ab}: & e_n - coalg & \rightleftarrows & coComm & : \imath
\end{array}$$

Here, the functor L is given by (2), U^n is the derived universal enveloping e_n- algebra the functor that is derived left adjoint to L, $\imath$ is the embedding of cocommutative coalgebras in e_n-coalgebras and Ab is its derived right adjoint.

The linear dual of a e_n-coalgebra is a e_n-algebra. If some e_n-algebra and e_n-coalgebra are related by Koszul duality, then the first one and the linear dual of the second one are called *Koszul dual e_n-algebras*.

The following statement generalizes the well-known fact that Hochschild homologies of Koszul dual algebras are dual to each other (see e.g. [18, Appendix D]).

Claim (Poincaré–Koszul duality) For a n-dimensional parallelized compact manifold without boundary M, the manifoldic homologies on M of Koszul dual e_n-algebras are linear dual to each other.

1.5.3 *n*-Weyl Algebra

We say that an element c of a e_n-algebra A is *central*, if the product map

$$e_n(k+1) \otimes \underbrace{c \otimes A \otimes \cdots \otimes A}_{k+1} \to A$$

factors through

$$e_n(k+1) \otimes \underbrace{c \otimes A \otimes \cdots \otimes A}_{k+1} \to c \otimes e_n(k) \otimes \underbrace{A \otimes \cdots \otimes A}_{k}.$$

The latter map is induced by the natural projection from $k+1$-ary operations to k-ary ones.

By a *e_n-algebra with curvature* we mean a e_n-algebra with a central element c of degree $n+1$. The condition on c may be relaxed by analogy with [15]. The new condition may be formulated in terms of the deformation complex of an e_n-algebra.

Conjecturally, one may define Koszul duality for e_n-algebras with curvature in such a way, that for $n = 1$, one recovers Koszul duality for algebras with curvature, see [15].

Let V be a graded vector space with a non-degenerate symmetric in the graded sense bilinear form q of degree $-(n+1)$. Let $S^*(V^\vee)$ be the free graded commutative algebra generated by the vector space dual to V. Denote by $S^*(V^\vee)^\vee$ the restricted dual coalgebra. By means of inclusion ι from (4) consider the pair $(S^*(V^\vee)^\vee, q)$ as a e_n-coalgebra with curvature.

Definition 3 For a graded vector space V with a non-degenerate symmetric in the graded sense bilinear form q of degree $-(n+1)$ we denote by $\mathfrak{D}^n(V)$ the e_n-algebra Koszul dual to $(S^*(V^\vee)^\vee, q)$ and refer to it as *n-Weyl algebra*.

Proposition 4 *For a n-dimensional parallelized compact manifold without boundary M, the manifoldic homology $HM_*(\mathfrak{D}^n(V))$ is one-dimensional.*

Proof By Statement 1.5, $HM_*(\mathfrak{D}^n(V))$ is linear dual to the manifoldic homology of the e_n-algebra that is Koszul dual to $\mathfrak{D}^n(V)$. Thus, one needs to prove that the latter homology is one-dimensional. The Koszul dual e_n-algebra is e_n-algebra with curvature $(S^*(V^\vee), q)$. The manifoldic homology of $S^*(V^\vee)$ is the free commutative algebra generated by $V^\vee \otimes H_*(M)$, where $H_*(M)$ is homology of M negatively graded. The curvature equips the underlying space of this algebra with a differential given by multiplication by an element of cohomological degree 1 and of homogeneous degree 2. This element represents the pairing induced by the tensor product of q on V and the Poincaré paring on $H^*(M)$. The cohomology of this differential, that is of the de Rham complex of a graded vector space, is manifoldic homology of the e_n-algebra with curvature. As the cohomology of the de Rham complex is one-dimensional, this implies the proposition.

