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Dissertação

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Abstract

As the field grows, History of Science has become wider-ranging than a purely progress-oriented view of the history of Science. The History of Mathematics, even though more resilient, has shown to follow the same development. The present dissertation tries to contribute to the general study by shedding some light on a book which has been belittled, misinterpreted or ignored altogether, *De Viribus Quantitatis*, one of the major historical recreational mathematics books, and its author Luca Pacioli. This text aims to provide a modern updated survey of the content of this book for related studies, as well as a résumé of its contents.

Keywords: De Viribus Quantitatis, Luca Pacioli, Recreational Mathematics, Popular Ciênce, History of Mathematics.

Resumo

Com o crescimento do ramo de História das Ciências este tem vindo a desenvolver um olhar mais abrangente que a clássica visão dedicada ao progresso das ideias científicas. A História da Matemática, embora mais resiliente, também tem vindo a mostrar interesse em expandir os seus horizontes. No presente texto tentamos contribuir para o estudo geral destas disciplinas estudando um tratado que pouca atenção tem tido até ao momento, sendo até mesmo mal interpretado. Trata-se *De Viribus Quantitatis*, sendo este um dos maiores compêndio de matemática recreativa no seu contexto histórico. O seu autor, Luca Pacioli, sendo uma personalidade de grande interesse e mais conhecido por outras obras suas. Nestas páginas tentamos fornecer uma versão atualisada da documentação relativa ao tratado tal como um resumo dos seus conteúdos.

Palavras-chave: De Viribus Quantitatis, Luca Pacioli, Matemática Recreativa, Ciência Popular, História da Matemática.

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Introduction

De Viribus Quantitatis[1] is a unique treatise. It is one of the first (if not the first) to gather multiple mathematical recreations, magical effects and "scientific" experiments, explaining and exposing these within the spirit of its age. It is the work of a mathematician and educator, but is very different to other textbooks written in its time, due to its dedication to recreations. These recreations can be found earlier in correspondence, literature or textbooks, mostly though individually or as interludes, and the sheer size of Pacioli's work places it in the spotlight. Like most other works of Pacioli the content is mostly mathematical at heart, but also includes other natural sciences and even, at the very end, some literary entertainment. Its discourse is guided by praise of the underlying "power" of mathematics, in its algebraic and geometric form. Various elements such motivation, disclosure, communication and education of science figure in it. The book is certainly a milestone in what today one might call "popular science".

But not only the mathematic inclined individual or the recreational mathematics enthusiast will find interest in *DVQ.* Many well-known effects appear on paper for the first time. This makes the book popular among magicians. The book is even named a *classic of Italian prestigitation* by some and even in Portugal many practitioners of the art of illusion will have heard of it. The book holds many illusionist secrets and tricks of the *trade* teaching many different aptitudes. It also describes and explains some seemingly-miraculous effects.

The recreations present in the book take little from the scientific and historic value of its content. It is very likely that a good deal of the book's sections were used for motivation during classes or education of a general public, while others seem present for the pure pleasure of their effect like a few pranks in the latter part of the book.

The author of the *DVQ*, Luca Pacioli, is a historical figure, known best for his *de Summa arithmetica, geometria, proportioni et proportionalita*[2], a landmark historic textbook on Algebra. With it, Pacioli provides a very general and embracing content for the student of mathematics. The *proportione: opera a tutti glingegni perspicaci e curiosi necessaria ove ciascun studioso di philosophia: prospectiva pictura, sculptura, architectura, musica e altre mathematice: suavissima sottile e admirabile doctrina consequeira: e delectarassi cõo varie questione de secretissima scientia*[3], is another commonly known work of his, unlike the *Summa* it discusses Geometry. The first treatise of this work focuses on the golden ratio giving the work its name. Although Pacioli's works are mostly collections and lack major scientific improvement, he is by no means unoriginal, adding to most of his materials and generalizing them. Pacioli is also of great interest for his impact on scientific education and the transmission of science.

Pacioli, however, is not too well known as yet, and much work is needed to get a comprehensive image of him. Jayawardene, in a review of the historiography until 1994[4], gives an eight point list of progress to be made to gain a greater

[1] *De Viribus Quantitatis* shall be referred simply as MS for the Manuscript or *DVQ.*
[2] This work has and will be referred to in short as *Summa.*
[3] This work has and will be referred to in short as *Divina.*
[4] Jaywardene, S.A."*Towards a Biography of Luca Pacioli"*, in Luca Pacioli e la matemática del rinascimento.

comprehension on Pacioli. Number 2 of that list are the publication of *Trattato d'Aritmética* and *De Viribus Quantitatis*. So far studies regarding DVQ have mostly only been accessible to the Italian reading public. For such an ambitious task some editing will be needed and possibly a modernization of the notation. Any additional viewpoint would also be helpful, the book having been tackled so far mostly by mathematicians and illusionists.

This dissertation envisions to aid in getting a better understanding of DVQ, by offering connections to other fields of study. Also, it tries to add to bring it into English. Further it is hoped that it may add to the field of History of Science and bring some new perspective to the topic. The content of this dissertation thus provides some analysis of DVQ's contents in its more recreational mathematics aspects, modernizes notation and terminology, and addresses some of the issues surrounding the text.

For a general understanding the DVQ a short annotated biographical chronology regarding Pacioli is given. Some of the major events in his life are listed together with some of his work. This is aided by a small historiographical sketch at the end, presenting some of the major names in the study of Pacioli.

Taking as understood the introductory sketch of Pacioli the proper core of this study, the DVQ, is then tackled. To begin with some general remarks and description of the book are provided. A historiography of the book is given, followed by the description and some general structural remarks on the DVQ and notation used. Given this preliminary contextualization, the DVQ is analyzed section by section in three parts echoing those of the DVQ. Finally some concluding remarks are made regarding the book, its contents and several smaller aspects not included in the sectional analyses.

Pacioli

Either because of the line of work and sometimes less orthodox style and content of his endeavours, or for other reasons, several controversies surround the author of DVQ. The first to is one about his name. Many different renditions have been used, for instance "Patioli", "Paciuolo", "Paccioli" or "Paciolus", last is found in Latin forewords by the author himself. [5] In recent times, however, the Tuscan form "Pacioli" is commonly used, probably popularized by E. Taylor's *Royal Road*.[6] Within the church order of the Franciscans of the time it was not uncommon to drop the family name, keeping instead only given name and a place of origin. Thus it is no wonder to find the signature of "Lucas de Burgo" or variations of this by the same person.

Luca Pacioli was born around 1445. His exact date of birth is unknown and like his name debated. In the Necrologium of the Cloister of Santa Croce of Florence his date of death at 70 is 1517. Lacking more documentation we cannot assume that his age was exact. He was born son of Bartolomeo. He was the nephew of Benedetto and had a brother named Piero, who had two children, Ambrogio and Siniperio. Little more is known about his family, and this knowledge is derived from two testaments, one from 9 of November 1508 and the other from 21 of November 1511.

Borgo

Of his childhood years little is known. These were spent in care of the Befolci family in Borgo Sansepolcro, now (May 2008 Census) a 16 thousand inhabitant town in the Tiber valley in the Arezzo province in Tuscany, Italy. It is asserted that in these years he had contact with Piero della Francesca (~1410 – 1492), fellow inhabitant of Borgo. Piero is often said to be one of Pacioli's teachers or tutors. If he instructed him, it was most likely in geometry and perspective. There is, however, room for doubt on this. On the other hand, there is some evidence of the inclusion by Pacioli in some of his works of several problems, commonly associated with Piero. These are the grounds for yet another controversy, regarding the possible plagiarism committed by Pacioli, started by Giorgio Vasari (1511 – 1574) in 1550, but it seems plausible that Pacioli might have used course materials, or used work he developed together with Piero, who he credits in the *Summa* as a great painter. Much has already been said on the topic and it shall not be the concern of this work.

It is plausible that Pacioli attended an "Abbaco school" in the post Leonardo Pisano (c. 1170 – c. 1250) tradition, influenced by the works of al-Khwārizmī. Here he would learn a more algebraic approach to numbers, along with other subjects, like the already-mentionedgeometry and perspective training.

Venice

In 1465 Pacioli is found living in Venice. Here he studied at the Rialto School around the time Domenico Bragadino was teaching there. Pacioli tells of this time in the *Summa*, mentioning Bragadino as great influence to his geometry and algebra. During Pacioli's time in Venice he stayed with the merchant Antonio Rompiansi in the Giudeca (Jewish Quarter). In the Rompiansi household Pacioli was probably employed as an assistant, as well as a tutor to the merchant's children Francesco, Paolo, and Bartolo. It is highly likely that Pacioli came into contact with several seafarers and

[5] Taylor, R. Emett (1944) "The name of Pacioli", in *The Accounting Review,* XIX, January, pg. 69-76.
[6] Taylor, Emmet (1942) *No Royal Road: Luca Pacioli and His Times*, Arno Press.

probably even made voyages himself. Pacioli's first treatise was dedicated to the Rompiansi children and finished in 1470. This book is lost today. These times amongst merchants and sailor seem to show influence on sections of *DVQ*, dedicated to the seafaring people.

De Computis et Scripturis

The *Summa* holds a large section dedicated to mathematical mercantile practicalities, namely, the chapter *De computis et scripturis.* In it double entry book keeping is described and given an educational use. In his *sui generis* manner, the author generalizes the "method of Venice", which was taught previously by cases. The pedagogic simplification and reduction of numeric use is characteristic, and anticipates modern practices.[7] These teachings of *De Scripturis* would be be of great importance for the next half century and are still used today. As James Don Edwards says "*This treatise caused Pacioli to be looked upon as the grandfather of double entry book keeping. The principles he set forth are still followed and have undergone but few changes in the past 468 years."*[8] This demonstrates the weight of this text for accounting and explains the great interest for the author within these circles.[9]

Rome

Probably after the death of Rompiansi senior, Pacioli moved to Rome, in 1470. Here he stayed with Leon Batista Albertihttp://en.wikipedia.org/wiki/Leon_Battista_Alberti (1404 – 1472) for about a year, possibly on recommendations of Piero. He then moved into the house of Cardinal Francesco della Rovere (later Pope Sixtus IV), which passed on to Giuliano (later Pope Julius II) after the coronation of his uncle. During his time in Rome Pacioli became a Friar of the Franciscan order. He pursued his studies up to the degree of "*Sacrae theologiae professore"*.

Perugia

Vat. Lat. 3129

From then on it seems that Pacioli lead the life of a wandering scholar, teaching at several institutions. The first of these is Perugia around 1476. Here he was the first to teach mathematics. From this time, between 1477 and 1478,[10] comes the text sometimes named *Tractatus mathematicus ad discipulos perusinos.* This mathematical textbook contains several problems, in 17 parts, for the education of his students. It includes about 38 mathematical 'business games' (similar to ones found in the *DVQ*). [11] One of his last paychecks from the Perugia University dates from 1480.

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[7] Sangster, Alan; Stoner, Gregory; McCarthy, Patricia (2007). "Lessons for the Classroom from Luca Pacioli", in *Issues in Accounting Education,* Vol. 22, No. 3, pp. 447–457.

[8] Edwards, J.D. 1960, "Early Bookkeeping and its Development into Accounting" in Business History Review, Vol. 34 (4) pp. 446 – 458.

[9] Given by the mentioned authors as well as for instance Brown, R. 1905, *A History of Accounting and Accountants*. T. C. & E. C. Jack, p. 119 mentioned also by Jayawardene, S. A. (1981). *Pacioli, Luca* in Gillispie, CHC. (Ed.). *Dictionary of Scientific Biography*. New York: Charles Scriner's Sons.

[10] Heeffer, Albrecht (2010) "Algebraic partitioning problems from Luca Pacioli's Perugia manuscript (Vat. Lat. 3129)" in *Sources and Commentaries on Exact Sciences*, 2010b, 11, pp. 3 – 52.

[11] Also known as MS Vat. Lat. 3129; a transcription can be found in Calzoni, Giuseppe and Cavazzoni, Gianfranco (eds.) (1996), *Tractus Mathematicus ad Discipulos Perusinos*, Città di Castello, Perugia.

Wandering years

In 1481 Pacioli went to Zara, now Zadar, Croatia, then under Venetian rule. Here, it appears, he produced another text while teaching. This written work is lost. Possibly, Pacioli made contact with Gian Giacomo Trivulzio (1440 – 1518), who is said to have wanted to employ him.[12] Several conflicts with fellow Franciscans are known of, and Pacioli is said to have been forbidden to teach young students. These conflicts might originate from Pacioli's playful approach to some subjects, from rivalries, given his good connections to the Pope, or some other reason. They are a likely reason for the temporary exile from Italy instructed by the Franciscan order. Later on the order, however, summoned him back to lecture once more in Perugia around 1486.

Among several places like Florence, Aquila, Pisa, he is said to have passed some time in Naples. In 1488 he stays in Rome, hosted by Bishop Piero Valletari. The time around 1487 in Perugia is said to be the date on which he started to work on the *Summa* work he would be completing in 1493 back in Sansepolcro. Next he travelled to Venice, once more, in 1494, to supervise the printing of the *Summa with the typography of* Paganino Paganini *(second half XV century – 1538).*

Summa

The Summa appears as the culmination, in content and refinement, of the prior treatises of Venice, Perugia and Zara. It distinguishes itself as a general school book, which did not target any specific lecture group, as was customary at the time. It gathers material from several mathematicians like Euclid, Boethius, and Leonardo Pisano. The manual would be one of the most influential for more than 50 years, and is mentioned by Gerolamo Cardano, Tartaglia and Bombelli. Besides the already mentioned *De Scripturis* it contains parts concerned with Algebra, theory and praxis, a summary of the *Elements,* and, more applied mathematics, such as conversion of weight measures or coin exchange rates of several regions.

Urbino

In 1495, Pacioli was probably employed at the court of Urbino, having Guidobaldo da Montefeltro (1472 – 1508) as benefactor and student. This suggested by the dedicatory letter to the *Summa*. This same pupil is said to be portrayed together with Pacioli in the painting traditionally credited to Jacobo de' Barbari (after 1460 – before 1516) dating to the same year (see Figure 1).

Pacioli's Portrait

In the picture, Pacioli, in the center, seems to be giving the younger pupil a lesson in mathematics. This pupil could also possibly be Albrecht Dürer (1471 - 1528) who later might have studied with Pacioli and was under Barbari's apprenticeship. However, Dürer's supposed eye color does not match that in the painting.

Some relevant features of the picture are the two solids, the tablet Pacioli is writing on, with Euclid engraved on it, several instruments, including the straight edge and a compass, and a book. The book is probably the *Summa*. Illustrations of the solids, a dodecahedron and a transparent rhombicuboctahedron filled with water, figure in the *Divina.* These objects hold some pedagogic value (and Pacioli repeatedly praises some

Figure 1: *Ritratto di fra' Luca Pacioli con un allievo* **by Barbari: 1495.**

[12] Puig, Albert Presas (2002). "Luca Pacioli, Autor der Summa de Arithmetica Goemetria Proportioni & Proportionalita, 1498", preprint 199 of the Max-Planck-Institute of history of Science.

of them in DVQ). Even in modern times sections of solids are taught making use of water filled transparent solids.

To add to this much credited portrait is another picture from 20 years prior by Piero in supposedly featuring Pacioli portrayed as Saint Peter Martyr (Second from the right see figure 2). In the center of the picture is Duke Federigo the father of the above mentioned Guidobaldo.

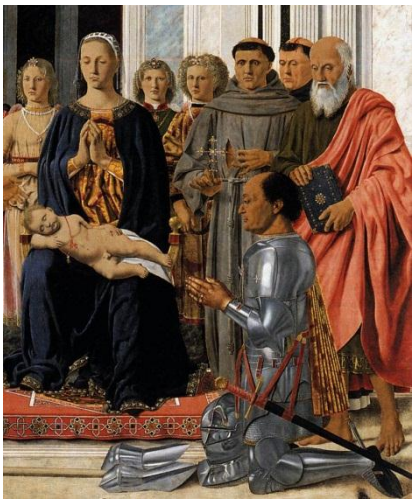

Figure 2: Montefeltro Altarpiece, between 1472 and 1474 bv Piero de la

Milan and the Divina

Pacioli was hired in 1496 by Duke Ludovico Sforza (1452 – 1508) to attend the court of Milan, where scholars and artists of all kinds were gathered. Pacioli filled the position of mathematics lecturer once more. The years in Milan can be characterized by the sharing of knowledge, which possibly influenced Pacioli's writings. Among many personalities gathered at this court was Leonardo da Vinci (1452 – 1519). They would work together, praising each other throughout their works. Leonardo would credit Pacioli in his writings as consultant in mathematical matters. Pacioli for his part, credits the polymath often and with great praise (namely in the *DVQ*). Paintings of Leonardo would even figure in the *Divina*. The *Divina* would be finished as early as 1497 in manuscript form, but would only be published in print in 1509, again by Paganini.

Divina

The *Divina* is three-parted. The name is derived from the first part of the book. In that part several relations are found in regards to the divine proportion, also known as the golden ratio. Some of these do also appear in *DVQ,* as will be seen. The second part is a treatise based on Vitruvius' work with regard to Architecture. The final part is a translation into volgare (the vernacular) of Piero's work *De corporibus regularibus. The parts of the book are dedicated respectively to* Sforza, to the people of Sansepolcro, and, to Piero Soderini (1450 – 1522).

Florence

In 1499 the French army conquered Milan and captured Sforza. Pacioli fled back south to Florence, at the time under Soderini's rule. It is possible that he was accompanied by Leonardo. The two are said to have shared quarters in the city. Leonardo, himself, would remain in Florence until 1506, with exception of an interruption in the service of Cesare Borgia (ca. 1475 – 1507).

During his time in Florence, from 1500 to 1506, Pacioli would lecture at several institutes. He seems to have taught both for the University of Florence and the University of Pisa. The University of Pisa had been shifted to Florence given revolts in its original town. In the university records Pacioli is accounted for until 1506 with exception of 1503. Besides this, he likely visited Perugia, and, held the position of *lector ad mathematicam at the University of Bologna for a year stating 1501. Here he might have met with* Scipione del Ferro (1465 – 1526). Scipione would solve the cubic equation, proving the statement of the contrary in the *Summa* wrong. *At that time* Pacioli would have been teaching the *Elements. In 1505, after Pacioli was elected superior of his order for the province of Romagna, he was accepted as a member of the monastery of Santa Croce in Florence.*

Final Years

In 1506 Pacioli travel to Rome as guest of Galetto Franciotti *(ca.* 1477 – 1508) crossing paths once more with Julius II, who gave Pacioli the authorization to own some goods even though he was a Franciscan.

Next Pacioli is known to have visited Venice once more, in August of 1508. Here he gave a speech on the Book V of the *Elements* at the Rialto Church. In December of the same year he would obtain the rights to publish five books for the following 15 years, namely a version of Campanus' of the *Elements,* the *Divina*, the *Summa,* the *De ludo scachorum* and the *DVQ*. The first two are known to have been printed in 1509 again by Paganini.

The Elements

The version of the *Elements* brought forth by Pacioli is based on the translation into Latin from the Arabic by Campanus of Novara (ca. 1220 – 1296), making its first appearance in 1482 Venice. Campanus' version had been criticized in comparison to a translation from the Greek in 1505 by Bartolo Zamberti. Pacioli blames the faults of the book on the publisher Eberard Ratdolt (1442 – 1528) and adds his own corrections and annotation to clear Campanus' name.

De ludo scachorum

This is a compendium dedicated entirely to games. It is composed of a treatise named *De ludis in genere, cum illicitorum reprobatione* which Pacioli himself calls *Schifanoia*, probably trying to match its title with the palazzo of same name in Ferrara. It discusses the game of chess and its play. The book is dedicated to the rulers of Mantua, Francesco Gonzaga and Isabella d'Este. A copy of the book or part of the book has only recently been found and studied.[13]

Pacioli was invited to lecture in Perugia in 1509 and called to the *Sapienza* in Rome in 1514 by Leo X (1475 – 1521). He was appointed *comissario* of his monastery in Sansepolcro, where he was found in 1510.

Finally, in 1517, shortly before his death, we have evidence of a petition by his fellow townsmen to make Pacioli minister for the Assisi province.

[13] Sanvito, Alessandro et al. (2007), *Gli Scacchi*, Aboca edizioni.

A Historiographical sketch of Pacioli's Biography

The main sources regarding Pacioli are his own works. They consist of three published texts, the *Summa*, the *Divina* and the latin *Elements* by Pacioli, as well as several unpublished documents, such as the *DVQ*; archival materials, such as university records, wills, and inquiries for publishing; There is also the report by Vasari, discussing of the life of Piero della Francesca including Pacioli, in which he accused him of plagiarizing Piero.

Based on these documents, several authors produced work regarding Pacioli's Life. In chronological order: Bernadino Baldi (1553 – 1617) wrote the first biographic note in his *Vita* of 1589. Baldassarre Boncompagni (1821 – 1894) published Baldi's *Vita* and several materials in 1879.[14] Hermann Staigmüller (1857-1908) wrote a biographical article.[15] Moritz Cantor (1829 – 1920) used Staigmüller's materials among others and spoke about Pacioli in his *Vorlesungen* of 1892.[16] Leonardo Olschki (1885 – 1961) in his 1919 survey of vernacular scientific literature devoted an entire chapter to Pacioli.[17] Amedeo Agostini (1892 – 1958) wrote a biographical entry for the Encyclopedia Italiana.[18] Ivano Ricci (1885 – 1966) produced a monograph surveying the literature on Pacioli's life and discussion the accusation of plagiarism, referring to further archive material.[19] Bruno Nardi (1884 – 1968) wrote several pages concerning Pacioli and his translation of the *Elements.*[20] Giuseppina Masotti Biggiogero (1894 – 1977) published a survey on Pacioli in regards to the *Divina Porportione*. In 1976 Paul Lawrence Rose published a general treatise concerning Renaissance Mathematics; in it he took it upon himself to produce a chronology and bibliography of Pacioli, yet to be published.

After this many more have joined the fray, among others S.A. Jayawardene, A. Presas i Puig, Elisabetta Ulivi and Enrico Giusti. Giusti, gathered several articles regarding Pacioli himself and related subjects in the 1994 Proceedings of the colloquium on account of the 500th birthday of the publishing of the *Summa*.[21]

Recently the great interest in Pacioli among historians of accounting, has lead to three international Conferences held in 2009, 2011, and 2013, for which several other papers have been produced and published by multiple authors.

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[14] B. Boncompagni, *In torno alle vite inedite di tre matematici (Giovanni Danck di Sassonia, Giovanni de Linneriis e fra Luca Pacioli da Borgo Sansepolcro) scritte da Bernardino Baldi,* in Bullettion di bibliografia e di storia delle scienze matematiche e fisiche», 12 (1879), pp. 352-438,863-872.

[15] Staigmüller, Hermann Christian Otto (1889) "Lucas Paciuolo. Eine biographische Skizze" in *Zeitschrift für Mathematik un Physik*, hist.-lit Abth.,34, pp. 91-102, 121-128.

[16] M. Cantor, *Vorlesungen über Geschichte der Mathematik*, 2nd ed., vol. 2, Leiptzig, Teubner, 1900, pp. 306-344.

[17] L. Olschki, *Geschichte de neusprachlichen wissenschaftlichen Literatur der Technik und der angewandten Wissenschaften vom Mittelalter bis zum Renaissance,* Heidelberg, 1919, vol. 1, pp. 151 – 249.

[18] Agostini, Amedeo (1935) "Pacioli", *Enciclopedia italiana.*

[19] I. Ricci, *Fra Luca Pacioli l'oumo e lo scienziato (con documenti inediti)*, San Sepolcro, Stab. Tip. Boncompagni, 1940.

[20] B. Nardi, *La Scuola di Rialto e l'umanesimo veneziano*, in *Umanesimo Europeo e umanesimo veneziano*. A cura di V. Branca, Venezia, Sansoni, 1963, pp 93-139.

[21] E. Giusti (ed.) (1998), *Luca Pacioli e la matematica del Rinascimento Atti del convegno internazionale di studi*, Sansepolcro 13-16 aprile 1994, petruzzi editore.

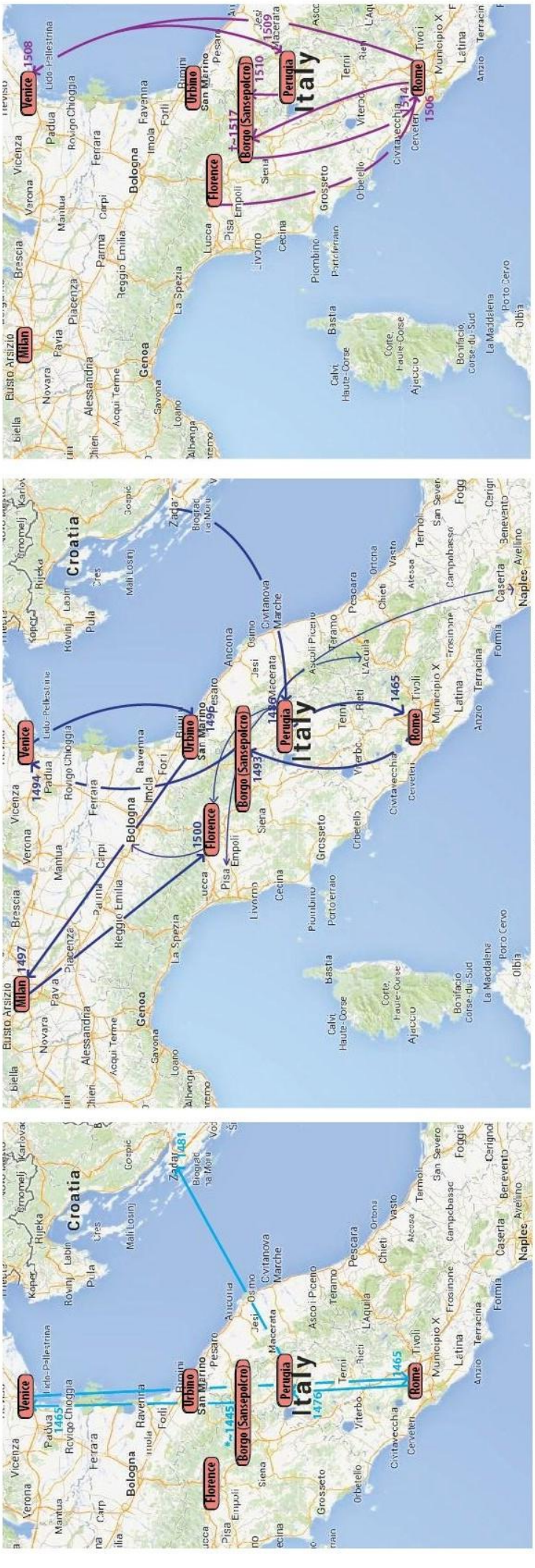

Figure 3: Pacioli's journey. Leftmost his stops before visiting Zadar. In the middle those before 1500. Right most his final years.

The De Viribus Quantitatis

Description

DVQ is known from codex 250 of the University library of Bologna, having most likely reached it from a private collection.[22] A copy of this manuscript figures as Ms. 4066 of the Library Casanatense of Rome. It is a 19th century facsimile of the Bologna manuscript. That copy was probably made from the Bologna version by the unknown Amadeo Caronti in honor of Baldassare Boncompagni in 1852. [23] The work refers to the 1496 *Divina* and figures in the 1508 petition for print of several works by Pacioli.[24] Thus its completion probably happened in that period. In addition a 1509 reference can be found in the book. It is well possible that the planning and a great deal of the writing of the book took place somewhat earlier.

The MS is in the writing of an amanuensis, easier to read than Pacioli's handwriting. Several numberings, illustrations, and other details appear to have been added later, some possibly by a third hand. This is evidenced by different tones of the ink and crammed spacing of some text blocks. Several blank spots seem to indicate prior spacing for later filling. Most likely images, titles, and special lettering are present in only a few sections. There is a frequent use of abbreviations common at that time, sometimes with unclear meaning. Numerals are Hindu-Arabic but sometimes ambiguous.

The folios, 24 by 16,5 cm in size, have been numbered in a different script from the amanuensis', most likely belonging to the hand of the archivist handling the book. There are 309 folios. All folios except the first are writtenon both sides, totaling the 614 page work.

The first two folios, not part of the book, are numbered '*a*' and '*b*' and hold written on them a hard to read note, F.*b*, and on F.*a*, a codex number 194, likely a reference in the collection of Giovanni Giacomo Amadei (+1768), canon of the Basilica St. Maria Maggiore in Bologna, whose name figures there.

The following thirteen folios, Ff.I-XIII, are the table of contents for the three parts. Several titles listed here differ from those used in the text. Many of the titles do not have a corresponding section in the text body, or the content they describe may be aggregated into another section. Some have no corresponding text at all. Here, unlike in the sections, the titles are indexed in Hindu-Arabic numerals. Those who have corresponding sections are also accompanied by a second less visible indexation.

Folios 1 and 2 are an introductory dedicatory letter whose addressee has been left blank. This letter was published in the source material on Pacioli by Boncompagni. In it Pacioli, who in his words is near the end of his life, tells of the printing of the *Summa* in 1494 and the completion of the *Divina* in 1496, for which Leonardo did the drawings. Leonardo and the time in Milan are highly praised. The translation of the *Elements* into the vernacular and the *Schifanoia* are also mentioned. With his work

[22] Agostini, Amedeo (1924), "De Viribus Quantitatis di Luca Pacioli" in *Periodico di Matematiche Vol. IV*, pp. 165 – 192.
[23] Montebelli, Vico (1998), "I Giochi Matematici nel De Viribus Quantitatis" in *Luca Pacioli e la Matematica del Rinascimento*, Petruzzi Editore.
[24] Luca Pacioli, (1508) "Suplica di fra Luca Pacioli Al Doge di Venezia in data 29 dicembre 1508. Per Ottenere un privilegio di stampa." In Notatorio dal Collegio dal 1507 a 1511 carte 34 verso e 35 recto, from Archivo Generale de Venezia.

Pacioli intends to divulge the "powers" behind the effects that are to amuse the reader, which he has gathered to that point. Further he apologizes for the use of vernacular, the language being a minor evil so that more people can benefit from the book.

The next folio contains a small prologue to the book. Here, Pacioli concentrates on sharing the miraculous powers of the mathematical entities. The premise "*the end justifies the means"* supports the less ecclesiastical nature of some of the "miracles*"* described in the book. Pacioli will make use of this premise throughout DVQ, especially when dealing with apparently less canonical effects. The *end*, at least in the first two parts, is it to share the mathematical marvels underlying the effects, problems and other amusements. Pacioli places these in the realm of the divine. Pacioli also explains that the book is to be used as compendium, and has been structured for easy reference.

The remaining 287 folios are the content of the book and shall be discussed section by section below. These sections are divided into three parts. The first part concentrates on algebraic matters. It is this part from which the book draws its name. Pacioli refers to it in vulgar as "*dele forze della quantita*" in the introductory letter. It spans Ff. 2v – 132v and covers 80 sections. The indexation is sometimes off and one section (what would be I.63,) is missing altogether. The index, however, lists 120 titles. It can be roughly divided into two groups Algebraic Effects and Numerical Games. The first of these contains more formal (classroom) content of the time, while the second describes more general mathematical recreations.

A smaller difference between the table of contents and the actual content is verified in the second part. There are 139 sections listed and 134 are part of the text body. The Second part covers Ff. 133r – 230v and is named "*della virtu et forza geometrica con dignissimi documenti*". The second part, too, can be roughly divided into two groups, Geometric Constructions and Geometric Marvels. The first is a practical guide to many Geometric constructions. The second is again general and not too tightly related to Geometry, as we understand it today.

Finally, the third part has an extra division into five chapters. It starts at F.231 and ends on the last folio. Unlike the other four the biggest chapter of the last part, Ff.236 – 261r, is divided like the first two parts into numbered sections. It is named *"Documenti et proverbii mercanteschi utilissime*". Like the first two parts, which are respectively dedicated to the number and the line, it is dedicated to Natural origins. The other chapters are of a more literary orientation containing poems, proverbs, riddles, jokes, and, other amusements.

Historiography DVQ

Most of the references regarding the book are based on secondary sources. Especially in the fields of History of Mathematics and in the History of Science authors seem unaware of the book's content and there are few dedicated studies. In general, little work can be found and most of it is in Italian. So it is not to wonder that misconceptions about the book and dismissal of its contents are common. However, there are some scholars who have paid more attention to the DVQ.

The first available study is the 1924 "De Viribus Quantitatis di Luca Pacioli" by Amedeo Agostini in *the Periodico di Matematiche*.[25] Agostini gives a brief description of the work, some notes on its sources, and then describes the contents of the first part. The paper stresses the influence on and relation to other scholars like Claude Gaspard Bachet de Méziriac (1581 – 1638) work of 1612. Agostini describes the first part section by section, but only mentions some of sections of the the second and third part. Each of DVQ's sections is succinctly discussed in the mathematics of Agostini's time, his notations and materials being used by several later scholars.

Only in the 1998 Proceedings of the 1994 conference, does the DVQ reappear in the spotlight. Vico Montebelli shared some insights from the book in regards to its historical context and the use of games in the Abacco tradition.[26] He categorizes some of the sections of the first part and discusses some of the mathematical themes present. At the end of the article, he relates some of the effects to the *Liber Abacci* by Leonardo Pisano (1170 – 1250) and other sources also mentioned in Agostini.

In 1997, Maria Garlaschi, under the edition of Augusto Marinoni, made available a transcription from the volgare, extending abbreviations.[27] Based on this transcription, in 2007, an unedited draft translation by Lori Pieper became available at the Conjuring Arts Research Center, New York. This translation keeps notes from the Garlaschi transcription and adds several of its own contextualizing some of the content and its references.[28]

In 2008, an edition of the manuscript by Paul Lawrence Rose was said to be[29] in preparation for the New York University Press. That edition has not appeared in libraries up to now. In the same year appeared two articles from David Singmaster and Vani Bossi.[30] Both articles figure in a book paying tribute to Martin Gardner. Singmaster describes several of the recreational mathematics problems present in the book. Bossi is more concerned with the card magic present in the MS.

Singmaster is also the author of an extensive and private sourcebook, still in progress, on the History of Recreational Mathematics.[31] The sourcebook discusses a vast

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[25] Agostini, Amedeo (1924). "De Viribus Quantitatis di Luca Pacioli" in *Periodico di Matematiche* Vol. IV, pp. 165 – 192.
[26] Montebelli, Vico (1998). "I Giochi Matematici nel de Viribus Quantitatis", pp. 312 – 330.
[27] Peirani, Maria Garlaschi Peirani & Marinoni, Augusto (ed.) (1997). Ente Raccolta Viniciana, Milano.
[28] Pieper, Lori (2007). *De Viribus Quantitatis: On the Power of Numbers*, Unedited Draft Copy of the Conjuring Arts Research Center, New York.
[29] "Pacioli, Luca." *Complete Dictionary of Scientific Biography* 2008, S.A. Jaywardene.
[30] Singmaster, David (2008). "*De Viribus Quantitatis* by Luca Pacioli: The First Recreational Mathematics Book" in *A Lifetime of Puzzles*, Taylor & Francis, and Bossi, Vani (2008). "Magic Card Tricks in Luca Paciolo's *De Viribus Quantitatis*" in *A Lifetime of Puzzles*, Taylor & Francis.
[31] Singmaster, David (2013). Sources of Recreational Mathematics personal notes 2013 version.

amount of recreations, including several from the *DVQ*. In the 2012 version one can find many of the recreations of the first and some of the second part in relation to other sources, similar recreations, and occurrences in other historical works.

In 2009 Aboca Edizioni published *Curiositá e Divertimenti con I Numeri,* by Furio Honsell and Giorgio Tomaso Bagni. The book provides a catalogue of the index listed effects side by side with the title of the content and a very brief description of its contents. Some sections are chosen and discussed in greater detail, tracing parallels to other currently known effects.

In 2010, Dario Uri made available to the general public the photographic copy of the original document on his website. The photo-facsimiles seem to be of better quality[32] than the microfilm in places.

In 2011, Franco Polcri contributed an article on the book in the Proceedings of the second International Accounting Conference on Pacioli, adding to the reviews of the book.

In 2012, Bossi's *Mate-Magica I Giochi di Prestifio di Luca Pacioli* was published by Aboca. It is the first book that explores the second and third parts a little more. It describes many of the effects described, illustrating them with images from other historic works when the original is lacking. Not all sections of the book are covered as the book restricts itself to the more illusionist aspects of the DVQ. It adds much information on the history of illusionism.

Finally, in 2013 a short paper appeared in the compendium *Religiosus Ludens*. It briefly gives a case study discussion of two effects described in the *DVQ* (I.27 and I.38) in a social and pedagogical context. The paper is written by Francesca Aceto, who is studying DVQ for her doctoral thesis at the École des Hautes Etudes en Sciences Sociales of Paris.

[32] Singmaster (2008).

Structure and notation used

This study's core is a discussion of the Bologna Manuscript of the *De Viribus Quantitatis*. It is based on the reading of the photographed folios by Dario Uri. The reading has been aided by the English translation of the text by Lori Pieper and existing studies such as *Mate-Magica* and *Curiosità e Divertimenti*. This study seeks to highlight the mathematical nature of the book, as well as discuss the sections often neglected, especially of the third part, which are closely related to popular science, as well as to aid reading of the original text. Some perform oriented aspects stressed by Pacioli have also been included, but where not the main focus.

The order of the sections, as they appear in the DVQ, is kept. These sections are titled in direct translation of the original. The sections of the three parts (I., II. and III.) have been numbered independently of the original in order of appearance (1,2,3,…). The third part has another five sub-parts, (i., …, v.), with the specific that its third sub-parts, (III.iii.), follow a similar structure as the first two parts and thus is numbered yet again. Additionally the fifth sub-part of the third part has over 200 riddles and jokes, (r.1,r.2,r.3,…) listed which are referred to by number of appearance .

For cross reference purposes, (II.84) refers to the 84th section of the second part on the construction of a bridge, and (III.iii.23) to the 23th section of the third sub-part of the third part, washing hands with molten lead. Finally, r.204 refers to the 204th joke of (III.v.)

To keep a one to one correspondence to the original as close as possible, blank sections have been included. Further fragmented sections have been joined when possible to their corresponding content. These changes are noted in the sections appropriately.

DVQ uses a mix of Roman numerals, Hindu-Arabic numerals and written out numbers for indexation. This mix follows no apparent rule through the sections. In the first part numbers are written out until the 16th section, in the second part numerals are used starting in the 2nd section of that part. It is also not clear when Hindu-Arabic numbers are used instead of the prevalent Roman ones. For this reason, in the translation of the titles all numbers have been written out to standardize the titles as much as possible.

The text is usually directed at the reader, who is to perform to someone else what he learns in the book. The author, Pacioli, serves as example and teacher. To simplify and ease understanding, the intervening characters were categorized into the *performer,* the subject who produces the effect, and the *participant* or *participants*, one or several actors with a passive role, to whom the effects are performed to.

Often Pacioli's style is exhaustive in explanation. He accompanies the reader through specific operations step by step in an almost recipe like fashion. To ease understanding and modernize script, these recipes have been condensed into symbolic notation when appropriate. With some regularity Pacioli reminds the reader how to apply his effects and games with diverse materials instead of a more abstract setting to one where concrete objects are used. These include apples, eggs, walnuts, chestnuts, coins, beans, spots on dice, cards, and many more. These shall not be stressed with the same regularity as Pacioli does and are left to the creativity of the

reader to be applied when appropriate. Often counters, or other artifices are used instead.

To try to make the sections, especially those of the algebraic part, as uniform as possible the following variables are used, unless otherwise specified. $n, m \in N$ and $s \in Q$[33] are initial numbers to which operations are performed. These are most often chosen by the performer and introduced surreptitiously. $a, b, c, d \in N$, and $p, q \in Q$ are parts chosen by participants, most often without knowledge of the performer. In some effects some constant is used, $k \in Q$; frequently this does not influence the effect itself. Throughout the sections it is to be kept in mind that negative numbers were not a commonly used and even zero was somewhat exotic, having the status of an artifice, fractions not being entirely common either.

Pacioli also makes many references to Euclid's *Elements*, probably in a version by Campanus. For easy reference and quick consultation however a modern version was used in the footnotes[34]. This makes for some slight inaccuracies in the indexation in relation to that used by Pacioli.

[33] Although in most effects where rationals are used, any number for which the usual operations are defined and make sense could be used instead.

[34] D.E.Joyce (1998). *The Elements* on the following Clark University website: http://aleph0.clarku.edu/~djoyce/java/elements/toc.html .

I. On the Powers of Quantity

Algebraic Tricks

After the introductory letters and dedication the book starts straightaway with the execution and explanation of the effects.

Equations (I.1-I.6)

1. First Effect: About a Number [divided] into two parts.[35]

Two or more participants are asked to divide among them a known number of coins (n), and then the performer asks them to do some calculations with the number of their share (*a* and *b* respectively). After revealing an apparently unrelated number resulting from these operations the performer predicts the hidden shares. Pacioli gives several settings for the effect. These can be applied to, and are repeated in part in the following five effects.

For instance, two participants hold 3 and 7 coins out of 10 they were given. The performer asks the first to double the number of coins he has. Next the second is asked to multiply his share by ten. Letting them consult in secret, the performer asks for the joined sum of the multiplications to be taken out of a pile of 110 coins. Looking at the remainder of the pile he guesses the share of each participant.

This effect is based on the equation given below. It results from the division of a number into two integer parts and the guessing of such without apparent transmission of information. The operations are given, operation by operation, granting the participant time for calculation. The effect is obscured due to its non-obvious algebraic nature, and also because some calculations are done mentally by the performer or use distraction devices.

$a + b = n,$

$$\frac{n(n+1) - (2a + nb)}{n-1} = a + \frac{b}{n-1}$$

Pacioli gives example with detailed calculations for $n = 10,\ a = 3,\ b = 7$ and $a = 7,\ b = 3$

Agostini observes that this and some of the following identities are found in Pisano's and Ghaligai's work.

2. Second Effect: About a number divided into 3 parts.[36]

Similarly here three parts (*a, b, c*), are chosen secretly by one or three participants from a known number (*n*). Again several calculations are performed with these parts resulting in the correct guess by the performer. The following equation gives the operations to apply.

$a + b + c = n,$

$$\frac{n(n+1) - [2a + nb + (n+1)c]}{n-1} = a + \frac{b}{n-1}, \text{ and } c = n - a - b$$

Examples are given for $n = 10,\ a = 2,\ b = 3,\ c = 5$ and $a = 2,\ b = 5, c = 3$

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[35] *DVQ* F.3v.
[36] *Ibid*. F.5r.

Here, as in the first effect, the author reminds the reader that these effects are to be done using integers. A way around the calculation for the uneducated is given with counters; this distraction device is used in several effects.

Giving Pacioli's example using these counters, coins in a pile: There are 110 coins; the participants have 2, 3 and 5 coins each. The first takes twice his share ($-2a$), the second ten times his ($-nb$), and the third eleven times his share ($-(n+1)c$). The performer enters the room, divides by nine the number of coins he sees on the table, $\frac{110-4-30-55}{9}$, and then guesses the shares of each participant.

3. Third Effect: Also about a number divided into 3 parts in another way.[37]

Another version is presented on how to produce the previous effect. This time the equation to be followed is,

$a+b+c=n$,

$$\frac{n^2-[2a+(n-b)+nc]}{n-2}=a+\frac{c}{n-1}, \qquad b=n-a-c$$

Example is given for $n=10, a=2, b=3, c=5$ and $a=2, b=5, c=3$

Pacioli adds and subtracts a number, k, to the denominator, amid operations. This is probably done to obscure the equation even further.

4. Fourth Effect: About a number divided into 3, and so on.[38]

Yet another way to find the three parts of a number is given, as well as a way to extend the conceal one of one of the operations.

$a+b+c=n$, and, $m-1>n$

$$\frac{n(m+1)-[2a+mb+(n+1)c]}{m-1}=a+\frac{b}{m-1}, \qquad c=n-a-b$$

An example is given using $n=10,\ a=2,\ b=3,\ c=5,\ m=12$ and $a=5,\ b=3,\ c=2,\ m=16$.

