BROUWER'S INTUITIONISM ;
a re-appraisal of Brouwer's contribution to the study of the foundations of mathematics

A THESIS BY
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submitted to the University of London
for the degree of
Doctor of Philosophy

30th May 1971

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A C K N O W L E D G M E N T S

I wish to express my sincerest thanks and gratitude to my promotor, Dr. G.T. KNEEBONE; his lectures inspired an interest in the subject of this dissertation, and its completion owes much to his encouragement and constant guidance. His readiness to give up his time on many occasions has been much appreciated.

I am grateful to all who through their stimulating interest and advice, or in any other way, have contributed. In particular I wish to mention Professor Arend Heyting and Professor Errett Bishop.

I must compliment my wife and children for their patience and forbearance during these years of study. I am particularly grateful to Judith, my wife, for her invaluable help in preparing and typing the manuscript.

I also thank my colleagues at Whitelands College, who kindly relieved me from some of my lecturing duties during the last few months.

A B S T R A C T

Brouwer's contribution to the study of foundations of mathematics is generally cepted, yet the greater part of his work has remained inaccessible because of language and for lack of a bibliography.

In this study of the fundamental concepts of Brouwer's intuitionism and philosophy use has been made of all Brouwer's published papers.

Chapter I

Some relevant biographical details are given, as well as a survey of Brouwer's foundational works. A largely forgotten work, Leven, Kunst en Mystiek, has been included as a useful source of information on Brouwer's character, his general views, and his mystical tendencies.

A bibliography of all Brouwer's work has been compiled, and is included.

Chapter II

An analysis is made of Brouwer's philosophy, which determined his intuitionism as early as 1905.

The central place in this philosophy is taken by Brouwer's theory of intuition and mathematics: intuition as the human mind acting independently of all data of experience, and mathematics as nothing else but this intuitive mental activity.

It is shown that this philosophy is the foundation of Brouwer's intuitionism; all his intuitionist theories and practices ultimately stem from his conception of mathematics.

Chapter III

In particular, Brouwer's criticism of classical mathematics, logicism, formalism, and Poincaré's neo-intuitionism is based on his absolute distinction between mathematics and language, i.e. expression of this mental activity in sounds or symbols. Neither language nor

logic, the post-factum analysis of this language can contribute anything to mathematics; any device, such as the Principle of the Excluded Middle which claims to produce mathematical results from a purely verbal structure, is suspect.

Chapter IV

Brouwer's conception of mathematics places greater emphasis on the active human role; this leads to an entirely new concept of the infinite sequence, of sets, and of the continuum.

A survey is given of these fundamental notions of Brouwer's analysis, especially in as far they diverge from classical mathematics.

Chapter V summarizes the main conclusions drawn in this work.

C O N T E N T S

CHAPTER I

BROUWER'S LIFE AND WORK

Luitzen Egbertus Jan Brouwer was born on February 27th, 1881 at Overschie, the eldest son of Egbertus Brouwer, a schoolmaster, and Hendrika Poutsma. He attended primary schools at Medemblik and Hoorn and received his secondary education at the H.B.S. at Hoorn and the Gymnasium at Haarlem.

On 27th September, 1897, at the age of sixteen, Brouwer was registered at the University of Amsterdam to study mathematics and science. His marriage to Elizabeth de Holle on 31st August, 1904 brought him an income that enabled him to devote all his time and energy to his studies. He completed his academic studies at the University of Amsterdam with the publication of his dissertation Over de Grondslagen der Wiskunde (The Foundations of Mathematics) and the public defence of 21 theses on 19th February, 1907.[1] He was awarded the doctoral degree 'cum laude'. He had earlier (1905) published his rather controversial book Leven, Kunst en Mystiek (Life, Art and Mysticism). These two publications are the only works of Brouwer that can be described as books.[2] Together with his other publications before 1907, which indicate his topological tendencies, they give a clear insight into Brouwer's complex personality and the wide range of his interests.

Leven, Kunst en Mystiek (1905) proves an embarrassment to Brouwer's serious intuitionist followers, who dismiss it as a ridiculous joke of a man of genius, written in a period of adolescent instability and depression, and conveniently forget it in their bibliographies. It is certainly not directly mathematical but it throws light on many aspects of Brouwer's life and views, including his views on mathematics, intuition, and science in general.

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1 This period of ten years is not excessively long by Dutch standards; the minimum period of undergraduate study is about six years. The writing of a dissertation follows the doctoral examination.

2 Other independent publications such as 1909A, 1912A, 1928C etc., are prints of single lectures; 1919B a reprint of 1908B, 1909A and 1912A.

Most of the views expressed in Leven, Kunst en Mystiek remained with him for the rest of his life. His major statements on the foundations of mathematics, such as 1929A, 1933 and 1948C, contain passages very reminiscent of those expressed in this book. At the height of his fame Brouwer himself recommended its reading, and towards the end of his life he made various attempts to have it translated into English and republished.[1]

Brouwer published it at the age of 24, eight years after he had started his mathematical studies at the University of Amsterdam and less than one and a half years before the publication of his dissertation. In fact he worked for some time on both publications simultaneously.

At first sight, it seems almost unbelievable that this was written by Brouwer at the same time that he was working on his Grondslagen. It is a confused, half-philosophical, half-mystical, romantic agglomeration of pamphlet-type articles on a whole range of subjects varying from life, intellect, science, truth, and language to politics, the debasement of society by socialism, and the inferiority of women compared to men, apparently written by someone with an axe to grind against society and especially those engaged in study, written sometimes in the most fanatical and coarse language. His main theme is that man has betrayed his noble animal nature, that his worst faculty is his intellect, which he has used to interfere with nature (even the Dutch are condemned for building dykes). Man is compared to a bird, 'arrogantly gulping up its own nest, interfering with mother earth, gnawing, mutilating her, making her creative power sterile until all life has been devoured ' (p.10). Schopenhauer's pessimistic outlook on life can be traced on almost every page; pages 47 - 65 are a Brouwer version of Schopenhauer's misogynistic Essay on Women in Parerga and Paralipomena, 'That undersized, narrow-shouldered, broad-hipped and short-legged race, the number two of the human race'): 'Between a

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[1] His housekeeper-secretary, A.W. Vermey, wrote to a prospective publisher on his behalf: 'As to Art, Life and Mysticism, this is a youthful series of lectures for three-quarters serious, one quarter blague.. but certainly giving an impression of the character and originality of his way of thinking. One chapter about language could perhaps give you an impression of this paper and Professor Brouwer agreed that I should try and translate it into English, which I shall gladly do..' Letter A.W. Vermey to A. Hill of 6th September, 1964.

woman in her innermost nature and an animal such as a lioness there is less difference than between two twin brothers' (p. 52) .. 'She must be humble and humbly take from his hands all ignoble work to leave him to enjoy the faculties of the body in which he walks this earth...... Without an 'Augenzucken' she will give her life in order to save his equilibrium' (p. 53). One cannot help reflecting on Brouwer's own life of study, one year after his marriage supported by his wife and the income from her chemist shops, especially when he writes: 'That money for one's livelihood is usually earned by the man is of as little importance as money itself; this happens to be so at present when earning money goes hand in hand with doing noble work; to the old Germanic tribes, tilling the land was regarded as ignoble work and therefore done by women; when all productive labour has been made dull and ignoble it will be done exclusively by women; in the meantime, men will occupy their time according to their ability and aptitude in sport, gymnastics, fighting, studying philosophy, gardening, wood carving, travelling, training animals and anything that at the time is regarded as noble work, even gambling away what their wives have earned; and this is really much nobler than building bridges or digging mines' (pp. 55 - 56).