Example 1 Let $n = 1$ and V is concentrated in degree 1. Then $\mathfrak{D}^1(V)$ is the usual Weyl algebra, that is the symplectic Clifford algebra generated by vector space $V[-1]$ with the skew-symmetric form on it. For $M = S^1$ the manifoldic homology is the Hochschild homology and Proposition 4 matches with the well-known fact about Weyl algebra:

$$\dim HH_i(\mathfrak{D}^1(V)) = \begin{cases} 1, & i = \dim V, \\ 0, & \text{otherwise.} \end{cases}$$

Note that this fact is crucially used in [5] and classes like (3) and (6) below restricted to the Lie algebra of vector fields are exploited there to present the Todd class.

1.5.4 Concluding Remarks

Finally, let us look at the morphism from Proposition 3 from the Koszul duality viewpoint.

For a commutative algebra C there is a canonical morphism $HH_*(C) \to C$. It may be generalized to manifoldic homology as follows.

Claim For a homotopy commutative algebra (= e_∞-algebra) C and for a n-dimensional parallelized compact manifold without boundary M there is a canonical map from manifoldic chain complex of C to C itself:

$$\pi\colon CM_*(\imath(C)) \to C. \tag{4}$$

Morphism π may be constructed by means of manifoldic homology of non-compact manifolds: every manifold may be embedded $\mathbb{R}^N$ and as commutative algebra may be considered as e_N-algebra, the embedding induces a morphism of manifoldic homologies, and manifoldic homology of C on $\mathbb{R}^N$ is C.

Diagram (4) shows that for a Lie algebra $\mathfrak{g}$ the e_n-algebras $U^n(\mathfrak{g})$ and $\imath(CE^*(\mathfrak{g}))$ are Koszul dual, where CE^* is the Chevalley–Eilenberg cochain complex. By Poincaré–Koszul duality (Statement 1.5) for a n-dimensional parallelized compact manifold without boundary M homologies $HM_*(U^n(\mathfrak{g}))$ and $HM_*(\imath(CE^*(\mathfrak{g})))$ are dual to each other. Morphism (4) gives $\pi\colon CM_*(\iota(CE^*(\mathfrak{g})) \to CE^*(\mathfrak{g})$ and composing with Poincaré–Koszul duality we obtain the map

$$H_*(\mathfrak{g}) \to HM_*(U^n(\mathfrak{g})). \tag{5}$$

Functors U^n and L from (4) being adjoint, there is as canonical morphism $U^n(L(A)) \to A$ for any e_n-algebra A. It induces a map on manifoldic homologies:

$$HM_*(U^n(L(A))) \to HM_*(A). \tag{6}$$

Claim The effect of the morphism from Proposition 3 on homologies is the composition of (5) for $\mathfrak{g} = L(A)$ and (6).

This morphism may be described even simpler in Koszul dual terms. The Koszul dual morphism is the composition

$$CM_*(A^!) \to CM_*(\iota(\mathrm{Ab}(A^!))) \to A^!, \tag{7}$$

where the first arrow is induced by the canonical morphism for a pair of adjoint functors ι and Ab and the second arrow is given by (4).

For our main example $A = \mathfrak{D}^n$ the Koszul dual e_n-algebra $A^!$ is a e_n-algebra with curvature and the formula (7) is not applicable directly. It is not clear, what the functor Ab means for such algebras. The question is interesting even for $n = 1$, where Ab is the derived quotient by the ideal generated by commutators.

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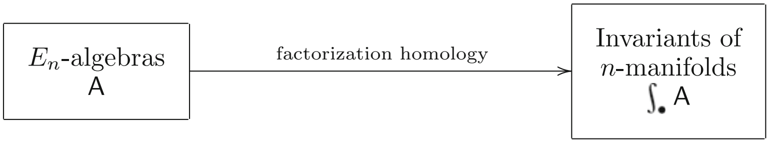
2 Factorization Homology and Link Invariants (by Hiro Lee Tanaka)

Abstract We define the notions of E_n-algebras and factorization homology, sketching how one can construct link invariants using a version of factorization homology for stratified manifolds. The work on factorization homology for stratified manifolds is joint with David Ayala and John Francis.