Pacioli then generalizes

$k<m-1$,

$$\frac{n(m+1)-[\mathrm{ka}+\mathrm{mb}+(\mathrm{n}+1)\mathrm{c}]}{m+1-k}=a+\frac{b}{m+1-k}$$

giving an example for $a=2,\ b=3,\ c=5,\ m=13,\ k=5$

Here the author goes into detail regarding the concretization of the tricks with physical props. He first suggests the application to guessing spots on two or three dice, the total of the roll being known to the performer. Then he discusses an application to playing cards through the number of pips or an agreed value for picture cards. This can be refined even further if each picture card is given its own number. This is for a standard deck.

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[37] *Ibid.* F.7r.
[38] *Ibid.* F.8r.

The numeration follows naturally for the *triomphi*. Pacioli speaks of these as 1 to 20 or 21 cards. As Pieper notes, it was common to have a 78 card deck, with 56 ordinary cards, 4 suits of 14 cards each, plus an extra 21 cards and a Fool. The card decks of the time would vary in size and kind depending on region.

An example for (I.1) $a = 4,\ b = 6$ is given here, as well as reference to multiple part effects with four or more participants (I.5) and (I.6.). Some artifices for further mystification are suggested, such as asking for multiplication by factors of a given number, one at a time, instead of the number itself, possibly changing their order in consecutive presentations.

5. Fifth Effect: About a number divided among 4, or, into 4 parts.[39]

This describes yet another artifice to conceal the transmission of information, through repeated calculations. The bracketed operations are calculated separately by the involved participant(s) and added together. The result is then shared with the performer. The performer then discovers one of the parts, c in the equation below. Then either he 'rotates' the parts and guesses another one or he applies one of the previous effects.

$$a + b + c + d = n,$$

$$n - 3n - [(a + b + c) + (b + c + d) + (c + d + a)] = c$$

An example is given for $(a, b, c, d) = (3,4,5,8)$ and one more example regarding cards is given. In that instance, Pacioli uses names in alphabetical order for the participants **A**ntonio, **B**enedetto, **C**ristofano and **D**omenico. This might be to facilitate keeping track of the various sets of calculations performed.

6. Sixth Effect: About a number [divided] into 5 parts.[40]

The idea is the same as in the previous section. The effect is first discussed for 5 parts and then generalized for any number of parts proceeding as in the previous effect.

$$a + b + c + d + e = n,$$

$$n - 4n - [(a + b + c + d) + (b + c + d + e) + (c + d + e + a) + (d + e + a + b)] = d$$

An example is given for $(a, b, c, d, e) = (2,3,4,5,6)$.

Modulo 4 (I.7, I.9, I.20)

7. Seventh Effect: Finding a whole number that has been thought of.[41]

A participant thinks of a number. He is asked to add half of his number to the number he thought of and tell if the result is an integer. If it isn't he is asked to round up. He is then asked to add half of this to itself, and again say whether the result is an integer or not, rounding up if not. Finally the participant divides the number by nine and announces the quotient. The performer guesses the number.

This effect is based on

$$n = 4q + r, \qquad q \in N, \qquad r \in \{0,1,2,3\}$$

[39] *Ibid.* F. 13v.
[40] *Ibid*. F.14v.
[41] *Ibid*. F.16v.

The effect is given by the following equation. The performer has only keep track of the rounding to know the remainder, r. This remainder is then added to the multiplication by 4 of the announced quotient.

$$\frac{\left\lceil \left\lceil n+\frac{n}{2} \right\rceil + \frac{\left\lceil n+\frac{n}{2} \right\rceil}{2} \right\rceil}{9} = \frac{n}{4} + r,$$

$$r = \begin{cases} 1, n+\frac{n}{2} \notin N \quad (\boldsymbol{i}.) \\ 2, n+\frac{n}{2}+\frac{n+\frac{n}{2}}{2} \notin N (\boldsymbol{ii}.) \\ 3, \quad (\boldsymbol{i}.) and\ (\boldsymbol{ii}.) \\ 0, \quad otherwise \end{cases} = \begin{cases} 0, n \equiv 0\ (mod\ 4) \\ 1, n \equiv 1\ (mod\ 4) \\ 2, n \equiv 2\ (mod\ 4) \\ 3, n \equiv 3\ (mod\ 4) \end{cases}$$

Examples are given using $n = 12,\ 5,\ 6,\ 2$ and 15, as well as a suggestion for the performer of dividing by the double or four times nine or ask to subtract bigger numbers and keep their division by nine in mind, examples given are 100 and 60 with respective result 11 and 1/9 and 6 and 6/9 respectively, in case of larger numbers to conceal the effect and aid the calculating participant.

8. Eighth Effect: When the number has a part [, is a fraction].[42]

Not as mathematically elaborate as the previous effect, this does include any number, fitting into the typical self-solving equation style divinations. As in the first effects, operations are asked for, one at a time and concealed by alternatively asking for the "double" or "quintuple" instead of the number "times 2" or "5" and concealing, when possible, the operations from the participant.

$$[(2q+5)5+10]10 - 350 = 100q$$

Examples are given for $q = 6\frac{2}{3}, 6\frac{3}{4}, \frac{3}{4}\ and\ 12.$

9. Ninth Effect: To find a number without parts [that is, an integer].[43]

This is a variation of (I.7.). Instead of rounding up after each division by two, one is to round down. This results in the following

$$\frac{\left\lfloor \frac{3}{2} \left\lfloor \frac{3n}{2} \right\rfloor \right\rfloor}{9} = \frac{n-r}{4}, \qquad r = \begin{cases} 3, \frac{3n}{2} \notin N \quad (\boldsymbol{i}.) \\ 2, \frac{3\left(\frac{3n}{2}\right)}{2} \notin N, (\boldsymbol{ii}.) \\ 1, (i.) and\ (ii.) \\ 0, otherwise \end{cases} = \begin{cases} 3, n \equiv 1 (\mathrm{mod}\ 4) \\ 2, n \equiv 2 (\mathrm{mod}\ 4) \\ 1, n \equiv 3 (\mathrm{mod}\ 4) \\ 0, n \equiv 0 (\mathrm{mod}\ 4) \end{cases}$$

Examples are given for $n = 5, 6$ and 7.

Modulo 2 (I.10)

10. Tenth Effect: On finding a number without parts [that is, an integer].[44]

This is a simplified version of (I.7) and (I.9), based on parity of a number.

[42] *Ibid.* F.19v.
[43] *Ibid*. F.20v.
[44] *Ibid*. F.21v.

$$\frac{3\left\lfloor\frac{3n}{2}\right\rfloor}{9} = \frac{n-r}{2}, \qquad r = \begin{cases} 0, n\ is\ divisible\ by\ 2 \\ 1, otherwise \end{cases}$$

The author offers examples using $n = 10$, and $n = 13$, then explains the principle behind the effect using 1, "the unit" and 2, "the binary". At the beginning of the effect reference is made to Euclid's *Elements*.[45] The effect ends with a variant without rounding down. This consists of keeping the fractions until the end and only rounding up after hearing the result with the fraction. Suggestion is made to combine this effect with one of the prior multiple part guessing effects (I.4 – I.6).

Binomial expansion (I.11, I.12)

11. Eleventh Effect: To find a number in all ways.[46]

A number divided into two parts is guessed. To achieve this some calculations asked for. These are the 2nd power binomial expansion in disguise. Each operation is made separately and in secret. The participant's part ends with the announcement of the sum of all separate calculations. The performer extracts the square mentally to guess the number.

$p + q = s,$

$$p^2 + q^2 + 2pq = s^2$$

The author refers to *Elements II, 4* as inspiration[47].

Examples are given for $(s, p, q) = (12, 4, 8)$and $(10\frac{2}{3}, 4\frac{1}{3}, 6\frac{1}{3})$

12. Twelfth Effect: A number in all ways.[48]

The same is given this time for the 3rd power binomial expansion. This implies, for the performer, the extraction of the cube. No source references are given here.

$$3qp^2 + 3pq^2 + p^3 + q^3 = o^3$$

Examples are given for $(s, p, q) = (10{,}4{,}6)$and $(10\frac{2}{3}, 4\frac{1}{3}, 6\frac{1}{3})$.

From the Elements (I.13 – I.19)

13. Thirteenth effect: To find a number in all ways.[49]

This time *Elements II, 2* is applied[50]. A number thought of is divided into several parts. Each part is multiplied by the thought of number. After all is added together the square root reveals the number.

$s, a_i \in Q,\ i \in \{1, 2, \dots, k : k \in N\},$

$s = a_1 + a_2 + \cdots + a_k,$

[45] "Any integer is ether even or odd". Elements Book IX, Propositions 21-34 discuss properties regarding odd and even numbers.
[46] *DVQ* F.23v.
[47] "If a straight line is cut at random, the square on the whole equals the squares on the segments plus twice the rectangle contained by the segments." Elements II, 4.
[48] *DVQ* F.24r.
[49] *Ibid.* F.25r.
[50] "If a straight line is cut at random, then the sum of the rectangles contained by the whole and each of the segments equals the square on the whole." Elements II, 2 in Algebraic and general form.

$$\sum_{i=1}^{k} a_i s = s^2$$

14. *Fifteenth Effect:* To find a number in all ways.[51]

This effect is based on the *Elements II, 3* masking[52] the divination of a number. The number is chosen and divided into two equal parts ($p + p = s$). A number known to the performer (k) is added to one of the parts ($p + k$). The participant chooses one of the shares. Then the performer asks for the following operations, as in the case of earlier effects.

$$p = \begin{cases} \dfrac{(\mathrm{p}+\mathrm{k})^2 + \mathrm{p}(\mathrm{p}+\mathrm{k})}{2\mathrm{p}+\mathrm{k}} - k, \text{if p is chosen} \\ \dfrac{\mathrm{p}^2 + \mathrm{p}(\mathrm{p}+\mathrm{k})}{2\mathrm{p}+\mathrm{k}}, \text{if } (\mathrm{p}+\mathrm{k}) \text{ is chosen} \end{cases}$$

In both cases the performer concludes $s = 2p$.

Example is given for $n = 12$.

15. *Effect Fifteen:* To find a number in all ways.[53]

Here use is made of *Elements II, 6.* [54]

2p = s,

$$\sqrt[2]{(p+k)^2 - (2p+k)k} = s$$

An example is given using $s = 12, k = 4$

16. Sixteenth Effect: To find a number in all ways.[55]

This time it is *Elements II, 7* which is applied. [56]

$s = p + q$,

$$\sqrt{2sp + q^2 - p^2} = \sqrt{s^2}$$

An example is given using $s = 12, p = 4, q = 8$

17. Seventeenth Effect: To find a number in all ways.[57]

Elements II, 8 is now applied[58]. A secret number is divided into two unequal parts ($s = p + q$). One of these parts is revealed (p). The participant is asked to do

[51] *DVQ* F.25v. In the manuscript a possible fourteenth effect is skipped. Instead we find two fifteenth effects. Some apparent later annotations seem to hint upon correction. Both "Quarto effecto" and in a faded script in the margin of another one. Another possibility is that these are cross-reference to the other effects.

[52] "If a straight line is cut at random, then the rectangle contained by the whole and one of the segments equals the sum of the rectangle contained by the segments and the square on the aforesaid segment." Elements II, 3.

[53] *DVQ* F. 27r. The title format from the MS is kept, italics highlight the repeated index.

[54] "If a straight line is bisected and a straight line is added to it in a straight line, then the rectangle contained by the whole with the added straight line and the added straight line together with the square on the half equals the square on the straight line made up of the half and the added straight line." Elements II, 6.

[55] *DVQ* F.27v.

[56] "If a straight line is cut at random, then the sum of the square on the whole and that on one of the segments equals twice the rectangle contained by the whole and the said segment plus the square on the remaining segment." Elements II, 7.

[57] *DVQ* F.28v.

calculations according to the following equation and to share the final result. The performer guesses both the number and the second part.

$$\frac{(s+q)^2 - p^2}{s} = 4q$$

An example is given using $p = 5, s = 12$

18. Eighteenth Effect: To find a number in all ways.[59]

Elements II, 9 is referred to in this effect.[60] As in the preceding section a number is thought of, divided into two unequal parts and calculations are done by the participant according to the equation below.

$p < q$,

$$\frac{p^2 + q^2}{2} - \left(q - \frac{s}{2}\right)^2 = \frac{s^2}{4}$$

Example is given $s = 12, p = 2, q = 10$

19. Nineteenth Effect: To find a number in all ways.[61]

Keeping the spirit of using Euclid's work, Pacioli uses *Elements II, 10*.[62] A number thought of (s) is guessed after having any other number (k) added to it and the following calculations done in accordance to the following equation.

$$\frac{(s+k)^2 + k^2}{2} - \left(\frac{s}{2} + k\right)^2 = \frac{s}{2}$$

Example is given for $s = 12, k = 8$

20. Twentieth Effect: To find a whole number thought of.[63]

This effect is a more extensively explained repetition of (I.9.).

Commutative Property (I.21, I.28, I.29)

21. Twenty-first Effect: To find a number in all manners in general.[64]

Here Pacioli discusses the Commutative property of multiplication to produce magic effects. He explains that given its disclosure in all *schools* it does form a less impressive feat of its own, but can be used as artifice or disguised by some misdirection (as suggested in some of the sections above).

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[58] "If a straight line is cut at random, then four times the rectangle contained by the whole and one of the segments plus the square on the remaining segment equals the square described on the whole and the aforesaid segment as on one straight line." Elements II, 8.

[59] *DVQ* F.29r.

[60] "If a straight line is cut into equal and unequal segments, then the sum of the squares on the unequal segments of the whole is double the sum of the square on the half and the square on the straight line between the points of section." Elements II, 9.

[61] *DVQ* F.30r.

[62] "If a straight line is bisected, and a straight line is added to it in a straight line, then the square on the whole with the added straight line and the square on the added straight line both together are double the sum of the square on the half and the square described on the straight line made up of the half and the added straight line as on one straight line." Elements II, 10.

[63] *DVQ* F.30v.

[64] *DVQ* F.32v.

An example is given taking 12, multiplying and dividing it in inverse order. Pacioli further exemplifies the process of using factors of the previous multiplied/divided by numbers and disguising these multiplications/divisions as successive sums/subtractions.

Chinese Remainder Theorem (I.22 – I.25)

22. Twenty-second Effect: To find a number thought of no more than 105.[65]

A participant is asked to think of a number and then asked for the remainder of the division by 3, by 5 and by 7. The performer guesses the number without further ado.

This and the following three effects revolve around the Chinese Remainder Theorem[66]. The participant thinks of a number, which is restricted to be no greater than 105, and gives the remainder of the three divisions, after this the performer guesses the number at hand.

$n < 105$, and given $n = is + r_i$, for some $s \in Q, i \in \{3,5,7\}$,

$$\frac{(70r_3 + 21r_5 + 15r_7)}{105} = s + \frac{n}{105}$$

One could also take $70r_3 + 21r_5 + 15r_7 - 105s = n$; this is however disregarded by Pacioli.

Examples are given for $n = 17, 104$ as well as 105, in which case the remainder will be zero for each divisor. Pacioli also makes the curious suggestion of generating the number by dice at the start of this section.

23. Twenty-third Effect: To find a number thought of no more than 315.[67]

Just as in the previous effect the remainder of the divisions by 5, 7 and 9 are asked for, and then the number is guessed.

$n < 315$, and given $n = is + r_i$, for some $s \in Q, i \in \{5,7,9\}$

$$\frac{(126r_5 + 225r_7 + 280r_9)}{315} = s + \frac{n}{315}$$

Examples are given for $= 34, 314, 30, 35, 63, 45$. As in the previous effect Pacioli discusses 315 as special case and dismisses null, as it is no "whole number".

24. Twenty-fourth Effect: One number, which divided by 2,3,4,5,6 has a remainder of 1, and divided by 7 has null.[68]

In this effect Luca Pacioli constructs a number with the properties mentioned in the effects title. Pacioli draws an analogy to these numbers and the *elmuarife* (quadrilateral) for which constructions can be given only for certain cases, as the construction of figures given in Euclid (which will be used by Pacioli in the second part).

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[65] *Ibid*. F.34v.

[66] Given $n_1, \ldots, n_r \in N$, pairwise coprime and $a_1, \ldots, a_r \in Z$ then the system

$$x \equiv a1 (mod\ n_1)$$
$$x \equiv a2 (mod\ n_2)$$
$$\ldots$$
$$x \equiv ar (mod\ n_r)$$

has a unique solution $mod(n_1 * \ldots * n_r)$

[67] *Ibid*. F.36v.

[68] *Ibid.* F.39r.

Pacioli tells the reader to start by using a common multiple of the divisors, 60, and to add 1. 61 ensures the remainder. Successively one is to add 60 until the remainder by 7 is nulled. Through this method 301 is identified as the least positive integer solution.

Pacioli then discusses some variants dependent on the choice of the remainders and divisors. Specifically he speaks of the solutions to "a remainder of 1 up to, but not including, some number and 0 for that number". This applied to the interval "2 to 11" results in 25,201 and 698,377,681 for the "2 to 23" case.

Again, the framework is present for the use of the Chinese Remainder Theorem simplifying the redundant remainders of

$$x \equiv 1\ (mod\ 2{,}3{,}4{,}5{,}6)\ x \equiv 7\ (mod\ 7)$$

The following is obtained,

$$x \equiv 1\ (mod\ 3), x \equiv 1\ (mod\ 4), x \equiv 1\ (mod\ 5), x \equiv 0\ (mod\ 7)$$

With general Solution $301 + 420s$

Pacioli remarks on the use of these kind of problems in class describing the above as a classical riddle "*a woman selling eggs in the piazza; someone who was playing ball accidentally broke them all, and when asked by the judge so that they could be paid for, she said that she did not know, but when she left home and reckoned them at 2 by the soldo, there was 1 left over; and at 3, there was still 1 left over*"… etc. up to 7 where the remainder is 0, the question is: How many eggs were there to begin with?

This form of the problem is related often to the seventh century text *Brahma-Sphuta-Siddhanta* by Brahmagupta.

25. Twenty-fifth Effect: To find a number which divided in 2 has a remainder of 1, in 3, 2, in 4, 3, in 5, 4, in 6, 5, [and] in 7, null.[69]

As in the preceding section the goal is to find a number whose remainders increase by one as the divisors do, up to, but not including, some number that evenly divides into it. The remainder should be 1 for divisor 2, 2 for the divisor 3, and so on up to the divisor 7 where the remainder is 0. As in the previous section, a common multiple is found, multiplying the prime factors of the intended divisors $2^2 * 3 * 5$, obtaining 60. This time 1 is subtracted, ending up with 59, assuring the remainders as before. Alternatively Pacioli suggests guesswork multiplication of 60 by some integer and then subtracting 1. The least such number is identified as 119, but several others are listed by the author.

A small, possibly misplaced (as It does not seem to follow from the text directly), paragraph mentions another rule,

$$\Big(\Big(\Big(\Big(\Big(\big((2+1)*3+2\big)*4+3\Big)*5+4\Big)*6+5\Big)*7+6\Big)*8+7\Big)*9 =$$
$$725751$$

which does not give a remainder of 0 by 7, which might be the reason for the annotation "*Revideas hanc regulam, que videtur claudi, cur?*" [70] after a double bar.

[69] *Ibid.* F.42r.

This is the number of eggs beforefore mentioned. These are otherwise not further mentioned in this section.

Once more the problem is generalized giving as examples the same cases as before, with the respective remainders, "2 to 11" and "2 to 23", respectively, obtaining 2519 and 4655851199.

Perfect Numbers (I.26)

26. Twenty-sixth Effect: To find a perfect number thought of.[71]

Pacioli begins by defining perfect numbers. He references *Elements IX*[72], redirecting the reader to his *Summa* as he wishes not to expand further on the topic in this treatise. Pacioli remarks that perfect numbers end in 6 or 8 by necessity. He also falsely conjectures that these last digits alternate. He would be proven wrong in 1588 by Cataldi's calculation of the 7th perfect number.
Pacioli then proposes the following effect: A participant thinks of a perfect number and limits it between two numbers. The performer then guesses that number. Alternatively one can boast to be able to find a number that is the sum of its factors given an interval. Either is based on Elements IX, 36:

$$\text{If for some } p \in N, (2^p - 1) \text{ is prime, then } (2^p - 1) * 2^{p-1} = c, \text{is perfect}$$

The somewhat artificial interval, $a < c < b$ exists to warrant the uniqueness of the perfect number in regards of the guessing. Pacioli exemplifies for (a , b) = (28 , 8128)

Pacioli, referring the *Summa* once more, also explains how to swiftly calculate the sums of the doubled numbers starting at the unit.

$$2^p - 1 = 1 + 2 + 2^2 + \ldots + 2^{p-1}$$

It is possible that this effect was used in class to teach both perfect numbers, Elements IX, 35 and 36 and sum of geometric progression. As it stands the first version of the effect does not seem too impressive, as the perfect numbers were likely known by heart to those who understood the concept, given the reduced amount of perfect numbers known at the time.

The second version however might have been put to practice by anyone who knew to sum and multiply. Pacioli describes the above using counters. One is to start with a stack of one and then double each following stack. When the sum of stacks is prime, multiply the sum by the number of counters in the last stack obtaining a perfect number. This works well for the first three perfect numbers.

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[70] Roughly translating to "Review this rule, which seems defective, why". Further a small lettered note is found in the margin nearby, too small to read from the copy of the folio used here.
[71] *Ibid*. F.44v.
[72] "*If as many numbers as we please beginning from a unit are set out continuously in double proportion until the sum of all becomes prime, and if the sum multiplied into the last makes some number, then the product is perfect.*" Elements IX, 36 is the underlying mathematics of this section.
"*A* perfect number *is that which is equal to the sum of its own parts.*" Elements VII, 22 is not mentioned but quoted.

27. Twenty-seventh Effect: To find a number by virtue of the unit.[73]

In this effect the participant is asked to do some calculation starting with a thought of, secret, number, sharing the operations but not the result. The performer forces the return upon the original number, or guesses it, through the "power of the unit". This is based on equality

$$A = B$$

Whatever the participant does to "his side" (A) of the equation the performer does on "his" (B). The performer is to take 1 and multiply it by the numbers used by the participant. In the end he only needs to ask for the inverse of operations to return upon the initial thought of number. This is similar to (I.21).

Pacioli initially restricts the effect to multiplications, but then discusses sums/subtractions. He alerts the reader to sum/subtract a corresponding ratio of the number added by the participant, for the equality to hold true. Even powers are discussed. Here an alert is made to force the first operation not to be a power, as the unit would remain one, and Pacioli recommends to avoid powers altogether for in his words powers are "*cose' sutili et maestre*".

Pacioli advises a focus on presentation, asking for the numbers which shall suffer the four basic operations and finally forcing a desired number to result from them. Many examples are given and the author repeats himself; also there seems to be a pedagogic note present, perhaps this material was once more to be used in class.

28. Twenty-eighth Effect: To find, spot on, the number thought of, in every way.[74]

Several operations are performed by the participant over a number he thought of. After the operations, which cancel each other out, the performer makes the result come back to the initial number. $\frac{\left(\frac{3s}{2}\right)*3}{9} * 2 = s$, is given as an example. A short explanation is given in regards to the unit, linking this effect to the previous one.

29. Twenthy-ninth Effect: To make any number appear, for a number thought of.[75]

At the beginning of this section there is a brief introductory paragraph dedicated to someone, whose name is left blank[76]. The reader is also informed of the author's limited allocation of free time as a friar, due to other duties.

This time the performer forces the result to be a certain number (in this case 100) through operations given a thought of number (n). At the end a short explanation is given regarding the power of "*repiego*", in vulgar, and scientifically "*comunicatia*"[77], of divisors or multipliers.

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[73] *DVQ* F.47r.
[74] *Ibid*. F.53v.
[75] *Ibid.* F.54v.
[76] In the MS "*a.u.sª*" can be read, possibly meaning *ad usum scolae* or more likely *ad uostra signoria* given the blank space.
[77] It is likely that what is meant is commutative property, although community is the direct translation.

$$\frac{(5n) * 4}{n} * 5 = 100$$

Dactylonomy (I.30)

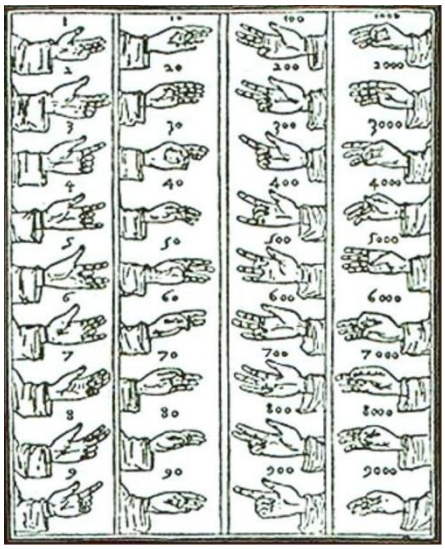

Figure 4: Methods of finger counting *Summa*, F.36v

30. Thirtieth Effect: On numbers thought of, in several rounds, which have been multiplied by diverse or the same [numbers]; to find again what you have asked for.[78]

The mathematical content deviates little from that of (I.27). However, the performer proceeds by multiplying together all products by which a secret number has been multiplied, then divides by the thought of number and to great astonishment reveals the result which has previously been written down.

This effect is of greater interest in regards to its description of a joint performance, where a child secretly gets a sheet with the results of the product to be revealed, or is signaled by the performer.

Pacioli suggests instructing the partner of the performer to learn his hand numeration (this Dactylonomy used by Pacioli might be a variant of that proposed by Saint Bede, or even date back to Arab tradition) present in the *Summa* (see Figure 4), and signal these behind one's back or otherwise discreetly. This way of encoding information is elaborated on and extended to several other effects throughout DVQ.

This kind of effect is exemplified by a story about a man from Ferrara named Giovanni de Jasone, who had instructed a boy to understand coded messages of several kinds and astonished several audiences to his credit.

Positional Writing (I.31)

31. Thirty-first Effect: For a thought of number, to let a friend perform operations, if for thousand years it lasted, and always know how much he has on his hands.[79]

The effect described in this section is similar to several already mentioned. A participant is asked to do calculations, at some point the performer dictates some additional operations of his own and then reveals the last digit or asks to continue only with the last digit.

Among others, the *cifra* becomes a crucial element. As Pacioli puts it, the *cifra* works as *articulo* for the effect. The key is that the performer introduces a multiplication of a power of 10, possibly disguised by factors[80].

$$\overline{a_1 \dots a_n} * 10^k = \overline{a_1 \dots a_n 0_{n+1} \dots 0_{n+k}}$$

The performer can then ask to sum any number and drop the first n digits or force another result. For instance, Pacioli offers this example: After a series of operations done in secret by the participant, the performer introduces the multiplication by a power of 10. Then the participant is asked to add 7. Consequently the remainder for a division of a power of 10 up to that used previously is known to the performer and can be used or revealed "miraculously".

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[78] *Ibid.* F.55v.
[79] *Ibid*. F. 60r.
[80] A bar over a set of symbols means that these are digits in positional notation. When possible these digits are used. The sub-index indicates the position of the digit.

Repdigits (I.32)

32. Thirty-second Effect: Of two numbers which when multiplied one by the other always result in a product with the digits you want.[81]

The performer intends to produce a number of repeated digits by ways of multiplication of two numbers. Pacioli begins by explaining how to achieve this for the repunit[82], of length 6, and then for one of length 12. The discussion consists mostly of case by case examples.

So for instance, Pacioli lays out how $143 * 777$ makes $111111 = R_6$ and then explains that repdigits can be obtained by multiplication of the result or any of the multiplicands by the desired digit. For instance, $2 * 143 = 286$ this multiplied by 777 gives $222222 = R_6^2$, or alternatively, $481 * 462$. For $111111111111 = R_{12}$ Pacioli suggests $900991 * 123321$. The factorization of R_6 and R_{12} clarifies these choice pairs of numbers.[83]

In between the discussion of the above repdigits the creation of pairwise repeating digit numbers is also approached, i.e. $\overline{ababab}$ (for a = 2 and b = 3 this would be 232323). Pacioli tackles these also for digit length 6 and 12. He suggests a method for length 6, however for lengths 12, he simply tells the reader to divide the desired number by 900991 and obtain the second multiplying factor that way. One is to take twenty one times (disguised in the text as twice the number times 10, plus the number) the to-be-repeated two digit string and multiply by 481, this is, $21 * 481 * \overline{ab} = 101010 * \overline{ab} = \overline{ababab}$. Pacioli doesn't discuss $112110 * 900991 * \overline{ab} = 101010101010 * \overline{ab} = \overline{abababababab}$, which could follow from the above.

At the end of this section, Pacioli once more declares that this effect concludes his exposition of purely numerical effects, and that others can be derived from these. He resolves to concern himself with mathematical games, again redirecting the more interested reader to his other works.

Cross Multiplication (I.33)

33. Thirty-third Effect: Take someone you wish and let him spend what he wants; to tell what number of things he bought.[84]

A participant, in Pacioli's example a servant, is sent out twice to buy two amounts of the same merchandise (a_1 and a_2), in the example apples. This has to be done: At the same rate of money per item, and, with two distinct fractions of a currency each time (or alternatively two currencies, p and d). How much money is taken at each instance is known only to the participant, but has to be the same in number of coins. The coinage used in each case, and exchange rate are known to the performer. For instance he takes 20 *picioli* and then 20 *denari* and buys 4 and then 20 apples.[85] The

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[81] *Ibid.* F.63v.

[82] These are numbers whose only digit is 1, these are commonly represented by R_n, where n is the length, the number of digits the number has. Similarly, repdigits, R_n^k repeat the same digit, k.

[83] The prime factorization of Repunits up to the length 1000 can be consulted at http://homepage2.nifty.com/m_kamada/math/11111.htm

[84] *Ibid*. F.67v.

[85] *Picioli, Denari, Quaterni, Bolognino, Grossi, Carlini*, etc. are coinage mentioned throughout the text. In this section some exchange rates between these coins are given. In the above

performer guesses the total amount of merchandise bought after either the first or the second instance's purchase has been revealed.

The results are obtained through cross-multiplication ($\frac{a_1}{p} = \frac{a_2}{d} \Longrightarrow a_1 * d = a_2 * p$). Pacioli gives examples for when the first or second purchase is revealed, the first coinage is more valuable than the second, or vice-versa. In all instances Pacioli assumes that all coins are spent for the purchases.

Subtraction Game (I.34)

34. Thirty-fourth Effect: To finish at any number before a companion; not to grasp more than a certain number.[86]

Pacioli gives an introduction in which he discusses the use of games that might appear to be evil, such as "*carti, tronfidati, tavle, etc.*" These are not to be judged harshly as their presence in the DVQ is to demonstrate the power of numbers and so better understand them. Similar to arguments in other sections the intention is what matters and since it is for the sake of amusement and understandings they do not conflict with Morality. To aid his argument he cites some Latin passages, the second of which is ascribed to Juvenal.[87]

Pacioli describes a game where two players alternate adding to the pot the amount of spots shown on a die. The die just limits the added amount to between 1 and 6. It is not meant to be rolled. The first to bring the pot to 30 wins. This is the discussion of a Subtraction Game which is a variant of Nim. Pacioli gives a winning strategy for the first to play. He should add up to the following number on each of his turns 2, 9, 16, 23, 30.[88]

Pacioli goes on to generalize. The first step of the winning progression in this game, p_0, is the remainder of the integer division of the pot, the last term of the winning progression, p_n, by one more than the maximum that can be taken each turn, in this game 7. This is,

$$p_0 \equiv p_n \ (\text{modulo } 7)$$

If the remainder comes out null, the winning player should go second, but Pacioli does not elaborate further. The other steps are obtained by summing the divisor of said operation to the first step. This is $P = \{p_0, p_0 + 7, \dots, p_n\}$.

Another example is given for a pile of cards. Here, cards are alternatively taken by two players and the player who takes the last card from the table wins.

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example, taken from the text and from several other discussed in this section the rates of 1 *grosso* = 21 *quattrini*; 1 *bolognino* = 6 *quattrini*; 1 *quattrini* = 4(or 5) *picioli* can be inferred.

[86] *Ibid*. F.73v.

[87] Pieper matches this partially to Juvenal's *Satires* 14, 109-19.

[88] This is an Impartial Combinatorial Game. In these games there is a winning strategy for one of the players. Commonly the players are denoted as *previous*, P, and *next*, N, in accordance to their turn of play. P, has a winning strategy which is denoted by the set of moves he should make in his play, in this case $P = \{2, 9, 16, 23, 30\}$ all other moves are losing moves, the set of *N*, since the previous player can add up to his next position.
For more on these games see for instance, João Pedro Neto and Jorge Nuno Silva (2007), *Mathematical Games Abstract Games*, printed by Publidisa for the Associação Ludus, pp. 137-166.

Permutations (I.35)

35. Thirty-fifth Effect: To know to find 3 distinct things divided among 3 people and 4 diverse among 4 and how many you wish, etc.[89]

The performer gives 12, 24 and 36 counters[90] to three participants, A, B, and C (A = 12, B = 24, C =36). These participants then choose secretly to distribute three objects, D, G, and Q, among them. They are asked to discard fractions of their total depending on the objects hidden, and give the performer the total remainder, S, of counters. The performer guesses who holds which object.

$$S = \frac{D}{2} + \frac{1}{3}G + \frac{1}{4}Q$$

It is easy to verify that there is one and only one remainder for all permutations of D, G, and Q in all 6 combinations.

Pacioli narrates an example for three volunteers: ***A**ntonio*, ***B**enedetto*, ***C**hristofano*. In it, each takes different kinds of coins, ***D**enari*, ***G**rosso*, ***Q**uatrino* for these participants to hide. Pacioli then goes through all possible resulting sums,

$$S = \{23, 24, 25, 27, 28, 29\}$$

with the associated distributions, respectively,

$$(D, G, Q) = \{(A, B, C), (A, C, B), (B, A, C), (B, C, A), (C, A, B), (C, B, A)\}$$

This implies who hid what. Similarly the effect can be produced keeping track of the discard pile instead. Pacioli states that he will put the effect into verse as a mnemonic aid. However, neither this, nor the generalization promised in the title, can be found in the manuscript.

Positional Writing (I.36)

36. Thirty-sixth [Effect]: The very same effect when each is given a number of one figure, with a digit.[91]

As in the previous section three objects are hidden among three people, this time however each has a different single number digit to choose from, or assigned, a, b, and c. In either case the digits are known to the performer. The performer guesses which participant holds each object after some operations have been performed according to who holds each object.

$a, b, c \in \{0,1, \dots ,9\}$ such that $a \neq b \neq c$,

$$(((2 * D + 5) * 5 + 10) + G) * 10 + Q) - 350 = \overline{DGQ}$$

The result is inferred by knowing:

$$(D, G, Q) = \{(a, b, c), (a, c, b), (b, a, c), (b, c, a), (c, a, b), (c, b, a)\}$$

An example for the same characters from before is given with (D,G,Q) = (7,5,9). The same could be done by rolling dice, Pacioli suggests. The reciprocal could as easily be done as it is in (I.41), to have three choose in secret either a digit, or a number of

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[89] *DVQ*. F.76v.

[90] These are to be thought of as abstract quantities, although one can read in the margin that "*desotto in mediate se da el modo a far con fave et monete in questo. XXXV effecto*", roughly "one could do this with beans and coins", which is then described in (I.37).

[91] *Ibid*. F.77v.

counters less than 10, and then successively ask them for operations guessing each number.

Note that although the effect might produce a result similar to the preceding one, guessing who has what object, it is intrinsically different. Here the positional writing shows the performer the result while in the above the permutations have to be known.

37. Thirty-seventh [Effect]: How the previous method can be done with fava beans and *quartaruoli*, etc.[92]

Pacioli demonstrates effect (I.35.) replacing the abstract counters by concrete objects. As in previous examples these are beans or coins, fava beans and *quartaruoli* (respectively), and are to be thrown literally into a pile.

Positional Writing (I.38)

38. Thirty-eighth Effect: To find the spots of two dice.[93]

Two dice are rolled by a participant in secret. After the performer has asked for some operations to be applied in turn to the numbers of spots, and having learned the final result, he guesses the spots on each die.

$a, b \in \{1, \dots, 9\}$,

$$(2a + 5) * 5 + b - 25 = \overline{ab}$$

This effect, like (I.36), relies on positional writing and the use of single digits. Examples are given for (a, b) = (6,6), (6,5), (4,3), (4,4). In case of a double it is impossible to say which die was which.

39. Thirty-ninth [Effect]: Of one who divides 10 ducats among two; to know how much he has in one, or how he divides them between hands.[94]

The author applies effect (I.38.) to someone who has taken coins or other objects into two hands or two people who have hidden objects among themselves. There should at most be ten of these objects to start from.

Parity (I.40)

40. Fortieth Chapter: Of two things, one per hand, divided among two, or, in unequal numbers; to know without question.[95]

Two objects are hidden among participants, or, by one in two hands, or in some equivalent fashion. A different price is given to each object. The performer asks to multiply the value of one hand and then of the other and to add these together. Learning the last digit of these operation he immediately guesses which object is where.

This effect works due to properties regarding parity, present in Euclid's *Elements.*[96] The values of the objects should be odd and even respectively. The values multiplied by should again be odd and even. If in the multiplication by the even number the other factor is odd the result is even and thus the sum with the other multiplication,

[92] *Ibid.* F.78v.
[93] *Ibid.* F.79r.
[94] *Ibid*. F.80r.
[95] *Ibid.* F.80v.
[96] *Elements* IX, 21 – 30.

also even, is even. If on the other hand in the multiplication by the odd number the other factor is odd the result is odd and thus the sum will be odd. In both cases the performer knows exactly where the odd and even valued objects lie.

Pacioli exemplifies for a pearl worth 7 and a ruby worth 10, each given to one of two participants, António and Benedetto. Antonio is asked to double, while Benedetto is asked to triple. Then they are to add the products together, and they give the result as 44. Since the result is even the one who had the odd value has doubled, so Antonio must have the pearl and Benedetto the ruby. This effect is suggested to be performed with cards and coins as well.

Positional Writing (I.41, I.42, I.43)

41. Forty-first Chapter: To find 3 numbers, or the spots of 3 dice, or 3 different things handed out. *Bella cosa.*[97]

This is a slight variation on (I.36). As the author himself says, the only difference is that here 10 is not added.

$$(((2 * D + 5) * 5) + G) * 10 + Q) - 250 = \overline{DGQ}$$

The variation already discussed in the above section is now presented. As in (I.36) this mathematical truth is used to emulate the immediately preceding effect, three people are given 3 objects with a value and the performer guesses who has what object. Pacioli once more suggests using dice but also names of cities, objects and other things can be guessed through this artifice.

An example is given for (D,G,Q) = (5,4,3).

42. Forty-second Chapter: To find one ring amongst more than one person and other things through the rule of 3 dice.[98]

The above used artifice is applied to find a ring in a round of people. A ring is hidden among up to nine people, D, organized so that the performer can keep track of them. One of the people hides the ring stuck on a finger, G, behind his back or in a pocket. Each finger is to be given a number starting with the smallest on the left hand up to the thumb of the right, 0 to 9. Even further the knuckle, Q, where the ring is stuck on is also guessed. Each knuckle is assigned a number from 1 to 3. In all that remains one is to proceed with the calculation as in the previous effect.

An example is given for (D,G,Q) = (6,7,2).

43. Forty-third Chapter: The same, in another way.[99]

This time the ring is hidden in the same way as before, but another digit enters play, thus there are people, D, finger, G, this time 1 to 5, hand, P, 1 or 2 and knuckle, Q. Accordingly the operations are:

$$((((2 * D + 5) * 5 + 10) + G) * 10 + P) * 10) + Q) - 3500 = \overline{DGPQ}$$

This might be the generalization referred to in the title of (I.35)

An example is given for (D,G,P,Q) = (3,2,4,2)

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[97] *DVQ* F.84r.
[98] *Ibid*. F.86r.
[99] *Ibid.* F.87v.

Equation (I.44)

44. Forty-forth Chapter: To know, without questioning, how many ducats or other a man has in hand.[100]

A participant has the same unknown quantity of counters in both hands, x. The performer asks him to move a particular amount from one into the other, a. The participant is then to empty the hand holding the lesser amount, followed by as much from the other hand. If the performer wishes he can further complicate matters introducing some counters himself, k, into his hand. The performer guesses the quantity remaining in one of the hands.

Once more a simple equation explains the trick.

$$x = x \Rightarrow x + a - (x - a) \pm k = 2a \pm k$$

At the end of the section Pacioli, once more, speaks of the Moral correctness of doing such tricks for their intellectual and entertainment value, this even being applied to small lie, or pretending not to know how the effect is produced.

45. Forty-fifth Chapter: To know, without further question, the number in hands of a friend.[101]

A participant is to distribute multiples of a known quantity, a, among several people, p. Each consecutive person is given as much more as far it is from the first. That is, the first gets a, the second gets $2 * a$, the third $3 * a$ and so on until the last, the performer, gets $p * a$. All is added together into a pot and half of this is thrown away. Next the remainder in the pot is shared equally among all except the last person, each getting the initial quantity, a. Finally the performer guesses how much is left in the pot.

$$\sum_{i=1}^{p} (i * a) - [(p-1) * a] = \frac{p * a}{2}$$

Pacioli illustrates this effect recalling a child's motivation talk "once, 12, for António, one more time, 24, for Benedetto and one more than that, 36, for the King of France". Pacioli tells the reader that the logic behind the effect is as in (I.27). The effect is to particularly good with the uneducated. "*Avenga chi a presso al vulgo et plebei stanno asai exstimati et presertim a pud mulierео.*"[102]

46. Forty-sixth Chapter: Of someone who goes to a teller and demands 3 things he wants satisfied at the same time.[103]

Pacioli tells a story about someone who went to the bank of the Spanochi, in Rome and demanded three things to be fulfilled in one run from a teller, Girolamo Savelli of Siena.[104]

The teller is:

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[100] *Ibid.* F.88v.
[101] *Ibid*. F.90r.
[102] Roughly, "very popular amongst commoners, especially women".
[103] *Ibid*. F.91v.
[104] This teller is mentioned by Pacioli as former disciple of his. Some ambiguity arises as later on Savelli has "been taught by *us*" likely referring to the institution where Pacioli taught instead.

(i) To match an unknown amount of coins in hand, *n*,
(ii) Add up to some given amount, *c*,
(iii) Give as a present another specific amount, *d*, of coins.

Pacoli proposes the following solution. The teller is to bring back $c + d$ coins. Of these he is to match the n coins which are now revealed, satisfying (i). There will be $c + d - n$ coins left. Of these $c - n$ are added to the already matched *n* coins, adding up to *c* the total of coins the customer now has, this satisfies (ii). Finally the teller gifts the remaining *d* coins, fulfilling (iii).

Pacioli provides an example using the asked for amounts (c,d) = (60,20) and (100, 27), the amount *n* varying between 10, 30, 12.

This might be a description of a failed change raising con. The effect survives in the present day and is performed even by professional magicians. [105]

Numerical Games

Two odds sum an even (I.47)

47. Forty-Seventh Chapter: Of a teller who places on a table some piles of coins for a "*bel partito*".[106]

Pacioli tells of a challenge by Carlo Sansone posed in Perugia by another disciple. 100 coins are piled in odd numbered piles of 1, 3, 5, 7 and 9 coins, so that there are four piles of each. This forms 5 groups of 4 piles, 20 piles in total. The challenge is to add an odd numbered amount of coins with an odd number of piles. In the DVQ the task is to specifically sum 30 coins picking up 5 piles. The reward for the task is to get all 100 coins.

It might be that there is supposed to be also some sort of fee, in coin placed as a new pile on the table, since mention is made of a second player who resolves the task. The new pile would enable the performer to succeed.

The challenge is impossible to solve. Pacioli uses the *Elements* explain this.[107]

The same bet is also suggested to be done with playing cards.