Although Leven, Kunst en Mystiek can hardly be described as an anthology of poetry[1], appreciation of poetry is evident from many quotations from Goethe, Zola, Sophocles, Shakespeare and others, as well as from the long-winded defence of art as the ultimate truth (Chapter VI, 'Immanent Truth'). Brouwer made attempts at poetry himself throughout his life and later moved frequently in literary circles.[2] Leven, Kunst en Mystiek leaves no doubt about his tastes; with his usual uncompromising vigour he condemns what he does not like: 'Transcendental truth cannot be found except in a few artists: Bach, Leonardo. Inflammatory and anarchical in the worst sense is practically everything in fashionable and popular music and the arts: Beethoven, Wagner, Rubens, Raphael, Rembrandt.' (p. 66)

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[1] As referred to by G. Kreisel, Biographical Memoirs, Royal Society, p. 39.

[2] Brouwer was a regular participant in the meetings of the literary section of the Dutch Royal Academy (KAW).

Strongest is the influence of the mediaeval mystics, Meister Eckhart - favourite of the German romantic movement - and Jacob Böhme, as well as the eastern semi-religious, mystic tradition; this is evident from the general tendency in the whole of Leven, Kunst en Mystiek, and the lengthy quotations from their works and from the Bhagavad Gîta.[1]

Though many of the views expressed in Leven, Kunst en Mystiek can be dismissed as irrelevant to Brouwer's mathematical thinking, others, and in particular his mystical leanings, have penetrated into his philosophical and mathematical thinking; his personal and passionate involvement in both fields of mysticism and foundations of mathematics made complete detachment impossible. In particular his views on language and logic, mathematical existence, and intuition which form the basis of the intuitionist philosophy of mathematics, find their ultimate motivation in Brouwer's mystical beliefs.

His views on science, as expressed in Leven, Kunst en Mystiek, seem preposterous and impossible in a mathematician and scholar. Chapter III, 'Man's fall through the intellect', condemns causality which in his Grondslagen he claims to be the basis of mathematical thinking: 'Intellect has done mankind a devil's service through linking the two phantasies of means and end.' (p. 19).[2] On medical science: 'The medical industry was with barbers and quacks in good hands; practised within the confines of the intellect, as medical science it is far less effective' (p. 19). On science in general, (the Dutch word 'wetenschap' does not just cover physical sciences but all systematic study or knowledge, including mathematics, philosophy, and even the arts): 'it does not stop with science serving industry; the means becomes an end in its own right and - horribile dictu - is even practised for its own sake' (p. 21). Interesting is Brouwer's interpretation on the contemporary epistemological debate and his views on foundation study: 'A scientific truth is no more than a certain infatuation of desire, living exclusively in the human mind. Every branch of science gets into more trouble as it climbs higher. It even climbs so high that

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[1] Cf. also 1948C, p. 1241.

[2] Later called 'the cunning act' (1948C, p. 1236); see further p. 68.

'scientific thought is taken as something independent outside man. Some even start searching for the foundations of their science and that soon becomes a science in its own right and they practise "a theory of knowledge". As they climb higher, trouble increases and they go completely crazy. Some in the end quietly give up. Having thought for a long time about the elusive link between the intuiting consciousness - which itself develops from the world of phenomena - and this world of phenomena (which again itself exists only through and in the form of the intuiting consciousness) - a confusion which originated in a sinful foundation of the world of intuition, they then plug the hole with the concept of the "I" which was self-created with and at the same time as the phenomenal world. And then they say, "Yes, of course, something must remain incomprehensible and that something is the "I" that comprehends... But there are others who do not know when to stop, who keep on and on until they go mad; they grow bald, short-sighted and fat, their stomachs stop working, and moaning with asthma and gastric trouble, they fancy that in this way equilibrium is within reach and almost reached...' He concludes the passage with: 'So much for science, the last flower and ossification of culture' (p. 22).

This disagreement or even contradiction between Brouwer's inner feelings and beliefs and what he was practising professionally as a mathematician, is solved in Leven, Kunst en Mystiek by a fatalistic acceptance of the inevitable.[1] On the other hand, the dominant theme of the whole work is the excellence of inner vision and conviction ('insight') and mistrust of over-rationalization, which made him in his philosophy of mathematics move to intuitionism, made him lose interest in topology when it became increasingly more algebraic, and caused his complete silence on foundations with the formalization of intuitionist logic towards the end of the twenties.

Chapter VIII, 'The Freed Life', sees the solution in dissension and escape from the world into the solitary 'Self': 'After this first escape, he will not feel at home among other people, whom he begins to irritate through his eccentricities which unwittingly follow from this liberation' (p. 84). It is this readiness to diverge from the usual

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[1] 'The inevitability of one's Karma', p. 65; the whole of Chapter IV, 'Appeasement'.

and to question what was commonly accepted[1], that is so characteristic of Brouwer's attitude in his work on the foundations of mathematics and that gave him the courage to oppose such formidable and established mathematicians as Hilbert, Cantor, and Zermelo, especially since their views ran so much counter to Brouwer's deep convictions and emotional inclination.

Both his tendency to mysticism, his emphasis on 'inner vision', and his readiness to disagree, determined the road he was to take in his dissertation, Grondslagen der Wiskunde (1907) (Foundations of Mathematics). In the then current controversy concerning the foundations of mathematics, Brouwer reacted strongly against the deductive-logical direction in which contemporary mathematicians were moving, in direct opposition to the importance he attached to inner vision. There is a certain sense of regret when Brouwer in his historical surveys[2] reports on the defeat of Euclidean Geometry as the basic discipline in mathematics as a result of the discoveries of non-Euclidean geometries. Rather than follow those who were looking towards axiomatic set theory or logic to fill the vacuum, Brouwer wanted somehow to preserve the central role of geometry.[3] While accepting Descartes's and Kronecker's arithmetization of space, Brouwer could not be satisfied with Kronecker's foundation of the number system. In this he found allies in the French mathematical intuitionists and especially Poincaré, who in the principle of complete induction had refuted this and other attempts at founding mathematics without appeal to synthetical a priori judgments but who had not provided a coherent alternative theory of foundations of mathematics. Neither was Brouwer satisfied with their preoccupation

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1 It would be an over-simplification, following Wilhelm Ostwaldt's analysis of genius, to attribute Brouwer's greatness solely to or to measure it in terms of this readiness to diverge from the usual or the expected. It has certainly contributed and is more significant at a time when the spirit of rebellion was not so fashionable. Neither is it correct to portray Brouwer as a detached mathematician, clinically and objectively studying contemporary mathematics and discovering flaws. The impression is rather that of someone who, disagreeing with or even disliking their findings, sets out to prove them wrong.

2 Cf. 1907, 1908B, 1909A, 1912A, 1952B, 1953.

3 Cf. The Nature of Geometry (1909B), further discussed on pages 20 ff.

with definability of axioms and freedom from contradiction. Searching for a seat of mathematical reality, neither empirical nor purely verbal, Brouwer was naturally attracted to the kind of intuitionism as presented by Bergson, who placed truth in 'consciousness', intuition, that part of the human mind which transcends the analytical intellect and grasps its object from within.[1]

De Grondslagen der Wiskunde (The Foundations of Mathematics)

As we shall consider Brouwer's views on the foundations of mathematics more in detail, it will be sufficient here to give only a brief outline of his dissertation, and consider the circumstances which contributed to his choice of subject.