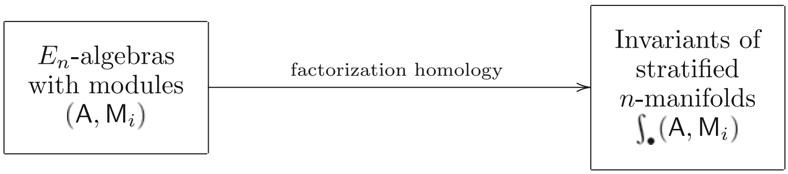
2.1 Overview

Factorization homology is a way to construct invariants of n-manifolds from one piece of algebraic data. This algebraic data is an *E_n-algebra* A, and the invariant associated to an n-manifold X is called the *factorization homology of X with coefficients in* A. We write this as $\int_X \mathsf{A}$.



The idea of using E_n-algebras to create invariants of n-manifolds has been in the air for some time, but ongoing work with Ayala and Francis [2] generalizes factorization homology to define invariants of *stratified manifolds*. For instance, one can define invariants for manifolds with boundary, for singular manifolds (such as graphs or cones), for singular manifolds with decorations (such as colored graphs), and for manifolds stratified by the image of an embedding. (As in the title, this includes the case of a link inside S^3.)

To define such an invariant we need more algebraic data than just an E_n-algebra. Roughly speaking, we need the data of E_n-algebras and modules over them.



There is also a physical motivation for factorization homology. Factorization homology is also called *topological chiral homology* (for instance, by Jacob Lurie in [13]) and this terminology is no accident. 'Chiral homology' is a concept familiar from conformal field theories—in studying conformal field theories, one inputs a chiral algebra, and chiral homology (i.e., the space of conformal blocks) is what one assigns to a Riemann surface.

Topological chiral homology is the topologist's analogue of this invariant—instead of a chiral algebra we input an E_n-algebra, and we produce an invariant sensitive to the diffeomorphism type of a manifold.[1] In other words, topological chiral homology should be the output of a topological field theory, instead of a conformal one.

More precisely, when one has a classical field theory defined on a space-time manifold X, the observables of the quantized field theory form a *factorization algebra*

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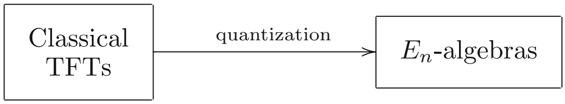
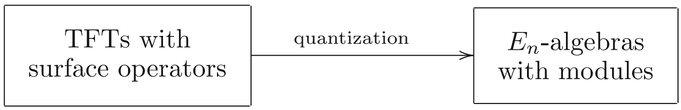
[1] As will be mentioned later, in this talk we create an invariant of manifolds *with a framing*. If we are interested in studying manifolds with some structure B (such as an orientation), there is a class of algebras (such as an E_B algebra) which defines invariants for all manifolds with B-structures.

on X. This is explained in Chap. 3. If the field theory is topological, a factorization algebra on a space-time is locally the same thing as an E_n-algebra.



The global observables of the field theory on X is precisely the factorization homology of X with coefficients in this E_n-algebra.

Finally, in a theory with surface operators (as in the work of Gukov and Witten in [10]), or Wilson loops (such as Chern-Simons theory) or ‘t Hooft lines, one expects the quantization to see the structure of embedded surfaces and embedded curves. In the case these field theories are topological, one can broadly call such theories “TQFTS with surface operators,” and we expect to produce E_n algebras with modules from such field theories:



One goal is to understand the E_3-algebras with modules that should arise from Chern-Simons theory.

2.2 E_n-algebras

In what follows, D^n denotes the open n-disk of unit radius. We first define a category Disk_n^{fr}, enriched over topological spaces.