The same proposition from the *Elements* is related by Pacioli to the "popular expression": To fit 20 pigs into 5 *botte*. The expression possibly means, to achieve something impossible. A solution however is given as a word game later on. Namely 20, *vinti* in volgare, is broken into *v-i-n-t-i*. This word play is found in the last sections of DVQ (III.iv.R.133).

One can observe that *P-o-r-c-I* works as well. This might be what Pacioli means here.

48. Forty-eighth [Chapter]: By which another places as many other piles for the "*bel partita*".[108]

This is a variation of the previous heap game. This time, only even numbered piles are used, and an uneven number is to be made with any number of piles. Pacioli credits Catano de Aniballe Catani from Borgo for this version. Catano is to have performed it

[105] This effect is sometimes known as "The trick that fooled Einstein". A performance can be seen at Scam School https://www.youtube.com/watch?v=PeFtx-lEQyI.
[106] *Ibid*. F.92v.
[107] "*If as many odd numbers as we please are added together, and their multitude is odd, then the sum is also odd.*" *Elements* IX, 23.
[108] *DVQ* F.93v.

during a Christmas feast in Naples in 1486. Once more an explanation based on the *Elements* is given.[109]

Jeep Problem (I.49)

49. Forty-ninth [Chapter]: Of two who carry apples, who ends up with more.[110]

The next four effects are Jeep or caravan problems[111]. In these, a character/vehicle has to cover a certain (straight) distance, *d*. It is to carry some sort of cargo, *c*. However some part of it is lost in relation to the route, *h*. The vehicle is only able to carry only a portion of the total to be transported, *p*.

In this section Pacioli tells the story of a citizen of Borgo, *A*, who sends someone to carry as many apples out of an initial collection of 90 apples ($c = 90$). These are to be delivered to a "*gentil humo*", who lives 30 miles away ($d = 30$) in Perugia, *B*. The carrier can however only be burdened with 30 apples at a time ($p = 30$). Further the carrier consumes an apple for each mile he travels towards his destination ($h = 1$).

Pacioli gives two solutions and refers to an illustration, which is once again missing.

The first solution is given. The bearer, loaded with 30 apples, travels to an intermediate point, *C*, 20 miles from the start. Here he deposits 10 apples, having lost another 20 along the way. He returns to *A* and repeats the venture two times more. In these three journeys a total of 60 apples get lost and 30 deposited at *C*. Finally a single trip is made from *C* to *B*, with a load of 30 apples. 10 apples get lost and a total of 20 apples arrive at *B* (see Figure 5)

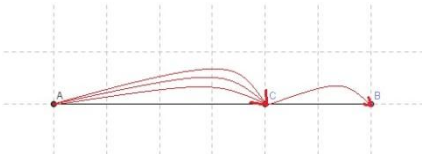

Figure 6: First Solution of I.49, 4 trips loaded with 30 apples.

The second solution uses new points to subdivide the trajectory. *C* is now 10 miles away from *A* and a new intermediate point, D, is introduced 15 miles from *C* and 5 from Borgo. First three fully loaded trips are made to *C*, storing the remainder of each trip here. This leaves 60 apples at *C*. Next two trips to *D* are made. This deposits 30 apples at *D*. A single final trip to *B* takes 25 apples to the final destination (see Figure 6)

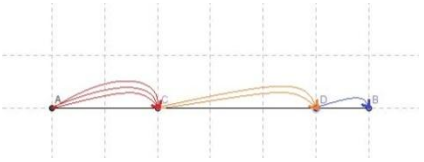

Figure 5: Second Solution of I.49, 5 trips loaded with 30 apples.

At the end Pacioli suggest changing the conditions of the problem (*d*, *c*, *p,* and *h*), for "something similar".

Besides posing the problem, Pacioli gives the apples a value, but never mentions it again. This is likely to stress the importance of maximizing the outcome of the Journey. Further Pacioli contextualizes all abstractions of the solution, begin and endpoints are Borgo and Perugia (A and B respectively) and intermediate points are made palpable as small localities in between the cities, a town called *Fratta* (*C* in the first solution), the bridge *"ponte moglio*" (*C* in the second solution) and "*capo cavallo*" (for *D*).

---

[109] "*If as many odd numbers as we please are added together, and their multitude is even, then the sum is even.*" *Elements* IX, 22.

[110] *DVQ* F.94r.

[111] More on this subject for instance in Gardner, M. (1961). *The Second Scientific American Book of Mathematical Puzzles & Diversions: A New Selection.* New York: Simon and Schuster, pp. 152 and 157-159 or on the web at http://mathworld.wolfram.com/JeepProblem.html .

50. Fiftieth Chapter: Of 3 ships passing 30 "*gabelle*", 90 measures.[112]

A problem is posed where some ships/mules are the form of transport for measures of grain. These have to pass various *gabelle*, or custom posts, before arriving at their destination. At each of these posts they are to pay 1 measure per ship still travelling. To begin with there are 3 ships with a total cargo of 90 measures of grain. Each ship can carry 30 measures. There are 30 posts to be crossed. How should they transport the cargo to their final destination so as to maximize the results?

This problem is equivalent in solution to the previous problem, $(d, c, p, h) = (30, 90, 30, 1)$. The first 10 payments are all removed from the same ship. This removes a total of 30 measures of grain, 3 per post, as there are 3 ships in the caravan. After the ship is empty it returns home or otherwise leaves the caravan. Two ships carry on. The next 15 payments are removed again always from one of them. Again after it has run out of cargo it leaves the caravan and one ship is left to clear the last five posts, now only paying 1 per post, as there is only one ship left sailing. It reaches the final port with 25 measures.

51. Fifty-first Chapter: About carrying 100 pearls, 10 miles, 10 a round and leaving 1 a mile.[113]

This is a Jeep problem for $(d, c, p, h) = (10, 100, 10, 1)$. Pacioli offers the following solution: Carry 10 pearls for 2 miles. Store the remaining 8 to return for another trip with 10. Repeat this until all pearls are stored 2 miles from the start. Then do this for every 2 mile interval until reaching the destination. After the final 2 miles, 16 pearls are left.

52. Fifty-second Chapter: The very same with more surplus.[114]

An alternate solution is presented to the previous Jeep Problem. Carry 10 half way, stack the remaining 5, return for another shipment, do this until there no pearls at the starting point; repeat this for the remaining interval. The result is 25 pearls at the destination.

Note that neither of these two solutions is the optimal solution. One could for instance, combine both and travel half way, then transport the remaining 50 pearls in two, two mile intervals, as in the above problem, leaving 32 pearls at one mile from the destination. The last mile is done four times, giving 28 pearls as a result to the problem.

Pacioli leaves this to the reader as he also refers here that he shall leave further examples and variations "*al tuo"* ingenious mind "*nel qual sempre me confide etc.*"

[112] *DVQ* F.95v.
[113] *Ibid*. F.96r.
[114] *Ibid.* F.96v.

Jug Problems (I.53)

53. Fifty-third Chapter: To split a barrel of wine: between two[115]

This and the next two sections are Jug Problems. [116] In a jug problem a volume of liquid is to be divided with the aid of "jugs", recipients of lesser volume than the first. One must however always pour the liquids between jugs so that it empties the jug one is pouring from or fill the one which is being filled.

The problem Pacioli gives, tells us of a situation where two brothers struggle to divide a cask of 8 *somme* (amounts) of wine equally. The conundrum consists of doing so with two smaller casks of 3 and 5 *somme* respectively. A friend, expert in numbers, comes to aid them.

The solution is promptly provided. Let's say the recipients are J3, J5 and J8, the last of which starts full. The procedure is as follows, pour from J8 to J3, this leaves J8 with 5 and J3 with 3 *sommes*; then from J3 to J5, followed by J8 to J3, then J3 to J5. This results in J3 with 1, J5 with 5 and J8 with 2 *somme,* $(J3, J5, J8) = (1,5,8)$. Next pour from J5 to J8, J3 to J5, J8 to J3 and finally J3 to J5 obtaining the desired outcome, J5 and J8 with 4 *sommes* each and J3 empty. This is, symbolically in obvious notation,

$$(J3, J5, J8) = (0,0,8) \rightarrow (3,0,5) \rightarrow (0,3,5) \rightarrow (3,3,2) \rightarrow (1,5,2) \rightarrow (1,0,7) \rightarrow (0,1,7) \rightarrow (3,1,4) \rightarrow (0,4,4)$$

There are several ways of simplifying both illustration and solutions of these problems, for example barycentric coordinates or trilinear coordinates. Here it has been decided to illustrate the problems using a graph (see Figures 7 and 8). The solution sequence is given following allowed directions starting at the origin (black vectors in the figures), filling each jug accordingly. The initial volume, J8, is implicit, and does not need to be represented in the graph, the surplus being assumed to be there. Note that Pacioli does not give the optimal 7 step solution nor does he mention the other solution.

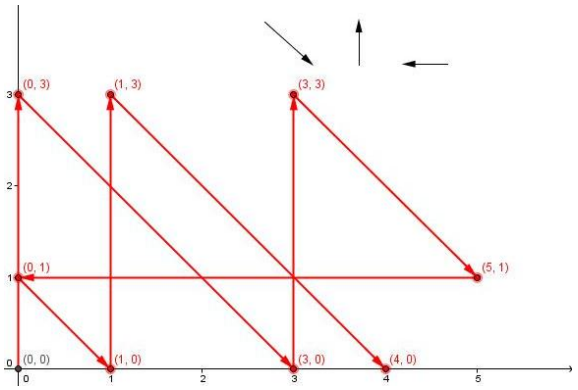

Figure 7: Graph representation of the 8 step solution given by Pacioli in I.53. $(x, y) = (J3, J5)$

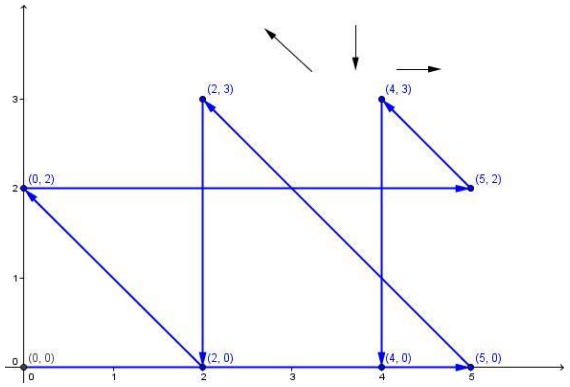

Figure 8: Optimal 7 step solution of the jug Problem of I.53 in graph representation. $(x, y) = (J3, J5)$

54. Fifty-fourth Chapter: On splitting another barrel amongst: two.[117]

Again two brothers struggle to divide the contents of a barrel equally. This time the largest volume of liquid holds 12, and, the initially empty Jugs are of 5 and 7 *somme*. As before we will use J5, J7, and J12 to aid the solving instructions. "*Fa cosi*":

$$(J5, J7, J12) = (0,0,12) \rightarrow (5,0,7) \rightarrow (0,5,7) \rightarrow (5,5,2) \rightarrow (3,7,2) \rightarrow (3,0,9) \rightarrow (0,3,9) \rightarrow (5,3,4) \rightarrow (5,0,7) \rightarrow (0,5,7) \rightarrow (5,5,2) \rightarrow (3,7,2) \rightarrow (3,0,9) \rightarrow (0,3,9) \rightarrow (5,3,4) \rightarrow (1,7,4) \rightarrow (1,0,11) \rightarrow (0,1,11) \rightarrow (5,1,6) \rightarrow (0,6,6)$$

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[115] *Ibid*. F.97r.

[116] Many variations of these problems are popularly found in puzzle books or similar. Recommended lecture include cut-the-knot (water), mathematica (water pouring), or for instance in Pfaff, Thomas J. and Tran, Max M. (2005), "The Generalized Jug Problem", @ Ithaca.edu.

[117] *Ibid*. F.97v.

Again this solution (see Figure 9) is not optimal. One can find a solution in 11 steps (see Figue 10). As in the previous section this results from having filled the smaller of the two jugs first.

In the title of this and the content of these two sections the number of people to divide for, "*doi*" is stressed. This gives the impression that a generalization, to divide among several some volume of liquid, was planned, for a later section. This is absent, but a brief note of three brothers dividing 18 *somme* among three is given at the end of this section. However, one of the jugs is of 6 *somme*, J6, which Pacioli reduces to the just discussed problem, ignoring the simple solution to use J6 repeatedly for the measure.

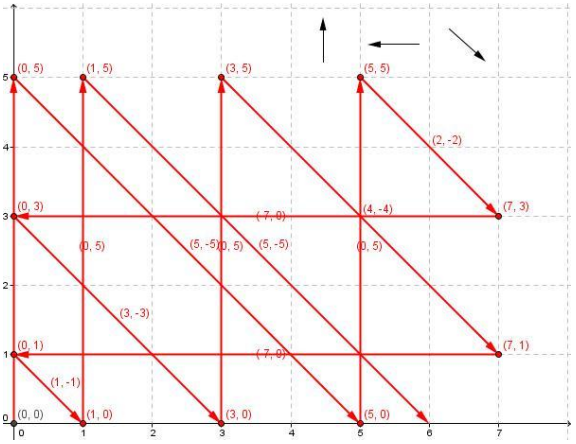

Figure 9: Graph representation of the 12 step solution given by Pacioli in I.54. $(x, y) = (J5, J7)$

Pacioli mentions a manifold of other similar problems derived from those proposed. A challenge which may be derived here is to obtain a challenging and interesting problem for more than two jugs.

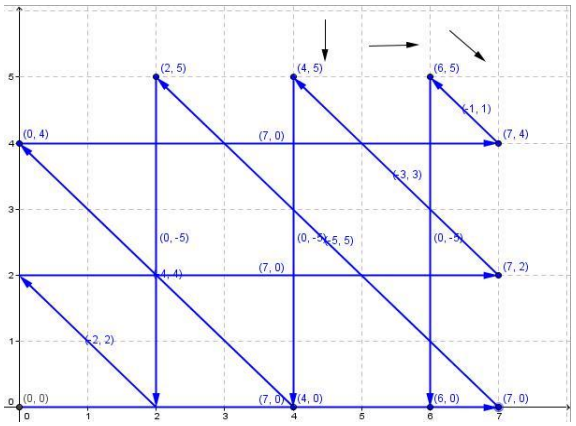

Figure 10: Optimal 7 step solution of the jug Problem of I.53 in graph representation. $(x, y) = (J3, J5)$

55. Fifty-fifth [chapter]: Of two other subtle divisions of barrels; as it will be said.[118]

This is the last Jug Problem. It poses the problem for J4, J6, and, J10. A little hint suggests that the reader will understand "*Et alo Idiota proposto sa fatigara in uano cercando lo impossibile*".[119] Since all the containers are even sized, an uneven quantity cannot be measured; Pacioli probably had the already mentioned parity propositions of Euclid in mind. Further it can be proven that

Given $Ja, Jb,$ and, Jc such that a, b are mutually prime naturals and $a + b = c$, any integer, q, such that $0 \leq q \leq c$, can be measured.

Since in this section 4 and 6 aren't mutually prime the proposition does not hold true, and one can easily verify that only multiples of two up to 10 can be measured.

[118] *Ibid*. F.98v.

[119] Roughly "And for the idiot the proposed will exhaust him, trying to achieve the impossible".

Josephus (I.56)

56. Fifty-Sixth [chapter]: Of Jews, Christians in different ways and rules; to make them as many as you want, etc.[120]

These next six sections are all variations on the Josephus problem.[121] These problems are named after the first century Jewish historian Josephus Flavius, and consist of a counting-out-game. It begins with a number of elements, *n,* arranged in a circle. A number, *m*, is counted and the respective element is removed. One counts always in the same direction, the number counted never changes, and counting restarts at the next element. Counting only ends once all but a certain number, *r*, of elements are left (traditionally $r = 1$).

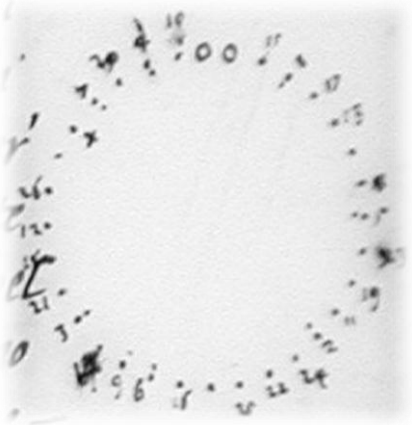

Figure 11: Josephus Game, illustration from the DVQ F. 100R. The cross marks the starting point. Dots are Jews, circles Christians, the numbers are the order of removal.

Usually these problems are contextualized with a round of people who are to suffer a grave fate when they are counted out. The problem thus is: where should one stand to be the last, or among the, m, last, to be removed? Unless otherwise noted, a clockwise counting and for an initial numbering, where the first position is the first person counted, is used as reference.

Again Pacioli advocates the usefulness of the knowledge of numbers and related contents such as the one presented. He sets a situation at sea where people have to be thrown overboard for the remainder to survive. To decide who goes and who stays in a "fair" manner a game is proposed. The game is mentioned side by side with drawing straws, which might mislead the reader into the impression that this game is one of chance.

The situation is as follows, a cargo ship with 30 Jews and 2 Christians is in a dire situation as mentioned above. To decide who stays on board they decide to count-out one by one those who don't. Standing in a circle every 9nth is thrown into the water ($n = 32,\ m = 9,\ r = 2$). The two Christians position themselves at 6^{th} and 7^{th} position in regards to the spot where the counting starts and are the last left standing (see Figure 11).

Pacioli gives a short "hands-on" method to determine the final standing positions. The idea is simply to do so by exhaustion, using pebbles or some other element in a circle one is to count according to the rules of the game to find the last positions. Pacioli suggests the use of different numbers each time and to vary the number of elements. The concrete use of 32 elements, in his example, as Pacioli explains, is due to the number of chess figures he used to obtain the solution. Little other mention of Chess is found in the DVQ, this might be to the dedicated work mentioned in the introduction.

Then solution for $m = 8$, in the same situation ($n = 32,\ r = 2$), is also given. This is, the Christians should stand in 17^{th} and 28^{th} positions, or, "to start counting after five, including one's self, towards the greater number of Jews from the Christian's perspective having 11 in between Christians on one side" as Pacioli puts it.

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[120] *Ibid*. F.99r.

[121] On this, see for instance wolfram (Josephus) or cut-the-knot (Flavius Josephus (http://www.cut-the-knot.org/recurrence/flavius.shtml)/ recurrence solution/ USAMTS 2005-2006), or for more dedicated literature Rouse Ball, W.W. and Coxeter, H.S.M. (1987). *Mathematical Recreations and Essays*, Dover or Graham, Ronald L., Knuth, Donald E., and, Patashnik, Oren (1994). *Concrete Mathematics*, Addison-Wesle.

In this section Pacioli speaks of his service under "*S. Antº rompiaci dala giuderia di venegia*", before the time he joined his religious order. From the passage it appears he travelled by ship often in that time, however, perhaps the passage is only reaffirm his authority on what happens on board a ship, and might be related to where/when he learned of the problem. Once more Morals are considered. Pacioli defends the knowledge of these games, in this dire and perhaps not entirely honest situation, suggesting it is wise to take the precaution of how this and other games work and in general be mentally fit in regards to the mathematical subjects, offering an analogy to the episode where St. Peter has a knife to cut off Malchu's ear (John 18:10 and 25).

A general solution, other than recursively calculating the position of the last man standing, or to extend a known final position to a bigger group, does not yet exist. For some special cases, however, as such as $m = 2, r = 1$, we can calculate the last elements starting position by the following equation[122],

$$2(n - 2^{\lfloor \lg(n) \rfloor}) + 1$$

Or in words: One should find the largest power of two in *n*, and subtract it from *n*. This is then to be doubled. Adding 1 more one finds the position of the survivor.

An application to this idea is found in the *Down/Under Deal* or *Australian Shuffle* consisting of doing the same as a $m = 2, r = 1$ count Josephus with playing cards. This often used in mathematical magic tricks.[123]

57. [Unnumbered chapter] Of 18 Jews and 2 Christians. [124]

This time $n = 20$, $m = 2$, $r = 7$ are taken. Chess pieces are again suggested for representation. The two last standing pieces should once more be placed next to each other at positions 2 and 3, in regards to the first counted person.

58. Fifty-seventh Chapter: Of 30 Jews and two [Christians] counting to 7, whose turn it is to go into water [takes a dive].[125]

This is the case of a Josephus for $n = 30$, $m = 2$, $r = 7$. Pacioli mentions the need of an interval of two Jews between the Christians, such that the initial positions are 2 and 5.

59. Fifty-Eight Chapter: Of 15 Jews and 15 Christians for 9 into the water.[126]

Fifty-ninth Chapter: *Quarter quinque, duo unus, tres unus et unus bis, duo ter, unus duo, duobus unus*

This time the Josephus problem is $n = 30, m = 9, r = 15$. A mnemonic verse is proposed and found in the next sections heading (see the above). The Latin worded

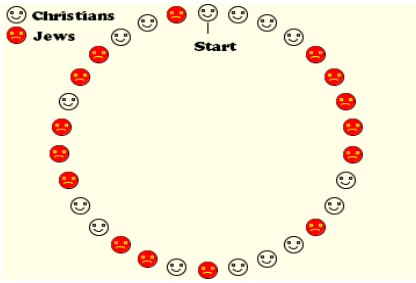

Figure 12: Josephus game implemented with the Java applet by Cut-the-Knot, with edited image by the author. Red unhappy smiles are the 15 Jews, white happy smiles are the 15 surviving Christians. For a Josephus $n = 30, m = 2, r = 15$ in regards to I.59

[122] A simple explanation can be found at http://www.exploringbinary.com/powers-of-two-in-the-josephus-problem/ .
[123] For a more detailed discussion of this and other mathematical card tricks see for instance CardColm (Australian Shuffle), or, Silva, Jorge Nuno (2006). *Os Matemágicos Silva,* Apenas editora; Mulcahy, Colm (2013), Mathematical Card Magic Fifty-two New Effects, CRC Press; Circo Matemático (to be published), MatheMagia com Cartas, Ludus.
[124] *DVQ* F.102r. This section is agglomerated with the Fifty-sixth, it has been opted to see it as a separate section given its content and other close by sections.
[125] *Ibid*. F.102v.
[126] *Ibid*. F.102v.

numbers are alternating, Christians and Jews. This is 4 (Christians), 5 (Jews), 2, 1, 3, 1, 1, 2, 2, 3, 1, 2, 2, 1 (see Figure 6).

60. Sixtieth Chapter: Upon by another verse, namely: "*populea irga mater regina reserra*".[127]

"*Populea irga mater regina reserra*" is another mnemonic, each vowel represents a number, a, e, i ,o , u respectively 1, 2, 3, 4, 5. This represents again the solution to the previous Josephus. Alternatively Christians and Jews are to be placed as above.

River Crossing Problems (I.61)

61. Sixty-first Chapter: Of 3 jealous husbands and 3 wives.[128]

In this section a classic river-crossing problem is described. In a river-crossing problem a group of elements is to traverse an obstacle, usually a river, although the scenario might change. Some elements may not be left alone with some of the other elements. Further, there is a vehicle that limits the amount of elements that can cross said obstacle. The most commonly known of such problems is the "Wolf, Goat and Cabbage" traverse. One at a time, these living beings have to be taken across a river by a Shepherd, but if the wrong two are left alone one eats the other. This problem figures both in Alcuin of York and Pacioli's work. However in the *DVQ* Pacioli uses a variation.

Three married couples, citizens of a city like Venice or Chioggia, experts in rowing, want to cross a river. Because the men are jealous, no wife can be left in the presence of another man without the presence of her husband. There is a single boat which carries up to two of them. Luckily an expert in numbers is among their ranks to propose a solution.

Pacioli presents a solution. *A*, *B*, and, *C* are labels for the men, and, *a*, *b*, and, *c* for their respective wives. Additionally Pacioli names the boat *D*, this does not further play a relevant role in his description. The traverses are then described by Pacioli, in order, (go and then return):

ac, *a*, *ab*, *c*, *AB*, *ab*, *Cc*, *c*, *ab*, *a* then finally *ac*.

An alternate solution found in Alcuin is:

Aa, *A*, *bc*, *a*, *BC*, *Bb*, *AB*, *c*, *ab*, *C* and finally *Cc.*

Pacioli suggests the reader should try for himself with the use of an image left in the margin, which is missing. Further varying the problem to include cases of 4 and 5 pairs of jealous spouses is suggested. In this case Pacioli mentions the boat size should also be increased to one less than the number of pairs, otherwise, Pacioli observes the variations to be impossible.

Double Counting (I.62)

62. Sixty-second Chapter: To guess a thing thought of or touched.[129]

[127] *Ibid F.103r.* Pieper translates to "*Queen mother, replant the popular shoot*" given Agostini's correction of the last word to "*resserat*". Either way the mnemonic works.
[128] *Ibid F.103v*.
[129] *Ibid*. F.105v.

A participant is asked to choose one object among, *n*, objects. After they are rearranged into a circle he has to count up to, and including, the object he chose from. He starts at any given object in the circle. He is asked to continue to count, this time in the opposing direction, restarting at the same place he did the first time, and again counting up to the object he chose. The performer guesses what number the participant counted (see Figure 13).

As Pacioli points out after giving several examples, the effect works on the fact that the chosen object and object one starts counting at are counted twice. The number will always be n+2.

Pacioli suggests obscuring the effect by shifting the place the participant starts counting the second time increasing or decreasing the result by as many as were shifted depending on counting and shift direction.

This artifice of double counting is often used to make quantities appear bigger or smaller than they are. A variation with the same principle behind it is, to have square with four piles of matches per edge, each edge summing to the same number of matches. A match is added and the performer is allowed to shift one match per pile to another one. After doing some movements like this total sum of matches per sides stays the same, one match seemingly having vanished. This works because the corners are counted twice.

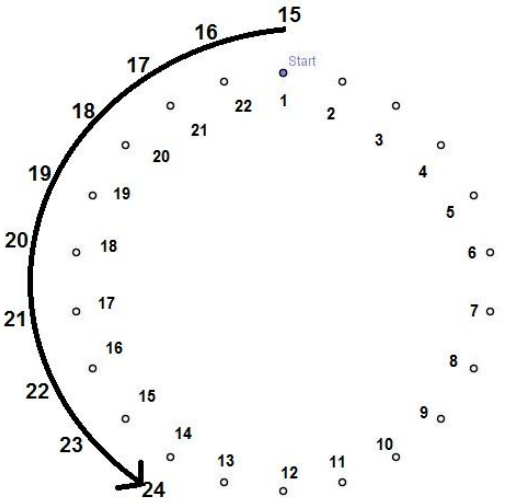

Figure 14: The participant, who thought of 14, is asked to count counterclockwise starting at the heap that has 1 counter. He is to count up to 24, which is 2 more than the 22 concealed piles.

63. This section is missing.

64. Sixty-fourth Chapter: Guess a number thought of through the use of a circle.[130]

A circle of several covered up heaps lies on a table. A participant is asked to think of a number and silently count from that number along the covered heaps up to another. The thought of number is uncovered. Several items can be used to implement this. Pacioli suggest grains of corn, coins or other counters. These should lie hidden under walnuts, bowls, or a sheet. Alternatively numbered paper or playing cards can be used face down.

The heaps are previously organized according to their positions. The first hidden heap has 1 object, the second 2, and so on. The thought of number should be smaller than the total number of heaps.

Then the performer has control, and knowledge of the disposition of the hidden heaps. So he can influence the total counted and the direction in which it is counted, like in (II.62) being able to vary starting point to further obscure the effect. Pacioli's instructions are for the participant to count up to two more than the total objects in the circle (n+2) starting at the first heap and going counterclockwise around the circle (see Figure 14 for an example with 22 piles and a thought of number 14).[131]

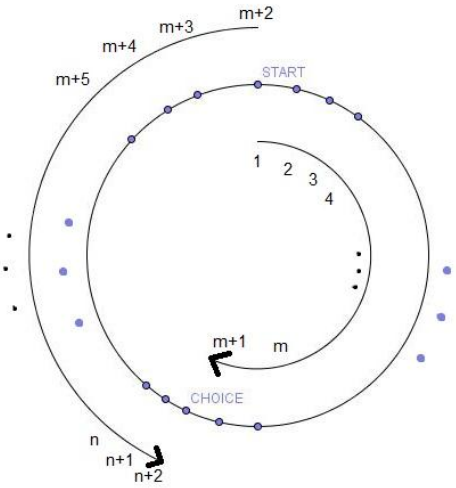

Figure 13: *n*, objects in a circle. The first *m* objects are counted up to the coin, clockwise, then the remaining *n* – *m*, counter-clockwise. The starting element is counted twice obtaining a result of *n* + 2.

[130] *Ibid*. F.108r.

Yielding the same amount by different Sales (I.65)

65. Sixty-Fifth Chapter: Of a merchant who has 3 foremen and sends all to a market with pearls. [132]

Three merchants, A, B, and C, are sent to the market with 10, 20, and 30 pearls, respectively. They all return with the same amount of money, namely 5 *denari*. How is this possible?

Pacioli explains one way this is possible. The one with 30 pearls, *C*, sells them for 5 *denari* setting the mark for the other two. *A* has sold 6 pearls for 1 *denaro* and needs to sell each of his remaining for 1 *denari* to complete the challenge. *B* has two pearls left over after the sale of 18 pearls for 3 *denari*, needing next to make 1 *denari* for each pearl.

(Pearl-/)Apple-seller's Problem (I.66)

66. Sixty-sixth Chapter [Document]: Of one who buys 60 pearls and resells them for as much as he paid [for them] and profits.[133]

This is a bookkeeping scam, similar to the missing dollar "paradox"[134]; given two exchanges of money an extra amount of money appears/disappears.

Someone buys 60 pearls for 24 ducats, that is 5 for 2 ducats. They are sold again in two goes first 30 pearls for 15 ducats, 2 for a ducat; and then the other 30 for 10 ducats, this is 3 for a ducat. Therefore, summing 3 for a ducat with 2 for a ducat, in total it seems as if like before the exchange rate was 5 pearls for 2 ducats. However the amount of ducats received is 25.

Unlike the missing dollar paradox where one purposely confounds debit with credit, here the erroneous reasoning lies elsewhere. The averages of prices are dealt with in a faulty way ($\frac{3}{1}+\frac{2}{1}\neq\frac{5}{2}$).

Coconut Problem (I.67)

67. Sixty-seventh Chapter: A lord who sends a servant to harvest apples or roses in a garden.[135]

A servant is sent into a garden having been told to bring back some apples (*n*). However, he has to pass (*q*) gates. At each gate he passes he has to pay a toll of half, or some other ratio, (*r*) of his total. Additionally he is to pay a fixed number of extra apples (*k*). How many should he start off with?

Pacioli mentions the method of *el cataym* given in the *algebra et al mucabala*[136], once more redirecting to his *Summa*. The problem can be represented algebraically as

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[131] Several variants of this effect exist. A version for cards and a broken clock is found in Ricardo, Hugo and Mendonça Jorge (2013) "O "Thesouro dos Prudentes" de Gaspar Cardozo de Sequeira", essay for the class of History of Recreational Mathematics, University of Lisbon given by Jorge Nuno Silva.

[132] *DVQ* F.119r. Four more sections similar to this one are listed in the index, two of which are referenced in this section, however they are unaccounted for.

[133] *Ibid*. F.119v.

[134] A discussion of such a problem can be found at MathWorld (http://mathworld.wolfram.com/MissingDollarParadox.html).

[135] *Ibid*. F.120r.

$$r^q\ x - q * k\ = n, \text{given } r, n, q \text{ and } k$$

Pacioli solves the problem using a recurrence relation. The reader is to think of the number of apples at the end and then back-track gate after gate to arrive at the initial number of apples, adding the desired quantities.

Examples are given for $q = 3$ and $q = 5$, with $n, k = 1, r = \frac{1}{2}$.

Octagram Puzzle (I.68)

68. Chapter Sixty-eight: Riddle of a city which has 8 gates, which it seeks to reinforce.[137]

Seven constables and their men have to enter a city to occupy 7 gates out of 8. They each are to enter through a vacant gate. Next they are to opt for one out of two paths, each leading to a different gate. The constables are to stop at this final gate occupying it and letting no-other through. The connecting roads make up an octagram. A diagram is mentioned but missing. However, a description can be found at the end of the section (see Figure 15 for reference).

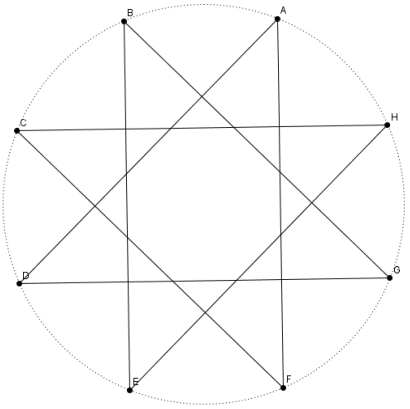

Figure 15: Octagram, the vertices A to H are the gates, the lines in between the streets, the circle the outer walls.

A brief introduction is given on the usefulness of the mathematical powers in the study of warfare. In this respect Pacioli mentions the works of Archimedes and Caesar *Commentaries*, specifically a bridge crossing over the Rheine[138].

The puzzle is contextualized by descriptions of a state of unrest due to two opposing factions thus justifying passage between gates as acts of discrete behavior. Additionally, stated alongside with the problem, there are wage bonuses for the constables who arrive first, starting at 200 and decreasing 50 for every consecutive arrival up to the 4th. This is likely to ensure the order of arrival in the problem, as the wages play no further part in the puzzle.

Pacioli gives the solution. The explanation is simple. As there are two paths to each gate once a gate is taken there remains a unique free path connected to it. Thus the others fall into place once the first is chosen. Concretely Pacioli proposes the consecutive occupation by the constables of *D*, *A*, *F*, *C*, *H*, *E* and *B*. *G* being the door left open and the last point of entry.

Singmaster observes that this puzzle is equivalent to the 7 knight's puzzle.[139] 7 knights have to be placed on a 3x3 board. The knights are to complete a move and then stay on the tile they arrived at. All but the first knight start their moves on the tile the previous knight was placed on.The first can start on any tile.

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[136] This is referent to a Rule of Double False Position, *hisab al-khata'ayn*, present in al-Khwārizmī's works this is an attempt deliberately to low and one deliberately to high and then finding the right result by adjusting in regards to the error.

[137] *Ibid*. F.112r.

[138] *Commentarii de bello gallico* 4:16-18, as Pieper observes. This might relate to the later section (II.84).

[139] See for instance http://people.cis.ksu.edu/~schmidt/300f01/Assign/assign3.html .

Binary Divination (I.69)

69. Chapter Sixty-nine: To find a coin among 16, that has been thought of.[140]

This is the predecessor of the "3 times 7", or "3 times 9" card trick. [141] A card among several spread out in two rows is chosen after sharing the row in which it lies three times, the performer guesses the card. Instead of cards Pacioli uses two piles of 8 coins.

One out of 16 coins is chosen. The coins are set out in two rows of 8. Pacioli illustrates the effect resorting to letters (see figure 16). The participant points out the row containing his coin. The performer stacks up the coins, column by column, from left to right, starting with the column that does not contain the chosen coin. Then the coins are laid out again in rows, first in first out. The process is repeated twice over with the difference that in the following iterations the columns are picked up starting with the one the coin is in. After laying out the rows a fourth time the coin will be the third counting from the end of the bottom row (note that in figure 16, as in the text, the last rows are inverted).

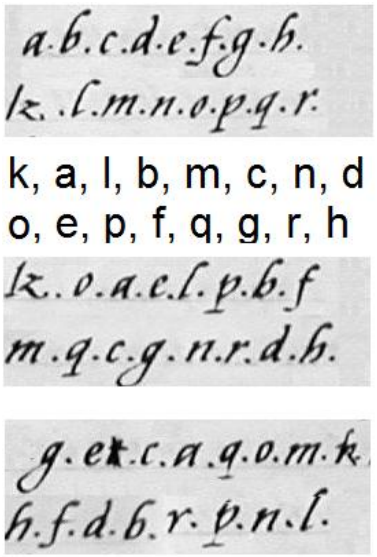

Figure 16: Example given as text by Pacioli, for the choice of *d*. Edited excerpt from FF.115-116v

Pacioli's description is somewhat obscure and his explanations do not completely match his lettered example. However the idea is simple and best illustrated with 8 cards, instead of 16 (see Figure 17). In this variation the cards are picked up right to left, column by column, always starting with the row the card is not in, and the dealing is done last in first out. Three iterations are needed. After the final one the performer knows that the card is the first of that row.

The effect works because at each iteration the performer leaves the card at a position x such that $x \equiv 1 \left(mod\ 2^i\right)$, i is the number of iterations performed thus far. In other words, he narrows down the position of the card and the order and direction in which the cards are picked up determine its position in the stack. In Pacioli's case the position should be $x \equiv 13\ (mod\ 16)$, thus another iteration is likely needed for the effect to work independent of the starting position.[142]

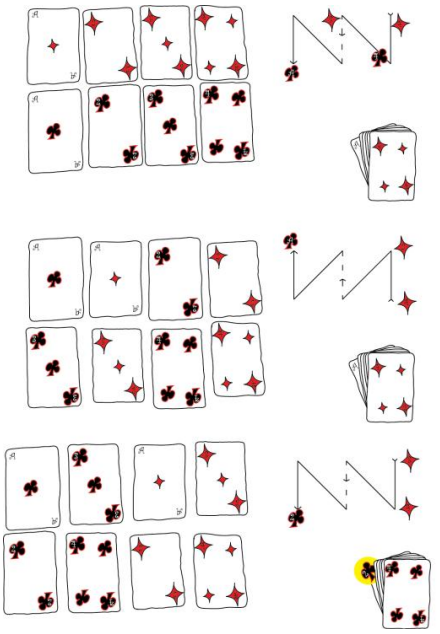

Figure 17: Binary Divination Card trick Illustration from Silva, Jorge Nuno, (to be published), Matemagia com Cartas.

Rearrangement Puzzle (I.70)

70. Chapter Seventy: Riddle about a priest who pawned the burse of the corporal with the pearl cross.[143]

A Priest pawns a burse[144], with a valuable pearl cross on it, to a Jew knowledgeable in the powers of numbers. The cross has 9 vertical pearls. Each arm is positioned such that counting from the bottom up and then along the arm 9 pearls are counted. As the priest returns to retrieve the cross two Pearls have been stolen in such manner that the above description still holds. How did the Jew do this?

Pacioli contextualizes and embellishes the puzzle by bringing into play a bill of sale which has the above faulty description of the cross. The priest tries to sue the Jew, who in the end must be absolved from any crime, given the lack of evidence.

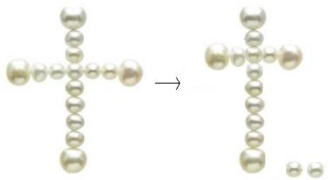

Figure 18: Pearl Cross, before and after the two pearls have been removed.

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[140] *Ibid*. F.114r.

[141] See for instance Silva, Jorge Nuno (2006). *Os Matemágicos Silva,* Apenas editora, pg. 23.

[142] The variant and general mathematical discussion of these effects are present in the upcoming book Silva, Jorge Nuno et al. (still to be published) *Matemagia com Cartas*.

[143] *DVQ* F.116r.

[144] *Corporas-case*, the container in which to store the corporal which is the cloth placed upon the altar for communion during the Catholic Eucharist .

The cross starts with 15 pearls, 9 vertical and two arms with 3 pearls defining the horizontal 4 down from the top. A pearl on each arm is removed. The horizontal is shifted up to account for the difference for the faulty description to hold. Pacioli suggests the use of other crosses for purposes of entertainment.

71. Chapter Seventy-first Document: A square of 3 for each line, diameter or side, and by adding 3, becomes 4 every line.[145]

Three coins are added to a square made of coins, where every side and the diagonals add up to 3 coins each. After this the sides and one diagonal all sum up to 4 coins. How was this done?

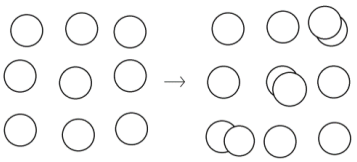

Figure 19: Coin square, before and after 3 coins have been added.

The diagram of a 3 by 3 square mentioned by Pacioli is missing. The solution Pacioli gives is to place the three coins along a diagonal. This makes all sides and the other diagonal sum 4 coins each. Bigger squares and greater number of coins are left for the reader.

Magic Squares (I.72)

72. Chapter Seventy-second: Of Numbers [arranged] in squares disposed according to astronomers, which for all lines sum the same, be it side or diameter. [They] Represent planets and are accommodated in many games and thus I insert them. [146]

This section discusses magic squares, n by n grids filled with numbers. These numbers when added following the same line, horizontal, vertical or diagonal, or in some cases special patterns always add the same.

Pacioli begins by mentioning the works of the great astronomers *Ptolemy*, *Albumasar*, *Ali*, *Alfraganus* and *Geber* and their work "giving the planets numbers". This correspondence is best known from the 1510 book *De Occulta Philosophia* by Heinrich Aggripa. These planetary magic squares are often related to ritualistic magic. Pacioli, however, suggests their use to produce entertainment and to use them for games. The 4x4 square is the same as the one used by Dürer for his engraving *Melancholia I*. The 8 by 8 square differs from Aggripa's, all others finding their counterpart in the other's work.

The bigger squares and the 4 by 4 are only partially provided. The first few lines of the magic squares are given, but remainder is to be found in the margin. It is clear that the text is supposed to be accompanied by images of the squares, but these are missing[147] (see Figure 7). The squares can be reconstructed given their properties and sums, which Pacioli mentions. Only the Mercury square seems to cause some problems given the first and last digit of the first line, possibly having been corrupted by transcription.[148]

[145] *Ibid*. F.117v.
[146] *Ibid*. F.118r.
[147] Unlike most other references to illustrations here the text is literally displaced as to leave room for these illustrations, two spaces are especially obvious in particular on Ff. 121r. and 122v.
[148] Bagni, Giorgio T. (2008) "Beautiful Minds - Giochi e modelli matematici da Pacioli a Nash", Treviso, Liceo Scientifico Leonardo da Vinci .

Saturn 3 by 3

8	1	6
3	5	7
4	9	2

Jupiter 4 by 4

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

Venus 7 by 7

4	35	10	41	16	47	22
29	11	42	17	48	23	5
12	36	18	49	24	6	30
37	19	43	25	7	31	13
20	44	26	1	32	14	38
45	27	2	33	8	39	21
28	3	34	9	40	15	46

Moon 9 by 9

5	54	13	62	21	70	29	78	37
46	14	63	22	71	30	79	38	6
15	55	23	72	31	80	39	7	47
56	24	64	32	81	40	8	48	16
25	65	33	73	41	9	49	17	57
66	34	74	42	1	50	18	58	26
35	75	43	2	51	10	59	27	67
76	44	3	52	11	60	19	68	36
45	4	53	12	61	20	69	28	77

Mars 5 by 5

14	10	1	22	18
20	11	7	3	24
21	17	13	9	5
2	23	19	15	6
8	4	25	16	12

Sun 6 by 6

1	32	34	3	35	6
30	8	27	28	11	7
20	24	**15**	16	13	23
19	17	21	22	18	14
10	26	12	9	29	25
31	4	2	33	5	36

Mercury 8 by 8

4 5

8	7	59	60	61	62	2	**1**
49	15	54	12	53	51	10	16
41	42	22	21	20	19	47	48
32	34	35	29	28	38	39	25
40	26	27	37	36	30	31	33
17	18	46	45	44	43	23	24
9	55	14	52	13	11	50	56
64	63	3	4	5	6	58	57

Figure 20: Five Magic Squares and respective Planets according to description by Pacioli. Black rimmed numbers are given by Pacioli. Bold numbers had to be altered for the magic square property to hold. The original numbers are left above the square.

Arithmetic Progression - Picking up objects(I.73)

73. Chapter Seventy-three: On taking 100 stones in a row.[149]

This section starts with a small tale about a military man called Benedecto dal Borgo, nicknamed Baiardo, and his military games. One day he is to have proposed the following problem to his men: "Which would be quicker: to walk 2000 paces, or, to pick up 100 stones in a row, one at a time, each one pace apart, and piling them up in the same place, one at a time?"