The end of the nineteenth century marks a period of fundamental crisis in philosophy as well as mathematics.

Kantianism was revived in Germany and Holland in reaction against nineteenth century materialism and intellectualism, again rejecting metaphysics and concentrating on the origin of knowledge and epistemology. By the turn of the century the role of philosophy itself was in question and its place contested by the rapidly advancing sciences.

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[1] Cf. also p. 73 ff.
Note on Leven, Kunst en Mystiek:

Brouwer's influence in general and that of Leven, Kunst en Mystiek in particular, extended also into the field of art. Joost Baljeu, (The Problem of Reality with Suprematism, Constructivism, Proun, Neoplasticism and Elementarism, p. 114), claims that the movement known as Neoplasticism was largely inspired by Brouwer's Leven, Kunst en Mystiek, and that the writings of Brouwer's friend, the mathematician Dr. H.J. Schoenmaekers, (The New Image of the World (1915) and The Principles of Plastic Mathematics (1916)), were entirely based on this work by Brouwer. Both Brouwer and Schoenmaekers lived at Laren and with the famous Dutch poet Frederik van Eeden and artists like Piet Mondriaan, formed a circle which often met. Mondriaan has indeed modelled his scenery after Brouwer; e.g. Scene I (Mondriaan 1919/1920) is taken from Leven, Kunst en Mystiek, p. 61: 'Alive in vizualization as one part of this polarity, it (i.e. inwardness of introspection) will not loosen its ties with that other part which provides eternal certainty, repose and wisdom. Thus, it will experience the vast blue sky as the exact counterpart of its own humility and contemplation; the immutable course of stars as the counterpart of the other colours and the flow of passions in its own blood ... ...this introspection sees the given environmental world as the karma which has to blame itself, and the confusion produced in the environmental world through thought and action as this karma's frivolous burden, which is why it will withdraw itself from the latter and no longer occupy itself with those phenomena in nature which came into existence through the arrogance of contemporary science, these being its main subject.'

While earlier philosophers had made their mark in the mainly social and psychological sciences[1], towards the end of the century practitioners of diverse sciences turned philosophers, anxious to propagate their thinking and the methods of their disciplines, as well as searching for justification of their basic assertions. (E. Haeckel, Helmholtz , Lotze - medicine; Bergson, Pearson - biology; Kirchoff, Mach, Meyerson, Hertz, Einstein and Eddington - physics; Husserl, Russell, Whitehead, Mannoury - mathematics). The predominance of mathematicians and especially mathematical logicians among those 'natural' philosophers is partly due to their traditional affinity with philosophy; it also reflects the growing awareness among mathematicians of the special problem of the foundations of mathematics.

The evolution of mathematical logic since Boole and de Morgan placed the basis of mathematics confidently in logic itself, this with varying degrees of emphasis to the point of actual identification of mathematics and logic. Others improved and extended the more traditional mathematical method of axiomatization to cover number theory and other fundamental theories of mathematics. Underlying both trends was the idealistic conviction of a Platonic 'real' world, which had found new expression in axiomatic set theory. Even Hilbert's programme of the formal axiomatic method (as expressed in Grundlagen der Geometrie (1899)), retained this idealistic element and a reliance on logic, although the emphasis was on internal consistency as the final mathematical arbiter.

The confidence of mathematicians who had thought that in this way the question of the ultimate basis of mathematics had finally been settled . was shattered by the publication of Russell's paradox. The attention of mathematicians all over the world was drawn to this crisis in foundation study.

In Holland, it was Brouwer's supervisor, Professor D.J. Korteweg, who promoted an interest in this current controversy. Although himself an applied mathematician, he encouraged research in foundation study and helped to create a chair in 'mathematics and the philosophy of mathematics' at the university of Amsterdam. This chair was occupied by G. Mannoury, a lifelong friend of Brouwer, and associate

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[1] Hegel - legal theory; Schopenhauer, Nietzsche - depth psychology; Auguste Compte - sociology.

in many of his foundational and 'signific' activities.[1]

Apart from these external circumstances which influenced Brouwer in choosing the foundations of mathematics as the subject of his dissertation, there is a certain personal preference. Any even superficial reader of Brouwer's work must be aware of the marked contrast between the technical, almost clinical approach of his topological papers and the emotional involvement so apparent in his foundational work. This personal preference is shown in his choice of topic whenever he spoke or wrote at an important occasion in his career: his dissertation (1907), at least 17 of the 21 theses for public defence on 19th February, 1907, his first public lecture (1909A), his inaugural address (1912A), and numerous lectures abroad or at international conferences. While his interest in topology or significs was temporary, his interest in foundation study remained with him throughout his life, even during the most prolific period of topological activity (1909 - 1913).

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[1] There is a strong element of common interest in language, logic and mathematics. Comparison between Brouwer's Grondslagen and Mannoury's first public lecture, Over de Beteekenis der Wiskundige Logica voor de Philosophie (On the Significance of Mathematical Logic for Philosophy), Rotterdam 1903, reveals a striking similarity of topics and suggests some influence on Brouwer, if only by inviting criticism and focusing attention on certain topics. Frequent discussions between Brouwer and Mannoury helped to sharpen and colour Brouwer's opposition to mathematical logic. Their views are sufficiently divergent not to pursue the point of mutual influence any further in this present work.

That feelings between both men were sometimes strained is shown in 1908C, which is a sharp reply to Mannoury's criticism of Brouwer's Grondslagen, published in Nieuw Archief voor Wiskunde, vol. VIII, pp. 175 - 180.

For further comment on Significs, cf.p.113.

G. Mannoury, in his first public lecture, deals with the development of axiomatics of geometry and arithmetic, the work by Peano and his school on mathematical logic, and with axiomatics in general. He defends the, at the time, highly provocative view that philosophy originates in linguistic problems, that neither logic nor philosophy can contribute anything to our knowledge that had not previously been supplied by 'sciences of observation', physics, sociology and psychology.

G. Mannoury became professor extraordinary in 1917, and in 1918 succeeded professor D.J. Korteweg in the chair of 'geometry, mechanics and the philosophy of mathematics'.
He retired in 1937 and died in 1956.
A Bibliography of Mannoury's work can be found in Synthese, vol. X a, which contains also various contributions on Mannoury and his work, by A. Heyting, E. Beth, D. van Dantzig, J.C. Destouches.

Over de Grondslagen der Wiskunde is divided into three chapters:
Chapter I - The Construction of Mathematics (De Opbouw der Wiskunde), pp. 3 - 78;
Chapter II - Mathematics and Experience (Wiskunde en Ervaring), pp. 81 - 123;
Chapter III - Mathematics and Logic (Wiskunde en Logica), pp. 125 - 180.