Definition 4 An object of Disk_n^{fr} is a (possibly empty) finite set S. The set of morphisms $\mathsf{Disk}_n^{fr}(S, T)$ is given by embeddings $f : (D^n)^{\sqcup S} \to (D^n)^{\sqcup T}$ such that, on each connected component, f is of the form

$$f(\mathbf{x}) = \lambda \mathbf{x} + C$$

for some fixed $\lambda > 0, C \in D^n$. The set $\mathsf{Disk}_n^{fr}(S, T)$ inherits a topology as a subspace of all continuous maps, which we topologize by the compact-open topology.

Note that Disk_n^{fr} has a symmetric monoidal structure given by disjoint union. Also, the fr stands for ‘framed’—see the remark after Definition 3.1.

Definition 5 (E_n*-algebra*) Let Chain_k be the category of chain complexes over some base field k. An E_n-algebra A is a symmetric monoidal functor

$$\mathsf{A} : \mathsf{Disk}_n^{fr} \to \mathsf{Chain}_k.$$

of categories enriched in topological spaces.

Remark 2 Chain_k admits an enrichment over topological spaces in a standard way, for instance by using the Dold-Kan correspondence on morphisms. Heuristically, the condition that A be a symmetric monoidal functor of categories enriched in topological spaces means:

1. $\mathsf{A}(\emptyset) = k$
2. A sends disjoint unions to tensor products
3. A sends (higher) isotopies of embeddings to (higher) chain homotopies,

where the last condition means A is a continuous map on morphism spaces.

Remark 3 One can obviously define what an E_n-algebra is for any target category $\mathcal{C}$ whose morphisms sets are spaces, and who has a symmetric monoidal structure.

Remark 4 Throughout, we write A for the functor, and we will write A for $\mathsf{A}(D^n)$.

Example 2 ($n = 1$) As we've heard before during the winter school, the $n = 1$ case recovers the notion of an associative algebra. Let me explain how.

First, we note that the inclusion of two disjoint intervals into a single interval gives a map

$$m : A \otimes A \to A$$

and the inclusion of the empty set into D^1 yields a unit map

$$1 : k \to A.$$

Composition of embeddings and the tensor product property (2) shows that 1 is indeed a unit for the multiplication m. Finally, factoring the inclusion

$$D^1 \sqcup D^1 \sqcup D^1 \to D^1$$

in two different ways yields the associativity condition on m.

However, there is a subtlety—because one can wiggle embeddings by isotopies, what we really find is that m ought to be an associative multiplication *up to higher homotopies*. So the correct statement is that any E_1-algebra is in fact an A_∞ algebra. We state this result for the record:

Proposition 5 *The category of E_1-algebras is equivalent to the category of unital A_∞-algebras in Chain_k.*

Remark 5 On a first pass, no real intuition is lost by simply thinking of A_∞ algebras as associative algebras. However, as the next example shows, for $n > 1$ we shouldn't be so cavalier.

Example 3 ($n = 2$) Let A be an E_2-algebra. Given a configuration of two disks inside the unit disk, we get a multiplication

$$m : A \otimes A \to A.$$

However, we see that there is an isotopy of embeddings taking this configuration to one in which the embedded rectangles have flipped labeles—i.e., there is a homotopy between

$$m(x_1, x_2) \quad \text{and} \quad m(x_2, x_1)$$

where $x_i \in A$ are elements of the algebra. This shows that m is in fact a commutative multiplication if you only remember m up to homotopy. However, as anybody who's studied the configuration space of points in $\mathbb{R}^2$ knows, there is a braid group hiding in this picture—namely, if you have j embedded disks in D^2, you can reconfigure them in ways that are homotopic, but not canonically so. (i.e., the configuration space of rectangles is connected, but has non-trivial topology.)

So remembering m only up to homotopy would discard the information of the braid group. (In fact, if the target category were vector spaces, rather than chain complexes, an E_2-algebra is the same things as a commutative algebra.) The conclusion is that E_2-algebras in fact encode a delicate system of multiplications, and sees the geometry of the configuration space of disks in D^2. This is precisely the reason that an E_n algebra should be expected to yield invariants of n-manifolds.