The solution is given by an arithmetic progression. So given that each successive pickup and deposit of the stones is given by the progression $2, 4, 6, \dots, 198, 200$, whose sum can be easily calculated, $50 * 202 = 10100$, it is preferable to walk 4 miles or more. Pacioli simply states that one is to multiply the distance by itself and add the total distance to it to obtain the total. He then proceeds explaining the accumulating of paces required. Further Pacioli suggests using the paradoxical appearance of this problem to make a competition out of it, with the participant doing pick up runs in competition to the performer who walks continuously. Non-linear routes are to be used to further confound the audience.

Coordinate System(I.74)

74. Chapter Seventy-four: Finding a coin, or other thing, touched by positioning it on a square.[150]

An object among several lined up into a square or rectangle is selected by a participant and after revealing in what line and column it is the performer guesses which object it is.

This is the use of a coordinate system $(x, y) \leftrightarrow (\text{column}, \text{row})$. To disguise the obvious intersection point rows should only be mentioned. Pacioli puts this saying the participant should tell him what row it is in counting up, and then what row it is in from the left.

Shifting viewpoints or even laying out the objects anew after having them piled up first might obscure the working of the effect, as is suggested in the next section. A reference to a prior effect is made, but it is unclear which is meant. Pacioli offers an example using the 6x6 magic square with 13 thought of.

75. Chapter Seventy-five: On finding a coin or other thing thought of in a quadrilateral in the most subtle and quickest possible way.[151]

This is the same effect as the preceding one, except that this time the objects are picked up and laid out so that the square suffers a 90º rotation, the spectator always pointing out in which row the object lies in (top to bottom for instance). An example is given with "*trionfy*", playing cards.

Geometric Progression (I.76)

76. Chapter Seventy-six: Of someone who doubles a quantity of coins or other things, suddenly tell him.[152]

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[149] *Ibid*. F.122v.
[150] *Ibid*. F.124r.
[151] *Ibid*. F.125r.
[152] *Ibid*. F.127r.

Pacioli introduces this effect with the story of a Jew who, when Pacioli was in the service of the Duke of Milan, "*Ludovica Maria*"[153], presented a divination of coins. The feat consisted of guessing the total number of coins in several hidden piles. He then proceeds to explain how to perform the effect.

A participant is asked to place piles secretly in a row after an initial one, known to the performer. Each pile is to contain twice (or any other ratio) the coins of the previous pile. The participant can make as many piles as he wishes. The performer upon seeing the piles, knowing their number, or, based on the knowledge of the size of the last pile, predicts the total amount of coins placed and/or the total piles.

This effect and its description revolve around the summing of a geometric progression. The sum is given by: $a(r^{m+1} - 1)$, where, *a*, is the initial pile, *r* is the ratio between the number of coins in consecutive piles, and, *m* the number of total piles.

Examples are given for $r = 2$ and $a = 1$ and 3, as well as $r = 3$ with $a = 1, \frac{1}{2}$ and $\frac{7}{3}$.

77. Chapter Seventy-seven: Of someone who quadruples.[154]

This is the discussion of $r = 4$ for the previous effect for $a = 1$ and 3

78. Chapter Seventy-eight: Of someone who quintuples.[155]

This is the generalization of the earlier effects. It starts with the discussion of $r = 5$ for $a = 1$ and 3 then leaving $r = 6, 7, 8, 9$, etc. for the reader.

Arithmetic Progression (I.79)

79. Chapter[156]**Seventy-nine: For a single rule, to know its [the progressions] sum, continuous or discontinuous, to know where it triggers to where it ends, *generalissima*.** [157]

Here the result of the sum of an arithmetic progression is discussed without further adornment. However, at the end of the section Pacioli suggests looking for an effect to aply this knowledge to expressing his confidence in the intelligence of the reader.

Pacioli describes the formula to obtain the result of the sum of an arithmetic progression

$$S_n = \frac{n(a_0 + a_n)}{2}$$

a_0 and a_n are the first and last terms of the progression. He then discusses the formula to find the number of terms

$$n = \frac{a_n - a_0}{r}$$

Here *r* is the rate of increment between consecutive terms.

Paciolis examples are for the arithmetic sequence $7, 10, 13, \ldots, 31$ $(a_0 = 7, a_n = 31, n = 9, r = 3)$.

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[153] Curiously the year is left out here.
[154] *Ibid*. F.128r.
[155] *Ibid*. F.128v.
[156] Here and in the next title "*CAPITOLO*" is written out in capital letters in the MS.
[157] *Ibid*. F.129v.

80. Chapter Eighty: On the gentleness that at times is made through a natural way without other calculations.[158]

A short story of two performing men named Francesco da la Penna and Giovanni de Iasone de Ferrara[159] whose performance estimating or guessing quantities is to have made a great impression at court.

Pacioli explains how to guess the number of objects or the weight of something. This time there is no explicit mathematical artifice at work, but no less a faculty very important to the field. The feat consists simply of having a keen intuition, for instance guessing the number of chestnuts in a hand or the number of nails given their weight. Simply put, it is to get acquainted with the weight and its equivalents and through empirical experience to train one's intuition.

Pacioli stresses the importance of being well prepared even for something one is to appear not to be prepared for.

81. Chapter Eighty-one: To make someone forcibly guess at Morra and cast in one's way the companion.[160]

Pacioli introduces an effect looking like a Morra[161] variant, but where one player always wins. Morra is a finger guessing game. Rules vary from region to region, but in general the goal of Morra is to call out a number predicting the number of fingers shown by two participants simultaneously. The predictions and showing happens as synchronously as possible. The game is commonly played between two, but can also be played with more players. Various bouts are commonly played in rapid succession until one of the players wins. Alternatively, each correct prediction can score a point and victory is achieved by reaching a fixed score.

In Pacioli's version only one of two players calls out. That player shows only with a single hand. The other player uses both hands but remains silent. Further the single handed player is restricted to predictions of 11, 10, 9, 8, 7 and is forced to show 5, 4, 3, 2, 1 fingers respectively to each of the predictions, this is, if he predicts 11 he has to show 5 fingers.

Pacioli asks which of the two players has a more likely chance to win, fares better in the game. He swiftly explains that it is the second one. To make sure of this all the two handed player has to do is to always hold out 6 fingers. Not to be caught in this ruse the player is advised to show the 6 fingers in different ways and to change the artifice to achieve further misdirection.

This is the last section of the first part.

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[158] *Ibid*. F.131r.

[159] The first of which supposedly having been mentioned in the eightieth effect, coinciding however with this very same one in the present MS

[160] *Ibid.* F.132r.

[161] For a modern game of Mora see for example http://www.youtube.com/watch?v=Ehk9uJ_71tk, or for more information in Camerano or Wikipedia .

II. On the virtue and strength of Geometry

The second part also starts with a small introduction. Pacioli reminds the reader of the unlimited recreations possible taking the work so far as base, using the power of numbers. For completeness sake, and as equal siblings, geometry ought to be equally treated in his treatise.

Like in the first part's introduction Pacioli gives a statement of structure. Each topic is divided into indexed sections. Here the sections are named *Documents*.

Pacioli speaks of the major sources for his work and the presupposed concepts therein. The fundaments of these concepts Pacioli leaves for reference in his own *magnum opus*[162]. Further the *Divina Porportione* is mentioned as additional reference. Like previously, Euclid plays a central role. Especially in the initial half several constructions of the *Elements* are given.

Pacioli lists the following concepts the reader should be familiar with: "*point, straight line and curve; obtuse, acute and right angles, be they curvilinear, rectilinear or mixed, this is, between curved, straight or both types of lines; straight or curved, concave and convex, surfaces*[163]*; and, finally, cubical, spherical, cylindrical, pyramidal solids (bodies) be they regular or dependent*[164]".

Further the following figures should be known: "*circle, triangle, quadrilateral and their variations (through angle and side length), semi-circles, diameter, circumference, center, arc, larger and smaller parts, perpendicular, equidistant and parallel*"; As well as Euclid's five postulates.

Introduction done, Pacioli proceeds with the different sections, like in the first part.

Figure 21: Drawing of a compass, F.134r

Figure 22: Illustration of an unmarked ruler, F. 134.r II.2

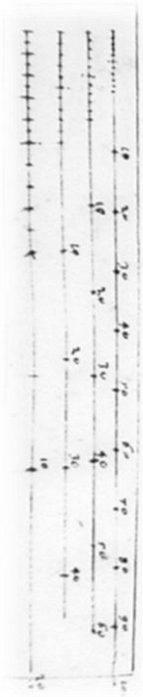

Figure 23: Illustration of a marked ruler, F. 134v, II.2

[162] Likely a reference to his translation of Campanus' *Elements*

[163] In the MS one will find in a different script and darker lettering what seems to read *spherical*. This seems to have been added posteriorly and is a special case of a convex surface.

[164] Pacioli uses "se ratile et dependente". Note that regular is not to be taken in the modern sense, this is, it does not necessarily imply that the figure has the same side and same angle.

Geometric Constructions

Instrumentation (I.1 – I.2)

1. Chapter One: About the instruments necessary for the practical construction of any superficial figures in the following documents.[165]

To perform the constructions that follow the practical geometer should be equipped with two fundamental tools: straightedge and compass[166]. These are described, and instructions are given on how to build them. Pacioli stresses the importance of sharpness of the straightedge and adjustability of the compass, for exact construction. The instruments are depicted in the manuscript's margin for reference (see Figures 21 and 22).

2. Chapter Two: On a model disposed according opportune points, for the mentioned universal constructions.[167]

This section instructs the reader on how to build a ruler, a straight lined piece of wood or brass, with several different scales (Pacioli recommends 3 to 6), of evenly sectioned intervals, disposed along straight parallel lines. Pacioli advices to make these based on experience and need. Again a picture aids construction (see Figure 23).

Construction of "regular" Polygons (II.3 – II.28)

Triangles (II.3 – II.5)

3. Chapter Three: How one could quickly make the first straight lined figure, in 3 of its kinds.[168]

Tools discussed, Pacioli proceeds with the constructions of geometric figures. He begins with the simplest, the equilateral triangle. Pacioli uses this opportunity to explain the calibration of the compass.

To obtain an equilateral triangle start by drawing a straight line segment, the side of the triangle. Then two same sized circles are drawn centered at each of the extremities of the segment, using it as radius. Either of the two intersection points formed can be chosen as third vertex of the triangle (see Figure 24).

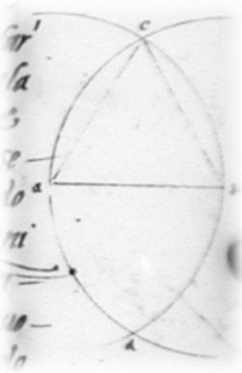

Figure 24: constructing an equilateral triangle, F. 135r, II.3

Pacioli poses the practical situation for a segment of length 10. Open the compass from 1 to 11 on the scaled ruler to obtain this length (there being no null position on it).

To clarify the idea of this difference Pacioli alludes to effect 67 which holds according to the index, (I.67) however, is unrelated. A similar discussion is found in (I.76).

4. Chapter Four: About the second kind of triangle with 2 equal sides, named *ysechele* (Isosceles).[169]

A small digression is made to the *Timaeus*[170] by Plato. He is to have stated that a square which is halved, along its diagonal, forms two right angled equilateral triangles.

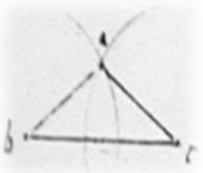

Figure 25: construction of an isosceles triangle, f136v, II.4

[165] DVQ F.134r.
[166] The compass is named *sexto* because with the same opening we can construct a circle and divide it's circumference into six equal parts.
[167] DVQ F.134v.
[168] *Ibid*. F.135r.
[169] DVQ F.136r. In the text below the triangle is called "*ysochele*" and vulgarly "*equicturo*"
[170] Pacioli's reference corresponds to 53c-55d of the mentioned book, found for instance http://www.anselm.edu/homepage/dbanach/tim.htm

This construction is left as exercise to be solved after learning the method to construct a square, later on.

Next, Pacioli discusses in which conditions the angle between the sides equal in length is obtuse (ambligonal triangle) or acute (oxygonial triangle). Given a line-segment, *bc*, and two sides equal in length of the triangle, *ba* and *ca*, take the sum of the square of their lengths, $|ba|^2 + |ca|^2$. Compared to the square of the third side, $|bc|^2$, the triangle is obtuse, if the sum is smaller, $|ba|^2 + |ca|^2 < |bc|^2$, acute, if it is bigger, $|ba|^2 + |ca|^2 > |bc|^2$, or right if it is equal, $|ba|^2 + |ca|^2 = |bc|^2$, than the square. This is formalized and discussed for any triangle in the next section. Pacioli makes mention of the second to last proposition of the second book of the elements[171] and second to last proposition of the first book of the elements[172].

To construct the triangle draw the line-segment, *bc*, then, centered on either of the extremities and the desired opening for the sides' length, make two circles in similitude to the above sections. The compass opening is measured, and it retains its opening if so desired in consecutive construction, it has "memory".

Pacioli does not discuss the case in which the equal sided segments are less than *bc*, it is assumed that they intersect (see Figure 25).

Like before, Pacioli exemplifies for $|bc| = 10$, as well as $|ac| = |bc| = 6$. Summing their squares, which gives 72, which is less than 100, and thus the triangle is obtuse.

5. Chapter Five: About the 3rd kind with 3 unequal sides, named "*stoleus*" [scalene].[173]

The sectioning of a tetragon, a rectangle, to obtain a right angled scalene, is mentioned. Again the construction itself is left to the reader. Boethius is mentioned as source of inspiration.[174]

Given a line-segment, *bc,* set the compass with opening of the desired length for one of the sides, *ba,* on the respective extremity of *bc* and draw a circle. Intersect this circle with the circle with radius of the other sides' desired length, *ca*, centered on the other extremity of *bc*, *c.* The intersection of the circles, *a,* is the sought vertex of the triangle *abc*.

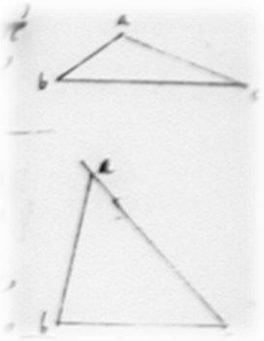

Figure 26: Construction of a scalene triangle, F. 138r, II.5

Again discussion of the different kinds of angles at *a* is given, and illustrated in the margin (see Figure 26).

Pacioli exemplifies with $|bc| = 10$, once for an obtuse angle, $|ab| = 6$ and $|ac| = 7$ and then for an acute one, $|ab| = 9$ and $|ac| = 6$.

[171] "In obtuse-angled triangles the square on the side opposite the obtuse angle is greater than the sum of the squares on the sides containing the obtuse angle by twice the rectangle contained by one of the sides about the obtuse angle, namely that on which the perpendicular falls, and the straight line cut off outside by the perpendicular towards the obtuse angle." P.12 B.2 *Elements*

[172] "In right-angled triangles the square on the side opposite the right angle equals the sum of the squares on the sides containing the right angle. " P.47 B.1 *Elements*, the "Pytagorean theorem"

[173] DVQ F.137v.

[174] Pieper relates this to *De Institutione Arithmetica* II, 26

Quadrilaterals (II.6 – I.9)

6. Document Six: On making the 2nd type of the rectilinear figures, named quadrilateral. [175]

Pacioli addresses the construction of tetragons (quadrilaterals). He categorizes these into 4 kinds: square, oblong rectangle, rhombus and rhomboid. All remaining four-sided polygons are designated by him as *elmuariffe*[176]. These four kinds are pairwise related. The 3rd kind, rhombus, is derived from the 1st, the square, and the 4th, rhomboid, from the 2nd, oblong rectangle, by shifting angles keeping opposite angles the same. The construction of the square is the first construction given.

To construct a square, take a line segment *AB* (the diagonal) and find its middle. This is done by drawing two circles with radius *AB,* centered at *A* and *B* respectively. The line segment which connects both intersections of the circles, *C* and *D*, intersects *AB* at the center of the square, *E*. Drawing another circle with radius *AE* it intersects *CD* at *F* and *G*. Thus the square *AFBG* is formed (See Figure 27)

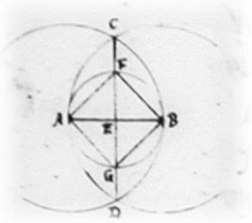

Figure 27: Construction of a quadrilateral, F. 139v, II.6

7. Document Seven: To make the 2nd kind of quadrilateral, named *tetragono longo* [oblong rectangle], or with a lateral long sides.[177]

Take any circle, divide it into two equal parts by its diameter, *AB*. Take two points, *C* and *D*, in different semi-circles such that $|AC| = |BD|$. *ACBD* will form an oblong rectangle (See Figure 28)

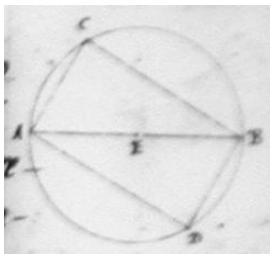

Figure 28: Construction of an oblong quadrilateral, F.140r, II.7

Pacioli makes reference to Dante as he explains that it is impossible to draw a triangle inscribed in a semi-circle that doesn't have a right angle, given that the diameter is one of its sides.

8. Document Eight: Forming the rhombus 3rd figure of the regular quadrilaterals.[178]

To construct the rhombus proceed like for the square (II. 6). Two more points, *H* and *K*, are to be found equidistant to *E* on *AB*. *HFKG* form a rhombus (See Figure 29)

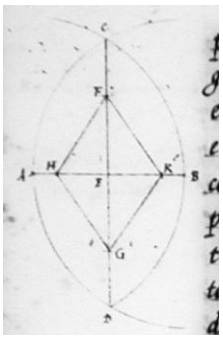

Figure 29: Construction of a rhombus, F.140v, II.8

9. Document Nine: To make the 4th figure of the regular quadrilaterals, named rhomboid.

Like before the construction is taken from a previous section (II.7), the construction of the oblong rectangle. Create point *H* on *BC* and *K* on *AD*, such that $|HC| = |KD|$. *AHBK* form a rhomboid. A side note warns to keep *HK* off the diagonals to avoid reduction to previous cases (see Figure 30)

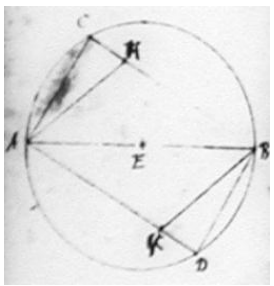

Figure 30: Construction of a rhomboid, F. 140v, II.9

Pentagons and some of their properties (II.10 – II.12)

10. Document Ten: When you want, doubtlessly, to form a [regular] pentagon. 3rd swift rectilinear figure.[179]

---

[175] DVQ F.138v. From here on "*Documento*" replaces the title of the sections, after it a mix of both is used.

[176] A likely Arab designation. Similarly *elmuaym* is used to describe a rhombus. These terms also figure in the *Divina Porportione* in similar ways and are as well supposedly to be found in Leonardo da Vinci's work.

[177] DVQ F.139v.

[178] *Ibid.* F.140r.

[179] *Ibid*. F.141r.

Pacioli gives the method to draw a regular pentagon. Draw a circle, centered at *A*, as big as one desires. Draw the orthogonal diameters of the circle *BC* and *DE*. The midpoint of *AC*, *F*, serves as center for the circumference with opening *FD*. The circumference will intersect *AC* at *G*. *GD* is the side length for the pentagon. Inscribing the polygon is left to the reader.

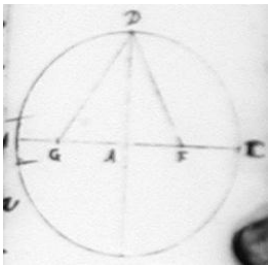

Figure 31: Construction of the length of the side of a regular pentagon, F.141r, II.10

Pacioli speaks of the scientific way of constructing the regular pentagon and its explanation, referring to Euclid IV, 11.[180] In the DVQ, as he explains relatedly. He however favors the practical way. The interested reader is to consult his *magnum opus* for more details. A quick argument is given based on Elements XIII, 7, of the accuracy of the inscription of the pentagon.[181]

11. Document Eleven: On the stupendous force of two lines, named chords, angled pentagonally [like the sides of a pentagon] or [also] pentagonal chords.[182]

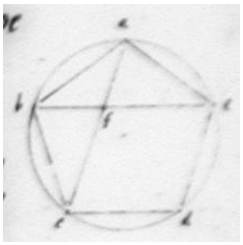

Figure 32: Illustration of the sectioning of chords inside a regular pentagon, F.142r, II.11

Pacioli mentions some remarkable properties of the chords connecting non-consecutive vertices of the pentagon. Namely, he illustrates *Elements* II, 11 (see Figure 32).[183] Intersecting diagonals of the pentagon,*ae* and *bc*, section each other at the golden ratio. This is, the smaller section, *af* or *bf*, by the whole length, *ae* or *bc*, of the diagonal gives the larger section's square, $|fe|^2$ or $|fc|^2$. Further he uses *Elements* XIII, 11 to argue that the diagonal is rational.[184]

The section concludes with the observation that the sum of the square of the side of the pentagon, $|ab|^2$, summed to the square of the diagonal of the pentagon, $|ac|^2$, equals five times the square of the radius of the circle, *r*, this is $|ab|^2 + |ac|^2 = 5r^2$. Proof is said to be found *Elements* XIV, 4.[185]

12. Document Twelve: On the other marvel derived from said pentagon, useful for everything.[186]

Elements XIII, 10 is paraphrased.[187] Pacioli thus relates the length of the sides of the regular pentagon to the length of the sides of the regular hexagon and decagon.

13. Document Thirteen: About the quality of the sides of the equilateral and equiangular pentagon [in regards] of the diameter of its encirclement.[188]

---

[180] "To inscribe an equilateral and equiangular pentagon in a given circle." *Elements* IV, 11.
[181] "If three angles of an equilateral pentagon, taken either in order or not in order, are equal, then the pentagon is equiangular." *Elements* XIII, 7.
[182] *DVQ* F.141v.
[183] "To cut a given straight line so that the rectangle contained by the whole and one of the segments equals the square on the remaining segment." *Elements* II, 11.
[184] "If an equilateral pentagon is inscribed in a circle which has its diameter rational, then the side of the pentagon is the irrational straight line called minor." *Elements* XIII, 11
[185] Note that there is no official 14th book of the Elements. However, at the time some extended versions circulated. Pacioli likely used one of these versions as he had Campanus' text as likely base. It is no less possible that there was some transcription mistake as *Elements* **IV**, **14** also address properties of circumscribed pentagons, but this yields no light on the claimed property.
[186] *DVQ* F.142v.
[187] "If an equilateral pentagon is inscribed in a circle, then the square on the side of the pentagon equals the sum of the squares on the sides of the hexagon and the decagon inscribed in the same circle." *Elements* XIII, 10.
[188] *DVQ* F.143r.

Pacioli states Elements XIII, 11 and goes on to give a short explanation of the meaning of "rational" and "irrational", "*quali sone de grandissima abstractione*", this is, which are of great abstraction and are treated in length in *Elements* X, from where he also quotes the 71st proposition[189] to clarify the concept.

Hexagon and Properties (II.14 – II.20)

14. Fourteenth Document: On the fourth rectilinear figure, named hexagon. [190]

Pacioli gives instructions how to construct a hexagon. First one should draw a circle. Next, use its radius to divide the circle into six equal parts. Starting with another circle centered anywhere on the circumference of the first one, six other circles are drawn so that each intersects the first one at the center of two other of the six circles. These six intersections, with the first circle, are the vertices of the hexagon (see Figure 33)

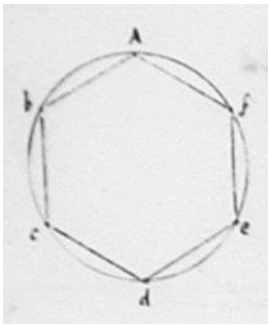

Figure 33: Construction of a regular hexagon, F. 143v, II.14

Pacioli refers to Elements IV, 11 for more detail regarding this construction.[191] He also observes that the hexagon sides equal to the radius length.

15. Fifteenth Document: About the force and marvel of the side of said hexagon in respect to the triangle.[192]

Pacioli shows that the square of the radius (and side of a regular hexagon in the same circle) is $\frac{1}{3}$ of that of one whose side is that of an equilateral triangle inscribed in the same circle.[193] (see Figure 34)

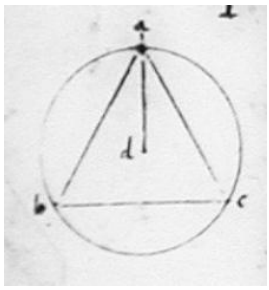

Figure 34: Construction of an equilateral triangle inscribed in a circle, F. 143v, II.15

A brief example is given for an equilateral triangle with side 10 units. From this follows that square of the hexagon has 33 1/3 square units.

16. Sixteenth Document: On another marvelous force of the hexagon.[194]

Lining up the sides of the Hexagon and the Decagon the golden ratio is found. It is *Elements* XIII, 9 which warrant this as Pacioli points out.[195]

17. Seventeenth Document: On the force and convenience which the hexagon and the decagon have together in respect to the pentagon.[196]

The content here is the same as that of (II.12.). This time, however, Pacioli states the property as equality.

Be *AB*, the side of a regular hexagon, *CD*, that of a regular Decagon, and, *EF*, that of a regular pentagon, all of which are inscribed in the the same circle then,

$$|AB|^2 + |CD|^2 = |EF|^2$$

18. Eighteenth Document: On another advantageous marvelous glory.[197]

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[189] "If a rational and a medial are added together, then four irrational straight lines arise, namely a binomial or a first bimedial or a major or a side of a rational plus a medial area." *Elements* X, 71.
[190] *DVQ* F.143r.
[191] "To inscribe an equilateral and equiangular hexagon in a given circle." *Elements* IV,11.
[192] *DVQ* F.145r.
[193] "If an equilateral triangle is inscribed in a circle, then the square on the side of the triangle is triple the square on the radius of the circle." *Elements* XIII, 12.
[194] *DVQ* F.144r.
[195] "If the side of the hexagon and that of the decagon inscribed in the same circle are added together, then the whole straight line has been cut in extreme and mean ratio, and its greater segment is the side of the hexagon." *Elements* XIII, 9
[196] *DVQ* F.144r.

Given the conditions of the previous section, Pacioli states that the perpendicular segment from the side of the pentagon to the circle's center is equal to the sum of the halves of the hexa- and decagon added together. Pacioli references Elements XIV, 1.

19. Nineteenth Document: About the force if the side of the divided hexagon.[198]

This section is the observation that the section by extreme and mean ratio of the side of the regular hexagon gives the side of the regular decagon inscribed in the same circle.

20. Twentieth Document: On another occult and marvelous force of the lineal virtue of the side of the hexagon.[199]

Pacioli mentions *Elements* XIV, 4 which states that "the square of the side of the regular hexagon is $\frac{1}{5}$ of the sum of the squares of the side, and, of the diagonal of the regular pentagon inscribed in the same circle".

Heptagon (II.21)

21. Twenty-First Document: On the way to form the 5th rectilinear figure by one opening of the compass, named heptagon.[200]

Pacioli gives a short introduction covering the difficulties of understanding the construction of uneven sided rectilinear figures greater than the pentagon. He proceeds to give a method to construct a circumscribed heptagon. For the pentagon the length of a single side is constructed.

Be, *bc*, the side of the regular hexagon inscribed in the circle in which it is desired to inscribe the hexagon. Then the orthogonal segment, *ad*, from the center of the circle to *bc* is of the length of the side of the heptagon (see Figure 35)

Alternatively half of the side of the equilateral triangle inscribed in the same circle can be used (see Figure 36)

Note that this heptagon will not be a regular one, as it is impossible to construct such only with straightedge and compass. This can only be achieved by using a marked ruler. The last inscribed side will end up to long in the first case and short in the second.

At the end of this section Pacioli starts discussing how to construct an Octagon, this clearly belongs to the next section.

Octagon (II.22)

22. Twenty-second Document: To form the octagon, the 6th rectilinear figure, that is with 8 sides, by one opening of the compass.[201]

To form the Octagon Pacioli tells the reader to find the midpoint of the arcs between vertexes of a square (II.6). These midpoints together with the vertices of the square form the regular octahedron inscribed in the same circle as the square (see Figure 37).

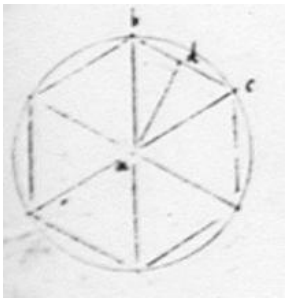

Figure 35: Construction of the approximation of a regular heptagons side to be inscribed in a circle, F.145v, II.21

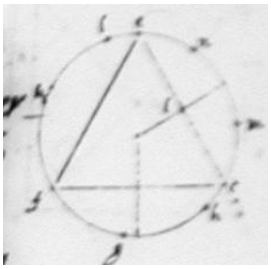

Figure 36: Alternate construction of the approximation, F. 146r, II.21

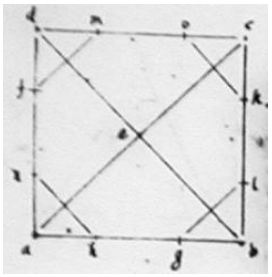

Figure 37: Truncating the square to obtain a octagon, F.146v, II.22

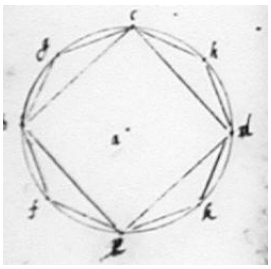

Figure 38: Construction of an octagon inscribed in the circle, F.146v, II.22

[197] *DVQ* F.144v.
[198] *Ibid.*
[199] *Ibid*. F.145r.
[200] *Ibid*. F.145v.
[201] *DVQ* F.146v.

Pacioli quotes Elements III, 28 warranting the same length for all sides of said octagon.[202]

An alternate method is that given in the previous section. The octagon is constructed by truncating a square by an eighth of its diagonal (see Figure 38).

Nonagon (II.23)

23. Twenty-third [Document] to make the 7th rectilinear figure, named nonangle [Nonagon], which has 9 sides; Difficult.[203]

Like in the case of the heptagon (II.21) only an approximation of the side is obtained. The length of the side of the nonagon is given by the difference of the side of an equilateral triangle, *bc*, and that of a regular hexagon, *cf,* both inscribed in the same circle. Pacioli again focuses the difficulty of these constructions.

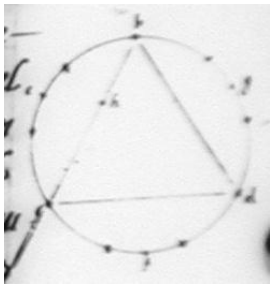

Figure 39: Construction of the approximate length of the side of a Nonagon, F. 147r, II.23

Decagon (II.24)

24. Twenty-fourth Document: Of the 8th rectilinear figure, named decagon.[204]

Elements XIV, 3 are used to give one way of creating the decagon. [205] However, as the means of doing the golden section are left for later (II.41) another method of construction is provided.

Given a pentagon the midpoints of the arcs are found, in semblance to the octagon. Added to the vertices of the pentagon they form the Decagon (see Figure 40).

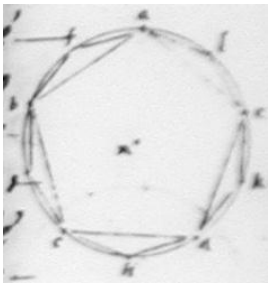

Figure 40: Construction of a regular decagon, F. 148r, II.24

Uneven Sided Polygons (II.25 – II.28)

25. Twenty-fifth Document: On the 9th rectilinear figure, named undecagon.[206]

Like in the previous cases of problematic polygons (II.21) or (II.23), an approximation of the lengths of the side of this polygon is obtained. Here $\frac{1}{6}$ of the radius plus $\frac{1}{3}$ of the equilateral triangle's side divided by the golden ratio is used to find the desired side. "*Per te prouare faccendo con diligentia ditta divisione"* (see Figure 41).[207]

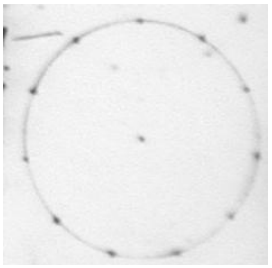

Figure 41: Approximation of a regular eleven-angle, 148r, II.25

26. Twenty-sixth Document: On [a polygon with] 13 [sides].[208]

Pacioli instructs how to divide the diameter of the circumscribing circle of the polygon in question into extreme and mean ratio, and take $\frac{5}{8}$ of the larger section. Again this is but an approximation (see Figure 42) .[209]

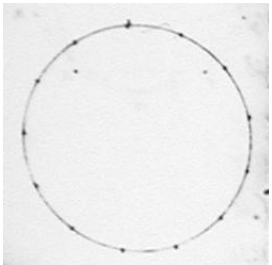

Figure 42: Construction of the 13-agon, F. 148v, II.26

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[202] "In equal circles equal straight lines cut off equal circumferences, the greater circumference equals the greater and the less equals the less." *Elements* III, 28.
[203] *DVQ* F.147r.
[204] *DVQ* F.147v.
[205] Pacioli quotes the *Elements* XIV, 3 as reading that "If you divide the radius of a circle by the golden section the longer part will be the side of the regular decagon inscribed in that circle."
[206] *DVQ* F.148r.
[207] "As you shall prove for yourself by dillegently making this division."
[208] *Ibid*. F.148v.
[209] As mentioned in case of the heptagon, this and the two other uneven sided polygons mentioned since cannot be constructed with ruler and compass. In fact Gauss conjectured what is now known as the Gauss-Wantzel Theorem, which states:

A regular n-gon is constructible with ruler and compass if and only if $n = 2^k p_1 p_2 \dots p_t$ where k and t are non-negative integers, and the p_i's are distinct Fermat primes.

27. Twenty-seventh Document: About the quindecagon, the figure with 15 sides.[210]

Pacioli makes use of *Elements* IV, 16[211] to construct the quindecagon, although here too sectioning of the diameter is suggested, taking $\frac{1}{3}$ of it as side.

The quindecagon is constructed by taking half the arc of the difference between the arcs defined by two consecutive vertices of the regular pentagon, and, triangle inscribed in the same circle (see Figure 43, the larger chord belongs to the equilateral triangle, the shorter one to the pentagon).

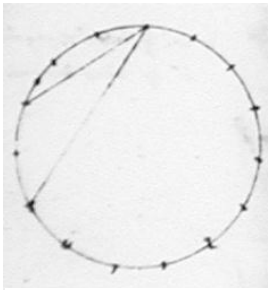

Figure 43: Construction of the 15-agon, 148v, II.27

28. Twenty-eight Document: About the 17-angle, the figure with 17 sides.[212]

The heptadecagon is obtained, like the previous approximations, through sectioning of other segments. The text seems corrupted and the parts to be used are unclear, but the golden ratio and the side of an equilateral triangle play their part.

This section ends the construction of polygons. Pacioli remarks that with those figures several other can be obtained as mentioned at the end of *Elements IV*.[213]

Basic Constructions (II.29 – II.40)

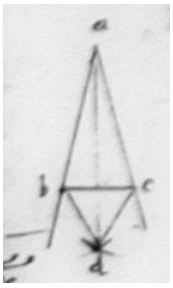

Figure 44: Bisecting an angle, F. 150r, II.29

29. Twenty-ninth Document: Divide a right angle in 2 equal parts.[214]

A way of bisecting an angle is given.

Given an angle at a point, *a,* it is contained by two rays. Let these rays be defined by equal sized line segments *ab* and *ac*. *bc* as base, construct an equilateral triangle, *bcd*. Join *ad* and the angle will be bisected, as proven by *Elements* I, 9 (see Figure 44) .[215]

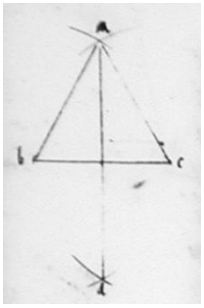

Figure 45: bisecting a segment, F.150v, II.30

30. Thirtieth Document: To divide a straight line in 2 equals.[216]

Here Pacioli discusses how to divide a line segment, *bc*, equally. This is, construct an equilateral triangle, *abc*, and bisect the angle at *a*. The bisecting ray will split *bc* equally as Elements I, 10 proves (see Figure 45) .[217]

31. Thirty-first Document: To know how to raise a perpendicular from a straight line.[218]

Given a line segment upon which we wish to raise a perpendicular at some point *c*. The practical geometer is to form an equilateral triangle, *abd*, such that $|ac| = |cb|$ and *ab* lies on said line segment. *cd* will be perpendicular to *ab* as proven in Elements I, 11 (see Figure 46).[219]

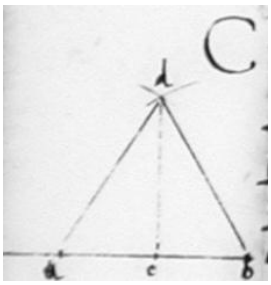

Figure 46: Raising a perpendicular, F. 150v, II.31

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[210] *Ibid*. F.148v.
[211] "To inscribe an equilateral and equiangular fifteen-angled figure in a given circle." P.16 B.4 *Elements*
[212] *DVQ* F.149r.
[213] "And further, by proofs similar to those in the case of the pentagon, we can both inscribe a circle in the given fifteen-angled figure and circumscribe one about it." *Elements* IV, 16 Corollary
[214] *DVQ* F.150r.
[215] "To bisect a given rectilinear angle." *Elements* I, 9.
[216] *DVQ* F.150r.
[217] "To bisect a given finite straight line." *Elements* I, 10.
[218] *DVQ* F.150v.
[219] "To draw a straight line at right angles to a given straight line from a given point on it." *Elements* I, 11

32. Thirty-second Document: On a given exterior point, make the perpendicular to a proposed line.[220]

the next topic is to raise a perpendicular to a line given an exterior point, *a*. To do so, draw a circle centered at *a* so that is intersects the line forming a segment, *bc*. Finding the midpoint, *d*, and drawing *ad* solves this problem according to Elements I, 12.[221]

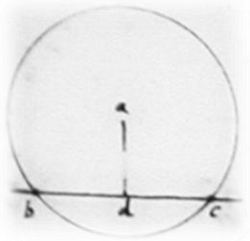

Figure 47: Raising the perpendicular through a point, F. 151v, II.32

On Angles (II.33 – II36)

33. Thirty-third Document: Understanding the kinds of rectilinear angles.[222]

Using Elements III, 31 as base Pacioli tackles the three kinds of angles formed in a triangle inscribed in a semi-circumference, in similitude to (II.4).[223]

To determine the nature of the angle at hand, subtend the angle to the base of a triangle. This base serves as diameter to the semi-circumference. Depending, if the vertex, whose angle is to be found lies on the semi-circumference, inside it, or, not, determine if it is right, convex, or, acute, respectively (see Figure 33).

Here Pacioli takes the opportunity to introduce an ingenious way to form a right angled triangle, in a practical situation. For this one only needs to make use of a measured string, or rope, making use of the Pythagorean triplet 3, 4, 5.

The string is to be divided such that 12 equal parts are sectioned into segments of 3, 4, and, 5 parts each. One end is to be bound to a stake or a nail, for instance on a field, the string is to be strung between stakes, dividing the sections, once all sections tight and straight so that both ends of the rope are tied to the same stake the angle between the two shorter sections will be straight.

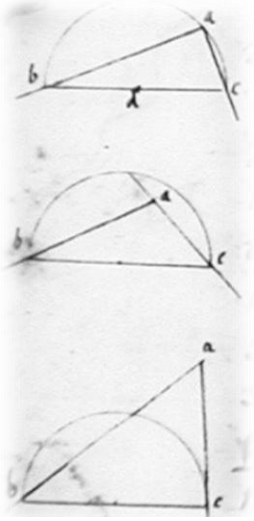

Figure 48: Illustration of angles at *a*, F.151v, II.33

34. Thirty-fourth Document: To make the acutest angles of acute angles.[224]

Elements III, 16 is mentioned.[225] Pacioli constructs the tangent to a circle. The angle mentioned in the title is the space formed between this tangent and the semi-circle formed by the orthogonal to the tangent passing through the center of the circle (see Figure 34).

As Pacioli says there is much discussion among the "philosophers" about it.

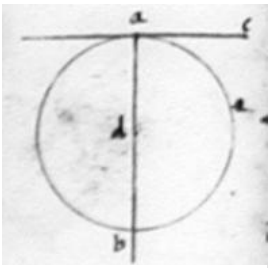

Figure 49: Tracing a tangent, F.152v, II.34

35. Thirty-fifth Document: To make the broadest angle of the acute rectilinear ones.[226]

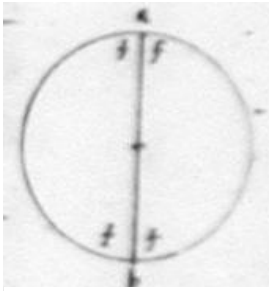

Figure 50: The diameter of a Circle, F.153, II.35

[220] *DVQ* F.151r.
[221] "To draw a straight line perpendicular to a given infinite straight line from a given point not on it." *Elements* I, 12.
[222] *DVQ* F.151v.
[223] "In a circle the angle in the semicircle is right, that in a greater segment less than a right angle, and that in a less segment greater than a right angle; further the angle of the greater segment is greater than a right angle, and the angle of the less segment, is less than a right angle." *Elements* III, 31
[224] *DVQ* F.152v.
[225] "The straight line drawn at right angles to the diameter of a circle from its end will fall outside the circle, and into the space between the straight line and the circumference another straight line cannot be interposed, further the angle of the semicircle is greater, and the remaining angle less, than any acute rectilinear angle." *Elements* III, 16.
These discrepancies are most like due to a offset of the Elements used by Pacioli, likely Campanus' version of it.
[226] *DVQ* F.153r.

Like in the prior section the angle isn't between two straight lines. The angle mentioned is the 'complementary' to the one mentioned in the previous section. It is formed between diameter and the semi-circle (see Figure 50).

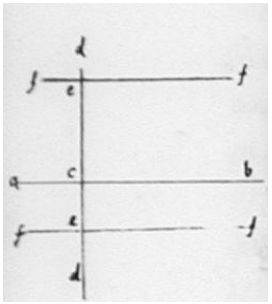

Figure 51: Drawing a parallel with respect to a line, F.156v, II.36

36. Thirty-sixth Document: On how to draw a parallel in regards to a given line.[227]

A small introduction on the importance of parallels, in geometry and in pictorial art is given. Pacioli refers Elements I, 31, and focuses itself on construction.

To draw a perpendicular on a perpendicular line so that the last line is parallel to the first (see Figure 51).

Pacioli supports his construction with Elements I, 27 and 28. [228] He gives an example and mentions a simpler method, which he will give (II.40).

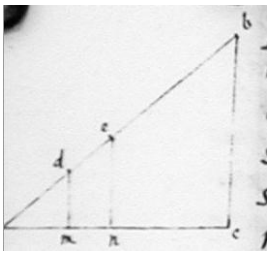

Figure 52: Dividing a line according to the proportion of another, F.156v, II.39

37. Thirty-seventh Document: On point marked outside of the line, to draw a parallel to a given line through an external point.[229]

Pacioli proposes an application of (II.36) to (II.32). The construction is justified by the propositions in the above section (see Figure 52).

Pacioli suggests the use of squares made of different materials to quickly draw right angles.

Proportions (II.38 – II.45)

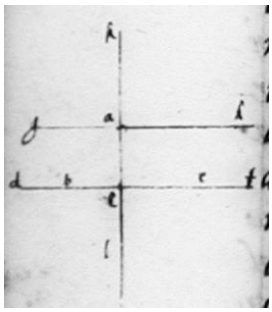

Figure 53: Drawing a parallel through a point, F. 157v, II.37

38. Thirty-eight Document: To take a part or more of a straight line to one's liking and necessity.[230]

This section is the practical application of *Elements* VI, 10, Tales Theorem.[231]

This documents instructions are somewhat dubious, as many points appear which seem unnecessary and do not figure in the margin (see Figure 53).