Chapter I supports the view of those 'constructive' mathematicians who - if not contemptuous of foundational theory - maintain that this adds precious little to their mathematical practice and that Brouwer's contribution to constructive mathematics can be seen as quite separate from his theory on mathematical foundations.[1] Apart from some minor philosophical degressions on the union of the continuous and the discrete, and a preference for Helmholtz's philosophical analysis of space over Hilbert's naive treatment, Chapter I is an elementary-constructive treatment of some fundamental parts of arithmetic and geometry. Without any speculation on the nature of number, Brouwer starts with the natural numbers as known and given, and proceeds to construct the order types ω and η, and the elementary propositions of algebra and geometry, cf. the opening lines of Chapter I: 'One, two, three, ... we all know by heart the sequence of these sounds (spoken ordinal numbers) as a sequence without end, i.e. a sequence which will for ever proceed according to a known fixed rule. Apart from this sequence of sounds, we know other intuitive sequences (Du. Voorstellingsreeksen) which proceed according to a certain rule, e.g. the sequence of written symbols 1, 2, 3, ... (written ordinal numbers). These are intuitively clear.'[2]

At the end of Chapter I, Brouwer explains what this constructive 'intuitive' treatment entails:

1) Starting with units given in intuition, and with these constructing new ones;
2) Carefully observing which theses are allowed by intuition and which are not.[3] It is here (page 77) that Brouwer explains the whole

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[1] Cf. E. Bishop, Foundations of Constructive Analysis, Chapter I, 'A Constructive Manifesto'; also Kreisel,'Biographical Memoirs ' L.E.J. Brouwer, R.S., p. 54 et ff..

[2] 1907, p. 3.

[3] Brouwer did not elaborate further on the grounds or guide lines for such a decision. That he relies on some intuitive common sense is supported by what he wrote in 1921A p. 798, quoted here on p. 23.

programme and aim of his Grondslagen: Chapter I shows how the fundamental parts of mathematics 'can be constructed from the units of intuition'; Chapter III will show that 'this is the only possible way that mathematics can be constructed; that no mathematics can exist unless it has in this way been intuitively constructed ... and that every other way of founding mathematics is bound to fail'.[1]

Chapter II

While Chapter I could be described as an example of intuitionist mathematics, both Chapters II and III are Brouwer's first contributions to the intuitionist philosophy of mathematics, metamathematical in the sense that they reflect on the system itself and its fundamental assertions. Significant is Brouwer's placing of intuitionist mathematics (Chapter I) before these philosophical reflections. Whereas Chapter III is largely polemical, and critical of the then current analytical attempts, which concentrate on the mathematical reasoning itself, in Chapter II, Brouwer expounds his own views on the nature of mathematics, its relation to the natural sciences, objectivity of mathematical knowledge, and the problems of time and space.

True to the Kantian tradition, Brouwer conducts his search for the origin of mathematical truth within the process of knowledge itself. He sees his own contribution as a bringing-up-to-date of Kant's teacing on a priority of time and space, following earlier attempts by Riemann and Helmholtz[2], and later by Russell in his Essay on the Foundations of Geometry.[3]

He summarizes his own views and those of Kant and Russell at the end of Chapter II (p.121) in the following way:

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1 1907, p. 777.

2 1907, p. 94.

3 Cf. p. 125.

	In Kant's *Transcendental Aesthetic*	In Russell's *Foundations of Geometry*	In this work (i.e. *Grondslagen*)
Inseparably connected with external experience	Euclidean three-dimensional space and non-measurable time	Euclidean three-dimensional space and the measurable time-coordinate	Nothing
Necessarily occurring in the mathematical receptaculum of experience because of:			
a) the organization of the human intellect	Euclidean three-dimensional space and non-measurable time	Projective space, free mobility in space and the measurable time-coordinate .	The fundamental intuition of mathematics, or the intuition of time
b) experience	Nothing	the three-dimensionality of space and the parallel-axiom of Euclid	Nothing

Chapter III

As quoted above, Brouwer's object in Chapter III was to show that the constructive methods of intuitionist mathematics, as practised in Chapter I, based on a philosophy which starts from the mental activity proper to mathematics (Chapter II), are essential to all mathematics and 'that every other attempt is bound to fail'. After dismissing logic as a post-factum analysis of the 'language of mathematics' and proclaiming the independence of mathematics from logic, Brouwer submits to a searching criticism:

1) The foundation of mathematics based on axiomatization (pp.133 - 142),
2) Cantor's theory of sets and finite numbers (pp. 142 - 159),

3) Peano-Russell attempt at founding mathematics on logic (pp.159 - 169),
4) The logico-formal foundations of mathematics according to Hilbert (pp.169 - 175).

Commentators generally agree on a certain immaturity in the Grondslagen.[1] Brouwer's attempt in Chapter I at constructive mathematics has only historical interest. But most of his views and criticisms in Chapters II and III (even if later further developed or sometimes altered), have later been proved right and have been universally accepted.

Practically all his later critical views can be traced back to his Grondslagen, even those on the Principle of the Excluded Middle and choice sequences.[2] In his later foundation papers, Brouwer always refers to his dissertation and with some pride uses it to prove the authenticity of his views.[3] 'Addenda en Corrigenda over de Grondslagen der Wiskunde' (1917A), published ten years later, is largely a list of references to further developments in later papers; only on two points does Brouwer want to withdraw statements made in the Grondslagen.[4]

Period of topological activity

The years following the Grondslagen and especially the period 1909 - 1913, are usually regarded (by topologists) as the most fruitful of Brouwer's life. The early appointment as 'privaat-docent' (unsalaried lecturer) at the University of Amsterdam was in recognition of his more orthodox mathematical abilities. He was 'licensed' to teach geometry and appropriately started his course on geometry with a public lecture, 'Het Wezen der Meetkunde' (The Nature of Geometry, 1909A).

In contrast to his publications, his lectures were very orthodox and remained so throughout his university career. His revolutionary views in the fields of foundations and even topology were not reflected in his lectures. Heyting admitted 'that he very seldom lectured at the University on those parts of mathematics that he himself had helped

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1 Freudenthal and Heyting, 'Levensbericht L.E.J. Brouwer', Jaarboek der Koninklijke Akademie van Wetenschappen, 1966 - 1967, p. 1.
Beth, The Foundations of Mathematics, p. 410.

2 Cf.pp. 144 and 177.

3 Especially 1928A.

4 On the PEM, and on logical operations.

to create. This is why he had only few direct followers'.[1] And even more strongly: 'He never lectured on his own investigations; on topology - as far as I know - he has never done it, on intuitionism only in his later years'.[2]

Even within the more restricted Dutch academic tradition of a nationally prescribed syllabus for undergraduate courses, there would have been ample opportunity to digress into areas so close to his mandate. But Brouwer was very shy by nature, he hated teaching and lecturing.[3] It is in his writings that Brouwer's views have to be found.

Het Wezen der Meetkunde ('The Nature of Geometry'), 1909A, is important as a further extension of Brouwer's investigation into the foundations of mathematics, his views on geometry and especially those on topology. It is a statement of policy and a programme, and provides a link between the two areas of his work usually seen in complete isolation.

Iterating his conclusion of 1907 that the only a priori on the basis of experience is the intuition of time, he expresses his firm conviction that with the arithmetization of space the a priori validity of geometry is no problem any more: 'The exactness of coordinate geometry is not a mystery any more, since it is now completely explained by the exactness of the operationsof number' (p. 6). The complete freedom of geometry, earlier proclaimed by H. von Helmholtz[4], is based by Brouwer on the absence of any a priori element other than the intuition of time. After questioning whether there is still a case for separating out a branch of mathematics under the name 'geometry',

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1 H. Freudenthal and A. Heyting, op. cit., p. 6.