2.3 Factorization Homology

Definition 6 Let Mfld_n^{fr} be the topologically enriched category whose objects are smooth n-manifolds X together with a *framing*, i.e., a choice of isomorphism $\phi_X : TX \cong X \times \mathbb{R}^n$. A morphism in $\mathsf{Mfld}_n^{fr}(X, Y)$ is a pair (f, h) where $f : X \to Y$ is an embedding, and h is a choice of homotopy from $f^*\phi_Y$ to ϕ_X.

Note that Disk_n^{fr} can be viewed as the full subcategory of Mfld_n^{fr} consisting of objects which are diffomeorphic to disjoint copies of D^n. This is because the space of rectilinear embeddings is homotopy equivalent to the space of framed embeddings.

So given an algebra A, the question is whether we can extend the functor A to the whole of Mfld:

$$\begin{array}{ccc} \mathsf{Mfld}_n^{fr} & \overset{?}{\dashrightarrow} & \mathsf{Chain}_k \\ \uparrow & \nearrow_{\mathsf{A}} & \\ \mathsf{Disk}_n^{fr} & & \end{array}$$

There is in fact a general way of doing this since Chain_k itself is a well-behaved category:

Definition 7 (*Factorization Homology*) Let $\int : \mathsf{Mfld}_n^{fr} \to \mathsf{Chain}_k$ be the *left Kan extension* of the functor A along the inclusion $\mathsf{Disk}_n^{fr} \to \mathsf{Mfld}_n^{fr}$. We call the resulting functor *factorization homology*, and write

$$X \mapsto \int_X \mathsf{A}$$

for any n-manifold X. We say that $\int_X \mathsf{A}$ is *the factorization homology of X with coefficients in* A.

Remark 6 (**Left Kan extensions**) Intuitively, any n-manifold is understood by seeing how it is glued together from many copies of $\mathbb{R}^n$. So one can express a manifold as a gigantic diagram of embedded copies of $\mathbb{R}^n$, together with gluing maps. To define the left Kan extension, one simply writes down the same diagram in the category of chain complexes, and glues along the corresponding maps given by the functor A. (i.e., one takes the colimit of the corresponding diagram.) Also, the left Kan extension we take is not a naive left Kan extension, but the ∞-categorical Kan extension. Equivalently, one takes the homotopy left Kan extension.

Remark 7 If one begins with an E_n-algebra in a general symmetric monoidal, topologically enriched category $\mathcal{C}$, one can still define factorization homology as left Kan extension so long as $\mathcal{C}$ admits enough colimits.

2.4 The Main Theorem

We first record some properties of factorization homology:

Theorem 1 *Factorization homology satisfies the following properties:*

1. *It sends disjoint unions of manifolds to tensor products of chain complexes.*
2. *It sends (higher) isotopies of embeddings to (higher) homotopies of chain maps.*
3. *It satisfies excision. That is, if a manifold X can be written as a union*

$$X = X_0 \cup_{Y \times D^1} X_1$$

where $X_0 \cap X_1 \cong Y \times D^1$ as framed manifolds, then the factorization homology of X is given by the bar construction

$$\int_X \mathsf{A} \cong \int_{X_0} \mathsf{A} \otimes_{\int_{Y \times D^1} \mathsf{A}} \int_{X_1} \mathsf{A}.$$

Remark 8 *(Excision)* The difficulty of the theorem is not in (1) and (2), but in the excision property. As discussed before, any E_1 algebra is an A_∞ algebra. And we see that the natural inclusion

$$\{id_Y\} \times \mathsf{Emb}((D^1)^{\sqcup i}, D^1) \subset \mathsf{Emb}((Y \times D^1)^{\sqcup i}, Y \times D^1)$$

gives the structure of an E_1 algebra to $\int_{Y \times D^1} \mathsf{A}$.