Given two rays starting at *a*, line segment *ab* and another segment *ac*, also given the segment *ad* on *ac,* draw *bc*. Next, draw a parallel to *bc* through *d*. The intersection with *ab, f*, forms the segment *af* in proportion to *ad* on *ab.*

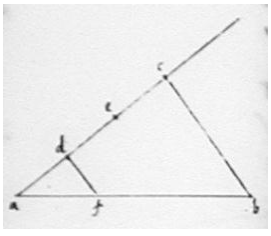

Figure 54: dividing a line into proportioned segments, F.155v, II.38

Pacioli uses thirds as an example and proposes the same exercise with 4 or more parts.

39. Thirty-ninth Document: Dividing a line into proportional parts in regards to another line divided as might be.[232]

Pacioli teaches how to divide a given straight line similarly to another segmented line.

Given a line segment, *ab*, which is segmented into three parts at points *d* and *e*, join the to be divided line segment *ab*, at any angle. Next join *bc* so that it forms a triangle

---

[227] *DVQ* F.153r.

[228] "If a straight line falling on two straight lines makes the alternate angles equal to one another, then the straight lines are parallel to one another." *Elements* I, 27.
"If a straight line falling on two straight lines makes the exterior angle equal to the interior and opposite angle on the same side, or the sum of the interior angles on the same side equal to two right angles, then the straight lines are parallel to one another. *Elements* I, 28.

[229] *DVQ* F.154v.

[230] *DVQ* F.155r.

[231] "To cut a given uncut straight line similarly to a given cut straight line." *Elements* VI, 10.

[232] *DVQ* F.156v.

and draw parallels to *cb* at *d* and *e*. These parallels will intersect *ac* at *m* and *n* respectively (see Figure 54). This seems like a generalization of the application of Tales Theorem to uneven sized segments.

40. Fortieth Document: To be able to draw an equidistant to the 3rd side of a triangle which intersects the other two sides.[233]

Pacioli describes how do draw a parallel with the use of a triangle.

Given triangle, *abd,* divide line segment *ad* at *c* so that it is in proportion to the section at *f* of line *bd*. Joining *cf* the parallel to *ab* is constructed. And vice versa. This is Elements VI, 2 (see Figure 56).[234]

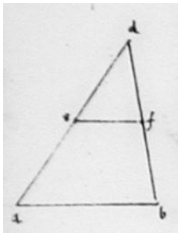

Figure 55: Drawing a parallel to the base of a triangle, F.158v, II.40

Extreme and Mean Ratio (II.41)

41. Forty-first Document: Dividing a line according to the proportions of the "*mezzo et doi extreme*" [the golden ratio].[235]

This section stresses the division into extreme and mean ratio, the golden ratio. As Pacioli puts it "it is the greatest power of the line". Several propositions are mentioned regarding the golden ratio such as *Elements* IX, 16;[236] XIII, 6;[237] and VI, 29, as *Elements* II, 11, which serves as a general method[238] to obtain the golden ratio.

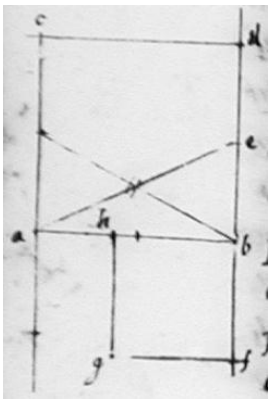

Figure 56: Finding the golden Ratio, F. 159v, II.41

The following construction (see Figure 56) is said to aid the practical geometer to understand this "maxim of geometry". Line segment *ab* is given to be divided into extreme and mean ratio. First, square the segment as to obtain the square *acdb*. Then find the midpoint, *e*, of one of the adjacent lines to *ab*, without loss of generality *bd* is used. Extend *db* so that it intersects the circle with center *e* and opening *ea* at point *f*. Join *f* to the closest vertex of *abcd*, *b*. Construct a square, *bfgh*, with side *bf* adjacent to *abcd*. The side that both squares share *hb*, divides *ab* into extreme and mean ratio.

If *hg* were extended, so to cut *abcd* at a point *k*, the rectangle *ahck* would have the same area as *bhgf*. This is left as exercise to be cut out with paper by Pacioli for the inquisitive reader.

This section is highly credited as maxim of geometry.

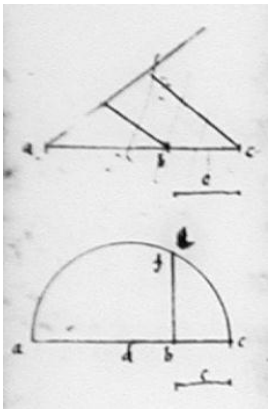

Figure 57: Extension by the mean of two lines, F. 16v, II.42

42. Forty-second Document: With two proposed straight lines to know how to find a third in the same proportionality.[239]

Instructions are given in how to find the mean proportional. Given two line segments that have been joined on a straight line, *ab* and *bc*, find the midpoint of *ab, d*. With center *d* and opening *dc* raise a semi-circle. Next raise a perpendicular do *ac* at *b* and intersect it with the semi-cirle at *f*. *bf* is the mean proportional to *ab*, *bc.* This is *ab* is to *bf*, as *bf* is to *bc* (See Figure 57).

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[233] *DVQ* F.157v.

[234] "If a straight line is drawn parallel to one of the sides of a triangle, then it cuts the sides of the triangle proportionally; and, if the sides of the triangle are cut proportionally, then the line joining the points of section is parallel to the remaining side of the triangle." *Elements* VI, 2.

[235] *DVQ* F.158v.

[236] This does not seem match any content related to the golden ratio in any of the used *Elements*, rather the proposition around IX, 16 are about prime relations.

[237] *Elements* XIII, 6.

[238] "To cut a given straight line so that the rectangle contained by the whole and one of the segments equals the square on the remaining segment." *Elements* II, 11

[239] *DVQ* F.160r.

This construction matches *Elements* VI, 13.[240]

There is space for confusion between the "third line put into proportion" and "the third proportional", which might explain the crossed out drawing next to the correct one is in the margin. The Third Proportional is explained in the following section.

43. Forty-third Document: For two proposed lines to know how to find a third in the constant proportion.[241]

Given two segments, *ab* and *ac,* in proportion one to the other, extend *ab* by $|ad|$, to *e*. Join *bc*. Draw a parallel to *bc* from *e* and intersect it with the extension of *ac*, at *d*. The resulting segment, *cd*, will be the third proportional. This is, *ab* is to *ac*, as *ac* is to *cd* ($\frac{|ab|}{|ac|} = \frac{|ac|}{|cd|}$) (see Figure 58).

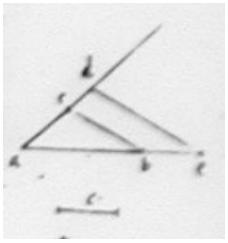

Figure 58: Extension of a line given the proportion of two others, F.161r, II.43

This is the construction of the third proportional as in *Elements* VI, 11.[242]

44. Forty-fourth Document: For three proposed lines to find a forth to which the third stands, as the first [does] to the second.[243]

In this section the construction of the fourth proportional is discussed, *Elements* VI, 12.[244]

Given three lines in proportion to each other, like in the previous example, a fourth line is found in proportion to the third, following the same construction as above.

There are no accompanying images.

45. Forty-fifth Document: To add to the three lines a 4th in the constant proportion, and to the 4th the 5th, and to 5 the 6th, etc.[245]

Pacioli generalizes the method above to produce a fifth line segment in proportion to the fourth, a sixth segment in proportion to the fifth, and so on.

The construction is the same as in (II.43) and works for any number of newly added line segment put in proportion with any previously found segments already in proportion with each other.

[240] It does however match "To find a mean proportional to two given straight lines." *Elements* VI, 13.
[241] *DVQ* F.160v.
[242] "To find a third proportional to two given straight lines.]" *Elements* VI, 11.
[243] *DVQ* F.161r.
[244] "To find a fourth proportional to three given straight lines." *Elements* VI, 12.
[245] *DVQ* F.161v.

Areas (II.46 – II.59)

46. Forty-sixth Document: How the lines are multiplied with glory.[246]

Pacioli introduces the four basic operations, addition, subtraction, multiplication and division, as geometric concepts. He stresses their importance in the geometric practice. Further he emphasizes the continuous nature of the "linear" quantities opposed to purely numerical ones.

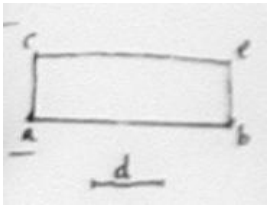

Figure 59: Area of a rectangle, F. 163r, II.46

The concept of multiplication of lines is defined by its result, a surface (area) limited by the interior of a rectangle generated by two given lines as side (see Figure 59).

Example is given for $|ab| = 6$ and $|ad| = 4$, which results in a surface of 24 square units.

47. Forty-seventh Document: To divide one [line segment] by the other, or, how to measure.[247]

Division, the inverse operation of multiplication, is used by Pacioli as measure process. This is, Pacioli defines that a segment is numerable (measurable) by another, smaller segment, if the latter divides the whole "exactly" (a finite number of times). Pacioli mentions *Elements* VII, 4 ensuring that this measuring is always possible.[248]

Practically, Pacioli explains how one can measure a line segment with ruler, compass or a piece of string.

Example is given for $|ab| = 12$ and $|c| = 3$ resulting in 4 measures of 3.

Pacioli cautions the reader to pay attention when he is using a fraction of a unit as measure, as confusion might arise.

48. Forty-eighth Document: On summing the straight line with straight line.[249]

The concept of addition is the extension of a line segment by another. Pacioli explains how to protract a segment by a certain length. He refers to a picture in the margin, which is missing.

49. Forty-ninth Document: To subtract a straight line from a straight line.[250]

Subtraction, the inverse of addition (protraction), (II.48), is discussed in this section. Pacioli stresses that it is necessary that the segment to be removed is smaller than, or equal to, the to-be-shortened segment.

50. Fiftieth Document: To divide a surface by a line, [both] being rectilinear.[251]

The area of a rectangle is divided by a segment to obtain another (smaller) surface. The construction is based on Elements II, 1.[252]

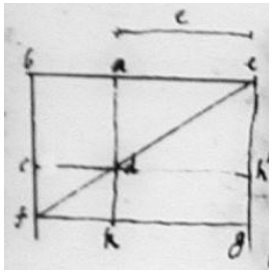

Figure 60: Find the segment, that given another makes given area, F. 165v, II.50

[246] *DVQ* F.162r.
[247] *DVQ* F.163v.
[248] "Any number is either a part or parts of any number, the less of the greater." Elements VII, 4.
[249] *DVQ* F.164r.
[250] *Ibid.* F.164r.
[251] *Ibid.* F.165r.
[252] "If there are two straight lines, and one of them is cut into any number of segments whatever, then the rectangle contained by the two straight lines equals the sum of the rectangles contained by the uncut straight line and each of the segments." *Elements* II, 1.

Given a rectangle *abcd* extend one of its sides, *ba*, by the segment which is to divide, *ae*, obtaining *be*. Join its end point to the closest vertex, *d*. Protract the thus obtained segment, *ed,* until it meets the extension of the only side not involved so far, *bc*, thus obtaining another segment, *cf*. This last segment is the result of the division of the area of *abcd* by the segment *ae* (see Figure 60).

Example is given for $|ab| = 6, |bc| = 4$ ($|ab||bc| = 24$). Dividing by $|ae| = 8$, to obtain $|cf| = 3$.

In case of areas of other shapes one is to reduce them to to a rectangle first.

51. Fifty-first Document: To divide a line by a surface, as they may be.[253]

This is the 'reciprocal' of the preceding section, to start with a line and divide it according to a squared surface.

Given a squared surface, *A*, ($|a|^2 = A$), and a segment, *b*, longer than the side of the squared surface (*b*>*a*), construct a square, *B*, of side *b*. The square root of the number of times the smaller square tiles the bigger one, is the result of the division.

Reference is made to subtraction, but is left to be consulted in the *magnus opus* without further reference.

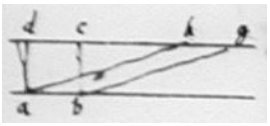

Figure 61: Two same area parallelograms, F.167v, II.52

52. Fifty-second Document: To make a surface of equidistant sides equal to another similar one proposed.[254]

In this and the next sections Pacioli discusses how to 'reshape' surfaces (this is, transforming them maintaining their area).

Given a rectangle *abcd*, extend two parallel sides, *ab* and *cd*, then draw two new parallels lines starting at the extremities to *ab* until they intersect the protracted line of *cd* at *kg*. *abgh* has the same area as *abcd* (see Figure 61).

Given the parallelogram first, the construction is the inverse of that already discussed. The same holds for two parallelograms on equal bases and equal parallels. These are the constructions of Elements I, 35 and 36.[255]

Pacioli quotes John Duns Scotus' second book of the *Sentences* to highlight that: parallelograms share a base and 'height' also have the same area. Further he instructs the reader when converting a parallelogram to a triangle to convert it to a rectangle first, as discussed below (II.54).

In the MS Pacioli uses two additional points, *e* and *f*, making reference to an aditional image, which is absent.

53. Fifty-third Document: To make a triangle in equal to any other.[256]

This time Elements I, 37 and 38 are applied to transform triangles maintaining their areas. [257]

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[253] *DVQ* F.166v.
[254] *DVQ* F.167r.
[255] "Parallelograms which are on the same base and in the same parallels equal one another." *Elements* I, 35.
"Parallelograms which are on equal bases and in the same parallels equal one another." *Elements* I, 36.
[256] *DVQ* F.168r.

Given a triangle, *abc*, in semblance to the previous document, protract one of the sides of the triangle, *ab*. Next, draw a parallel to the protracted line passing through the yet unused vertex *c, ch*. Any triangle formed by *ab* and with a vertex on *ch* will have the same area as *abc*. Likewise a triangle defined by an equal base and equal distanced parallels.

Pacioli refers to two images, both of which are missing.

54. Fifty-fourth Document: To make a surface of equidistant sides equal to any sort of triangle.[258]

This section discusses the transformation of rectangles into triangles, keeping their area constant, and vice versa. These propositions are *Elements* I, 41 and 42.[259]

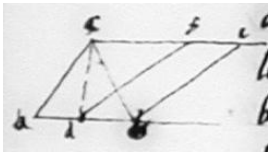

Figure 62: Same area parallelograms and triangles, F. 168v, II.54

Given a triangle, *abc*, extend its base, *ab,* and draw a parallel to it passing through *c*. Find the midpoint, *d*. Any parallelogram with the base *ad* or an equal base to *ad* between the parallels or between equal distanced parallels, will have the same area as *abc* (see Figure 62). And vice versa, any triangle with a common base and between the two defining parallels of a parallelogram will have half the area of that parallelogram.

55. Fifty-fifth Document: Grow a square up to another square, or any other proposed figure.[260]

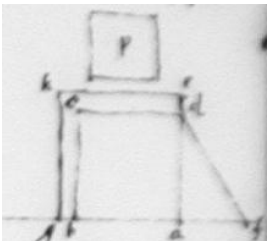

Figure 63: Extension of the area of a square by another area, F.170r, II.55

It is the purpose of this section to explain how to augment a square so that it includes a given area. This area can be given by a square, other figure, or, sum of figures. The last two can be reduced to squares as will be discussed further on.

Given a square, *abcd,* and the area, p^2, extend one of its sides, *ab,* in both directions. Find a point, *f*, on *ab* such that $|af| = p$, *opposed to* b. The segment given by the closest vertex and *f*, *df*, will be the side of the augmented square. (See Figue 63)

The difference of sides, of the original and the enlarged square (the "L" shaped area), is named *gnomon*. A brief discussion regarding the relative sizes of the squares whose areas are to be added is given making reference to Elements I, 29 and 46[261].

56. Fifty-sixth [Document]: Knowing how to make a square equal to a proposed triangle and more.[262]

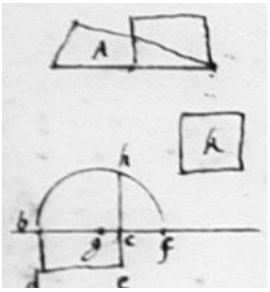

Figure 64: Squaring a triangle or rectangle, F. 171v, II.56

This section begins by lauding the square and cube as fundamental objects to understand all things geometrical. Given this, the practical geometer should be able to convert any given surface to a square. For this the reader should make use of the

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[257] "Triangles which are on the same base and in the same parallels equal one another." *Elements* I, 37.
"Triangles which are on equal bases and in the same parallels equal one another." *Elements* I, 38.

[258] *DVQ* F.168v.

[259] "If a parallelogram has the same base with a triangle and is in the same parallels, then the parallelogram is double the triangle." *Elements* I, 41.
"To construct a parallelogram equal to a given triangle in a given rectilinear angle." *Elements* I, 42.

[260] *DVQ* F.169v.

[261] "A straight line falling on parallel straight lines makes the alternate angles equal to one another, the exterior angle equal to the interior and opposite angle, and the sum of the interior angles on the same side equal to two right angles." *Elements* I, 29.
"To describe a square on a given straight line." Elements I, 46.

[262] *DVQ* F.170v.

previous sections, to convert a triangle into a rectangle and apply (II.42) to form a square.

Given a rectangle, *bcde*, join one of the longer sides, *bc*, and a segment equal to the shorter side, cf. Next find the midpoint, *g*, of *bf*. Raise a circle with center *g* and raise a perpendicular at *c* so that it intersects the semi-circumference at k. The square formed by *ck* has the same area as *bcde* (see Figure 64).

This is *Elements* II, 14.[263]

57. Fifty-seventh Document: On making a square equal to a rectilinear figure, in whichever kind or form.[264]

Having established the square as tool to measure areas, and, how to obtain it from triangles and rectangles, Pacioli now turns to other polygons. To easily square these they should be sectioned into triangles first. Pacioli notes that this is possible into *n-2* triangles, where *n* is the number of sides the polygon has (this can easily be proven by induction). Next each of these triangles are transformed into a square. Finally these squares are added together to obtain a single square.

Pacioli gives the example of a pentagon, *abcdef,* sectioned into three triangles, which result in squares of side *g*, *h* and *k* (see Figure 65). To add the squares together apply Pythagoras Theorem, *Elements* I, 47. [265] This is, take side *g* and *h*, and join them at a right angle. Join the other extremities, to obtain the hypotenuse, *m*, of a right angled triangle. *m* is the side of the desired square.

To add a third or more squares repeat this process (*k* in the figure).

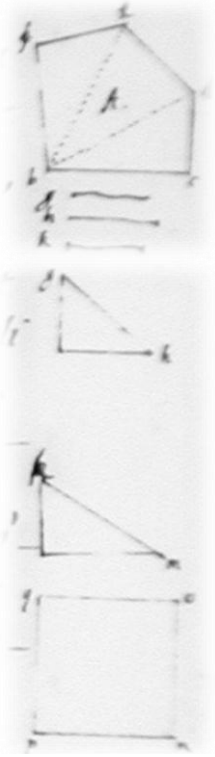

Figure 65: Slicing polygons into triangles and converting them into a square, F. 172r, II.57

58. Fifty-eighth Document: On making a square double or triple to one proposed; quadruple, quintuple and the likes, infinitely.[266]

The above method to sum squares is generalized. To obtain a multiple of a certain area one is simply to repeatedly sum its squares. In case of multiples of two one can simply repeatedly take the diagonal of the initial square. To hasten the sum other multiples of squares one can add the squares of smaller multiples.

For example: to obtain double said square take its diagonal for the new square. If three times the square is the desired result join the side of the square to its diagonal at a right angle like above. If four times said square is desired join two lengths equal to its diagonal, and so on (see Figure 66).

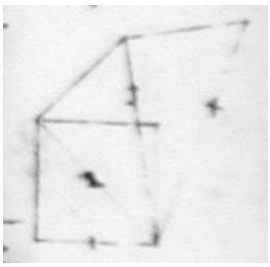

Figure 66: Converting a tetragon into a square, F. 173r, II.58

Pacioli stresses the decomposition of other polygons mentioned in previous sections here too.

59. Fifty-ninth Document: To know to extend the side of triangle not changing any other without shrinking it.[267]

This construction is an effect to astonish the "*idiota*". Pacioli takes a triangle and keeps two sides and extends or shortens the third one, to obtain a same sized triangle (in terms of area).

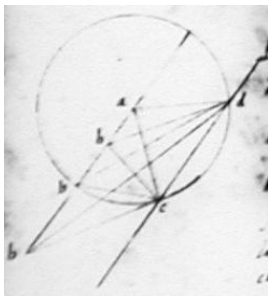

Figure 67: Construction of same sized triangles, F. 173v, II.59

[263] "To construct a square equal to a given rectilinear figure." *Elements* II, 14.
[264] *Ibid.* F.171v.
[265] "In right-angled triangles the square on the side opposite the right angle equals the sum of the squares on the sides containing the right angle." *Elements* I, 47.
[266] *DVQ* F.173r.
[267] *Ibid*. F.173v.

Given a triangle, *abc,* find the midpoint, *d*, of one of the sides, *bc*. Draw *ad*, this segment should be greater than *bd* and *dc*, for the construction to be possible. Extend the segment *ad* to double its length, *ae,* this will be the new base of the triangle. Join either of the vertices to *e* to obtain *abe* or *ace* with the same area as *abc*.

The accompanying figure is more complex in construction (see Figure 67), likely related to the comment found at the bottom of the page.

Circles (II.60 – II.74)

60. Sixtieth Document: to find the square radical [square root] of a number spot on, through line and with infinite precision.[268]

Discussion is given on the way to calculate the square root of a given number, and the impossibility to do so accurately through fractions. Fractions result in a surd, an irrational radical. Pacioli uses *Elements* VI, 13 to find the square root of a number geometrically.[269]

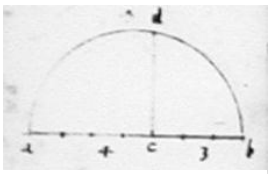

Figure 68: Finding a radical, F. 175v, II.60

Given a number one is to decompose it into two factors. Then construct a diameter so that it is made up of the two segments with length of the factors. The orthogonal segment from the joining point of the segments to the semicircle will be the desired radical (see Figure 68).

Pacioli refers the 7th book of the *Elements*, but no proposition in particular.

61. Sixty-first Document: To find the center of a proposed encirclement, as we shall say.[270]

Pacioli gives a brief introduction of the following sections. They are about curved lines. He begins by teaching the reader how to find the center of a given circle.

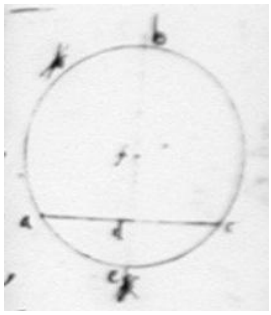

Figure 69: Finding the center of a circle, F.176r, II.61

Given a circle, draw cord, *ac*, and find its midpoint, *d*. Draw an orthogonal line at *d* so that it intersects the circle at two points, *b* and *e*. The midpoint of *be*, *f*, is the center of the circle (see Figure 69).

This is the construction of *Elements* III, 1.[271]

62. Sixty-second Document: Dividing an arch of a portion equally.[272]

Pacioli refers (II.30) using it to section an arc, *ec*, of a given circle.

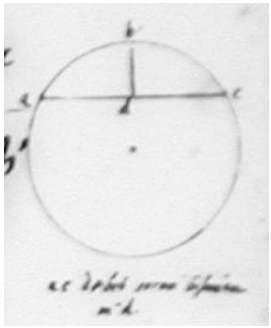

Figure 70: Splitting an arch equally, F. 177r, II.62

Find the midpoint of the line segment ec, *d,* then raise an orthogonal at *d*. The point where the orthogonal intersects the arc, *b*, is the midpoint of that arc (see Figure 70).

Pacioli justifies this with Elements III, 29.

63. Sixty-third Document: Given a portion, to finish the encirclement.[273]

Given an arc, Pacioli gives a method on how to find the center of the circle it is contained in, and thus, to be able to complete it.[274]

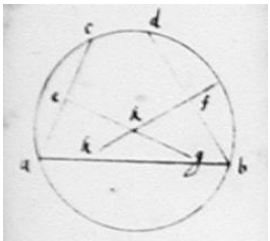

Figure 71: Completion of the circle given an arc, F.177v, II.63

[268] *Ibid*. F.175r.
[269] "To find a mean proportional to two given straight lines." *Elements* VI, 13.
[270] *DVQ* F.176r.
[271] "To find the center of a given circle." *Elements* III, 1.
[272] *DVQ* F.176v.
[273] *DVQ* F.177r.
[274] "Given a segment of a circle, to describe the complete circle of which it is a segment." *Elements* III, 25.

Given an arc, *ab*, draw two line segment, *ac* and *bd*, contained in the arc. Find their midpoints, *e* and *f*. Then draw orthogonals, *k* and *g,* to these cords passing through *e* and *f*. The intersection of *k* and *g, i*, is the center of the arc.

Pacioli emphasizes that the chords ought to contain the extremities of the arc, which is not a necessary condition. However, this might be useful to discuss the case in which chords are parallel to each other. In that case, Pacioli notes, the mid-point between the parallel chords is the center. Pacioli justifies the construction with Elements III, 1 and 24.[275]

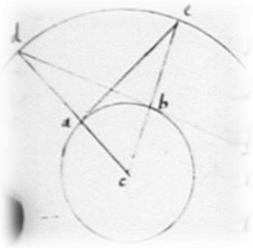

Figure 72: Drawing a tangent from a point, F.178v, II.64

64. Sixty-fourth Document: How to draw a contingent [the tangent] of the encirclement.[276]

This section gives the construction of *Elements* III, 17[277].

Given a circle with center, *c*, and a exterior point, *d*, we wish to draw a tangent at a point of the circle, *b*, passing through *d*. Join *d* to *c.* Next draw a concentric circle through d. Raise a perpendicular line at the intersection of *dc* and the original circle, *a*. The intersection of this perpendicular with the circle passing through *d*, *e*, is to be joined with *c*. Finally the intersection of *ec* with the original circle results in the desired point *b* (see Figure 72).

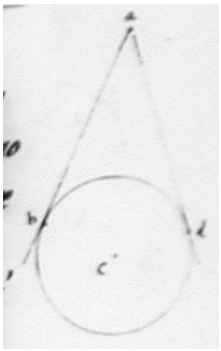

Figure 75: equidistance of the two tangents, F.179r, II.65

65. Sixty-fifth Document: Contingents [tangents] of a given point are equal.[278]

Pacioli states the unicity of intersection of two tangents of a circle through a given outside point, and, that the point-circle segments are of equal length. *Elements* III, 35 prove this as Pacioli mentions. The accompanying figure illustrates the proposition (see Figure 73).[279]

Circumscription/Incircle (II.66 – II.71)

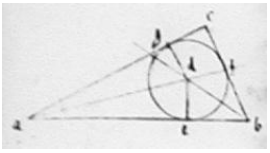

Figure 73: Encirclement by a triangle, F.179v, II.66

66. Sixty-sixth Document: To make circumference within the triangle.[280]

As Pacioli states, the following documents are concerned with how to inscribe and circumscribe various polygons.

In this section the construction of *Elements* IV, 4 is described.[281]

Given a triangle, *abc,* in which a circle is to be inscribed. Bisect two adjacent angles, at *a* and *b*, the resulting rays will intersect at point *d.* Next raise a perpendicular from all sides passing through *d*, these segments have the same length, thus the radius of the inscribed circle is centered at *d* (see Figure 74).

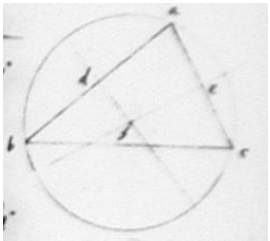

Figure 74: Circumscription of a triangle, F. 180r, II.67

67. Sixty-seventh Document: To make a circumference around the triangle. [282]

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[275] "On the same straight line there cannot be constructed two similar and unequal segments of circles on the same side." *Elements* III, 23.
[276] *DVQ* F.178r.
[277] "From a given point to draw a straight line touching a given circle." *Elements* III. 17
[278] *DVQ* F.179r.
[279] "If a point is taken outside a circle and two straight lines fall from it on the circle, and if one of them cuts the circle and the other touches it, then the rectangle contained by the whole of the straight line which cuts the circle and the straight line intercepted on it outside between the point and the convex circumference equals the square on the tangent." Elements III, 36
[280] *DVQ* F.179r.
[281] "To inscribe a circle in a given triangle." *Elements* IV, 4.
[282] *DVQ* F.179v.

Pacioli describes the construction of the circumscription of a triangle found in *Elements* IV, 5.[283]

Given a triangle, *abc,* find the midpoint of two of its sides, *d* and *e*. Raise two perpendicular lines passing through *d* and *e* until they intersect, at point *f,* the center of the circumscribing circle. Draw a line to any of the vertices, *ab*, this will be the radius of the circle (see Figure 75).

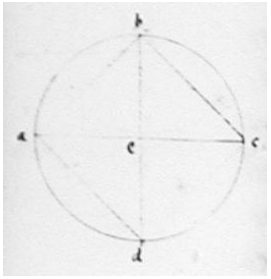

Figure 76: Inscribing a Square inside a Circle, F.180v. II.68

68. Sixty-eight Document: To make a square inside a circumference, swiftly.[284]
This time it is the construction of a encircled square of Elements IV, 6.[285]

Given a circle centered at *e*. Draw the diameter, *ac*, and raise a perpendicular diameter *bd*. Join the extremities *abcd* to obtain the circumscribed square (see Figure 76).

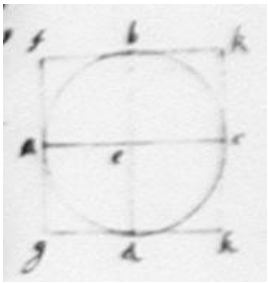

Figure 77: Encirclement by a square, F.181r, II.69

69. Sixty-ninth Document: To make a square around a circumference.[286]
This is the construction of Elements IV, 7.[287]

Given a circumference centered at *e*. Draw the diameter, *ac*, and raise a perpendicular diameter *bd*. Draw perpendicular line segments at *a*, *b*, *c* and *d* until they intersect. The intersection *f*, *g*, *h*, and *k* form the encircling square *fghk* (see Figure 77).

Mention is made, in regards to tangents related to a corollary of the *Elements*, to assure the last intersection holds[288].

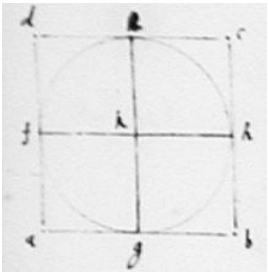

Figure 78: Encircle a square, F. 181v, II.70

70. Seventieth Document: To make a circumference within a square, swiftly.[289]
The construction of *Elements* IV, 8 is described.[290]

Given a square, *abcd*, find the midpoints of its sides *e*, *f*, *g,* and, *h*. Join them so that they intersect at point *k,* by drawing *eg* and *fh*. *k* is the center of the incircle and *fk* or any other midpoint joined with *k* is the radius (see Figure 78).

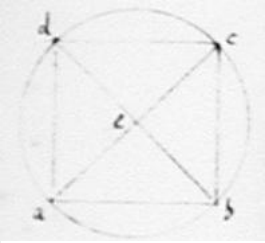

Figure 79: Circumscribing a square, F. 181v, II.71

71. Seventy-first Document: To make a circumference around a proposed square.[291]
Here the circumscription of a square is given as per *Elements* IV, 9.[292]

Given a square, *abcd*, find its diagonals, *ac* and *bd*. The intersection of the diagonals, *e*, is the center of the desired circle and *ae*, or any other vertex connected to *e*, is the radius (see Figure 79).

Intersecting Lines (II.72 – II.74)

72. Seventy-second Document: Of the miraculous force and virtue of two straight lines that intersect inside the encirclement.[293]

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[283] "To circumscribe a circle about a given triangle." *Elements* IV, 5.
[284] *DVQ* F.180r.
[285] "To inscribe a square in a given circle." Elements IV, 6.
[286] *DVQ* F.180v.
[287] "To circumscribe a square about a given circle." *Elements* IV, 7.
[288] Most likely *Elements* III, 16 already mentioned above.
[289] *DVQ* F.181r.
[290] "To inscribe a circle in a given square." *Elements* IV, 8.
[291] *DVQ* F.181v.
[292] "To circumscribe a circle about a given square." *Elements* IV, 9
[293] *DVQ* F.181r.

Pacioli formulates Elements III, 35.[294]

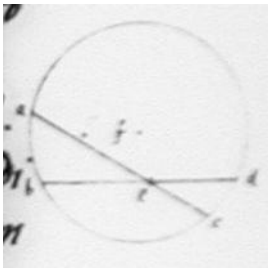

Figure 80: The intersection of two lines inside a circle, F.182r, II. 72

Given two chords, *ac* and *bd*, intersecting at a point, *e*, of a circle, centered at *f*. The areas of each of the rectangles, formed by the sections of the lines, have the same size. The line segments can meet in three ways: both chords pass through the center, one passes the other not, or neither do, additionally the sectioning can be into equal sections or not. So there are five possible cases. Should the chords both pass the center of the circle by all sections have the same length and the above holds true. If one chord passes the center and splits the the second in half it does so at a right angle (as per *Elements* III, 3) thus the sections form same are rectangles (as per *Elements* II, 5). The other two cases are stated to hold true and any other possibility of sectioning discarded according to Euclid, but not gone into detail.

Pacilio refers five pictures according to each arrangement, but only one is present (see Figure 80).

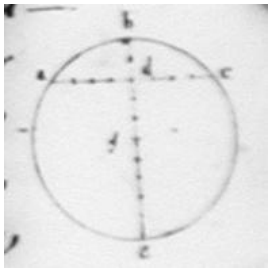

Figure 81: Given arch to find the center and diameter, F. 183r, II.73

73. Seventy-third Document: The grand gentleness and usefulness which follows of the previous document.[295]

Pacioli explains how to find the center, f, of a circle, given an arch, its chord, *ac*, and the "*saetta longissima*" (longest arrow)[296], *db* (see Figure 81).

To find the center and calculate the diameter one is to square half the chord's length and divide this by its longest arrow to find the opposing arcs longest arrow, *de*. Add these together to find the diameter, *be*. Succinctly, $\frac{|ac|^2}{|db|} + |bd| = |be|$. At half of the diameter is the center of the circle.

This is a concrete application of the construction discussed in (II.60) and (II.61). Example is given for $|ac|$=8, $|bd|$=2.

The example further discusses the calculation of the whole circumference and length of the segment, *df*. $|df|$ is found by subtracting the long arrow from the radius. To calculate the circumference, having found the diameter, one is to multiply it by 3 and 1/7[297] (approximately 3.143).

Figure 82: Given arc, to know the diameter, F.184r, II.73b

As Pacioli puts it, this is a practical application of the previous document. This section gives the so far geometrical results a numerical significance.

73rd [Document]: The other [thing] which comes from this for the practical [geometer]. [298]

The calculations and results of the above section are applied to a stretched rope.

One of the rope's ends is fixed in unknown distance. Let it be the case of the other side of a river whose width is to be measured. Use the riverbank and the rope, to

[294] "If in a circle two straight lines cut one another, then the rectangle contained by the segments of the one equals the rectangle contained by the segments of the other." *Elements* III, 35.

[295] *DVQ* F.183r.

[296] This is the perpendicular to the chord with the greatest length to the arc, in particular, the bisecting segment of the chord, from chord to arc.

[297] Which corresponds to Archimedes $\frac{22}{7}$ approximation of π.

[298] *DVQ* F.183v. This section is numbered separately and in different script using hindu-arab numerals. It might have been part of the previous document as the content is sequential to the topic.

define an arc. Measure the straight segment of riverbank between the spots where the straightened rope's end meets the water, this will be the cord. Measure the longest stretched segment of rope one can produce outside of the water, the "longest arrow". This is all information necessary to calculate the lengths of the fixed rope according to the above document (see Figure 82).

Other examples are: to calculate the height of a bell tower using the rope of the bells, the depth of a well, or, the depth of an anchor of a ship.

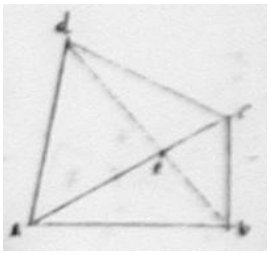

Figure 83: Proportions inside a tetragon, F.185r, II.74

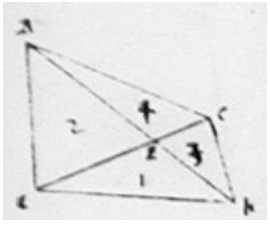

Figure 84: Proportions inside a tetragon, F.185v, II.74

74. Seventy-fourth Document: Of the force of two straight lines which intersect diametrically inside a quadrilateral.[299]

In this section Pacioli focuses on the intersection of diagonals of quadrilaterals. The four triangles formed by this sectioning are in proportion to each other (see Figure 83). Pacioli names five proportions: inverse (*conversim*), permutated (p*ermutatim*), conjunct (*coniuntime*), disjoint (*disgionta*) and opposing (*adversim*).

He proceeds to make each of these proportions explicit and justifies using *Elements* VI, 1 as argument. Given equal heighted triangles their areas are to each other as the bases they are constructed upon are to each other. The diagonals act as base line and the vertices as height of the triangles.

So for instance the first proportion, is given algebraically by,

$$\frac{A_1}{A_2} = \frac{A_3}{A_4}$$

Where A_1, A_2, A_3, A_4 are respectively the areas of the triangles *abe*, *ade*, *bec*, *dec* of the quadrilateral *abcd,* were *e* is the intersection of *ac* with *bd* (see Figure 84).

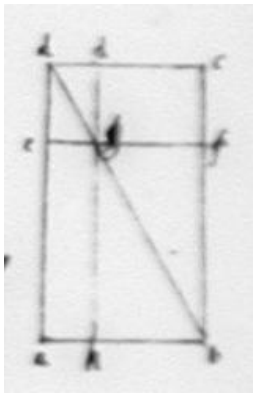

Figure 85: Scaling a parallelogram, F. 187r, II.75

75. Seventy-fifth Document: Draw a lessened parallelogram according to its width and height in proportion.[300]

Instructions are provided on how to scale down a parallelogram given a segment.

Given a parallelogram, *abcd*, that is to be scaled down proportionally so that one of its sides gets shorter to a certain length, *ed*, where *e* is a point on the side *ad*, draw a parallel to the adjacent lines to *ad*, *dc* and *ab,* at point *e*. The intersection with the fourth side is the point *f*. Next draw the diagonal at the vertex which includes the to be shortened segment, *db*, which will intersect *ef* at *g.* Raise a parallel to *ad* passing through *g,* which will intersect *dc* and *ab* respectively at *h* and *k*. *edhg* defines the scaled down parallelogram (see Figure 85).

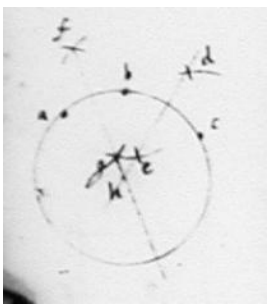

Figure 86: Passing a circumference through three given points, F.187v, II.76

76. Seventy-sixth Document: To draw 3 points inside a circumference.[301]

This is an extraction of the Elements IV, 5, which Pacioli reminds the reader has already been addressed previously (II.67).

Given points *a*, *b* and *c*, which are to lie on a circumference. Center the compass on one of the points, *b*, draw a circle so that one of the other points lies on its circumference,(*c* in the figure) while the third point is contained in the circle. Next, keeping the opening of the compass, draw two circles having the other two points as centers, *a* and *c*. Draw lines passing through the intersections of the latter two with

[299] *DVQ* F.185r.
[300] *DVQ* F.186r.
[301] *DVQ* F.187v.

the first circle, *f* and *g* and *e*, so that they intersect, at point *h*. This last point is the center of the desired circumference (see Figure 86).

77. Seventy-seventh Document: To know how to make a material set-square right, at once, without compass.[302]

Pacioli stresses the importance for the practical geometer to be equipped with a *gnomon* so "named by the philosophers" (a set-square and not the sun dial component). These are made of various materials.

For the case that such should not be available Pacioli gives instructions on how to quickly create one with a piece of paper or how to measure the right angle with a length of string, in the likes of (II.33).

To obtain a right angle with a piece of paper, fold it once to form a straight line. Next fold the line upon itself so that the two line segments are overlaid and a second straight line segment is folded, orthogonal to the first.

Pacioli gives a more geometrical example after this explanation with corresponding image in the margin (see Figure 87). The first fold of a sheet of paper *abcd* is *ef*, where *e* and *f* are points laying on two opposing sides of the sheet, the second fold, *gh*, splits *ef* at point *h* so to overlay the so formed segments *eh* and *fh*. As a result *gh* and *hf* or *he* form a right angle.

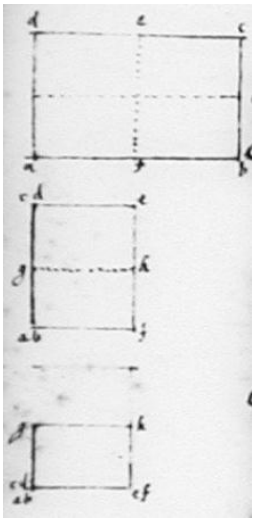

Figure 87: Folding Instructions for square angled paper, F.188v, II.77

78. Seventy-eighth Document: How to of measure surfaces, solids and numbers on the line is treated.[303]

This small section is a disclaimer. These documents, as well as those of the first part regarding numbers, ought to be accompanied by the *great printed work*, as copious use has been made of it in the MS.[304]

This is likely reference to Pacioli's transcription of the *Elements* as there are no detailed proofs included in this work. This ends the first, more scholastic, half of the second part.

[302] *DVQ* F.188r.
[303] *DVQ* F.189v.
[304] The text itself funnels downwards to the bottom of the page, perhaps to give this section some more visibility as there seems to be no reason elsewise as there are no illustrations mentioned nor would they make sense here.

Geometric Marvels

Staircase Cutting (II.79)

79. Seventy-ninth Document: A tetragon, to know to elongate it by tightening it, and broaden it by shortening it.[305]

Here start the less mathematically rooted documents of this second part.

Pacioli begins by telling of his visit to Ferrara, in 1466. He was there for the feast of St. George, duke Borso.[306] The duke wanted a brocade with a golden rim, 32 by 3 units, for the great *palio*[307], but only rectangles of other dimensions where available. The solution was found by cutting one of the rectangles, 24 by 4 units, into two equal pieces and knitting them together. Pacioli proceeds to explain how.

Given a rectangle, *abcd*, it is desired to reshape it so that its sides change forming a new rectangle. Start cutting away 1 unit away from one of the vertices, *c*, on the shorter side of the rectangle, point *e*. Cut 8 units parallel to the long sides, up to point *f*. Here, cut parallel to the shorter sides another unit in the direction moved away fro the vertex, to point *g*. Proceed in this staircase cut until the rectangle is split into two pieces (see Figure 88).

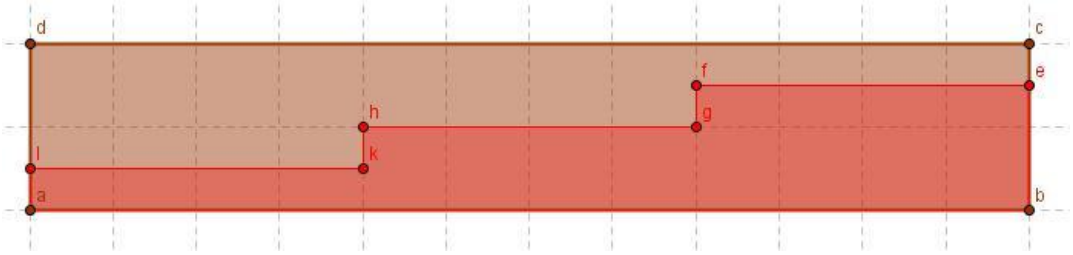

Figure 88: Step section of *abcd* into two congruent pieces.