2 A. Heyting, Memorial address to Royal Dutch Academy on 24th February, 1968. Cf. also 'In memoriam L.E.J. Brouwer', Algemeen Nederlands Tijdschrift voor Wijsbegeerte en Psychologie, vol. 59, April 1967; also: 'Luitzen Egbertus Jan Brouwer', Yearbook of the American Philosophical Society, 1967, pp. 116 - 119.
Attempts to obtain lecture notes of Brouwer have been unsuccessful, but all students approached agreed that Brouwer had never lectured on these two topics.

3 Cf. Heyting, op. cit.

4 'On the origin and Significance of Geometric Axioms (1870), reprinted in The World of Mathematics, (ed. J. R. Newman) London 1956
Helmholtz is not quoted or mentioned here by Brouwer.
His rejection of physical a priority based on accuracy of measuring is also reminiscent of von Helmholtz. (p.9)

Brouwer suggests a definition of geometry as the invariant theory of a transformation group, very much on the lines of Felix Klein's Erlanger Programm of 1872[1]: 'Geometry is concerned with the properties of spaces of one or more dimensions; in particular it examines and classifies possible point sets, transformations, and transformation groups in these spaces', (p.13). The reference to 'point sets'[2] here does not imply that Brouwer saw points as simplexes of geometries, (i.e. lines or surfaces made up of an infinite number of points as 'atomic' elements, as in point set topology); spaces to him are continua, his simplexes are parallelepipeds.[3]

Brouwer specifically excludes from 'geometry' 'finite and denumerable sets of points' as well as 'the more abstract spaces of Veronese and Hilbert' (p. 14). On the other hand, he regards operations of real and complex numbers as part of geometry (p. 14).

Most of his argument for classification of geometries on the basis of transformation groups is in line with current trends. His definition of analysis situs as the geometry invariant under the group of continuous one-one transformations (p. 16) - with the restriction as to the fundamental elements applying to all geometries - points towards Brouwer's 'combinatorial' line of approach to topology as distinct from set-theoretical topology.

He advocates the use of topological methods in other branches of mathematics, as had been done by von Staudt in projective geometry; in particular, he mentions the possibility of starting function-theory from topology once a way has been found to eliminate the difficulty of the metric. ('under continuous one-one transformations of the plane, finite lengths and areas (Du. lengten en inhouden) can become infinite or reduced to zero', p.20). It is, however, the visual-perceptive simplicity of this geometric topology that Brouwer wanted to emphasize.

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1 'A geometry is the study of those properties of a set S which remain invariant when the elements of the set S are subjected to the transformations of some transformation-group Γ '.

2 For the use of the word 'set' (Du. verzameling, Germ. Menge) by Brouwer, see further Chapter IV.

3 'These spaces are built up from one or more connected Cartesian paralellepipeds' (p. 14).

In the final lines he equates 'topological' to 'geometric' and 'formula-less':

'Therefore also in other theories (i.e. besides projective geometry) - even if one succeeds in founding them on analysis situs - coordinates and formulae need not entirely be banned; but the "formula-less", the "geometric" treatment will be the starting point, while the analytic treatment becomes a dispensable expedient. It is to the possibility and the desirability of the priority of the geometric treatment - also in parts of mathematics where this is not yet done - that I have wanted to draw your attention' (p. 20). The elegance and simplicity of his later topological findings and proofs, such as the fixed point theorem, the invariance of dimension and domain, are largely due to this geometric approach.

Apart from being a remarkable exposé of Brouwer's general views on geometry and topology and their place in mathematics, Het Wezen der Meetkunde contains many suggestions he was to follow up during the coming years such as the extension of the Jordan Theorem (p. 18), and also his reservations as to the sufficiency of Schoenflies' concept of accessibility (p. 19).

In his many topological publications in this short period of 1909 - 1913, Brouwer showed his originality and proved his mathematical mastery. (An excellent analysis of these works is given by M.H.A. Newman in Biographical Memoirs of L.E.J. Brouwer, Royal Society, 1970, pp. 46 - 53). They gave Brouwer international fame.

He took part in the IVth International Congress of Mathematics in Rome in 1908, where he contributed 'Die Theorie der endlichen kontinuierlichen Gruppen unabhängig von den Axiomen von Lie' (1908B). He often visited Paris after 1909 and had frequent contacts with Poincaré, Borel and Lebesgue.

After 1911, he regularly visited Göttingen. He took part in the International Congress of Mathematicians in Cambridge in 1912 (1912B), the Deutschen Naturforscherversammlung, and the congress of the Deutsche Mathematiker Vereinigung at Karlsruhe in that same year. He developed a friendship with Schoenflies, who with Brouwer's help revised his earlier Bericht über die Punktmannigfaltigkeiten.[1]

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[1] Cf. 1913B

In recognition of his outstanding abilities, a new chair was created at the University of Amsterdam in set theory, function theory and axiomatics, which he occupied from 1912 until his retirement in 1951.

Loss of interest in Topology?

Freudenthal and Heyting blame national isolation during World War I for Brouwer's 'loss of interest in topology'.[1]

Other important factors are: the increasing need for intricate algebra in the development of higher-dimensional topology, quite out of step with the 'visional-perceptional' simplicity of the geometric treatment so much emphasized by Brouwer in 1909A; and the impact of set theory on the notion of geometry itself following Maurice Fréchet's introduction of abstract spaces which no longer fitted neatly in Brouwer's classification on the basis of transformation-groups.

Brouwer's attitude to topology after 1913, or even after 1919, can hardly be described as desertion or loss of interest. It is the growing awareness of these developments that made Brouwer concentrate his efforts on the more fundamental notions of sets, the continuum, and function, which underlie topology as well as the whole of mathematics. The first major publication of these efforts is 'Begründung der Mengenlehre unabhängig vom Logischen Satz vom ausgeschlossenen Dritten'[2], presented to the KAW in November 1917. His continued interest in what topologists would regard as 'topology proper' is shown in his work with his students (assistants), even after 1925, (P. Alexandroff, K. Menger, Vietoris, H. Hopf, H. Freudenthal, and W. Hurewicz), as well as in some later publications. It would be an oversimplification to draw a sharp distinction between these two areas of Brouwer's activities, if such a distinction in the general mathematical scene at the time is at all justified.

His 'proper' topological work - even if it is appreciated and acceptable to topologists of various schools - is partly the result of some self-imposed fundamental restrictions, as testified by Brouwer in Het Wezen der Meetkunde and later in 1921A: '.... in my non-philosophical mathematical papers I have regularly used older methods, but always

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[1] Op. cit. p. 4.

[2] 1918A.

'careful to derive only such results as could be expected to find their place in the new system after the systematic construction of an intuitionist set theory.'[1]

'De onbetrouwbaarheid der logische principes', published in 1908, and the inaugural address, 'Intuitionisme en Formalisme' of 1912, are proof of Brouwer's deep involvement in foundation study right through the period of topological activity.

The extraordinary combination of 'set theory, function theory and axiomatics' for his chair in mathematics (his own agreed choice), indicates the principle area of activity as Brouwer saw it already in 1912.

Foundations after 1907

Brouwer's increased prestige, his proven mathematical worth in 1912, added great weight to his intuitionist views and drew attention to his earlier work on the foundations of mathematics - his thesis and 1908B - which had passed almost unnoticed.

The most revolutionary conclusion from his general views on logic as expressed in Grondslagen was taken in De Onbetrouwbaarheid der Logische Principes (The Untrustworthiness of the Logical Principles) (1908B), an article in a philosophical journal, and concerned the principle of the excluded middle (PEM). In the course of the general argument on the limited role of logic in philosophy, science and mathematics[2], Brouwer claimed:

1° The identification of the problem of the solvability of every mathematical problem and the PEM;

2° That the PEM is not a reliable principle in an infinite system;

3° That, since even improper use of the PEM does not lead to contradiction, non-contradictority is no guarantee of truth.