Moreover, we have a family of natural embeddings

$$(Y \times D^1) \sqcup X_0 \to X_0, \qquad (Y \times D^1) \sqcup X_1 \to X_1$$

which, by the monoidal property of factorization homology, give rise to maps

$$(\int_{X_0} \mathsf{A}) \otimes (\int_{Y \times D^1} \mathsf{A}) \to \int_{X_0} \mathsf{A}, \qquad (\int_{Y \times D^1} \mathsf{A}) \otimes (\int_{X_1} \mathsf{A}) \to \int_{X_1} \mathsf{A}.$$

In other words, the decomposition gives $\int_{X_0} A$ and $\int_{X_1} A$ the structure of a right- and left-modules over $\int_{Y \times D^1} A$, respectively. Thus the bar construction makes sense in (3).

Conversely, let $\mathcal{H}$ be the category of all functors $H : \mathsf{Mfld}_n^{fr} \to \mathsf{Chain}_k$ satisfying the properties (1)–(3) in the theorem above. There is a clear map

$$ev_{\mathbb{R}^n} : \mathcal{H} \to \; E_n\text{-alg}$$

given by evaluating $\mathcal{H}$ at the manifold $\mathbb{R}^n$. The following recognition principle was proven by John Francis in [7]:

Theorem 2 (Francis) *$ev_{\mathbb{R}^n}$ is an equivalence of categories. An inverse functor is given by factorization homology.*

My joint work with David Ayala and John Francis [2] replaces Mfld_n^{fr} by a category SMfld_n of *stratified* n-manifolds.[2] The point is that, even for a stratified n-manifold X, we know what the local structure of X looks like. For instance, a graph locally looks like an interval or an i-valent vertex for some i. And an embedded submanifold $A \subset B$ looks locally like an open neighborhood of B, or like the tubular neighborhood of an open patch in A. Such local pieces form a category Loc—this is in analogy with the case of smooth manifolds, where the local pieces form the category Disk_n^{fr}. Roughly speaking, Loc is a category whose objects are disjoint unions of local pieces, and whose morphisms are embeddings between them.

Then a functor $\mathsf{A} : \mathsf{Loc} \to \mathsf{Chain}_k$ is called a Loc-algebra, and these often give structures that look like modules over an E_n-algebra, where n is the top dimension of pieces in Loc. (I will give examples in the next subsection.)

Once more one can define factorization homology, for stratified manifolds, by taking the left Kan extension

$$\begin{array}{ccc} \mathsf{SMfld}_n & \overset{\int}{\dashrightarrow} & \mathsf{Chain}_k \; . \\ \uparrow & \nearrow_{\mathsf{A}} & \\ \mathsf{Loc} & & \end{array}$$

[2] The notation is somewhat misleading, since there is not a unique category of stratified manifolds—one can choose to include or exclude certain kinds of stratifications, but this is irrelevant to the philosophy of this talk.

This functor still satisfies excision, and the main result of our joint work is the following generalization of the previous theorem:

Theorem 3 (Ayala-Francis-T) *Let $\mathcal{H}_S$ be the category of functors $H : \mathsf{SMfld}_n \to \mathsf{Chain}_k$ which satisfy excision, are monoidal, and send (higher) isotopies of embeddings to (higher) chain homotopies. Then the restriction map to Loc induces an equivalence of categories*

$$\mathcal{H}_S \cong \mathsf{Loc}\text{-}alg.$$

The proof of this theorem appears in [2].

2.5 Examples

Example 4 (*The circle and Hochschild homology)* Recall from Chap. 13 that the basic example is when $n = 1$ and $X = S^1$. Then the excision axiom tells us that

$$\int_{S^1} \mathsf{A} \cong A \otimes_{A \otimes A^{op}} A.$$

The right-handside of this equivalence is a well-known object—it is the *Hochschild homology of A with coefficients in A*, or in short, the Hochschild homology of A. It is the derived 'abelianization' of A, in that H^0 of the right-hand-side recovers the group

$$A/[A, A].$$

Geometrically, one can see this as the ability to collide two points from the left, or from the right, on a circle.