The case given uses a 24 by 4 rectangle to be cut entwine to form a 32 by 3 rectangle. However, further examples are mentioned. These include a 12 by 8 to form a 16 by 6, a 98 by 1 to form a 48 by 2, given a 48 by two to 8 by 12, all having the same area of 96 square units, as Pacioli points out. He further puts emphasis on the proper divisibility of the sides, and, the parallel cuts.

Pacioli mentions several arrangements in the MS, but given his description and the numbers provided one is to cut stepwise and in proportion to the desired tetragon in all cases. A visual proof can be found at foot of the page (see Figure 89). [308] Here the rectangle, *abcd*, is cut into three pieces, the triangle *cdg*, a smaller triangle *cef*, and the remaining pentagon *agfeb*. The pieces fit one by one on the rectangle *hkbe*. By construction *gdc* is congruent to *hkf*, and, *cef* is to *agh*.

Figure 89: Image acompanying the stepsectioning, possibly misunderstanding the text, F. 190v, II.79

[305] *DVQ* F.189r.
[306] Most likely Dorso d'Este (1413 – 1471).
[307] A regular horse race in Ferrara.
[308] The images are missing here too, with exception for the 32 by 3 case. At the end of the section the text funnels like in (II.78) here most likely to make room for the various illustrations. Similar staircase-cuts can be found at https://projecteuler.net/problem=338 .

Arrangement puzzle (II.80)

80. Eightieth Document: How it is not possible for more than 3 points, circles or spheres touch each other on the plane.[309]

This document discusses the impossibility of more than three points, discs or spheres on the same plane to mutually touch all others.

This puzzle can be formulated the following way: Is it possible place four mutually touching coins? Pacioli explains that this is impossible on a flat surface, as can simply be verified. This can be shown using fingertips. In case we leave the plane, there is a simple solution: simply stack the coins.

In case square or cube objects are used, such as dice, it becomes possible to lay out four of them so that they touch at their vertices. Pacioli credits Averroes for this idea.

Illustrations mentioned by Pacioli are missing. Two of these are said to depict four circles and a four sphere pyramid to illustrate the above (see Figure 90). [310]

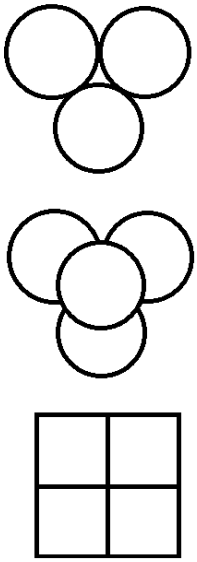

Figure 90: Three arrangements of flat objects. Circles and squares on the plane and coin stack.

81. Eighty-first Document: To know how to say, how much snow and water falls , and, rains on top of the universe; [in] one night.[311]

Pacioli describes the measuring and estimation of rainfall. Archimedes and his estimation of the grains of sand are mentioned as a source of inspiration.

A container, with known dimensions, is left out in the open, to collect water. The precipitation is measured and related to the interval exposure as well as the area the container collected water from, i.e. rinse from a roof. By knowing the area of a city, the total amount of water that fell on that place can be estimated.

82. Chapter Eighty-two Document: To fill a square window with three square stones as one may, and may not.[312]

This section discusses a puzzle and word game, which can be phrased as: “Tile a square with [3] squares”. The number of squares can vary, Pacioli suggests 5 or 7. It is left open if the squares are all of different sizes. The objects used can further be square frames or filled squares, as the *volgare* allows for both interpretations. It roughly discusses Squaring the Square.

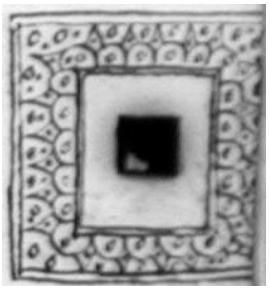

Figure 91: Fitting of four square stones into a square, F.193r, II.82

Pacioli gives the solution of the word game, using square frames; the puzzle becomes easy (see Figure 91).

If the squares are allowed to be same sized, the square can be tiled a square number of times, i.e. have 9, 1 by 1 unit side squares tile a 3 by 3 unit square. The problem becomes trickier if we admit that the squares should be of different sizes. According to Pacioli that way the problem has no solution. It can easily be verified that it is so, for 3, 5, and, 7 squares.

---

[309] *DVQ* F.191r.

[310] A side note remarks on the case of different sized circles. The section, however, assumes even sized objects are used. Similar puzzles are well known such as to place 5 coins mutually touching (see the solution at https://richardwiseman.wordpress.com/2013/06/24/answer-to-the-friday-puzzle-211/) or the six/seven mutually touching cigarettes/cylinders arrangement in Gardner, Martin (1988). *Hexaflexagons and Other Mathematical Diversions: The First Scientific American Book of Puzzles and Games*, University of Chicago Press

[311] *DVQ* F.192r.

[312] *DVQ* F.193r.

However, there is a solution. This version of the problem was considered impossible up into the 20th century, and it was only in 1939 that Roland Sprague (1894 – 1967) came up with an example of a squared square.[313]

83. Chapter Eighty-three Document: Someone with 4 short beams, in a rectangular house, makes a ceiling, without other tools.[314]

A ceiling is to be built on top of four long beams that make up a square room, *ab*, *bc*, *cd* and *ad*, however, there are only bars that are much shorter than the beams. How can this be done?

Pacioli gives an example with beams of length 4 and bars of length 3. The solution is, instead of trying to cross the distance lengthwise, to place the shorter bars at an angle over the corners (see Figure 92).

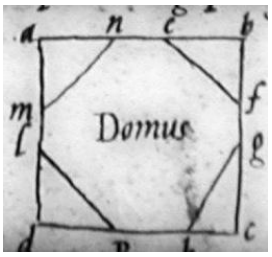

Figure 92: View from above of a plan of the house ceiling supports, F.193r, II.83

Leonardo Bridge (II.84)

84. Chapter Eighty-four: Document: 1 river 24 wide and with logs only 16 long to make a river without another [support].[315]

Cesare Valentino, duke of Romagna[316] leading his armies, comes to a river wishing to cross it. The river spans 24 units and there are only logs of 16 units. The noble military engineer traveling with the army solves the problem without resorting to ropes or other tools. The question posed is: how did he do it?

Pacioli, proceeds to explain aided by a drawing in the margin (see Figure 93). The idea is that part of the logs serve as weight on the margin, lying parallel to the river, the others are extended into the river and meet with others as Pacioli describes "like splinters" in the middle. In the picture, lines *mn, op,* and, *qr* are logs that weight *ef, gh* and *kl* down ¼ of their extend being on land, *st* and *ux* are extensions to the other side). The description seems incomplete.

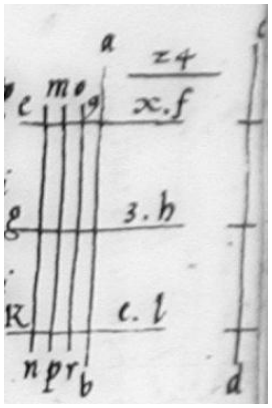

Figure 93: Scheme of bridge, F.194r, II.84

This might possibly be the description of one of Leonard's bridge designs like the commonly named Leonardo-Bridge (see Figure 94). Leonardo served some time as military engineer likely serving as inspiration to this section.

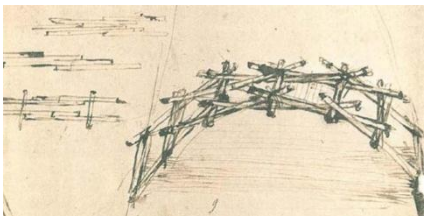

Figure 94: Leonardo Bridge from Codex Atlanticus, Volume 1. pages 69r and 71v. (1483 – 1518 Ambrosian library in Milan)

85. Chapter Eighty-five: Of a square stone, make 3 pieces without leftover [which when] placed on top of each other do not exceed each other.[317]

One is to make 3 congruent pieces out of a square stone. Once more the meaning of square is ambiguous. Pacioli uses on the border stone of a square well, or, the mouth of a cistern, this is, a square frame.

The problem can be expressed the following way: Cut a square frame into 3 congruent pieces.

Figure 95: Tri-part a square rim into equal sized pieces, F193v, II.85

Pacioli illustrates this geometrically (see Figure 95) and compares it to (II.83) this problem too being said impossible to solve.

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[313] For a comprehensive history on this topic, and several solutions to this problem visit http://www.squaring.net/history_theory/history_theory.html .
[314] *DVQ* F. 193r.
[315] *DVQ* F.193v. This document is listed after the next in the index and the title is misplaced in regards to its text. It has been opted to keep it here.
[316] Most likely Cesare Borgia, also named il Valentino (1476 – 1507).
[317] *DVQ* F.193v.

However, it is not obvious that there is no solution. A section of the frame can be tiled by a multiple of three. This might make it possible to find a three piece tiling. It is also open for what number one can find congruent pieces. Further questions arise when other holed polygon are considered.

Figure 96: Illustration of the cut of the circular sections, F. 194v, II.86

86. Chapter Eighty-six Document: To slice a circle, hollow like the mouth of a well, [and] in two slices make six pieces.[318]

This section describes how to section an annulus into 6 pieces with only two cuts.

To do so slice the annulus in half and align the pieces so that the next cut will divide it into 6 pieces, either by placing them side by side, by laying them one inside the other or one on top of each other (see Figure 96).

In the margin a picture of how to make 8 parts with two slices of a disc is shown (see Figure 97). This is, however, not mentioned in the text. The idea of the cut is the same.

Figure 97: Slicing a Circle into 8 Pieces with two cuts

These are Circle Division problems, with some degree of freedom. A classical question is: What is the maximum number of pieces you can form with 3 cuts, 4 cuts, and so on.[319]

Physical Experiments (II.87 – II.99)

87. Chapter Eighty-six Document: To find the north without a compass, in any place, at sea or on land.[320]

The following sections cover several physics experiments starting with, roughly termed, 'Seafarers' Knowledge'.

To begin Pacioli exalts the necessity to preserve oneself and thus to be able to navigate. In order that one always has a point of reference, he should be familiar with a compass. The compass works due to a magnet, which is placed inside. Pacioli proceeds to describe some of the characteristics magnets: pointing to the [geographic] north, attracting iron, and, if broken, the pieces share magnetic properties among other things.

Having established the magnetic properties, Pacioli briefly describes the compass in its general appearance and how to use it. It is divided according to the "four winds" (N, S, E, W), and has further subdivisions a magnetic iron needle. One should align this needle with the north wind to be able to know where the other directions are.

In the event of losing a compass he gives instructions on how to build a makeshift one.

Place the magnet (with previously marked poles) in a bowl, or, another object that floats. Place this into a bucket of water, or other water filled container. The magnet will spin until it is aligned north-south.

This section, further mentions maps and globes, and praises them as another mathematical marvel of the line. He further praises the translation by Francesco de Nicolò Berlinghieri of the work of Ptolemy. [321]

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[318] *Ibid*. F.194r.
[319] See for instance http://mathworld.wolfram.com/CircleDivisionbyLines.html .
[320] *DVQ* F.194v.
[321] Francesco de Nicolò Berlinghieri (1440 – 1501) translated Ptolemy's *Geographia* from latin into vulgar

Of particular interest, is a short note after the description of the compass, claiming that Pacioli might include some problems regarding areas, in this work. Which problems are meant is not mentioned. This furthermore hints at the rug-tapestry nature of the book and that the section is likely to have been removed from some other source.

Pre-Galilean notions of velocity

88. Chapter Eighty-eight Document: [While] being under cover, on a ship without compass or map, not seeing sky or water, to be able to say how much the ship travels; to be spot on.[322]

Pacioli describes the use of a pendulum mounted vertically at a right angle at the center of a ship through which, aided by cross-multiplication, the speed of a ship is calculated.[323]

A great deal of focus is spent on the precise size and position of the measuring instruments. The instruments are a ruler with horizontal rod at which a pendulum is secured, a plummet on a rope, an hourglass, which is to be stopped measuring the time of fall of the plumet, and, a drafting compass to measure the distance the plummet deviates from the expected center of impact (see Figure 98). The plummet is to be cut and both, the time of its fall and the distance from the dead center, are to be meticulously measured. The distance of the plumet to the dead center is to the mile as the interval of the falling plumet is to the hour.

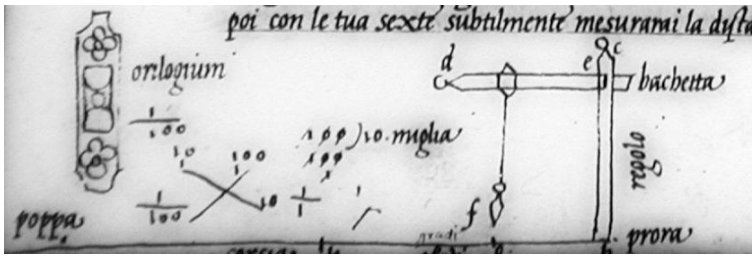

Figure 98: Illustration of the measurement of speed on a ship, F.196v, II.88

89. Chapter Eighty-nine Document: Being on land, to say this ship goes [at this speed or at that speed].[324]

The concern of this section is the same as the previous, to measure the speed of a ship. This time the observer is not on board. The calculating artifice of the "*regola del 3*" (cross-multiplication, see Figure 99) is stressed here. This highlights the same from the previous section.

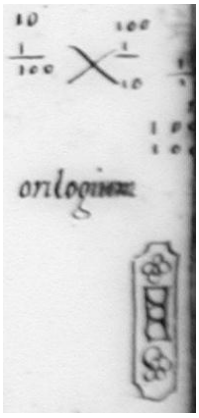

Figure 99: Cross-multiplication and illustration of a ship, F.197r, II.89

The method is, to observe the ship from two points, *d* and *e*, on the margin, *ab* (see Figure 100). Stop a watch for the interval it takes the ship to travel the distance of the

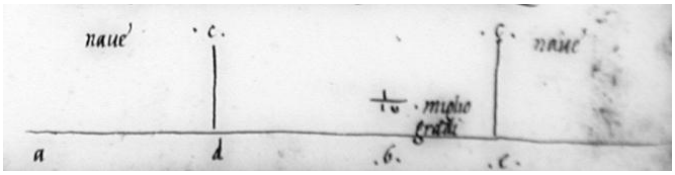

Figure 100: Shoreline and points on which to measure the speed of a ship, F.197r, II.89

[322] *DVQ* F.196r.
[323] This is pre-galilean physics and most likely to work due to the movement of the air if it worked but unrelated to the vertical issues pointed out by a note in the margin which reads "*Res haec est dubia et incerto si porpendiculo imprimitur motus navis*"
[324] *DVQ* F.197r.

lines of view orthogonal to the river bank. Next, measure the distance *ed*. Now, the speed of the boat can be calculated with cross-multiplication.

Pacioli suggests that the observer either runs along the shore to do these measurements, or, for higher precision gets assistance by someone who signals or also measures each standing at *e* and *d* respectively. These measuring methods are also recommended to calculate the relative speed between boats. Pacioli makes mention of using this for any moving object. A great deal of stress is laid on the necessity of precise measurements and measuring instruments for rigorous results, this is, the timing and clock should be the best there are to get the best results.

90. Chapter Ninety Document: To make a clock practical for seafaring.[325]

This section describes the making of an hourglass.

Pacioli begins by discussing some of the benefits and disadvantages of the substance used for the hourglass. The materials proposed are sand, water and quicksilver. Quicksilver seems to be the most favorable as it stays level, giving precise readings, unlike sand, which forms mounds, and, water, which can rot or freeze.

To build the clock one should join two bottles by their necks and secure this on a structure.

Throughout the section some more elaborate clocks are mentioned, such as ones with weights and gears. Archimedes is cited[326] a as well as Leonardo as authorities on the art of clock making. Leonardo's name is written in a different script highlighting it from the remaining text.

Hydraulics (II.91, II.92)

91. Chapter Ninety-One Document: Empty any large body of water with two spouts, by the force and virtue of the line.[327]

Here Pacioli describes the siphoning process.

The siphoning process can be used to empty a container of water of any size. Pacioli speaks of well. He suggests using hollowed reeds, as uniform as possible, joined together and insolated by cloth, pitch, and, wax, for the piping. One end is placed inside the vessel and the other outside. Great notice is given that the length of piping inside the vessel is to be shorter than that outside (see Figure 101).

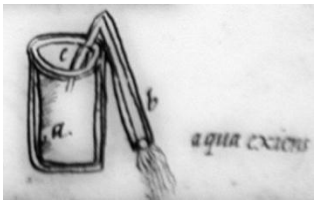

Figure 101: Illustration of the siphoning process of a bucket, F. 199r, II.91

92. Chapter Ninety-two Document: Emptying water, another way.[328]

Yet another way to transport water by the use of piping, made of *cana* (hollowed reeds), is explained.

This is again the syphoning process, this time, however, a single reed is bent so that one end, inside the water, is higher than the other end (see Figure 102).

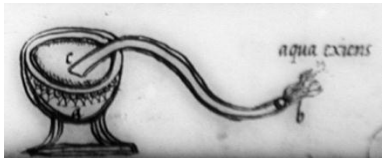

Figure 102: Alternate process of syphoning illustrated, F.260r, II.92

The section ends with a remark on other materials used for the piping. Pacioli mentions the use of these water draining techniques by the military, possibly to extinguish fires.

[325] *DVQ* F.198r.
[326] This time most like in regards to his *On the Sphere and CylinderI.*
[327] *DVQ* F.199r.
[328] *Ibid.* F.200r.

Aristotelian Natural Philosophy (II.93)

93. Chapter Ninety-three Document: On certain doubts which circulate among commoners & even learned theoreticians of two vessels.[329]

Discussion is made of the question: Given two equal containers filled with water and placed at different heights, which contains more water?

Pacioli claims that commonly the answer is that they contain equal amounts of water, but that the right answer is that the upper one does contain more water. He then proceeds to argue in favour of this within the Aristotelian system. The crux of the argument is that a more expanded body of water (the higher one) holds more substance (see Figure 103).

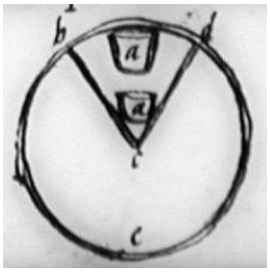

Figure 103: variation of density of the water according in the Aristotelian spheres, F.201r, II.93

Center of Mass (II.94 –II.99)

94. Chapter Ninety-four Document: To make a knife stand away from a table, [carved into] its scabbard or other stick.[330]

This and the following five sections are concerned with matters regarding the displaced center of mass.

The section starts by discussing how to place a dagger carved into a wooden plank in such way that when placed on the edge of a table it does not fall (see Figure 104).

The same principle is used, referred to in the description of a feat, where a man stands outside a building only supported by a loose plank and then jumps inside the building.

Again the explanation of underlying causes is given within the Aristotelian framework. This is used partially to demystify funambulism (tightrope walking) with the use of weighted poles.

Figure 104: Illustration of a knife balancing outside a table, F.202v, II.94

95. Chapter Ninety-five Document: To make a filled basket, no matter how big, attached to a knife stay on a table.[331]

Pacioli describes how a basket is hung from a knife, which blade rests on top of a table or wall without tying them together.

The effect is accomplished by placing a stick from the bottom of the bucket to the handle, likely to be done in secret. The tension formed between the downward pull and resistance of the stick, locks the stick in place and forms a rigid hook when balanced from a table (see Figure 105).

Picking the basket up breaks the tension and the knife drops to ground.

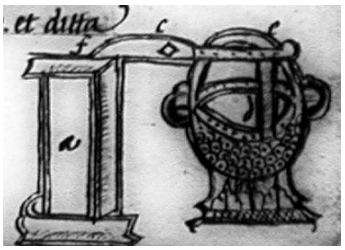

Figure 105: Illustration of a balanced basket, F.204r, II.95

96. Chapter Ninety-six Document: A strap mounted on a stake on top of a finger; keeping itself [in balance], with other things attached.[332]

This effect describes one more feat of balancing.

A stake is placed on top of a finger or a table with a belt on top of its furthest end, so that the ends form a right angle with the tip of the finger (see Figure 106). A reference of similitude is made in regards to the preceding document. As it stands, the set

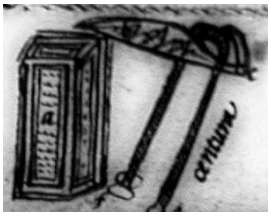
Figure 106: Illustration of a balanced stake, F.204v, II.96

[329] *Ibid.* F.201r.
[330] *Ibid.* F.202v.
[331] *Ibid.* F.203v.
[332] *Ibid.* F.204v.

seems unlikely to balance out. Either this is a feat of skill, or some auxiliary mechanism is needed.

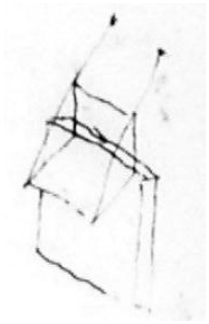

Figure 107: Illustration of the balanced roof, F.205r, II.97

97. Chapter Ninety-seven Document: To make a roof that keeps itself balanced.[333]

The principle described in (II.94) is applied to the construction of a roof.

The idea is to build the roof in such a way that the long beams, *a* and *b*, balance out the remainder of the structure due its center of mass (see Figure 107).

98. Chapter Ninety-eight: Document: at the tip of a needle sustain a stick with two or more knives balanced.[334]

A stick with two (or more) knives carved into it is made to balance on the tip of a needle.

Figure 108: Balanced knifes, F.205v, II.98

Two or more knives are to be carved into a stick; the stick itself is to be placed on top of a finger or needle. The knives ought to be longer than the stick (see Figure 108).

This is then proposed to be performed large-scale using a large pole with larger knives and the stick for larger audiences. Putting the stick in a circular spinning motion is said to be greatly appreciated by the audience.

Although, Pacioli does not mention it, the knives ought to form a V shape as one can find in balancing-toys.[335] The idea to use the center of mass is common to the well-known party trick of balancing two forks on a glass using a coin, toothpick or match.

99. Chapter Ninety-nine Document: A knife with a rock or other weight staying balanced on one end, on top of a needle, [in] another way.[336]

A knife's handle is balanced out by tying a weight onto the tip of the blade. This is suggested to be set to rotate on top of a needle by the use of a magnet to amuse spectators (see Figure 109).

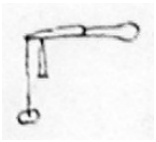

Figure 109: Counterbalanced knife, F.205v, II.99

It is likely that both the counter weight, at the tip of the blade, and, the magnet are to be kept secret from the audience.

Topological Puzzles (II.100 – II.132)

100. Chapter One-hundred Document: Hollow a stake by a thread through three holes.[337]

This effect and the following ones are Vexier or Disentanglement puzzles[338].´

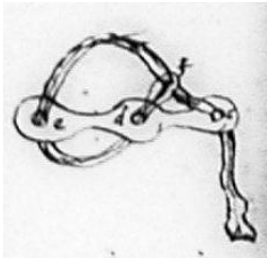

Figure 110: Disentanglement puzzle, F.206r, II.100

Pacioli describes how to set up the puzzle (see Figure 110). A loop of string, *f*, is placed through a three holed piece of wood. The holes are respectively c*, d,* and, *e*. This is done so that it loops around itself, between two holes. The doubled loose ends are handed to someone, or tied to something. The challenge consists of removing the piece of string from the wooden slab without releasing the un-looped ends.

To do so, pass the loop through hole *c* and over the piece of wood then pull it free.

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[333] *Ibid.* F.205r.
[334] *Ibid.* F.205r.
[335] See for instance Balancing Butterflies, at the virtual museum of Grand Illusions ltd.
[336] *DVQ* F.205v.
[337] *Ibid*. F.206r.
[338] This puzzle is analogous to a looped string puzzle.

Solomon's Seal (II.101)

101. Chapter One-hundred-one Document: With another thread through 3 holes in a stake with piece of amber, to remove it make it move all at once.[339]

Again a slab of wood and a thread are presented in form of a puzzle. This time one or several pieces of amber are strung to the thread (see Figure 111).[340]

A loop is made through a hole in the middle of the slab and the ends of a thread are passed through it. Before securing each end of thread on each side of the slab, by knotting it to another hole, a piece of amber (or other holed disk) is secured on one of the loops formed. The challenge is to pass the amber piece secured on one side of the thread, locked in-between the knotted end and the loop, to the other side. The pieces should not fit through the hole in the middle to make the puzzle challenging.

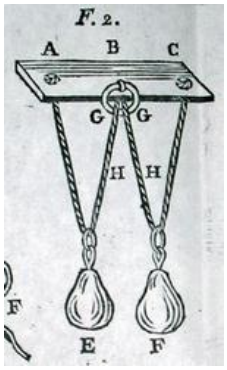

Figure 111: Solomon's Seal in Pietro Rusca (1743), *Il Maestro de' Giuochi Piacevoli*.

Pacioli calls for secrecy in regards to the solution of the puzzle '*divulgata non dilecta*', yet gives a solution nonetheless. The solution becomes hard to follow as the accompanying image is missing. The trick lies in passing the whole slab through the looped middle of one of the loops. As Pacioli says, martial experience will make the solution clear to you.

102. Chapter One-hundred-two Document: Another sight, remove two buttons, with a split string in-between [them] and looped tips [from each other].[341]

Two four-holed buttons are tied up with a circular piece of string and the challenge is to remove them from the string. Alternatively, one can place them on the string.

The string passes through all four holes of the buttons, but the end loops around the length between two holes in like in (II.100) and is the same in solving.

103. Chapter One-hundred-three Document: Join, with the leftover of the said split string, two shoe soles both in the same way, *bella cosa*.[342]

This section uses the idea of (II.100) to tie two show soles together.

A circular string is set up so that both ends loop around the two lengths passed through the holes in the shoe soles. The challenge is to remove the string, or, alternatively, to place tie the shoe soles together.

Pacioli suggests the use of four soles and two circular lengths of string to make this a competition between two youths, one having to set it up, the other to take it apart.

Cherries Puzzle(II.104, II.105)

104. Chapter One-hundred-four: Document: take and place 2 cherries in a letter split in half.[343]

Two cherries are strung to a piece of paper cut in a particular way and are left as a puzzle to be removed. This is an impossible object.

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[339] *DVQ* F.206v.
[340] This puzzle is known by several names, Dario Uri relates the name to Salomon's Sigil, but it is also known as Wedding Vows among others, a modern version made out of straws is called "Missing to Kissing"
[341] *DVQ* F.207v.
[342] *DVQ* F.209r.
[343] *DVQ* F.210r.

Pacioli makes reference to a missing illustration while explaining how to remove the cherries.

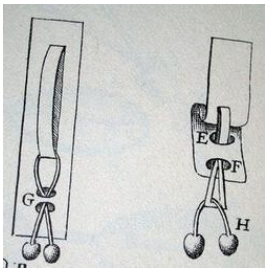

Figure 112: Connected Cherries from Giuseppe Antonio Alberti Bolognese (1795). *Giuochi Numerici e Fatti Arcani*.

It is most likely that this puzzle is the following: Take a piece of paper, cut it so that an oblong rectangular slip is created. By one of its shorter sides, make a hole next to the slip of paper (see Figure 112). The cherries, or a string with two rings attached, are placed on the slip by folding the long rectangular bit through the hole. After unfolding the piece of paper again, the stem of the cherries secures them to the strip of paper.

105. Chapter One-Hundred-Five Document: To loosen a cherry knotted to another, one of the two attached ones, without undoing the knot.[344]

This is a dexterous challenge to free a cherry stem from another knotted to it. The idea of this trick is to screw the stem through the knot.[345]

106. Chapter One-hundred-six Document: To loosen a strong knot made with a belt; beautiful and subtle ingenuity for the youth.[346]

This is another unknotting problem.

A knot is tied into a belt, which has been doubled. The looped end of the doubled belt is then passed through the buckle of the belt. After this the other end of the belt is fitted through the loop. The challenge is to undo the knot while someone else securely holds the end with no attachments. Once more Pacioli stresses the secrecy of this simple solution.

Chinese Rings (II.107)

107. Chapter One-Hundred-seven Document: Take and place a secured strenghetta from a few secured rings; a difficult case.[347]

Figure 113: Zhuren, Zhu Xiang (~1821) ***Little Wisdoms***

Pacioli speaks about the puzzle commonly known as Chinese rings (see Figure 113).[348]

Pacioli starts by describing the gadget, a piece of wood with various posts. These posts consist of a wooden pillar, fixing a ring in place, granting it up, and, down movement. Each of the rings circle the next post, with exception of the first. The number of posts varies, but ought to exceed 3 for the puzzle to pose a challenge. Through the rings fixed on the posts a looped string is drawn. The challenge is to remove the string, or, to place it so that it passes all rings leaving the posts in the middle.

Pacioli describes how to place and remove the string. Pacioli again remarks on the importance of practice in order to understand what he is writing about. Again a competition between youngsters is left as a didactical challenge.

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[344] *DVQ* F.210v.

[345] The idea present in the section reminds of a trick with two straws orthogonal to each other one knotted around the other which are then separated, sometimes known as Set Free.

[346] *DVQ* F.211r.

[347] *Ibid.* F.211v.

[348] This puzzle is also known as Cardan's Rings and Baguenaudier and can also be seen with static rings stepwise displaced and interlinked. A mathematical discussion can be found in Józef H. Przytycki and Adam S. Sikora "Topological Insights from the Chinese Rings", Cornell University Library archive (http://arxiv.org/pdf/math.GT/0007134.pdf) or on the wolfram page (http://mathworld.wolfram.com/Baguenaudier.html). For digital implementation of a solution one can visit Jill Britton's Website (http://britton.disted.camosun.bc.ca/patience/patience.htm). A list of patents can be found on Dario Uri's site (http://www.uriland.it/matematica/DeViribus/2_113.html) discussing this section.

108. Chapter One-hundred-eight Document: Remove a large ring from two [rings] linked to a rod, through its head.[349]

This time a rod fastens two loops of cord, one at each end, each with a ring secured by the loop. The puzzle is to remove a ring that that hangs on the rod between rings.

On one of the ends the ring was placed inside the loop, the other end ties onto itself so the ring is kept in place by a hitch. The second ring is strung over the other loop as well, so that it is kept in place by the first ring. Finally the third and bigger ring, large enough to pass over the smaller ones, lies on the rod. It is able to pass over chord and ring, but seems knotted in. Pacioli once more, after having described the gadget, gives the solution and then proceeds to discuss it in terms of presentation. Telling the reader that the bigger ring only serves as decoy and how to assemble the knot properly, so the puzzle is challenging.

The concept of this puzzle is next applied to tie a purse, in a manner only easily accessible to one who knows the solution to the puzzle. Instead of having a rod and ring, a circular length of string is threated through various holes of a purse around its opening. The looped ends on opposing sides serve to place the rings mentioned above, or equivalent. The looped ring can be replaced by a metal rod big enough for the second ring not to fit through.

109. Chapter One-hundred-nine Document: Loosen a bag or button linked to a belt, which is looped to its buckle.[350]

A purse tied to a belt passed through its buckle is given as one more way to seal and secure a purse. The description unaided by illustration is dubious. However, opting for the simplest solution that fits the description it describes how to untie and then tie a lark's head hitch around the buckle, taking into account that the cord which ties the hitch is secured to a purse.

110. Chapter One-hundred-ten Document: A button [on the string] of a crossbow, or of two cherries and a crossbow.[351]

A description is given of a string of a crossbow, or other bow-like shape, with a string in it.

Looped on the string are two cherries secured by their stem, other buttons tied to a string, on the other side of a *Pater Nostro,* or another piece with a small hole just large enough to pass the string through.

This is analogous to (II.104), the string of the crossbow needs to be flexible enough to fit through the *Pater Nostro*, doing so one can place or remove the cherries.

Self Untying knots (II.111)

111. Chapter One-hundred-eleven Document: Make the knot named loose [self-untying] as used by smiths in valeting, for horses and [other] beasts.[352]

This section discusses a self-loosening knot.[353]

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[349] *DVQ* F.213r.
[350] *DVQ* F.215r.
[351] *Ibid.* F.215v.
[352] *Ibid.* F.216v.

The knot is presented as used to tie down unwary or sick animals, with several loops around their heads, which, if necessary, can quickly be released. Pacioli describes how to form such knot. The description is hard to read and alludes to illustrations that are missing, the exact knot or hitch is unknown.

It is possible that this is the description of a Highwayman's Hitch.

112. Chapter One-hundred-twelve Document: Write a difficult to read letter.[354]

Here Pacioli describes how to encrypt a letter with the use of a ruler or bar.

The idea is to write along the length of a ruler, around which an oblong rectangular piece of paper has been spiraled. This ruler has its duplicate at the receptors side.

A second version where two bars are used instead of the rulers is also proposed. Other means of codification in the next part of the book are also hinted upon, however, are not included as they are not related to the geometrical matter of this part.

Origami envelopes

113. Chapter One-hundred-thirteen: To seal a letter without any wax.[355]

Pacioli proposes to give instructions on how to fold a sheet of paper in such a way that it becomes its own envelope.[356]

Three variations are proposed for a single-sheet letter followed by instruction on how to fold an envelope and remark on a possibility to conceal a letter. The lack missing illustration and the *volgare* obscure the folds described in this section. The following descriptions seem to match those of Pacioli.

Given a rectangular sheet of paper, one is to fold it widthwise so to obtain a strip of paper. Both ends of the paper are bent in such a way to obtain a similar trapeze standing out to each side of the strip. The side of the trapeze gives the next fold, which is to be folded over until both ends are close enough that after folding them into each other they form a square shape.

The second method is by far simpler. Start with a square paper. Fold it diagonally. Next, tuck one of the acute angled tips of the triangle formed into the fold of the other. The right angled tip is then tucked between them and possibly even secured by a single stitch[357] where all of the tips overlap.

Another method is to have the letter wrapped around a round piece of leather. It is closed. The way it is closed is not to clear. However, Pacioli, stresses that there are

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[353] A very simple example is a string with 3 knots in a row. Passing one of the ends through the knots unties these, http://www.youtube.com/watch?v=MvOHV5cARM8. Some examples of untying hitches can be found on Peter Suber's page, http://legacy.earlham.edu/~peters/writing/explode.htm.

[354] *DVQ* F.217r.

[355] *DVQ* F.218r.

[356] The folds at hand are a very close variant to the one presented on wiki how (http://www.wikihow.com/Fold-Paper-Into-a-Secret-Note-Square)

[357] Laurie Pieper notes that the nizza mentioned in the MS, is a flap that would be drawn with a proper tool through the paper to seal it.

special tongues with rounded tips to crease the letter shut. Upon opening the crease marks will be obvious making it a hard task to restore the letter to shut state.

Finally, Pacioli remarks on how to create a false bottom for the sending of concealed messages to prisons or in a war so that they are not discovered by the wrong hands.

114. Chapter One-hundred-fourteen Document: Of three castles and three fountains, a beautiful case.[358]

This is similar to an ABC Connection Puzzle.[359]

Three castles within a circular wall, which, when going to war, each has their own way to their own gate. Their paths may not cross. The challenge is to find a solution so that every castle has access to its gate. Pacioli gives the two symmetric solutions and discusses them. The same can be set with of three conflicting monasteries which get their water from three wells.

This can be brought into a more abstract version. Three points, one of which is on the edge of an encirclement are to be connected to their pairs, which are situated on the opposing side of the encirclement. This should be done without crossing lines (see Figure 114).[360]

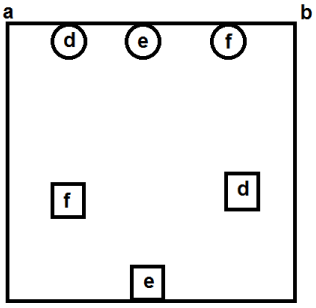

Figure 114: Linking Problem, join the corresponding letters.

Modernly, a similar looking variant is known, the adaptation of Kuratowski's Theorem, to a problem involving a $K_{3,3}$ graph[361] and the concept of planar graph.

115. Chapter [One-hundred-] fifteen Document: Burning a candle in water, namely to its end.[362]

Pacioli describes how to make a candle float in water and how to use this for the amazement of the general public.

One is to take a candle and sharpen it in such a way that it forms a cone shape with the wick opposing the point. At the point one is to fasten a weight, for instance a coin to weight the candle down, thus establishing vertical buoyancy (see Figure 115).

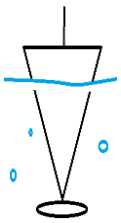

Figure 115: Floating cone shaped candle attached to a coin

Pacioli elaborates on how this trick astonishes the commoner as it is a candle and not some floating material. He suggests lowering the burning candle with a bucket into a well in secret and then calls assistance to produce an astounding effect.

116. Chapter [One-Hundred-] sixteen Document: To unveil a given coin in a basin.[363]

This section discusses an effect based on the refraction of water.

Pacioli uses this to eulogize the power of the line. After a brief introduction in regards of optical illusions and once more panegyrizing both the duke of Milan Ludovico Maria Sforza, and Leonardo da Vinci he proceeds to describe the effect.

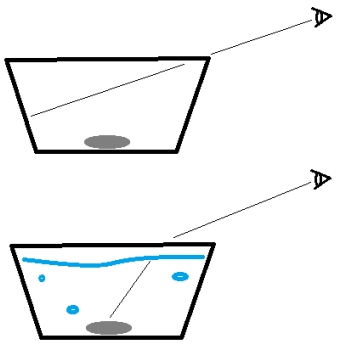

Figure 116: Refraction Principle.

[358] *DVQ* F.220r.

[359] For the rules of such puzzle see http://rohanrao.blogspot.pt/2009/05/rules-of-abc-connection.html

[360] A variant of this puzzle named "Twisted Wires" is often ascribed to Clifford Pickover (a link on his page was not found but it is hinted upon here http://forums.xkcd.com/viewtopic.php?f=3&t=62517).

[361] See for instance http://topologia.wordpress.com/2010/09/21/el-problema-del-agua-la-luz-y-el%C2%A0gas/

[362] *DVQ* F.221r.

[363] *DVQ* F.222r.

A coin is placed in an empty basin or bucket in front of a participant in such manner that he cannot see it. The question, if one can make it visible to the participant without moving the coin is asked, and then shown as possible by filling the container with water and thus revealing the coin (See Figure 116).

A mirror is suggested to read the inscriptions and thus produce even more elaborate effects. Pacioli hints on making the coin invisible underneath water with a layer of something else, once more hinting upon the trick through refraction.

117. Chapter [One-Hundred-] seventeen Document: To move a string from the hand, and [do the same with] a ring.[364]

This is the description of a well-known effect where a ring gets removed from a string. Pacioli discusses two variants of presentation, one with, and the other without, a ring.

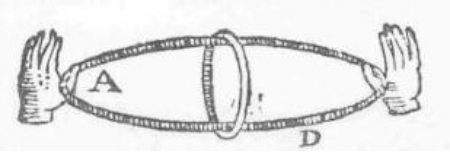

Figure 117: Schwenter, D. (1686). *Deliciae Physico-mathematicae oder Mathematische undPhilosophische Erquickstunden*

A circular piece of string is stretched between the two thumbs of a participant's hands. The spectator is then asked to pinch index and thumb together. The performer places two crossed fingers in between the two lengths of string and picks them up. This is done so that the bottom finger picks up the length of string furthest away from the performer, and, the finger on top the closest one. He then untwists the crossed fingers keeping the string secure and lopped in the fingers. Next the performer passes one of the loops through the participants pinched fingers closest to that hand. After releasing the other finger, the string falls off the hand.

The effect can be implemented with a ring strung on the string (see Figure 117) in the same fashion. Catching the string and stretching it produces the illusion that the ring passed through the string.

Modernly, most variants of this effect use two stretched fingers. If the string is secured again on the finger upon releasing the illusion that the string did not leave the fingers is created. It will appear as if a neutral twist has been undone and the ring was passed through the chain, falling off.

118. Chapter [One-hundred-] eighteen Document: Of knowing how to do the labyrinth with diligence according to Virgil.[365]

This text is incomplete. A folio or more might be missing here. The five introductory lines present tell of the Aeneid's episode with the noble bull called the minotaur imprisoned in the portmanteau, the agglutination of two or more words to form a new one, joining of *labor intus* (usually labyrinth is said to derive from the greek *labrys* double edged).

This section likely describes either how to construct a labyrinth to be solved by someone, or some sort of artifice on the well-known yarn used to mark the path of Theseus to escape the labyrinth. It can be taken from the index that the effect is concerned with the form and quality of the labyrinth.

119. [Chapter one-hundred-nineteen Document: To make and loosen the circular knot from the handkerchief and other cloths, handy in many cases.][366]

This section discusses a special kind of knot.

[364] *DVQ* F.222v.
[365] *DVQ* F.223v.
[366] *DVQ* missing page The title is taken from the one listed after the preceding one in the index.

The beginning of this effect is missing. The text starts in the middle of the explanation on how to tie a certain knot. It isn't clear which knot is meant.

The knot is described as hard to undo. It tightens itself further when the ends are pulled in an incorrect way.

It is suggested that one uses it to pull pranks on sleeping friends, whilst a companion goes swimming, or, women are in their underclothes. Pacioli also tells the story of an unhappy lodger who after using the knot gets his money back from the host, who is unable to solve the problem.

A noose or the like, for instance a Windsor knot, might be what's meant.

120. Chapter One-hundred-twenty Document: To make the long knot of *Benducio*[367] and loosening it.[368]

Pacioli describes how to tie yet another vexing knot to intrigue the fool.

For this he uses a *venducio (*bundle*)*[369], rolled up like a map, or a will. One of its tips is tied to a rope. The rope in turn passes lengthwise through the middle of the roll. The other end is rolled up in the opposite direction, folding over the bundle and rolling up around the knot in the cloth and forming a clump.

This proposes a challenge to be untied.

121. Chapter [One-hundred-] twenty-one Document: Remove a button and more, from two strings.[370]

This is a rope trick.

Two pieces of string are doubled and passed through a whorl so that both loops come out the same side of the whorl. One loop passes through the other and doubles back onto itself. Pull the loops tight so that the doubled loops hang together and pull them into the whorl. Do this so it appears you are tying a square knot.

The two ends of one of the strings are handed to a participant the other two are kept or tied. The performer easily undoes the loops causing surprise when the whorl is released from the strings.

Slicing Fruit (II.122-126)

122. Chapter One-hundred-twenty-two Chapter: Cut an apple into four parts and bring them back together.[371]

This section discusses a puzzle made out of the cutting of an apple. Once more the content is obscured by the lack of imagery. The idea is to cut an apple into four parts so that is hard to return them to the original shape of the apple. The cuts are made through the middle and from every direction.

A likely cut would be to slice the apple entwine vertically, from stalk to stamen. These should be halved again, one half, by a horizontal, the other, by a vertical cut. The

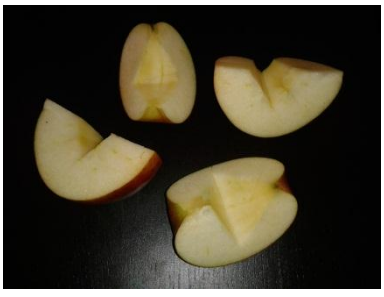

Figure 118: 4 apparently equal quarters of apple, two horizontal, two vertical.

[367] Likely as one would tie a bundle, like for firewood.
[368] *DVQ* F.224v.
[369] From the context this seems to be some sort of cloth, in other contexts it is related to the sale of lumber, perhaps it is referent to some typical cloth to bundle the wood.
[370] *DVQ* F.225r.
[371] *Ibid.* F.226r.

three cuts are thus orthogonal to each other. For a more difficult puzzle the core should be removed alike in all pieces, stalk and stamen as well (see Figure 118).

123. Chapter [One-hundred-]twenty-three Chapter: Of another cut, into two pieces.[372]

In similar fashion to the preceding section, the apple is sliced in only two pieces.

One of the cuts should be made horizontally, between stalk and stamen. Pacioli suggests further cutting the apple into several uniform pieces.[373]

A non-trivial solution is a puzzle in itself (see Figure 119). In both effects Pacioli notes that the apple should be as homogeneously colored and round as possible, to be more challenging.