Also mentioned for the first time is Brouwer's definition of negation in terms of a contradiction.

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[1] 1921A, p. 798.

[2] In the final summary, the PEM is not mentioned: 'Summarizing:
In wisdom there is no logic.
In science: logic is often but not permanently effective.
In Mathematics: it is not certain whether or not logic is permissible, neither is it certain whether it can be decided whether all logic is permissible or not.' (p. 12).

That Brouwer was not fully aware of the implications of his rejection of the PEM and his stronger definition of negation (or was not fully confident) is shown by the fact that neither are mentioned in his inaugural address.

Intuitionisme en Formalisme (1912A, 1913A)

In the inaugural address intuitionism is presented as a school of mathematical philosophy fundamentally opposed to formalism in its views on mathematical existence and the place of intuition. It is the first time that Brouwer uses the terms 'intuitionism' and 'intuitionist'[1].

To a great extent it repeats or summarizes the views already expressed in 1907, the revision of Kant's view on intuition, the nature of language and its role in mathematics, the rejection of Cantor's set theory and transfinite numbers.

No sharp distinction is made between logicists, axiomaticists and formalists. The greater part of the offensive is directed at Zermelo's axiomatic treatment of set theory in 'die Grundlagen der Mengenlehre'[2]. Whereas in 1907 Brouwer's attack on axiomatics was based on its general lack of intuitive content, Intuitionisme en Formalisme singles out two of Zermelo's axioms, 'Auswahl' and 'Aussonderung' to which Brouwer refers as the axiom of choice and the axiom of comprehension[3]. He shows that in particular, the axiom of comprehension leads to contradiction (Burali - Forti paradox), and claims that, even in its modified form[4], which avoids known paradoxes, it forms the basis of a totally meaningless and unacceptable theory of potencies of sets, of numbers and the continuum.

In 'Die Intuitionistische Mengenlehre' (1920B) and 1921A) Brouwer claimed that the rejection of the PEM (of 1908B) and of the axiom of comprehension (of 1912A) had been the two guiding principles in his systematic construction of an intuitionist set theory.

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[1] See further pp.73 ff.

[2] MA, vol, 65 (1908), pp. 261 - 281; also, 'Sur les ensembles finis et le principe de l'induction complète, Acta Mathematica, vol. 32, pp. 185 - 193.

[3] Translated by Professor A. Dresden (1913A) into 'axiom of selection' and 'axiom of inclusion'.

[4] Zermelo, op. cit. p. 263.

Begründung der Mengenlehre unabhängig vom logischen Satz vom ausgeschlossenen Dritten (Foundations of Set Theory independent of the logical Principle of the Excluded Middle).
(Part I (1918A) presented to the Royal Dutch Academy in November 1917 and published in 1918, Part II (1919A) presented to the Royal Dutch Academy in October 1918 and published in 1919).

The publication of 'Begründung der Mengenlehre' not only marks the beginning of a decade of intentse activity by Brouwer in foundation study but also an important shift in his general approach and in his treatment of set theory. The first part of Brouwer's programme[1], 'the first act of intuitionism', was the liberation of mathematics from the shackles of language and logic; most of his earlier work, accordingly, was negative, critical of the classical tradition and of the prevalent trends of logicism and formalism.

Set theory, which had earlier been dismissed as trivial and non-mathematical, now becomes the foundation of an intuitionist mathematics; in its special form it is later referred to by Brouwer as 'the second act of intuitionism'[2].

In spite of the title, the Principle of the Excluded Middle is not mentioned anywhere in either Part I or Part II, nor is there any reference to classical mathematics. 'Begründung der Mengenlehre' is not an exercise in re-writing Cantorian set theory, carefully avoiding the use of the PEM and the axiom of choice. It is a positive, non-polemical attempt to find a firm basis for the 'non-separable parts of mathematics'[3], an attempt to rid mathematics of the Cantor set and replace it by a completely new concept. The Cantor set not only provided a foundation for the natural numbers, it claimed to bridge the gulf between the discrete and the continuum by the relation element and set. The primordial intuition of time had provided Brouwer with an alternative basis for the natural numbers; his views as to the impossibility of representing the continuum as a totality of discrete elements had also been strongly expressed in 1907, 1908A, 1912A (1913A). He now gives up conformity with the general definition of a set as a totality of elements (still accepted in 1908A) and reserves the word

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[1] 1952B, p. 140; cf. also p. 108.

[2] Inidem.

[3] Ibidem.

set (Germ. Menge) for a novel tool in the treatment of the continuum.

The Brouwer set is a generalization of the organic structure of the continuum in which there is no discrete and where the elements, even if referred to as points of the continuum, can never be treated as Euclidean, finished discrete points. The elements of the Brouwer set are essentially unfinished, 'becoming', as is the Brouwer set itself, infinite sequences in their most general form, proceeding in time with or without restriction.

The alleged incomprehensibility[1] of Brouwer's treatment and definition of sets in 'Begründung' (but also in 1920B, 1921A and 1925D) is mainly due to his attempt to incorporate within one definition (and one sentence) the complex character of the set and its elements as well as their generation, and that without introduction or further explanation and without reference to the purpose it was to serve (i.e. the continuum).

The nearest equivalent to the classical set, the species (Du. soort), is introduced in one line as 'a property of mathematical entities previously acquired'[2]. Both the Brouwer set and the species will be discussed in more detail in Chapter IV.

Part I (1918A), 'General Theory of Sets', is divided into three sections: the cardinal numbers, the ordinal numbers, and the well-ordered ordinal numbers. Even if ordinal numbers are treated after the cardinal numbers, cardinality is defined in terms of the ordered sequence of natural numbers whose existence is presupposed in the very first line of 'Begründung' (also 1925D).[3]

The distinction between various degrees of equipotency, to which Brouwer attached great importance at the time[4], is just one example of his critical analysis of concepts which in language are covered by

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[1] A. Fraenkel and Y. Bar-Hillel, Foundations of Set Theory, p. 250; also S.C. Kleene and R.E. Vesley, The Foundations of Intuitionistic Mathematics, p. 43.

[2] 1918A, p. 3; cf. 1925D, p. 245.

[3] The relation of equipotency is defined as a one-one correspondence (see further 'Splitting of equipotency') ; the cardinal numbers are introduced after what Brouwer calls 'The Fundamental Property of Finite species': 'For any representation of the one-one correspondence between a finite species E and the set of numerals of an inital segment of ζ , in short, for any way of counting E (Zählungsweise von E), the same initial segment of ζ is used' (p. 5).

[4] Cf. 1920B, p. 205; 1921A, p. 799.

a single word.

With Part II (1919A), 'Theory of Point Sets', a beginning was made with the construction of an intuitionist analysis on the basis of the Brouwer set and species.

Accepting the possibility of a treatment of point sets of other dimensions, plane point sets are singled out for treatment throughout. The starting point is the 'set Q of squares (Quadrate) in a rectangular coordinate system, whose corner points are the coordinates $\frac{a}{2^n}$ and $\frac{b}{2^n}$ and sides (parallel to the axes) are of length $\frac{1}{2n}$ or $\frac{1}{2n-1}$. A point of the plane is an indefinitely proceeding sequence of squares of Q, each of which is contained in the interior of its predecessor'[1].