Example 5 (*Hochschild homology with coefficients)* Now let us consider the category SMfld whose objects are smooth 1-manifolds with marked points. Then Loc is a category generated by two objects: The open interval, and the open interval with a single marked point. (Any 1-manifold with marked points can be constructed by gluing disjoint unions of these 'local pieces' together.) Let us suppose we have a Loc-algebra $\mathsf{A} : \mathsf{Loc} \to \mathsf{Chain}_k$. Then to the interval with the marked point, we associate a chain complex M, and to an open interval with no marked point, we associate a chain complex A. A is an A_∞-algebra as before.

Given an interval $(-1, 1)$ with a marked point at 0, one can include a copy of the interval $(0, 1)$ on either side of the marked point. These two inclusions induce maps

$$A \otimes M \to M, \qquad M \otimes A \to M$$

and one can check that a Loc-algebra in this case is precisely the data of an A_∞-algebra A, and a *pointed* bimodule M. Pointed simply means that there is a map $k \to M$ compatible with the A action. (This corresponds to the inclusion of the empty set into the interval with a marked point.)

Now let $X = (S^1, t_0)$ be a circle with a marked point t_0. Then by excision, we see that

$$\int_X \mathsf{A} \cong M \otimes_{A \otimes A^{op}} A.$$

The right-hand-side is the chain complex giving rise to *Hochschild homology of A with coefficients in M*. In general, a collection of k marked points on the circle will have factorization homology equal to Hochschild homology of A with coefficients in $M^{\otimes k}$.

Example 6 (*Hochschild homology with more coefficients)* More generally, if we let SMfld contain one-manifolds with *colored* marked points, each color will correspond to a different bimodule M_i over A, and the circle with various colored, marked points will yield Hochschild homology of A with coefficients in the appropriate tensor powers of the M_i.

Example 7 (*Link invariants)* Now let Loc be the category generated by two objects: $\mathbb{R}^3$, and a copy of $\mathbb{R}^1$ linearly embedded into $\mathbb{R}^3$. We will refer to the latter by the pair $(\mathbb{R}^3, \mathbb{R}^1)$. The stratified manifolds which can be built out of such pieces are precisely 3-manifolds with embedded links. Moreover, one can describe the structure that a Loc-algebra A has. Let us denote $A := \mathsf{A}(\mathbb{R}^3)$ and $M := \mathsf{A}((\mathbb{R}^3, \mathbb{R}^1))$.

Clearly, the stratified embeddings of copies of $(\mathbb{R}^3, \mathbb{R}^1)$ into itself give the structure of an E_1 algebra to M, and A as usual has the structure of an E_3 algebra. Moreover, the inclusions

$$\mathbb{R}^3 \sqcup (\mathbb{R}^3, \mathbb{R}^1) \to (\mathbb{R}^3, \mathbb{R}^1)$$

yield maps

$$A \otimes M \to M$$

which are compatible with all multiplication maps. Hence, M is an E_1 algebra *receiving a compatible action* from the E_3-algebra A.

Remark 9 Though we have not talked about the notion of push-forward, one can take a generic map from $L \subset \mathbb{R}^3$ to $\mathbb{R}^2$ to obtain a link diagram in $\mathbb{R}^2$. This is a stratified manifold, and its factorization homology is the same as that of the link itself. One verifies easily that the Reidemeister relations hold for this invariant.

Remark 10 As Witten explained in his seminal paper [19], it was his and Atiyah's aim to give a definition of a link invariant which is manifestly three-dimensional; that is, one that does not crucially rely on the Reidemeister relations, and is closer in philosophy to an embedding invariant. One can view this formulation, in terms of factorization homology, as a continuation of this arc.

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