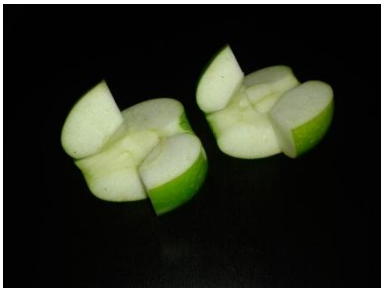

Figure 119: Two symetrical halves of an apple.

124. Chapter One-hundred-twenty-four Chapter: Slicing an apple inside, without cutting the peel, and similarly for a peach [and] orange.[374]

Pacioli describes the process of slicing a fruit without peeling it.

Take a thin copper wire and pierce one of its ends, preferentially, through stalk or stamen; then carefully move it around so that it loops along the inside of the peel. After getting both tips outside the peel, the cut is produced by pulling the ends out carefully. A ripe fruit is recommended as it facilitates the slicing.

An alternative method is mentioned, yet unclear. It is to have been shown in Empoli on a 8th of August in front of Signore Soderini[375], his brother's ,Piero, wife, and nephews Thomaso and Giovan Baptista.

This is done commonly with a banana and a needle.[376]

125. Chapter One-Hundred-twenty-five Document: To peel an orange or even a peach into an intertwined chain that does not break.[377]

The effect describes the peeling of an orange or peach, in such way that the peel remains whole and the pulp is eaten. The peel should be marked in a zigzag way previously. With overlaying depths of cut this line is to be followed to create two halves of peel. The inside is to be scraped out, for instance using the previous section. The whole description is obscure referring to a missing image. The reader is to find out himself through practice.

Button-hole Puzzle (II.126)

126. Chapter One-Hundred-twenty-six Document: Remove a belt from the arm placed in its buckle without removing the other end from the hand.[378]

A belt is tied around the arm and pulled tight through the buckle, while the other end stays firmly in the hand and is not to be moved.

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[372] *Ibid.* F.226v.
[373] Some another apple puzzle can be found at http://www.cutefoodforkids.com/2011/04/3-d-apple-puzzle.html.
[374] *DVQ* F.226v.
[375] Likely Cardinal Francesco Soderini (1453 – 1524) and the wife of Piere Soderini (1452-1522), his brother and their children.
[376] See for instance http://www.wikihow.com/Slice-a-Banana-Before-It-Is-Peeled.
[377] *DVQ* F.227r.
[378] *Ibid*. F.227v.

Alternatively, a loop attached to a stick is described. The idea is to remove this string and loop from an arm or certain position.

This last variant seems to be the Button-hole Puzzle.

127. Chapter One-hundred-twenty-seven Document: One that is forced to walk 50 miles continuously taking 5 steps forth and 5 steps back.[379]

This is a trick question.

Can someone who alternatingly walks backwards and forwards reach their destination, always walking the same distance in each direction? Yes. He should move turning towards his goal, then when turned walk backwards towards his goal.

128. Chapter One-Hundred-twenty-eight Document: To twist a needle with a handkerchief.[380]

Pacioli gives instruction of how to bend or break a needle.

To do so wrap two pieces of tightly doubled-up tissue at each of the ends. Next turn each into opposing directions, twisting them like a screw. The needle will bend naturally without too much effort.

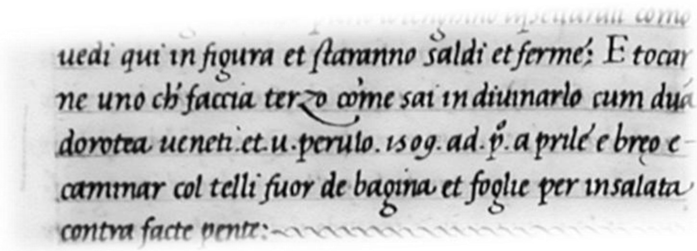
129. Chapter One-hundred-twenty-nine Document: To cross three knives at their half their edges.[381]

Figure 120: Interlocking knives supporting filled tea cup.

Pacioli describes the feat of locking three blades together so that on them a carafe can be placed.

To do so, interlock the knives. Each blade lies under the tip of the previous knife and on top of that which comes after, forming a triangular shape with the blades. Have each of the handles supported by the edge of a bowl, or three cups (see Figure 120).

This seems to have been shown to Pacioli on the first of April of 1509. The passage ,however, is cryptic. The section in general seems uncharacteristically written, almost like a short note. The text is rather unclear especially at the end of it (see Figure 121).



uedi qui in figura et staranno saldi et ferme: E tocar
ne uno ch' faccia terzo come sai in diuinarlo cum dua
dorotea ueneti.et.u.perulo.1509.ad.p.a prile e breo e
caminar col telli fuor de bagina et foglie per insalata
contra facte pente:

Figure 121: Excerpt DVQ, F228v

Laurie Pieper points out that this section has one of the first references to the custom of playing tricks on April Fool's Day, as this seems to have been performed by a Hebrew on a salad bowl before an audience on the first of April.

The date present "1509" is also not very clear and may just be an abbreviation, as Peirani's transcription shows interpreting it as "*isog.*".

The other names mentioned hold further room for speculation. Singmaster suggests that the '*dorotea*', might be an occupational reference, e.g. nuns, and, *perulo* might be

[379] *Ibid*. F.227v.
[380] *Ibid.* F.228r.
[381] *Ibid.* F.228r.

a common name. Pieper on the other hand proposes that both could be a currency (although, no further reference to this currency is found in the manuscript). Another possibility is that "*dorotea venti*" is the plants name, and "*perulo"* is close to *perula* (bud scale). Leaves are mentioned further on which might support this. In either case it seems like the Hebrew performed the effect with aid of plants and knives.

130. Chapter One-Hundred-thirty Document: Break a porphyry marble slab with the fist.[382]

Another stunt of illusion is proposed.

The performer breaks a porphyry or serpentine marble or other stone into pieces with his fist. To do this the performer should hide a smaller piece of the same stone, of 2 to 4 fingers width, under a mantle placed on the bigger flat stone, given the pretext to protect the hand. Then hit the smaller stone to shatter the slab, the smaller stone unnoticed in the debris.

131. Chapter One-Hundred-Thirty-one Document: To make three points on your hand which turn to 6.[383]

Three dots on the hands of the performer turn into six.

Make three ink spots close to one of the main folding lines of the hand. Before the ink dries, using some misdirection, close the hand, letting the ink touch the opposing side of the line duplicating the spots.

132. Chapter One-Hundred-Thirty-Two Document: About the puerile solace named *bugie* [lies].[384]

Here Pacioli describes a gadget best known as Jacobs Ladder.[385]

Pacioli describes how to construct the device with two wooden slabs and 3 straps, forming a "wallet". Place a straw in between the single strip and close and open the wallet again in the other possible way to produce the straw trapped by the other two straps. Pacioli tells how old people entertain infants with this device (as captured by contemporary painter Luini, Figure 122)

Pieper points out that this is the earliest known written reference of this kind of gadget, and further that Luini's portrait was first ascribed to Leonardo. As Singmaster says, the multiple piece variant would only appear in the eighteenth century. Bossi discusses four strap variants, with crossed straps, and application of this device by Leonardo to build a theatre set.

Figure 122: *Puttino che gioca*, Luini Bernadino, ca. 1500, from fundatione Federico Zeri

[382] *Ibid.* F.228v.
[383] *Ibid*. F.229r.
[384] *Ibid*. F.229r.
[385] For one of many articles on the topic see for instance Donald Simanek, "Toys, Tricks and Teasers." (http://www.lhup.edu/~dsimanek/TTT-rings/rings.htm)

Two other effects (II.133, II.134)

133. Chapter One-Hundred-Thirty-three Document: To know how many circular surfaces as big as the revolving sun, fill well the circumference of its ecliptic.[386]

This section discusses the measurement of how many discs of sun fit into the course of the sun during the day.

Pacioli instructs to time the sunrise (or sunset), from the moment the sun appears on the horizon until the whole disk is visible (or the other way around), as well as the time from sunrise to sunrise (or sunset to sunset) with one or two clocks, as described in (II.90.), for greater rigor, turning them alternately. He suggests taking count of the turn with acorns or other things inside a jar. By dividing the later measurement by the first, the desired result shall be obtained.

134. Chapter One-Hundred-Thirty-four Document: To toss a needle with a string and have it stay in the door or other wood.[387]

Pacioli finishes this second part with the description of a needle with a string through its oar thrown at a wooden object, *"la natura da se lo fa*[,] *se tu con buon brachio la tirerai*[,] *perch' sempre andara ritta*[,] *perchel filo la guida come penna altra veretta*[,] *et ficarsse sempre".*[388]

[386] *DVQ* F.229v.
[387] DVQ F.230r.
[388] F.230v, "*Nature will make it happen, if you with good strength*[good Arm, sic.] *throw it, because it will always go straight, because the string guides it as the feathers do other rods* [arrows], *and it always sticks."*

III. Other Documents

Unlike in the previous two sections Pacioli does not give any starting introduction. This final, and third part, is fragmented into five sub-parts.

i. Moral Documents very useful as proverbs.[389]

This first sub-part the reader will find 23 proverbs. These proverbs are concerned mostly with the good conduct of men. To list two examples:

"***Non si po dare a figlioli melior parte chi dar li buon costume e porli al arte.***"[390]

"***Con Falista é inganno se vive la ½ parte del ano con ingano é falsta se bive laltra metad***"[391]

ii. Lament of a lover addressed to one maiden.[392]

This sub-part is constituted entirely by a poem. The poem has twenty-seven stanzas. It is a declaration of a devotee and passionate lover to his mistress. Both addressee and addresser are not identified throughout the poem leaving room for speculation on who they are based on the text.

Ignoring the introductory line the acrostic reading of the stanzas, this is taking the first letter of each, is the ordered alphabet.

iii. Mercantile documents and proverbs most useful.[393]

This sub-part is split yet again. The first part is a collection of several proverbs related to mercantile functions. Many of these proverbs are explained, commented on or contextualized. Examples are:

"***Gli meglio dare é pentiré ch' tenere e pentire.***"[394] Is contextualized, by Pacioli as settling for a sale rather than clinging onto it, for instance while travelling with goods.

"***Chi non robba non fa robba.***"[395] Pacioli's comment is that it is simply a terrible proverb.

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[389] *DVQ* F. 231r

[390] "*You can not give children a better boon than to teach them good costumes and train them in an art [trade]*"

[391] "*If you live with falsehood and deceit for 1/2 of the year, with deceit and falsehood you live the other half.*"

[392] *Ibid.* F.232r,232v

[393] *Ibid.* F.233r – 235v

[394] "*It is better to regret giving, than to regret having.*"

[395] "*He who doesn't steal makes no profit*" a pun done with *robba* both meaning to steal as well as goods.

Natura Magistra

This half of the third sub-part Pacioli lists 83 documents. These are in the same fashion of the first two major parts of this book (I. and II.). They describe other marvels not included prior but "fitting of this work", as Pacioli notes. The ongoing and subjacent reason for the working of the marvels described below, in similitude to arithmetic and geometry in the first parts, is *Nature*.

Secret Messages (III.iii.1 – III.iii.12)

The first twelve effects describe how to conceal messages so that only those with the knowledge of a shared secret can read them. The first three sections are concerned with the invisible writing which appears by exposure to heat[396].

1. First Chapter Document: Of the force and natural virtue of writing.[397]

Pacioli instructs to write with a mixture of ammoniac salts and generous amount of water. This ink will reveal itself on paper when the parchment is heated to produce black lettering. Several other concoctions produce the same effect, however, with a different coloring, those Pacioli mentions are fig milk, onion, orange, lemon and citron juices on their own or mixed.

Depending on the solution the process which renders the ink visible may vary. One possibility is to use substance that reacts to heat. For instance the heating might oxidize the organic substances applied to the paper.

2. Second Chapter: To write in such a way that it is not seen.[398]

A second method of invisible writing is proposed, this time by writing with the acidic liquids mentioned already, or alternatively with fat rich liquids. When dried the letters appear by spreading a pulverized substance like charcoal, or, dust over the surface containing the ink, thus uncovering the ink.

Pacioli's examples include the mention of prostitutes, who are written on with urine in times of war and sent as unknowing messengers, and napkins blotted with milk sent into prison with food.

Another process, with which one can unveil the ink is to have it react to some other chemical or other revealing agent. Be it ultraviolet light that uncovers faint fluorescence of certain substances, an acid/base indicator, or, like described by Pacioli, dust that sticks to grease.

3. Chapter Three Document: Writing that does not appear unless in water.[399]

As a third method Pacioli suggests writing with tallow, or, another greasy substance. The letters appear by exposing the written on surface to some sort of strain. For instance, one is to place an inscribed page in cold water, only the coated surface will remain untainted. Similarly it can be done on stone, bathing it in an acid liquid, like

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[396] For a comprehensive history of various kinds of secret messages see for instance Macrakis, Kristie ().*Prisoners, Lovers, and Spies: The history of Invisible Ink from Herodotus to al-Quaeda*, Yale University Press. For a simple modern take on how to write with lemon juice, see for instance http://www.wikihow.com/Make-an-Invisible-Ink-Message .

[397] *DVQ* 236r.

[398] *Ibid*. 236v.

[399] *Ibid.* 237r.

vinegar. This will corrode the stone more on the uncoated surface leaving the message to be read.

Here Pacioli briefly suggests using this message as effect to trick the commoner. The performer is to prepare a parchment or other surface with a message, symbol or other thing and then opportunely reveal it to the unsuspecting participant.

Yet, another way to conceal a message might be, to disturb the fabric of the parchment. The substance coating the paper might be acidic in nature and thus corrode the surface of the paper such that when heated or directly held against a light source this becomes visible. Or the other way around, it might protect the specific spot from influence of a mixture.

4. Chapter Four Document: To make letters of gold or silver, copper or brass.[400]

Here Pacioli describes an ink, which will turn the color of the metal it is rubbed with after written on parchment. The idea is similar to that of a touchstone, a stone with a finely grained surface on which soft metals will leave a visible trace. These stones, like slate, where commonly used to assay metal alloys, as Pacioli mentions by goldsmiths of his time.

One is to crush a crystal and mix it with egg-white, similarly to cinnabar. The ink derived from *Dracaena Cinnabari* is likely meant here, and not the crystal from which mercury can be derived. It is named Dragon's Blood which is made reference to further on, in another context.

5. Chapter Five Document: Writing in another way that does not appear.[401]

The milk of a plant is proposed for the same purpose as in (III.iii.2). This might possibly be Euphorbia serrata, a commonly found plant of the Mediterranean, used even nowadays for body painting.

Pacioli explains that the ashes hold onto the script due to its viscosity.

6. Chapter Six Document: Write with clear well water on a white sheet and the writing comes out black.[402]

Pacioli discusses how colorless liquids become visible. Two ways of presentation are mentioned. The first, is to write with water on parchment and the letters appear in black. In the second, likely a prank, hands are washed in two different clear liquids becoming black.

To achieve the first parchment should be coated in the powdered mixture of 1 part gall and 2 parts roman vitriol (copper sulphorate)[403]. Alternatively tortoise gall and liquid from fireflies can be used in the mix instead.

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[400] *Ibid.* 238r.
[401] *Ibid.*
[402] *Ibid*. 238v.
[403] $Cu(OH)_2$ reacts in combination with bases forming a blue particles. If then heated these will turn black as CuO is formed. This could have occurred in similitude to previous writing methods.

For the second effect Apulian gall is mixed with water. The mixture is filtered repeatedly until it becomes clear. The other mixture to be used is roman vitriol filtered in the same way. The filter used consists of layers of felt or wool.

Using these as prank is mentioned in (III.v.r.222.

Not fitting the description in the title, this section contains a method to write golden. To do so, use the crushed seeds of Lupin pods.

7. Chapter Seven Document: To make varnish to write well.[404]

To make varnish, mix 1 part rosin, or, juniper gum, with 12 parts of ground eggshell, or, marble powder. The varnish can be used to coat a sheet of paper, or, a pen to prevent them from becoming runny.

Both, eggshells and marble, are essentially composed by Calcium hydroxide. Likely the surface this "varnish" was applied on would be rough and more adherent, as well as absorbent. Possibly this mixture might also have been used to amend writing mistakes hiding the ink it underneath the mixture.

8. Chapter Eight Document: To order letters from a sheepskin sheet, this is, erase them.[405]

To remove the mistaken letters soak them in Lemon, or, Orange juice. The acid will decompose the ink.

To make small balls of white paper, as said in another part. [Unaccounted for in the index, but titled]

Paper is coated in a mixture of dissolved Alum or *sabsci* (some form of soap might be meant, this would also work). The balls made of this paper are said to float.

Pacioli suggests a performance or scam where some participants are given the coated balls and others not. In the scam, those who don't manage to make their balls float have to pay.

Alum is commonly used to make things waterproof.

9. Chapter Nine Document: Writing that can't be read unless with a mirror.[406]

Mirror writing is discussed as yet another method to conceal messages. Pacioli mentions Leonardo who was already famous at the time for the mastery of this kind of writing.

10. Chapter Ten Document: Writing on a rose and other flowers.[407]

The idea is to use a stencil made of perforated or, cut paper. The paper is used to cover the flower. Next, spray it sulfur containing perfume. Alternatively, spray the whole flower covering the place where letters are meant to appear.

Sulfur oxidizes organic substances turning them black or dark brown, like coal.

To produce golden or, silver letters use the dust of the respective metal. Apply it onto glue with which was written on the flower. Fig milk or *scisa* are suggested as glue.

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[404] *Ibid.* 239r.
[405] *Ibid*.
[406] *Ibid.* 239v.
[407] *Ibid*.

11. Chapter Eleven Document: Writing on metal with liquid, engraving.[408]

Like in previous sections, letters are made by shielding some of the surface of the metal from corroding (either by oxidation or rust). Pacioli suggests the use of of vinegar and arsenic to corrode the metal. For the covering protection, use wax. The wax is later to be carved free.

Alternatively, corrosion can be obtained with rock salt and ammoniac or, verdigris.

12. Chapter Twelve Document: Of writing in cipher. How it's done.[409]

Next Pacioli addresses encryption[410]. There are two kinds of encrypted messages, ciphers and codes. As Pacioli explains, a cipher is an agreed alteration of a message according to a rule known by both recipient and sender. Modernly, ciphers are distinguished from codes. While the first acts upon specific characters, the latter acts upon meaning of the message. Pacioli gives several examples.

Starting with substitution ciphers, these substitute individual letters for other letters, digits or symbols. Examples of these are 'Bartus – Felipo' and, 'p-lines – vowels' substitutions (see Figure 123). In the first all instances of the letters *B*, *A*, *R, T, U* and *S* are replaced by *F, E, L, I, P* and *O* respectively. The encryption of "Barnabeu" would be "Felnefao". The receiving end needs to do the same substitution to read the message.

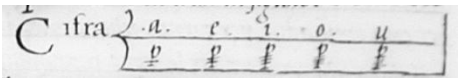

Figure 123: 'p'-cipher, F.241r, III.iii.12

The principle of the second substitution is the same. The vowels are replaced with p's with lines according to the Figure above. Other substitutions are mentioned by Pacioli, musical notes, or, numeric substitutions, make each letter correspond to a musical note (a – do, b – re, c – mi, …), or, number (a – 1, b – 2, …). Illustration of the musical substitution is missing albeit being mentioned.

Word–Letter and *Symbol–Word* substitutions are also mentioned. Pacioli exemplifies, respectively: instead of using the letter *b* one is to use "*coltello"* (knife), and, instead of writing "*franco*"(free) one might use the letter *e*.

Pacioli also mentions some codes: changing specific words for other words, or, words for symbols. This is exemplified if "*carne*" (meat) or Δ stand for, respectively, the kings of Naples or France. Speaking in jargon like *calmone*, is yet another way to conceal meaning exemplified by Pacioli.

Pacioli notes that ciphers like those mentioned can be broken with due time as he has done together with those of the Cardinal of Capua[411], first known as Perusino, nicknamed Lopis in lower orders. Messer Lorenzo Giustini da Castello[412] is also mentioned. He is to have written a twenty page booklet with several ciphers such as the Aragonese, Venetian, Florentine and papal ciphers.

The dactylonomy from (I.30) is an example of signaled code.

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[408] *Ibid.* F.240r. Pacioli uses *aqua*, water, generally referring a liquid, like in (III.iii.6.) when speaking of "*con aqua del verme chi di nocte luce*".
[409] *Ibid*.
[410] For a comprehensive history on this topic see for instance Singh, Simon (1999). *The Code Book: The Science of Secrecy from Ancient Egypt to Quantum Cryptography*, New York by Doubleday.
[411] Most likely Juan López (~1455 – 1501) as he held the post of Archbishop of Capua from 1498 to 1501 prior having been bishop in Perugia.
[412] Lorenzo Giustini (1430 – 1487)

Recipes For Inks (III.iii.13 – III.iii.22)

From here on several quick recipes are listed, many are to cryptic or use unclear compounds to completely comprehend or explain them.

13. Chapter Thirteen Document: To make good ink, if you follow this.[413]

Take ¼ Roman vitriol, ½n[414] Arabic gum, 1 Perugian Gall and 8 of falernian wine to make good ink. If the quantities, 1, 2, 4 and 8 are used instead double proportion is yielded. An alternative recipe uses 1, 2, 3, and, 30 as quantities of the above ingredients.

The chemical process here is similar to that in (III.iii.6).

14. Chapter Fourteen Document: To make very strong glass glue.[415]

To produce glass glue, add equal proportions of pulverized quicklime and mastic to the varnish mentioned in (III.iii.7).

15. Chapter Fifteen Document: To Remove the oil of a stained book.[416]

To remove oil from a a book spread powdered ashes on the stain and compressed the book. Keep changing the ashes.

The oil gets "sucked up" by the ashes, similar to how oil stains can be removed in clothing with flower, or, cornstarch.

16. Chapter Sixteen Document: To make Purple [ink].[417]

2/3 n melted tin, 1/3 n mercury, 2 halves of pulverized sulfur, 2/3n powdered ammonia salts are to be mixed in a heated flask until no more vapors come out. The cooled liquid is mixed with egg-white.

Alternatively, a *dragmam* crystal and orpiment, in equal amounts, are mixed with 1n of sulfur and ground on marble, then brought to a boil until golden foam rises. The mix is then to be diluted in cold water with tragacanth gum.

Both, the tragacanth and egg-white, where likely used as thickening substance for the ink.

It isn't clear what exactly happens, but it was usual to obtain purple by blending blue and red colored substances, such as Lapis lazuli and cinnabar. Cinnabar is obtained through mercury salts and ammonium polysulfide.

17. Chapter 17 Document: Dye bones and hair and wood.[418]

5 lb of water, 1lb of powdered litharge (PbO) and 1 to 2 pounds of oak ash are to be boiled down up to 1/3 or 1/5.

PbO is red. Any resulting paste used for coloring should have this color as well.

18. Chapter Eighteen [Document]: To make fragrant birds [incense] of cypress.[419]

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[413] *DVQ* F. 242r.
[414] Pieper notes that the ounce in this book is the troy ounce, which will be abbreviated like Pacioli does elsewhere by *n*, 12n stand to 1lb.
[415] *DVQ* F. 242r.
[416] *Ibid*.
[417] *Ibid* F.242v.
[418] *Ibid.* Here the numeral is written in roman numerals like in some upcoming sections.

Crush 3 parts benzoin cypress, 1 part storax, ½ part charcoaled willow. Mix this into rose water, or, distilled rose leaves. Mix in a pinch of fine riverbed sand, ½ part aloe wood, and, rose-water-soaked tragacanth, until a paste forms. Shape it into a candle. This candle can then be lit once dried like incense.

For a simpler ambient perfumer, use light bark of the *pino paradise*, sandalwood, moss, incense, cinnamon, or, cloves. They are best left to dry in the shade.

19. Chapter Nineteen Document: A paste for impression of any figure.[420]

A paste of 1/3 Arabic gum, 1/3 *gersa*[421], 1/3 painters gypsum is mixed. The paste can be kept moist by wrapping it in cabbage leaves, or, lambskin. The can be used to form statuettes. Colors and perfumes can be incorporated for color and smell, respectively.

Alternatively, for a paste that sticks to stone and metal, mix 2/3 resin pitch, 1/3 wax and a bit of oil. Boil the mixture and place it inside a mold, on top of the desired spot.

Both, pastes will become rigid once dried or cooled down.

20. Chapter Twenty Document: To make earth or other powder for impressing.[422]

2/3 Blacksmith residue, or, soot, and 1/3 well crushed pumice are mixed. They should be shaped dabbling clear urine on it. Pacioli tells the reader to throw the mixture.

Given the paste there is no specific need for urine, and the shaping could be done with any liquid. This might be a prank.

21. Chapter Twenty-one Document: To make good smelling garments and cloth.[423]

Several plants like Sage, Mint, or, other herbs[424] are suggested to be left scattered with the clothing. This gives the clothes a nice smell.

22. Chapter Twenty-two Document: To make milk of eggshells for beautiful skin.[425]

Skinless eggshells, or, other shells[426], are to be dissolved in lemon juice over several days, the resulting tincture is to be applied to the skin.

This recipe results in a white coloring tincture due to its calcium carbonate. The acid solution solves the solid particles better than water.

The skin would be covered in a white base, likely fashionable and associated to health.

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[419] *Ibid*.
[420] *Ibid*. F.243r.
[421] Pieper notes this to be make up similar to face powder made of rice or starch
[422] *DVQ*. F.243v.
[423] *Ibid*.
[424] Pieper identifies Wintercress, southernwood, wormwood in her translation.
[425] *DVQ*. F.243v.
[426] Pieper translates to cowrie and indian shells.

Apparent Miracles (III.iii.23 – III.iii.34)

Leidenfrost Effect

23. Chapter Twenty-three Document: Washing the hands in melted lead.[427]

This section is the description of a stunt. The performer dips his hands into boiling lead and comes away unharmed.[428]

To do this, soak the hands in fresh well water prior, leaving them there for a while. The effect works even better if some alum is diluted in the water. Pacioli repeatedly reassures the reader that this is no prank.

If the heat is high enough, the surface water on the wet hands evaporates so quickly that the water-gas bubble 'shields' the hands from scolding. This physical effect is commonly known as Leidenfrost effect (see Figure 124).

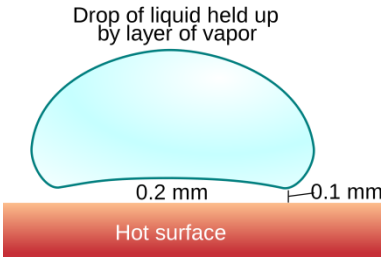

Figure 124: Droplet of water on hot surface. From Wikipedia by Vystrix Nexoth

Violent Chemical Reactions

24. Chapter Twenty-four [Document]: To make fire light itself in water.[429]

Pacioli explains how to produce a flaming reaction in water.

Fill an eggshell with saltpeter, quicklime, quick sulfur, and, ammoniac salts, in equal parts.

The saltpeter (HNO_3) is an acid that likely corrodes the eggshell over time. This grants the contact of the other substances with the water, staying afloat. Once the sulfur and quicklime get in contact with water they start a strong exothermic reaction. This in turn can produce a flame when coupled with flammable materials. The ammoniac works both as yet another acid and grants color to the flame. This is a possible recipe for Greek fire.

Combustion Color

25. Chapter Twenty-five Document: To make men appear as the dead in light.[430]

The effect described is to create a light which makes people appear to be pale.

For this, use brandy-impregnated tow as torch. Alternatively salted white wine is set to a boil on embers until it lights.

Brandy burns blue, salt blue green, while white wine should have a neutral color. In either case this should light the surroundings in a spooky dim light.

26. Chapter Twenty-six [Document]: To quickly make fire signals by hand.[431]

The hands are covered in rosin, or, varnish. Then, with a burning candle in hand, the signals are made without risk of scalding. Pacioli tells of military use of this protection.

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[427] *DVQ*. F.243v.
[428] See Mythbusters Season 7, Episode 23, for an implementation of this stunt.
[429] *DVQ* F.244v.
[430] *Ibid.*
[431] *Ibid.*

The cover serves as an isolating substance.

27. Chapter Twenty-seven Document: To make a medicine that whatever is put in it burns.[432]

Sulfur, *tartaro sarco* (some sort of skin calcification perhaps), glue, cooking salt, petrol, and, common oil are set to a boil. Anything placed inside this mixture catches flame. The mixture is said to be extinguishable only by urine or vinegar.

The sulfur is likely only a constituent here because it reacts easily. Urine and vinegar possibly extinguish the flame because they react with the sulfur, creating hydrogen sulfide (H_2S) which might. The hydrogen sulfide could choke the flame.

Endothermic Reaction

28. Chapter Twenty-eight Document: Boil an egg in a well without fire.[433]

The effect is achieved by filling a canister with quicklime and placing an egg in its midst, lowering the canister into a well, or, other body of water.

As seen above the quicklime has an endothermic reaction when in contact with water. The heat released is enough to boil an egg. Modernly, this is commonly used for meals and drinks.

29. Chapter Twenty-nine Document: Burn a rock in water.[434]

Camphor is set ablaze in water as effect to amaze the onlooker.

Pacioli, in similitude to (II.115), suggests lowering the burning rock-like terpenoid into the well before calling any spectators.

Camphor is easily flammable. It could also be brought to light in 74°C or hotter water.

Heat Transfer

30. Chapter Thirty: Document: thread thrown into fire doesn't burn.

A thread is wrapped around an egg. Both are placed into an open flame. Pacioli claims that the thread won't burn until the egg is cooked. This is amazes the fools.

31. Chapter Thirty-one Document: Cooking eggs, fish, meat in a paper pan.[435]

Pacioli describes how paper can be used as a frying pan.

The paper is to be folded and closed off with pins or glue, so that it can be used as a pan. Fill it with oil. The food products are carefully placed into that oil. The pan is placed on top of a metal grid. Pacioli recommends careful usage to keep the paper from rupturing.

This is similar to a popular science experience. A balloon with a little water at the bottom is put over a burning flame. The balloon does not pop. The heat is transferred through the rubber to the water, leaving the rubber intact.

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[432] *Ibid.*
[433] *DVQ* F.244v.
[434] *Ibid*.
[435] *Ibid.*

32. Chapter Thirty-two Document: To fill a vessel with [solid] matter and the same water.[436]

Filling a watertight vessel full of ash with a same volume of water is presented as a great marvel.

Density

33. Chapter [Thirty-three] Document: To fill a vessel of water and then add silver.[437]

Pacioli observes how 1/3 the amount of gold, compared to silver, makes a container, filled to the brim, pill over. This is, having established that even though the container is filled to the brim some coins of silver can be added without it spilling.

According to Pacioli this happens due to the porosity of the two metals. Pacioli tells the story of Archimedes' experiment from the *Floating Bodies*. The fraud of a silver alloyed crown is discovered and the weight of gold in it calculated, using the above observation. The weight of gold is obtained through cross-multiplication.

The observation above is known as Archimedes' Principle.

34. Chapter Thirty-four Document: One drinks from a deep Well by ingenuity.[438]

Archimedes is called upon again. This time, the Greek uses stones to make a well spill over and thus produce water for a thirsty company.

Pacioli tells of birds that have been seen drinking when it would otherwise not have possible. This is, by bringing stones to flood the water in a cup. Pacioli reflects on the benefits of mimetism of the natural world. He uses another example of this. In Ambrose's Hexameron a stork drinks saltwater to purge itself. This is the idea appropriated by doctors for enemata.

Animal rites (III.iii.35 – III.iii.37)

35. Chapter Thirty-Five: Document: To make it that ants don't go a certain place.[439]

Pacioli describes how to keep ants away.

The insects won't cross charcoal, like that of a willow. Pacioli warns of charlatans. The charlatans might fool the idiot by saying incantations, keeping the ant circled inside a ring made of charcoal, or, on the other side of a line, making it look like a miracle.

36. Chapter Thirty-Six: Document: Of slicing the head of a pigeon by knife and it doesn't die, or a chicken or other bird.[440]

Pacioli describes how a dove or other bird gets stabbed in the head and afterwards is still able to eat saliva moistened bread, which it is fed.

The instruction is to pierce the knife in lengthwise through the neck. Either this is to be done underneath the beak, or so that both brain halves of the bird are unharmed.

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[436] *DVQ* F.245r.
[437] *Ibid*.
[438] *DVQ* F.245v.
[439] *DVQ* F.246v.
[440] *DVQ* F.247r.

The moisture of the bread closes the immediate wounds. Again charlatans might make it appear as if they have mystic powers.

Pacioli focuses the precision of the cut to keep the bird alive

While there are reports of decapitated poultry surviving an abnormal time span, and it is not uncommon for the body still to act without the head, most birds would likely die of the wounds. It is reasonable to assume that the birds do not die immediately; they would likely do so due to infections, trauma, or, blood loss over a short time.

37. Chapter Thirty-seven Document: Killing a pigeon hitting it on its head with a feather.[441]

A Pidgeon gets tapped on the head with a feather and dies.

The effect can be produced by crushing the bird so that its heart explodes. This can be done in absence of onlookers, while picking up the bird. The hit on the head is nothing more than an act. Again, various charlatans (*Camufatori*), use this to fool the unknowing and unaware.

Glue

38. Chapter Thirty-Eight Document Attaching the cup or bowl to lips.[442]

Pacioli gives instruction on how to produce a glue for tricks and pranks

To produce the glue make a tincture of fig milk and Arabic gum, one ounce each and let it set for a night.

Smearing the rims of glasses with this ointment will make it stick to the lips. To solve the glue use vinegar soaked bread.

Producing a Mirror

39. Chapter 39: Document: to make a mirror of burnished Steel.[443]

This section discusses the production of a Mirror

Melt and mix 1 lb brass copper, 2 lb fine tin, a bit of marcasite and 2/1 of something which is abbreviated "*iiij.*" ("mj.", "iuj." or similar). The melting process can be repeated to get rid of impurities. The hot mixture is to be poured into a mold on a stone slab and left to cool. The cold slab should be sanded.

Again, like in (II.92), Euclid's and Archimedes' works of concave mirrors are mentioned. They are said to have burned down Marcus Marcellus' ships at Syracuse. Pieper refers that this might be found instead in Eutochius. In both of the sections Pacioli mentions the *De speculis comburentibus.* Elsewhere reference to Gerardus of Cremona can be found.[444]

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[441] *DVQ* F.247v.
[442] *Ibid*.
[443] *DVQ* F. 248r.
[444] This work is available in the infothek Alcuin of the University of http://www-app.uni-regensburg.de/Fakultaeten/PKGG/Philosophie/Gesch_Phil/alcuin/work.php?id=21286 .

40. Chapter Forty Document: To make an egg walk over a table.[445]

An egg moves untouched over a table.

The trick lies in filling a hallowed egg with leeches and sealing the opening with wax. By making splashing noises the leaches get attracted to the water it and move in that direction.

Magnetism

41. Chapter Forty-One Document: Make a coin rise and fall inside a glass.[446]

A coin rises and falls inside a cup.

The coin and a thumb should previously be covered in magnetic dust. Having done this, place the coin inside a cup of vinegar.[447] The coin is then made to rise and fall inside the glass by the motion of the hand.

42. Chapter Forty-Two Document: Of one who has an egg go up a lance.[448]

An egg moves up a lance.

Pacioli describes how to hollow out an egg. Next one should fill the egg with morning dew and seal it with wax. This done, place it at the bottom of a lance, so that it is angled with its tip towards the sun. The egg is said to rise as the sun shines onto it.

It is a mystery how this could work. The effect seems to be missing some element. It might also be the case that it is some special kind of lance. It is possible that the egg would stick to the lance due to magnetic powder, or, using the dew on the lance. Assuming that the egg is only partially filled with dew, the water steam might make a difference and make the egg rise. Further the curvature of the lance might play a role in the rising of the egg.

43. Chapter Forty-Three Document: To make a cooked chicken jump on the table.[449]

A cooked chicken is brought to move.

Secretly place a flask filled with quicksilver and magnetic powder within the chicken. As the flask heats up it will move violently and animate the chicken.

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[445] *DVQ* F.248r.
[446] *DVQ* F.248v.
[447] It isn't clear if the cup is filled with vinegar or if the coin is coated in vinegar and then placed in a cup filled with water. In the second case, an oily coating seems more plausible, perhaps olive oil.
[448] *DVQ* F.248v.
[449] *DVQ* F.249r.

Threads

44. Chapter Forty-Four Document: To make a coin dance inside a cup.[450]

A coin begins to move inside a cup in reaction to the movements of a person.

The tip of a long hair should discretely be glued the coin. This can be done with wax, or glue. The other end is kept handy. The coin is dropped somewhere. Moving the hair sets the coin to move as well.

Pacioli suggest doing this in a dark environment. He describes a scene where a glass is placed on the ground amongst people. The performer having fastened the hair to his shoes brings it to move by jumping around.

45. Chapter [Forty-Six] Document: Not to be able to blow a bit of coal out of a circle.[451]

A bit of coal is placed inside a circle. A participant is challenged to blow it out of the circle. It proves to be impossible.

Like in the previous section the coal has been tied to a string of hair. It is secured in the middle of the circle. Again, this should be performed in a dark environment.

Feats with Fire

46. Chapter Forty-Six Document: Eating tow and spitting fire.[452]

Pacioli describes how a piece of cotton or tow is chewed and secretly replaced by a burning counterpart. The performer spits out a burning cotton ball. Alternatively, keeping the burning ball in his mouth he is able to blow embers and smoke.

It is a common trick, among fire eaters, to light cotton balls and use them in various ways. The moisture of the mouth and breathing techniques can keep the heat from burning the mouth.

Chapter [Forty-Seven] Document: Lighting a wax torch in hand, so it is not seen. [453]

A candle gets relit miraculously after it had been extinguished through rotation.

The performer should have a cone of paper hidden in the palm of his hand.

This can be coupled with the physical effect of relighting a candle through its fumes.[454]

47. Chapter Forty-Seven Document: Placing a burning torch into the mouth without harm.[455]

This is another fire eating feat.

Pacioli plain and simply instructs the reader to try to eat a torch adequately sized for his mouth. As the palate is wet no harm shall come to the fire eater if he closes his mouth.

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[450] *Ibid*.
[451] *DVQ* F.249v.
[452] *DVQ* F.250r.
[453] *DVQ* F.250v.
[454] For a slow motion gif and physical explanation visit http://www.itsokaytobesmart.com/post/34760974156/light-candle-using-smoke .
[455] *DVQ* F.250v.

48. Chapter Forty-Eight Document: Make a snow torch that burns.[456]

Snow is apparently lit on fire.

The effect is produced by secretly incorporating a waxed, or alternatively brandy-soaked, piece of paper, or wick, in the snow. It is also possible to simply swap the snow for a cotton ball.

In the index this is the only section regarding fire effects. The following two effects are not mentioned in the index.

49. Chapter Forty-Nine Document: To make a wick that never wears out for the lantern.[457]

To make a wick that does not burn out, use a flaky talc piece instead of the wick.

Also here the construction of making a candle float in water from (II.115) is repeated and an apparently misplaced bit of text belongs to the following effect.

50. Chapter Fifty Document: To make the cross turn in water.[458]

An oat spikelet, which is all dried up and has twisted itself like a screw, is secured vertically to a light straw cross. This can be done with wax. Fill a cup to the bottom of which the spikelet is secured. The floating cross is set to spin.

Another variant uses the spikelet in such manner that two coins are fastened on each end, when the spikelet gets moistened the coins start moving. Here the spikelet serves as axis. Pacioli tells of gypsies who act as enchanters resorting to this effect.

The above misplaced section uses a spikelet to produce motion of several figurines that have been secured with wax on to top of a box. It is here that Pacioli mentions the secret moistening of the spikelet to produce the movements described above.

Tricks with eggs

51. Chapter Fifty-One Document: To make an egg stay behind the ear.[459]

To stick an egg behind the ear, use a hollowed out egg. Wet the egg it with saliva and it will stick behind the ear.

52. Chapter Fifty-Two Document: To make an egg stand up straight without anything else.[460]

How to stand an egg on its tip? Many solutions to this puzzle can be found. For instance place the egg in a bed of salt. Or just balance it skillfully. [461]

The solution, Pacioli gives, is to strike it skillfully tip first so that the end is smashed in enough to support it, while going unoticed. Pacioli credits it to Florentine architect Brunelleschi.[462] This challenge is known as Columbus Egg, and the discoverer is usually credited for coming up with it (see Figure 125).

Figure 125: Columbus Breaking the Egg' (Christopher Columbus), by William Hogarth, from wikimedia commons an original can be found in National Portrait Gallery, London

[456] *Ibid*.
[457] *DVQ* F.251r.
[458] *Ibid*.
[459] *DVQ* F.251v.
[460] *Ibid*.
[461] See for instance http://www.wikihow.com/Balance-an-Egg .
[462] Likely Fillipo Brunelschi (1377 – 1446). Pieper calls to attention that Vasari, too, tells of this story. This can be read in Martin Roberts Longman (1994), *Italian Renaissance,* Addison-Wesley Longman, Limited .

53. Chapter Fifty-Three Document: To float an egg on the water surface [of a] full [bucket].[463]

The same egg is put into two different buckets filled with clear water, in one it floats in the other it sinks.

One of the buckets is filled with salted water or alum in it. The explanation Pacioli gives relates the phenomenon to the 'viscosity' of the two liquids.

54. Chapter Fifty-Four Document: Finding the size of a bell.[464]

Pacioli gives instructions on how to measure different circles of a bell with the use of a pair of tongs and a string.

Although Pacioli does not mention proportions, the section ends with an implicit reference to π, as the perimeter by the diameter is constant "*et cosi in tutte*" (in all things).

55. Chapter 55 Document: Untying knotted hair with your closed fist. [465]

A multiply tied thin hair is placed in the crease of the hand, moistened with a bit of spit. By rubbing it and beating the closed fist onto the leg it is said to untie.

56. Chapter 56 Document: Removing the water of a watered wine, a fair thing.[466]

A cloth is placed between a filled and an empty wineglass to soak the wine. If the wine is watered the water passes through the cloth and fills the empty glass over time. This is said to be a common practice to check for watered wine.

Note that regardless of how watered down the wine is a part of the water will always drip through the cloth.

57. Chapter 57 Document: Removing the water underneath oil in a jar.[467]

To remove the water underneath a layer of oil on the surface, use a sponge previously coated in wax. Drive the sponge down into the water underneath the layer of oil, then break the coating with a stick so that the sponge can soak up the water. As long as the sponge is already filled with water the sponge shall not soak up the oil.

Vacuum Experiences

58. Chapter 58 Document: Placing a shelled egg into a bottle.[468]

An egg is placed on top of a bottle and gets sucked in.

Prior to placing the egg on top of the bottle a match is thrown lit inside, or, the bottle is held neck down over a fire. Pacioli suggests using a devilled egg.

The fire will consume the air in the bottle and create a difference of pressure. This difference of pressure is enough for the egg to be sucked in. The egg soaked in vinegar will become rubberlike so it doesn't squash as it is sucked into the bottle.

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[463] *DVQ* F.252r.
[464] *Ibid*.
[465] *DVQ* F.252v. The next eight sections are numbered in Hindu-Arabic numerals.
[466] *Ibid*.
[467] *DVQ* F.253r.
[468] *Ibid*.

59. Chapter 59 Document: Filling a bottle that sucks itself.[469]

This is the description of a bottle that fills itself.

The bottle is to be heated over fire, like in the previous section and then the mouth of it is to be placed on a bucket of water.

This is the vacuum-like phenomenon mentioned.

60. Chapter 60 Document: Breaking a bottle leaving the wine hanging.[470]

It isn't clear if a skin prevents the wine from falling out or if some sort of jelly is made to produce the effect. In either case tragacantha gum or pure caravelle glue is used in mixture with wine.

61. Chapter 61 Document: Putting a grape or peach into a small bottle.[471]

A whole piece of fruit is placed inside a bottle.

The bottle, preferably a round one with short neck, is to be directly placed onto the growing fruit to produce the impossible object.

62. Chapter 62 Document: Preserve fruits and grapes fresh for a year.[472]

To preserve the fruits place them in virgin honey. This produces amazement among those who do not know this method of conservation. Similarly olive oil can be used.