A theory of point sets is then developed within the restrictions imposed through avoidance of the PEM and the axiom of comprehension, further complicated by Brouwer's own distinction between sets and species.[2]

Section 3 of Part II introduces the concept of measure (Inhalt) and measurability of point species within the unit square. Measurability is first defined for outer limiting species[3], and a necessary and sufficient condition is given for measurability of arbitrary species: 'If for a point species Q there exist two measurable outer limiting species A_Q' and A_Q'' of resp. content i and 1 - i, such that an arbitrary point of A_Q' belongs to Q and an arbitrary point of A_Q'' cannot possibly belong to Q, then Q is measurable and its content is i'.[4]

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[1] 1919A, p. 3; 1920B, p. 206; 1921A, p. 800.

[2] E.g. that the intersection of two sets is not necessarily a set. See further: Sets, Chapter IV.

[3] First a limiting sequence (limitierte Folge) is defined as a fundamental sequence $i_1, i_2, i_3, \dots$ where each $i_\nu = \frac{a_\nu}{2^\nu}$ for positive integral values a_ν, such that for every ν there is positive whole number μ and $|i_{\nu+\lambda} - i_\nu| < \frac{1}{2^\nu}$ for every non negative λ.
'If the content of the first ν squares of a domain β (Bereich) is denoted by i_ν and $i_1, i_2, i_3, \dots$ form a limiting sequence then β is measurable and i the content of β.' (p. 26).
'Let M_ν be a finite set of squares x (for every ν). If then every $M_{\nu+1}$ belongs to M_ν, and the contents i_ν of the M_ν form a limiting sequence then $k = \cap(M_1, M_2, \dots)$ (das Bereichkomplement) is said to be measurable and i is the content of k.' (p. 26).
'If for an outer limiting species $A = \cup(k_1, k_2, \dots)$, every k is measurable and the contents i_ν of the k_ν form a limiting sequence i, then A is measurable and the content of A is i.' (p.28).

[4] 1919A, p. 31.

Finite additivity is proved, and countable additivity intuitionistically formulated as: 'If F is a fundamental sequence of a measurable point species such that the contents of the unions of their initial segments form a limiting sequence i, then the union of F is measurable and its content equal to i'[1].

The first few years following the publication of 'Begründung der Mengenlehre' were probably the best in Brouwer's life. His topological work had given him a place among the leading mathematicians of the time, now his contributions on the foundations of mathematics were acclaimed as classics. Hermann Weyl, one of Hilbert's most distinguished students, dramatically renounced his own earlier attempt 'Das Kontinuum', and hailed Brouwer as the revolution:('... und Brouwer - das ist die Revolution! '), the one mathematician who at last had solved the problem of the continuum which since ancient times had defeated even the greatest minds.[2] In a series of lectures 'On the new crisis in the foundations of mathematics', given in the Mathematical Colloquium at Zürich in May 1920, (published in 1921[3]) Weyl expounded Brouwer's views on the continuum, the PEM and set theory and pronounced a crisis in mathematics of which the antinomies had only been symptomatic. In America these views were similarly acclaimed by Arnold Dresden in 'Brouwer's Contribution to the Foundations of Mathematics'[4]. As a result of the first world war, the Göttingen Society had been disbanded and Hilbert was now approaching 60; it now seemed natural that Brouwer should take over the lead and Amsterdam become the centre of foundational activity.

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1 1919A, p. 33.

2 Although Weyl recognized that his own atomistic continuum did not faithfully reflect the intuitive continuum (op. cit. p. 47), he was satisfied that for the purpose of analysis it would be sufficient. More alarming to him was Brouwer's revealing the weakness of the notion 'There is' (PEM): 'An existential statement such as "there is a number" is indeed not a proposition in the strict sense ... "There is a number" is only a propositional abstract (Urteilsabstrakt)' p. 54.

3 Mathem. Zeitschrift, vol. 10 (1921), pp. 39 - 79.

4 Bulletin of the American Mathematical Society, (1924), pp. 31 - 40. (Presented to the Society 1923).

'Begründung der Funkionenlehre unabhängig vom logischen Satz vom ausgeschlossenen Dritten' (1923A) (Part I)

It becomes already apparent in 'Begründung der Funkionenlehre' that reconstruction of mathematics within Brouwer's restrictions and on the basis of the Brouwer set and species not only caused considerable limitations but also produced results widely different from orthodox analysis.

After giving his definitions of real functions on Brouwer sets, continuity at a point and continuous functions, Brouwer almost immediately derives his uniform continuity theorem for all full functions.

'By a real function or just function we understand a rule which correlates to certain x [1], (which we denote by χ and which form the "domain of definition" of the function (Definitionsbereich)) always one y [1], which we will denote by $\eta = f(\chi)$ and which in this way usually generates a plane point species. We limit ourselves to those real functions f(x) where χ belongs to the closed unit interval' (p.3).

'A function is said to be continuous for the value χ_0 (stetig für den Wert χ_0) or continuous in the point (χ_0, η_0) if for every ϵ an α_ϵ can be indicated, (and therefore a positive rational α_ϵ), such that for $|\chi - \chi_0| < \alpha_\epsilon$ the inequality $|f(\chi) - f(\chi_0)| < \epsilon$ holds '

'A function which is continuous for every χ is referred to in short as a continuous function' (p.3).

'A function is said to be uniformly continuous, if for every ϵ a positive α_ϵ can be indicated such that for $|\chi_2 - \chi_1| < \alpha_\epsilon$ the inequality $|f(\chi_2) - f(\chi_1)| < \epsilon$ always holds '. (p. 4).

'A full function is a function whose domain of definition is the whole of the closed unit interval '. (p.5).

Uniform continuity theorem

'Every full function is uniformly continuous' (p.5).
A proof is hardly given at this stage. Brouwer confessed in 1927: 'This theorem has since 1918 frequently been mentioned by me in my lectures and conversations ... I did not, however, succeed in proving it until much later (1924H, 1924I)'[2]

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[1] x : the point cores of the X axis '; y : the point cores of the Y axis'.

[2] 1927B, p. 62.

A proof was attempted in 1924H 'Bewijs dat iedere volle functie gelijkmatig continu is', (Germ. version 1924I) and again in 1927B. (See further p. 34).

With the publication of Part I of 'Begründung der Funktionenlehre' the programme of systematic reconstruction of mathematics on intuitionistic lines virtually ended; Part II never appeared. At a meeting in Hamburg in 1922, Hilbert started the defense of the PEM and Cantor's set theory and transfinite numbers, 'What Weyl and Brouwer do comes to the same thing as to follow in the footsteps of Kronecker! They seek to save mathematics by throwing overboard all that which is troublesome ... If we would follow such a reform as the one they suggest, we would run the risk of losing a great part of our most valuable treasures'.[1] Brouwer answered in 'Over de rol van het principium tertii exclusi in de wiskunde, in het bijzonder in de functietheorie' (1923B)(On the role of the PEM, especially in the theory of functions) (Germ. version 1924A; cf. also 1925A). He again states his case of 1908B: the Principle of the Excluded Middle is acceptable for finite domains because of its verifiability but has no a priori validity. A priori validity was presumed by classical logic and the validity of the PEM was extended in mathematics to infinite domains. Brouwer's main concern in 1923B, however, is with the impact of the PEM on classical mathematics and the consequences of rejecting the PEM for mathematics and the theory of functions in particular. Brouwer's real number, defined as a convergent sequence dependent on the occurrence of a given finite sequence in the decimal expansion of π , introduced for the first time in 1921B as an example of a real number which has no decimal expansion[2], made him realize that the continuum cannot be ordered (this had been presumed in 1918A . Because of this, and because of the splitting of

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[1] 'Neubegründung der Mathematik', Abhandlungen der Hamburgischen Universität, vol. 1 (1922), p. 157; also 'Die logischen Grundlagen der Mathematik, MA, vol. 88 (1922), pp. 151 ff.
Cf. C. Reid, Hilbert, London, Heidelberg, New York 1970, p. 155.