63. Chapter Sixty-three Document: Make worms appear on cooked meat.[473]

Harp or lute strings are cut up into long lengths and then sprinkled onto hot meat. The heat makes them move and cringle up.

Pacioli suggest eating the meat to causes further commotion among onlookers.

64. Chapter Sixty-four Document: Making one or more knifes jump out of a pot.[474]

Pacioli proposes a trick in which one or several knifes are placed inside a pot and then jump out of it.

The highest jumping knife is called "*grillo*" (cricket). It is unclear if this is just a trick or if it is some sort of bet, where several people give knives and the one that jumps the highest pays for dinner. In either case several knives are placed in the pot.

The trick is achieved by placing a longer blade bent inside the pot as spring. The pot should be filled with saltwater, this either serves as a diversion, or the water causes some sort of reaction which brings the metal to spring.

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[469] *DVQ* F.253v.
[470] *Ibid*.
[471] *Ibid*.
[472] *DVQ* F.254r.
[473] *Ibid*.
[474] *DVQ* F.254r.

65. Chapter Sixty-five Document: Removing a cup from another without touching it.[475]

A cup is placed inside another cup. The challenge is to move the first cup into another one without touching any of the cups.

This is achieved by blowing into the first cup. It is likely that the cup to be blown out is smaller than the rest. This trick is similar to a bar bet where an egg is made to jump between cups.[476]

66. Chapter Sixty-six Document: Cut around a glass like a screw.[477]

Pacioli describes the process of cutting a glass into two pieces, like a screw and the effect of pulling them apart. This changes the apparent size of the cup.

The technique used to snap cut the glass apart, using a hot iron rod and wet cold knife. After the cut has been made, pull from bottom and mouth in opposite directions. The winding or unwinding the halves is said to produce a 'beautiful' effect.

Surface Tension

67. Chapter 67 Document: Floating a coin, have it sink and then make it return to surface, without touching it.[478]

Pacioli describes how to float a coin on a water surface. With the aid of a glass it can be pushed down and then brought back up.

The coin floats due to water's surface tension (see Figure 126). The same can be done with paperclips.

Figure 126: Floating Yen coin. From Wikimedia commons, picture by Eclipse2009

68. Chapter Sixty-eight Document: To make an egg walk over the table.[479]

Pacioli explains another way to make an egg move.

This time the egg is moved like the coin from (III.iii.44.) The egg is to be hollowed out and sealed with wax with long hair is attached to it.

69. Missing

This chapter is missing. The index makes note to an effect of moving figurines which seems similar to the one of this previous section. At first sight it might look as if the section of (III.68.) present on F.255 might not be the counter part to the ending on F.256. The second part, however, clearly makes reference to the shell of an egg and the other effects which could include an egg, such as III.70, are accounted for. This makes it likely that a mistake was made with the indexation.

70. Chapter Seventy Document: Pick up an egg of end of a long rug.[480]

An egg is placed on one end of a rug. A participant, standing on the other end, is challenged to pick it up without walking over.

The performer solves the challenge by retracting the rug.

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[475] *DVQ* F.255r.
[476] See for example http://eggs.ab.ca/kids-stuff/leaping-egg or http://spoonful.com/family-fun/make-ping-pong-ball-jump
[477] *DVQ* F.255r.
[478] *DVQ* F.255v.
[479] *Ibid*.
[480] *DVQ* F.256r.

71. Chapter Seventy-one Document: Making wine stay on top of water.[481]

A sheet of paper or thinly sliced bread is used to keep the wine and water separate.

Pacioli suggests this effect to be performed at a distance so that onlookers don't see through the sheet.

72. Chapter Seventy-two Document: To make water stay atop of wine, a beautiful thing.[482]

This is a continuation of the previous section. One is to take a pitcher or the likes and flip the glass so that the wine is now on the bottom.

Illusions

73. Chapter Seventy-three Document: Fooling one's sight, deceiving him.[483]

An optical Illusion is described by Pacioli.

Given two equally long straws one form a T or ⊥ (see Figure 127). Most people, Pacioli explicitly mentions 9 out of 10, will say that the vertical bar is longer. Pacioli says that the explanation of why this occurs is a hard one.

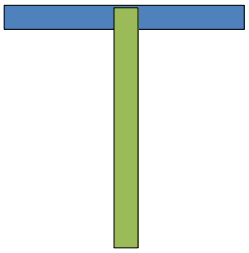

Figure 127: The rectangles are congruent to each other.

Bossi relates this description to Leonardo's anamorfosis studies and the study of artists and sculptures of the times.

74. Chapter Seventy-Four Document: Fooling one's sense of touch, making one seem two.[484]

A tactile illusion is described next.

Ask a participant to cross his middle and index finger, so that the middle finger is on top. Place a ball in between the fingers. Not seeing his finger it will appear to him as if there are two balls touching his fingers.

Bossi tracks this illusion back to classical Greece as *Aristotele's Ilusion*. An illustration, from the *De Homine* by *Descartes* (see Figure 128), shows this principle.

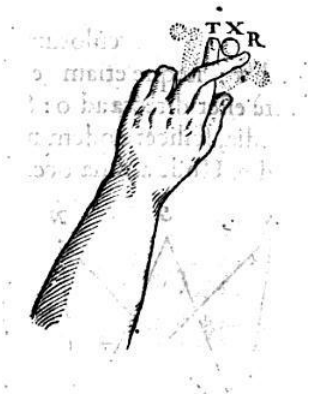

Figure 128: Fig. 27, pg.62 of *De Homine* by R. des Cartes

Pulleys

75. Chapter Seventy-Five Document: Pulling a weight alone which 10 wouldn't manage.[485]

Pacioli describes the use of pulleys to lift a weight that several men wouldn't manage to lift.

He establishes a relation between the pulleys and the men needed to lift the same object. If for 1 pulley 10 men are needed, 3 pulleys reduce the number to 2, and if 4 pulleys are used this comes down to 1. The more pulleys that are added make the easier lifting becomes. This is an open problem as Pacioli ends with "tantum causa non probata est" ("the cause of which has not been proven"). A side note adds *physicalis et nominalis*.

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[481] *Ibid*.
[482] *DVQ* F.256v.
[483] *Ibid*.
[484] *DVQ* F.257r.
[485] *DVQ* F.257v.

76. Chapter Seventy-Six Document: Make a coin appear better in water. [486]

Pacioli says that a silver coin looks better in water. The explanation of which, is again an open problem sought to solve by the natural philosophers.

76.A Chapter Seventy-Seven Document: Make its reciprocal.[487]

No further text is added to the given title. This might be a simple reminder that the inverse of the above is also true. The coin becomes less visible if water is taken away.

Both phenomena can be related to optics.

77. Chapter Seventy-Seven Document: To make parsley germinate within an hour.[488]

Pacioli describes a phenomenon in which parsley seeds germinate as they come into contact with heat.

To achieve this, the seeds are to be soaked for 10 days in wine or brandy. Pacioli suggests using dragon's blood and goat's blood (two wines according to Pieper). After this has been done spread the seeds on top of a hot piece of bread. Alternatively, they can be placed on top of fine earth underneath of which quicklime has been hidden, or sprinkled over meat left in the oven to be kept warm.

Some more Recipes

78. Chapter Seventy-eight Document: Artificially make blue without much expense. [489]

A recipe is given to produce a blue pigment.

Burned marble pieces are soaked in horse dung for a day, ground and soaked in *spuma dei tentore* (literally dyers' foam, the exact compound is unclear). This is then crushed and incorporated several times to produce a beautiful blue.

Halloween Pumpkin

79. Chapter Seventy-nine Document: To make a brute head appear at night. [490]

The carving of a gourd lit from the inside with a candle is suggested to scare people from afar.

80. Chapter Eighty Document: To make gauzy paper or what seems paper. [491]

Animal glue or fish glue, or both in equal quantity, are applied to a thin sheet of tin-plated iron. After cooling remove them carefully to produce gauzy paper.

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[486] *Ibid.*
[487] *Ibid.*
[488] *DVQ* F.258r.
[489] *Ibid.*
[490] *DVQ* F.258v.
[491] *Ibid.*

Carbon Copying

81. Chapter Eighty-one Document: To write and counterfeit every letter.[492]

Pacioli describes how to make a carbon copy.

The process is done by taking a white sheet of paper, and on top of it, making several layers with a solution of water and carbon powder. The carbon coated side of the paper is laid onto a white sheet, the copy. The original is laid on top of the other side of the carbon coated sheet, making a three sheet stack. Next, trace the contents of the original with a blunt pen. If desired the carbon copy can then be reaffirmed with ink.

To know how to retrace any leaf, especially those with nerves.

An analogous process is used to copy a leaf. One of the sides of the leaf is covered lightly with ground charcoal or soot black, as used for printing, and then used as a stamp. Pacioli goes into detail how to make the leaf look lifelike.

Gunpowder

82. Chapter Eighty-two Document: making Lombard powder, as fine as any.[493]

Pacioli gives a recipe of how to make gunpowder.

He starts by describing the difficulty, given the secrecy surrounding the making of gunpowder, to find a good recipe. He then proceeds to explain how one can deduce the components of the gunpowder.

The idea is to probe for the ingredients. This is done in a quantitative aspect, as Pacioli assumes the components to be known. First, solve the whole powder in water and carefully extract the undissolved sulfur and coal to weight them. After having mixed them in again, extract the sulfur only and weight it. As a result the saltpeter can be inferred, given the total weight of the sample.

The section ends with Pacioli's dosing for the powder. He uses 1/2 willow charcoal 1/3 salpeter, 1/6 sufur.

83. Chapter Eighty-three Document: to make very thin *verzino.*[494]

The recipe for ruby red ink is given.

After shaving the resin from the sappan wood or brazilwood and thinly slicing it; dissolve it in strong white vinegar together with a bit of the bark of the tree and a sprinkle of rock alum. Then, carefully boil it with rock alum and leave it to soak one more day in vinegar. Boil it once more. Finally, seal it once it cooled down. Similar to the previous inks the thin filaments of the tree can also be incorporated with egg-white.

The section ends with two riddles likely intended for the next part.

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[492] *DVQ* F.259r.
[493] *DVQ* F.260r. The Lombard was an early mortar like projectile weapon; this is taken broadly as gunpowder.
[494] *DVQ* F.260v. Verzino is a red ink. For more details you can visit http://lem.ch.unito.it/didattica/infochimica/2008_Il_Rosso/HTML/verzino.html .

Brazil wood was a popular agent to produce dyes and was the origin of Brazil's name. The name for the wood likely derived from "*brasa*" (pt., ember) being related to its color. It likely came to Italy only post 1500's colonialization of Brazil, although it is possible that previous excursions had brought some back.[495] Pacioli probably had good relations with seafaring merchants, given his youth, and would be knowledgeable of goods soon after they were introduced into European markets.

This section ends the Natural Miracles.

[495] See for instance Allan, Chris, "Brazilwood: A Brief History", in James Ford Bell Library's - Trade Products section (https://www.lib.umn.edu/bell/tradeproducts/brazilwood)

DE PROBLEMATIBUS ET ENIGMATA

Riddles for the Litrate

iv. Of learned Problems and Enigmas

About 83 riddles (counting variations) are listed for learned people. The condensed way of writing, separation done only by capital letters, and, the absence of numbering, leave room for speculation where one riddle or rime ends, and the other begins.

This sub-part is almost exclusively written in Latin. The Riddles cover several kinds of word games. Below several examples of some of the families of riddles are listed below:

Riddles using homophone or homograph words to produce double meanings, such as,

Si Lupus est agnum, non est mirabile magnum.
aludit est pro comedit agnum.[496]

Charades,

DOminus quis est illi qui oritur sine pelle moritur cantando et non videture ille.
Dicas trullum seie crepitum ventris., and,

SSet avis unica quae animal parit et lacte nutrit queritur qusit
Dicas noctuá seie vesperti[-]lionem pro plinium. [497]

Akrogramma, where one or several letters of a word are disregarded to form another,

Nascitur in nemore nigro vestita colore
si capul abstuleris erit alba nimis
Aludit capul pro prima sillaba, dicitur cornix, nix.

Crasi, where a differente reading of silabes reveal a different meaning,

Comomo lodasti Bergamo, viz. Como, Lodi, Asti, Bergamo
quatuor ciutates Lombardie.[498]

Acronyma, taking the first letters of a word to form another, such as,

DOcet <u>saligia</u> quae sint peccata nociva
Sae septem peccata moralia per 7 Irás habemus dictionis caligia.

The palindrome,

Roma tibi subito notibus ibit amor.[499]

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[496] "*If a wolf is (est) a lamb, it is not greatly to be wondered at. Est alludes to the wolf eating (edo) the lamb.*" Translation Pieper. *DVQ* F.263r .

[497] "*Master, what is it, which is born without skin, dies singing and it is not seen? You say fart or rumbling of the belly.*" and "*It is the only bid which gives birth to an animal and nourishes it with milk; guess what it is. You say the night bird or bat according to Pliny."* Translation Pieper. *DVQ* F.263v.

[498] "*It is born in the woods, dressed in black, if you take off the head, it will be too white. The 'head' alludes to the first syllable, it is said,* cornix *(crow),* nix *(snow).*" And "*How you have praised Bergamo – they are [also] Como, Lodi, Asti, Bergamo – four cities in Lombardy*" and "*Rome, love will immediately go out to you from those who know you*." Translation Pieper. *DVQ* F.262v.

[499] "Saglia *teaches what are deadly sins. That is: seven mortal sins by 7 letters; we have the word* Saligia." Translation Pieper *DVQ* F.264r.

Further kinds of word games are also present.

Word Puzzles

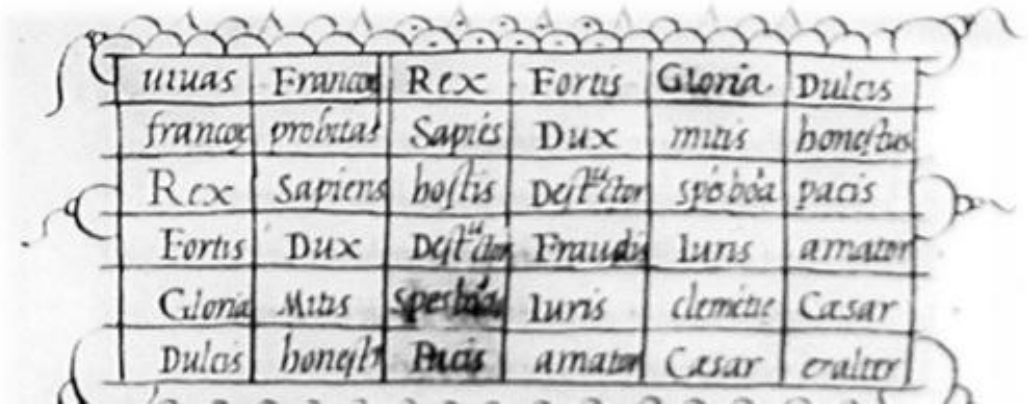

Figure 129: Sentence-Magic-Square, F.264r.

Most notably is the sentence-magic-square midway through the word games (see Figure 129). The same can be read both horizontally (top to bottom) or vertically (left to right). Further combinations with other meanings seem also possible.

Just below, lays a syllable puzzle. The various syllables are joined by lines that form a rhombus grid adorned with various flowers (see Figure 130). The text can be read alternating diagonals (L-ex- ra-pit- …), revealing a poem exalting justice.

Figure 130: Rhombus Grid with Syllables F.264v.

Ten Horses in Nine Stalls

Related to (I.) a conundrum of placing 10 horses in 9 stalls is given. Pacioli gives a trick solution double counting the first horse. The conundrum can also be solved by symbolically filling 9 boxes, the stalls, with the letters spelling T-E-N-H-O-R-S-E-S. This solution would also work in the *volgare* version *1-0-C-A-V-A-L-L-I*, but is not mentioned by Pacioli, albeit it is inferable from (I.47).

A variation of this kind of problem can be found in the next sub-part (III.v.r138). Here it appears as trick question where 3 horses have to be placed in 9 stalls or 3 fishes have to be placed in 9 buckets. The solution mentioned there is to join every 3 stalls into one. In case of the fishes, it is to sell the 9 buckets to buy 3 larger ones. It is also possible that the numbers where swapped by the scribe and, instead of merging, the idea is to split the “cells”.

Proverbs

Towards the end of this sub-part some more small poems as in (III.iii.) give proverbial wisdom. This one specific to gambling:

Die mibi primas quis abstulit tibi vas
Per chris shesu abstulit mihi x et v.
Viz. xu ludendo ad taxillos dicens XV alavanzo.[500]

Genealogical Conundrums

Along the section several genealogic conundrums are posed, like the following one,

"Salve nepos frater"dixit filio suo matter.[501]

This is credited to father Egidio in memoriam. Several other names are also mentioned as source of inspiration like Thomas of Aquinas[502] and Nicolaus of Lira.

Also mentioned by name are the archbishops of Florence, likely St. Antonius (1389 – 1459), and of Milan. The first of these two wrote *De Scandalo* from which Pacioli cites mnemonic verses to keep track during the ember days. The latter one is likely Ippolito d'Este (1479 – 1520), given the date present in the text, 23rd of March 1499, and his term of office.

Riddles and Jokes for the common people

v. Common Problems to solicit the ingenuity and entertain

This last subsection begins with the apology of the author for his less appropriate words or interpretations in the then following jokes and riddles. These are made in cause of jolly amusement and are not to be taken seriously. What might be offensive in one dialect or region might not be in another.

There are around 220 numbered riddles, jokes, trick questions, conundrums, pranks and two poems.[503] A few of these word games are repeated elsewhere in the book, or variations them. Examples of the repetition are (r.29) and (r.218). Those word games most noteworthy that occur elsewhere have been mentioned at the respective location.

It is likely that the indexation of the word games was done subsequently by another scribe. The numbering is faint and done in small Hindu-Arab numbering in the margin.

The riddles are arranged in no noteworthy order, but show conceptual relations at times in regards to subject. For instance, (r.98) to (r.104) have donkeys as theme, (r.104) to (r.106) a blind person. Most of these riddles are "question-answer" structured. Some are meant to make the participant think, others meant to jest. Sexual double-entendres are found often, as questions, having a harmless and plausible answer.

Others are challenges to ingenuity to find a solution given certain restrictions such as in (r.78) in which two people have to exchange an apple without tossing it over or crossing a river. Some examples follow:

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[500] *"Tell me the first things the cup takes from you, by Christ Jesus it takes from me* x *and* u. *That is:* xu*, when playing on dice, meaning* XV *of your savings."* Translation Pieper. *DVQ* F.267r.
[501] *"'Greetings, nephew, brother' says the mother to her son."* Translation Pieper. *DVQ* F.263v.
[502] See Pieper, notes 336,339 pg. 244
[503] r.206 and r.218

Dimme como faresti tu a in segnare a uno cosa chi tu no lui non la sa: Dirai chi mesorarai in sua presentia uno distantia o uer longhezza (...)[504]

A joke,

Dimme perch' se sorbo [bove] el naso con la lengua? Dirrai perch sparramiare el fazoletto. [...][505]

Many of the jokes are directed at some profession, or regional aspect. Two examples of professional jokes, in spirit of the mathematical amusement are,

Dimme tu chi se abachista como farai á cavaré doi de uno senza prestar: Dirai mettere el naso in culo a un cane et tirerolo fuore e cosi aeverai doi buchi de un bucho. , and,

Dimme anchora quanti para fan 3 buoi: Dirrano ch' fanno 3 para de corna al comun detto. Peroch' comunamente se dica lui ha un paio de corna in capo et non senavede viz. Ma dicendo quanti paia son 3 buoi dirai 1 1/2 e pero alle proposte sappi destingue etc.[506]

The first might be meant as an insult for students, as the answer seems rather strong. Independently, the word game depicts the notion of non-positive integers held at the time. The second is a play of words in regarding both a pair, and, reference to a group of cuckolds. The double meaning of having horns and being a cuckold is repeated multiple times see for instance the following joke:

Dimme qual é el piu desgraciado animale chi sia o ver piu infortunato: Dirai el capretto perochi o lui morre giovene o vero douventa beccho.[507]

Other professions included in these riddles are painters (r.146), tailors (157.), humanists (r.164), theologians (r.170), natural philosophers (r.187), and many others. To give yet another:

Dimme confessore como se despera luomo: Dirai montando insu nun pero et lasciarse cascare quello se chiama desperare.[508]

Regards the regional jokes see for instance:

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[504] *"Tell me how you would teach someone something which neither you nor he knows. You will say that you would measure in his presence the distance or legth (…)."* Translation Pieper. r.203, *DVQ* F.289v.

[505] *"Tell me why the ox wipes his nose with his tongue. You will say, in order to spare the handkerchief (…)*" Translation Pieper. r.108, *DVQ* F.281v.

[506] *"Tell me, you who are an expert on the abacus, how will you subtract two from one without borrowing? You will say put your nose in a dog's ass and you will pull it out, and thus you will remove two holes from one hole.*", and,
"Tell me again how many pais 3 oxen make. You will say that they make 3 pairs of horns, in the common expression, because commonly people say ' he has a pair of horns on his head and does not see them' etc. But when you say how many pairs 3 oxen are, you will say 1 ½ , and therefore in the things that are proposed you must know how to distinguish."
Translation Pieper. r.128 and r.129 *DVQ* F.281r.

[507] *"Tell me what is the most unhappy or unfortunate animal there is. You will say the kid goat, because he either dies young or becomes a cuckold."* Translation Pieper. r.204 *DVQ* F.289v .

[508] *"Tell me, confessor, how man despairs. You will say by climbing on a pear tree* (pero) *and letting himself fall. That is called des-pearing* (desesperar)." Translation Pieper r.158 *DVQ* F.283v.

Dimme tu ch' se stato studtante a padua in collegio o uer conuenti, ch' menestra susa a far: Dirai la inatina rane et la sera navoni quod idem est elundi cauli e lalto verze idem: [509]

Other riddles further show the pranking nature hinted at throughout the last part:

Dimme qual é la piu genti lana chi sia: Dirrai la mufa de uno stronzo.[510]

This word game seems to be a prank question to get someone to touch a turd and get his hands dirty. Other examples challenge someone to lick the mold of a turd (r.176), a swineherd is to suck in the liquid inners of feces with a straw to play a "grandiose" prank on his pig (r.177), or, inflating the bladder of a cow inside the genital of the mistress of a bathing woman (r.178).

Even though the less orthodox examples of jokes the greater part are simple riddles. Topics range over a variety of subjects. Roughly categorized by their answers into: plants, such as an elder tree (r.29) or grapevines (r.59); animals, such as crawfish (r.34) or a fox (r.65); objects, of all kinds such as a shovel (r.13), a scale (r.27) or an oven (r.87), entities, human or other, such as a husband (among others r.21) or the fog (r.56).

Many of the riddles or variations of these are still used today. These include: "What goes first on 4, then on 2, and, before it dies on 3 legs" (r.4), "What does everyone have, and, no one goes without?" (r.23), a joke fashioned version of the proverb "If the mountain won't come to Muhammad (…)" (r.98), "What does one have that others use more than himself?" (r.184), "How to split 3 eggs among two sons and two fathers?" (r.191), "What is it that the more a man has of it the more he falls into it" (r.194), and, "What stall has [36] white horses and a red one that kicks them all" (r.209).

The last few numbered effects fall out of style in regards to the reminder of the riddles and the last four paragraphs lack numbering altogether. These last four effects are preceded by a crossed out illegible title. They resemble notes. The first of which is particularly incomplete and the second has a reflecting tone to it.

[509] "*Tell me, you who are a student in Padua at a college or convent, what food are you accustomed to cook? You will say in the morning* rave *(cabbage) and the evening* navoni *(savoy cabbage), quod idem est; and one day cabbage and the other savoy, idem.*" Translation Pieper. r.163 *DVQ* F.284r.

[510] "*Tell me what is the softest wool there is. You will say the mold on a turd.*" Translation Pieper r.175 *DVQ* F.285r.

Concluding Remarks

Genre

The *DVQ* is not a textbook, no was it made to teach, neither mathematics, illusionism, or, any other subject. The way the contents are presented speaks for itself. This can be compared to textbooks of the time and even Pacioli's own works intended for education. This does not mean that the sections don't have an interwoven sequence. Although to speak of continuity might be going a bit too far. Each section can be read on its own. When other knowledge is necessary the section usually makes the reader aware of such. It is a compendium for easy reference of effects, recreations and practical knowledge.

References

No less, when the knowledge can be deepened the author gives reference to other source material. Often these include his, more mathematics education oriented, works. These are namely the *Summa*, the *Divina*, his transcription and translation of the *Elements*. Not always does he specify the exact book. Instead they are referred among other ways as "*Magnus opus"* or "*grande opera"*. This leaves some room for doubt which work is meant exactly. Pacioli also refers to works of other thinkers as sources of inspiration and consultation material. To add to this sometime Pacioli refers works in plural, leaving doubt if multiple-authorship is implied and what the exact title he means.

Purpose

The books main purpose seems to be to share and preserve the amusement Pacioli gathered over the years as he himself claims in his opening letter. Further it seems to substantiate the recreations within the social and cultural context of Pacioli's time. This is evidenced with Pacioli's constant concern to make sure that none of the sections are misinterpreted. Often he leaves notes and justifications to more delicate sections. The number of sections is considerate and of several families of recreations sharing much information valuable to all kind of people.

Tone

The first two parts are clearly set out to expose the reader to the amazing properties and marvels of mathematics. This is done within a Thomist context. This is, the mathematical properties and methods used are in themselves taken as a supernatural entity. They are miraculous in their very own existence. Even god follows the rules of Mathematics. Thus the effects, tricks and games are more than just entertainment. They are the divine powers provided by numbers.

The *DVQ* also has a Pythagorean feel to it. All is number. And both parts are related to numbers. As Pacioli shares in the introduction of the second half, each of the first parts relates to numbers, the first to discrete and the second to continuous numbers. Thus the first part is substantiated mostly with sections related to the discrete mathematics, containing those effects closely related to Arithmetic and the algebra of the time, while the second half is dedicated to the continuum, the line, which contains several geometric teachings.

The last part, on the other hand, stands out. It ruptures from mathematics but keeps the theme of amusements and amazements for the pleasure and growth of the mind as well as the spirit to enlighten the reader. While in the first two parts magic was exalted, because it came as a result of the divine powers of mathematics, in the last part a different look is used regarding magic. The look is one of scrutiny and critical thought. In the mercantile sections great emphasis is given to natural causes and not to be fooled as well as jocosely fooling those who are unlearned people, and who will not understand the 'nature' of what they are presented with. The jokes, proverbs, conundrums and pranks further add to intellectual recreations.

There seems to be a mixture of a renaissance spirit, what isn't known is not yet known – Several natural phenomena Pacioli mentions are still sought by the natural philosophers, as he stresses – or can be explained given it enough thought, with the medieval scholar who above all collects phenomena and gathers curiosities.

Audience

At first glance the fact that the work is written in vernacular might make it seem to target the general reading public, which in itself is limited, however this might not all be the case. Pacioli stresses to keep secrecy through the whole work. The one who knows the ideas should not spoil the fun of others to find out (in case of the puzzles, games and riddles), to produce greater amusement (in case of magic and spectacular presentations), or, to take his amusement from the fact he knows (in case of the pranks, jokes). To understand a great deal of the work the reader needs some schooling and some ideas are assumed known. On the other hand Pacioli stresses many base ideas, especially in the mathematical and scientific parts. Also from the introductory letter the book is likely to be read in cultured circles, likely by one who is then to entertain others, or draw from the book to do so. It works a little as bridge between a scholastic, and, a performing world.

Reception & Propagation

Both, the restricted target audience and, the delicate matters of interpretation of a recreational book, likely played a role hindering the publication of the book and its little propagation. To add to this many of the sections seem to have been added over time not helping the organization of such a work. To add to this Pacioli wanted to print several books at the same time at the same time having a busy and much travelled schedule. These are but a few possible reasons why the book was only put to print with the Garlaschi transcription and has passed unnoticed for a long time.

Readers

Examples of possible readers, or at least people who enjoyed the same subjects the book discusses, are set throughout the work often by name. Some of these are also likely sources of inspiration to the author. The most noteworthy person and repeatedly mentioned is Leonardo. Some sections even find parallels in the polymaths' work, as has been pointed out by Bossi. Another person that finds great relevance is Ludovico Maria Sforza, who is panegyrized often together with Leonardo. Sforza is always mentioned as duke of Milan, title which he ceased to hold around 1500, dying incarcerated eight years later in 1508, this is influential when dating the time of writing of the *DVQ*. Sforza is also commonly linked to occult practices like Numerology, albeit the *DVQ* has few references in that regard (other than the magic squares).

People and Dates

The mention of Duke Cesare Valentino, duke of Romagna in (II.85) sets another date for the writing of sections of the book. Paciolo writes "*Casaro valentine. Duca de Romagna et al present signor de pionbino neli di pasaati captando aun fiume …".* Cesare Borgio (1476 – 1507) also called *il Valentino* proclaimed himself duke of Romagna in 1501. Piombino being under his lordship from 1501-1503 while the siege of the holdings ended in 1502.

Pacioli further writes "*con questi el suo nobile ingegnieri*", who solves the problem of (II.85). Cesare employs Leonardo da Vinci in 1502 as military engineer. Given Pacioli's close relation to Leonardo, this could account for the knowledge of said feat. Cesare's military ventures seem to come to a relevant end at latest in 1504.

Other dates and people are mentioned, although most of these haven't had a big enough historical relevance to easily retrace their footprints, and many have been discussed already. To mention some in order of appearance through the sections: Gonella (I.25); Giovanni de Jasone (I.30) possibly Giovanni de Verrazano (1485–1528); Girolamo Savelli de Siena (I.46); Carlo Sansone (I.47); Catano de Aniballe Catani (I.48.); Benedecto dal Borgo (I.73); Francesco da la Penna, Giovanni de Iasone de Ferrara (I.80); Juan López (~1455 – 1501), Giovanni Scoto (~810 - 877) (II.52), Dorso d'Este (1413 – 1471) (II.79); Cardinal Francesco Soderini (1453 – 1524) and the wife of Piere Soderini (1452-1522) (II.124); Lorenzo Giustini (1430 – 1487); Fillipo Brunelschi (1377 – 1446) (III.iii.52); (III.iii.13.); Nicolaus of Lira, Ippolito d'Este (1479 – 1520) (III.iv.).

The introductory letter makes believe that the *DVQ* stems from the time in Milan and possibly was even started in cooperation with Leonardo. The *DVQ* mentions the 1496 manuscript edition of the *Divina* which adds credibility to this estimative for a beginning. Beyond this, the time and the court diversity gathered by Sforza are likely to have been a good time and place to gather contents of the kind present in the book. Pacioli also mentions that he has refrained from disclosing the secrets in this book, and only does so as he is getting old. This is consistent with the appearance of dates as late as 1509 and the effort shown in getting printing done of his work in Pacioli's 1508 petition.

All these make it plausible that the book was written around 1502, +/- 6 years, Pacioli possibly having started or become motivated by his the stay in Milan and continuously adding sections to the main bulk. The intention to print would explain the script by the hand of an amanuensis instead of Pacioli's own writing which is hard to read. It also explains the difference between content and index as well as the rug-tapestry-like nature of the MS. To add to this, are the many transcription mistakes, and, the style of some of the sections, which closely resemble personal notes. All these hint upon the existence of a collection of separate notes in Pacioli's own script.

Educational Aspects and Influences

The use of people who star as participants and how they are named also sheds some light on the sources Pacioli uses. Evident examples are Antonio, Benedetto, Cristofano and Domentico, from (I.6). They might seem to be common names, but not only are they named in that order the initials A, B, C and D aid the mathematical abstraction and comprehension of the effect they appear in. The play between the abstract and the concrete is generally evident but becomes most evident in the first halves of the first two parts, the more formal sections likely inspired by the class room. In general

they are related to propositions of the elements. Here the recreations serve as a bridge for the educator. No less Pacioli clarifies their recreational use as well. As shown elsewhere many of these sections are drawn from the *Liber Abaci* among other sources. Although it is here that the fragmented course material becomes more than just "recreational relief" of the subject at hand, or, motivator and a rather a subject of its own.

Related also to the rug-tapestry nature of the work and collection of sections are many references to students. Passages as "*Dele quali forze mathematici in infinito se po trebbe procedure*" [511] and the open ending of many of the mathematic sections make this clear. There is left room for exploration and appropriation of knowledgethe sections ending with "*Ideo tu*" or "*etc.*".

Design and Corrections

A great deal of concern is displayed on presentational aspects. These have been secondary to this reading, but will be briefly made mention of. Here too Pacioli shows his pedagogue vein. Often hints and tips on how to present tricks are included, such as how to tell someone to make a multiplication, simplifying it to summing, or in the case of integer division, through successive subtractions, for those who do not know how to multiply or divide, or for those who are weak in calculations; How to be more convincing, and so on. Also a great focus is always given by the author to the practical geometer or architect who he seems to write the instructions for, assuming that he can construct them. These are some aspects to consider the large amount of imagery referred to and present in the work.

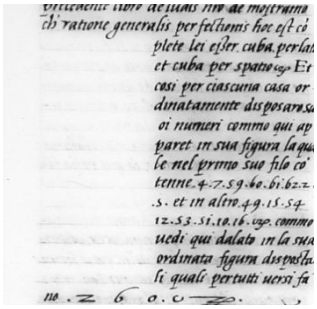

Figure 131: Vacant space for magic squares, F.121v, I.72

The images facilitate reading and understanding many images are said to be included. Most of the illustrations are, however, missing (see Figure 131); those present, are in their greater part geometrical constructions inferable from the text. These can be assumed commonly known among mathematicians, especially mathematicians with some training within the Euclidean work. Some of the descriptions do not match the illustration, like for instance in (II.38.). To this adds that some things are crossed out in favor of others written in the margin. Similarly some faded out text has been over written by the apparent newer (darker) scrip. For instance on F 139 "*magiori derecto ognuno et gli altro doi"* is crossed out, a small "^" pointing to *"chi angoli opposti equali*" written above the text. This seems to make it likely that the illustrations, and comments, have been added a posteriori. The initial absence of imaginary is not out of the extraordinary, especially given the transcription of an amanuensis; also it explains the absence of more dedicated drawings of some of the effects that do not result easily from a reading of the text.

Figure 132: stylised 'C', F.199r, II.91

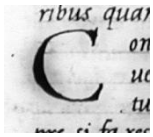

Figure 133: Unstilised 'C', F2v, Prologue

Peculiar is the great detail of some images that are present, as well as those simple ones that accompany mostly the geometric parts. As it wasn't uncommon for an illustrator to fill in blanks which are found throughout the text as vacant spaces for lettering and images, but as well apparently for titles (see Figures 132 and 133 for comparison of the details).

Comparing the second to the first part, Pacioli rarely makes reference to images present on the side. However, a great deal of imagery embellishes the first parts' pages (see Figure 134).

Figure 134: Drawing on top of the page, caracteristic cross in the last chapters, FF.239r.

[511] *DVQ* F.68r

Most of the illustrations seem to have been added a posteriori (after the manuscript had been copied), as often they are accompanied by a comment on the sections of the book as for instance in (II.80.), the scrip seems to differ often, and not always do they completely match description of the section itself specially if it is not entirely explicit or confusing in regards to its content and the content is not of a general knowledge, as for instance in (II.62.). The same happens with some of the titles of the chapter, as they appear in a lighter script and seem to have been written into a blank square space specifically left clear for them. This can account for some disparities such as the swapped titles of (II.84.) and (II.85.).

Further several effects have a stylized letter lattice of some complexity and diversity (see Figure 135). There is no obvious rule in which they appear and they do so along the whole text. Some authors mistake them for possible illustrations of the sections. It is possible that they were just an amusement of the copyist, or served to get rid of extra ink, but no more detailed discussion of this here or similar occurrences in other texts is known.

Figure 135: One of many, lattices below lettering, F.25r

Comments on the Content

The, possibly intentional, obscuring of the text, as well as the lack of illustration and amanuensis transcription defaults hinder the understanding. To this is to add the common gap in time and the distance in both notation and terminology, both in regards of mathematics and illusionism, which modernly have their own structure and framework knowledge associated. The above analysis tries to provide this and refresh the notation of some of the authors who have discussed the topic before, making use of the internet to also illustrate the discussed topics. It has however to be taken into account that the understanding and comprehension of the author of concept such as numbers themselves, don't equal those of present days. The translation of the titles and some of the content has tried to keep the spirit behind them in accord to that of Pacioli. In other cases this was not possible and generated another kind of difficulty, dubiousness in meaning.

For instance one can note that in the descriptions of the geometric constructions the letters have multiple meanings, so for instance a line segment can be designated as '*ab*', but as well simply as '*c*', while this single letter might be referent in turn to the vertex, or angle formed at a given vertex, or the likes. The last abbreviation of a single letter for a line is most often used when referring to the length of a given segment and usually further on used as a complementary segment of an already existing one starting at one of the extremes. Alphabetical order in regards to construction does exist, is however not necessarily in order of appearance or construction, although mostly it relates to the latter.

Similarly as example in the algebraic effects, all quantities are worded and often the effect itself is only comprehensible through the examples themselves. Often assumptions are made naturally which need further explanation modernly as assuming operations to be made with the bigger of two parts (I.18). Further due to the time it is set numbers are generally assumed positive, possibly containing a fraction, this is being rational. Irrationals are briefly discussed in the second part, but in general avoided. Similarly zero is used mostly as an artifice and has this status, not that of a number, or if so as a very special one.

Added to this is an uncertainty in regards to Pacioli's sources and references. For instance it is not certain which version of the elements Pacioli uses as reference in the book, and many reference do not match the modern version used. Reference is made several times to the fourteenth book, which only some versions of the *Elements* contain. In all likelihood Pacioli uses Campanus' version as a reference. This version contains a fourteenth book and Pacioli he himself published a version of it and was working on a translation into vulgar of it. A brief comparison to the 1482 print of *Elementa Geometriae* has been made, at the lack of availability of Pacioli's version and propositions match up nicely.

In regard to the non-mathematical aspects of the book some things can also be said. Pacioli seems not to be an expert illusionist, as some of the explanations of the effects seem rather rudimentary or incomplete. See for instance (II.117.), where a common illusion seems to be posed as a problem and explained, while focus is taken off the more elaborate illusion which could be achieved. Similarly Pacioli lists variations of many effects more in a fashion of giving an idea where one can see such and such effect at work or how such effect could established given illusion. It is more likely that he is trying to find a natural explanation to the effects given. As often the profanity of the art is negated in favor of natural or mathematical essence which is said to be miraculous, but this is to be understood as not supernatural. Phrases like "[...] *Et para gran facto*. *Non dimeno sia natural como fai*"[512] illustrate this.

Some of the less mathematical sections seem to be intended to be used as prank to others, like (III.iii.27.), this view seems to be shared by JP, one of the commentator in Pieper, the comment to (III.iii.33.) after setting Pacioli as the typical renaissance man, intent on demystifying and explaining as well as a renewal, she says **"(…) there is a fine line in his writing between his own enjoyment of these "pranks" and his will to defraud and reveal their nature.**" Perhaps due to these pranks, some of which are directed at the reader himself in (III.iii.24.) Pacioli reassures the reader he is not joking but speaking serious, as the effect seem incredulous.

Motivation and Publishing

As mentioned, Pacioli shows, in this work, traits typical of the renaissance, searching for answers of the supernatural in natural causes and elevating the thought as present in the mathematics to a divine level. No less a medieval tradition is very present, to start the compendium way of assembling the work, the exaltation of the supernatural and miraculous, the book is a collection of observations of wondrous effects, even though an explained one. The effects are very descriptive. Almost all effects do not stop at the descriptive and add something to them, they resemble recipes of actions one should follow more than they do an explanation.

In respect to the more standard mathematical effects, this is whose mathematics Pacioli was most likely aware of, a closer explanation is often given in form of a short explanation and often reference to other works. These works are mostly the *Elements* or Pacioli's own other works *Summa* and *Divina* or another part of the *DVQ*. Even the first half of the second part, which given the remainder of the collection is somewhat uncharacteristic, gives little explanation. It focuses rather on the production of geometric drawings and some of their properties than to exhaustively proving them.

[512] F. 228v

This is often substantiated by the author as practical geometry. The work rather attains the status of a compendium of mathematical miracles, and oddities. This somewhat dubious status of the work sets it apart in its own right and distinguishes Pacioli as a man of his times bordering two ages, middle ages and renaissance.

The most likely reason for Pacioli to have written these down is likely for his own enjoyment, and the memory of it. These recreation being even merrier when shared and perhaps in this the friar saw some means to an end to get into some court, or perhaps it was intended to honor some of his many patron over the ages. Reason enough to wish to publish it as he had other works of his. If it a courtesan view of the mathematician is accepted, which Pacioli would likely fit, having frequented many high society circles; this could easily fit the picture around in the time of possible printing in 1509 Pacioli got invited and promoted no longer needing a patronage and dedicating himself to other matters, it is also plausible that the printing of the *Elements* took a considerable amount of effort, perhaps more than Pacioli was willing to spend finishing the work, as he himself declares to have other duties to attend in (I.29). Another reason might have been the fear of possible consequences of the possibly misinterpretation of the work as impious given the illusionist effect which could have been met with great criticism in Pacioli's circles.

Propagation

It is not clear if the manuscript was spread – although if it was, it certainly was less than the printed works by Pacioli. Still, equivalent content is found throughout the literature relatable to the *DVQ*. Such are Fracesco Ghaligai's, 1521 *Summa de Arithmetica which contains several problems in the likes of Pacioli, the work of* Niccoló Fontana (1499 – 1557), Bachet de Méziriac(1581 - 1638) or even Vincenzo Filicaja (1642 – 1707). Although it is not clear if the presence of effects as those mentioned, originate from Pacioli's efforts, or if instead, these in the likes of Pacioli's work are the effort of collection from many distinct sources.

It is known that the works of Pacioli came as far as Portugal, as prominent royal cosmographer Pedro Nunes, considered one of the greatest mathematicians of his time, writes in one of his major works, from 1567, the *Libro de Algebra en Arithmetica y Geometria* that the books coming from Spain amongst which the *Summa* of Frey Lucas de Burgo have arrived and are worth consultation. Further poems using the golden ratio are written by the Duke D. Luiz (1506 – 1555) as a past time. In sequel of which one finds several very similar effects to those discussed in the *DVQ* in the works of the mathematician Gaspar Cardozo de Sequeira and his 1612 book *Thesouro de Prudentes*. Sequeira's book is divided into four treatises covering Astronomy, Medicine, Arithmetic, Geometry and Illusionism. Very strong similarities to some of Pacioli's card tricks are found here in.[513] These certainly were pioneering applications of the mathematical principles to card tricks. The spread and who read the books in Portugal remains an open question.

Elsewhere similar spread can be found. It does not seem by chance that Eberhard Welper a fellow mathematician publishes his *Das Zeit kurtzende Lustund Spiel-Hauss* in 1694 in which he both discusses several recreational problems like the purse lock of

[513] See a discussion of these in Ricardo, Hugo and Mendonça Jorge (2013) "O "Thesouro dos Prudentes" de Gaspar Cardozo de Sequeira", essay for the class of History of Recreational Mathematics

(II.109). Several other authors discussing similar effects are Pietro Rusca, Domenico Tancredi, Da Alberti, Da Schwenter, Da G. Schot, Filipo Calandri, in whose works some of the sections present figure.

Hopefully this text might have brought more readers to this text and provides a framework to build upon. So perhaps to the historian of mathematics, or of the history of science might look into it. What is irrefutable is that the effects, experiences, puzzles, and, other recreational marvels described by Pacioli survive until today, being used by science educators, magicians and even mathematicians in very similar situations as those Pacioli might have lived.

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World Wide Web

Most of the following have been use to illustrate or give a quick and general idea to the reader as well as, in many cases an interactive medium to experience some of the challenges, puzzles and so on first hand. Often the web-addresses have been added as a footnote, when considered fulcral, other times a hyperlink is found instead.

Bogomolny, Alex. Cut The Knot, http://www.cut-the-knot.org

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All webpages in the text have been last accessed on the 29.4.2015

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