[2] 1921B, p. 811; This real number becomes Brouwer's classic example refuting all cases of the use of the PEM; see further

the notion of convergence, also introduced in this article (1923B)[1], the whole of the classical theory of functions, including the notion of the Lebesgue integral, is now called into doubt: 'With this first fundamental property (i.e. that the points of the continuum form an ordered point species) falls the usefulness of the notion of the integral of the Paris School, the so-called L integral, it cannot even be applied to a constant function'[2]. Also rejected are: the theorem that every mathematical species (classical set) is either finite or infinite; the Bolzano Weierstrass theorem and the Heine Borel covering theorem.

The logical consequences of the rejection of the PEM were drawn in 'Intuitionistische splitsing van mathematische grondbegrippen' (1923C)(The "splitting" of fundamental notions of mathematics in intuitionism) (Germ. version 1925B). As a 'special case of the PEM the principle of reciprocity of complementary species' was rejected, 'the principle which for an arbitrary mathematical system derives truth drom the absurdity of the absurdity of a property'[3]. A calculus of absurdities was set up which started an international debate on 'the Brouwer Logic', which went on until the end of the decade.[4] Brouwer himself, however, stood aloof from the debate and was far more concerned with the growing crisis in intuitionism itself.

Enthusiasm for Brouwer's intuitionism began to wane; it became more and more apparent how much of classical mathematics had to be sacrificed and how complicated a reconstruction of mathematics on intuitionist lines proved to be. Earlier visions of a clear, constructive mathematics, freed from language and logic, gradually vanished. Weyl wrote later: 'It cannot be denied, however, that

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1 Positive and negative convergence, see further p. 219. Brouwer's student, M.J. Belinfante, showed that this distinction leads to different theories, positive divergence leading to a theory similar to the classical. (M.J. Belinfante, 'Zur intuitionistischen Theorie der unendlichen Reihen', Sitzungsber. der preuss. Akademie, Phys. Math. Klasse, 1929, pp. 639 - 660); cf. also J.G. Dijkman, 'Convergentie en Divergentie in de Intuitionistische Wiskunde, 's Gravenhage 1952.

2 1923B, p. 3.

3 1923C, p. 877; 1925B, p. 251.

4 For the debate on the Brouwer Logic, see further p. 150.

'advancing to higher and more general theories, the inapplicability of the simple laws of classical logic eventually results in an almost unbearable awkwardness. And the mathematician watches with pain the larger part of his towering edifice which he believed to be built of concrete blocks, dissolve into mist before his eyes'.[1]

Growing isolation was partly self-imposed. Brouwer was not an easy man to work with; he was an ambitious man and identified himself completely with intuitionism; his references to others are usually critical, and valuable contributions of even his most dedicated collaborators were never publicly acknowledged.

In order to retain support for the intuitionist cause and present intuitionism as a viable alternative, Brouwer needed to show how much of classical mathematics can be retained within intuitionist mathematics. A more positive approach can be seen in 1924F (1924G), 'Intuitionistic Proof of the Fundamental Theorem of Algebra', 1924J (1924K) supplement to 1924F, 1925C 'Intuitionistic proof of Jordan's Theorem of Curves', 1926E (1926F) 'The intuitionistic form of the Heine-Borel theorem'; (cf. also the intuitionist analogues of the classical properties of the continuum of 1928C[2]).

Meanwhile, Brouwer prepared a revised version of Part I of 'Begründung der Mengenlehre' which appeared as 'Zur Begründung der Intuitionistischen Mathematik' (1925D, 1926A and 1927A). Additions and alterations mainly reflect his changes of view concerning ordering and well-ordering and the development of his theory of functions.

'Über Definitionsbereiche von Funktionen' (1927B) (On the domains of definition of functions) is Brouwer's last contribution in the Mathematische Annalen. It summarizes his theory of functions and attempts to 'prove as lucidly as possible' what Brouwer regarded as the most fundamental theorem of intuitionist function theory, i.e. the uniform continuity theorem. An important step in the proof is the

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1 Philosophy of Mathematics and Natural Science, Princeton 1949, p.54.

2 See further p. 223.

Bar theorem and the Fan theorem[1], which play a major role in later intuitionistic analysis[2].

Brouwer's main argument of 1927B, (as in 1923A and 1924A) for his uniform continuity of all full functions ultimately centres round his definition of function and continuity (see above p.31), and follows from Brouwer's notion of mathematical existence. This requires for a function effective calculability. We could simplify this argument in the following way:

Let f be a full function, and x_0 be a point of the unit continuum whose decimal expansion is $.k_1\ k_2\ k_3\ \ldots$; let further $f(x_0) \equiv y_0$ and the decimal expansion of y_0 be $.m_1\ m_2\ m_3\ \ldots$ For an effective calculation of $f(x_0)$ to any finite decimal place (say $.m_1\ m_2\ m_3\ \ldots m_p$) a finite inital segment $.k_1\ k_2\ k_3\ \ldots k_q$ of the decomal expansion of x_0 must suffice. It then follows that for any x_1 whose decimal expansion has the same inital segment $.k_1\ k_2\ k_3\ \ldots k_q$, $f(x_1)$ will have the same inital segment $.m_1\ m_2\ m_3\ \ldots m_p$ in its decimal expansion as $f(x_0)$. The continuity requirement is therefore already implied in the definition of function:[3]

if $|x_1 - x_0| < .000 \ldots k_{q+1}$ then $|f(x_1) - f(x_0)| < .000\ldots m_{p+1}$.

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[1] The Bar theorem asserts: 'If with each element of a set M (i.e. a Brouwer set) a natural number β is associated, then through this association M is split into a well-ordered species S of subsets M_α, each determined by a finite initial segment of choices. With all elements of each M_α the same natural number β_α is associated' (as given in 1924H, p. 191).
In the (English) terminology of 1953, 'Every crude bar contains a well-ordered block' (p. 14).
The Fan theorem (so-called in 1953, p. 15): 'If with each element e of a finite set M a natural number β_e is associated, then a natural number z can be indicated such that β_e is completely determined by the first z choices generating e' (1924H, pp.191-192).

[2] A detailed analysis of Brouwer's Bar Theorem and its logical implications is given in Kleene and Vesley, The Foundations of Intuitionistic Mathematics, Amsterdam 1965, pp. 43 - 89; cf. also Charles Parsons, Introduction to a reprint of 1927B in J. van Heyenoort, From Frege to Gödel, H.U.P. 1967, pp. 446 - 463; C. Spector, 'Provable recursive functions of analysis; a consistency proof by an extension of principles formulated in current intuitionistic mathematics', Recursive Function Theory, Proc. Symp. Pure Mathematics, vol. V, Am. Math. Soc. RI, pp. 1 - 27; G. Kreisel, 'Consequences of Brouwer's Bar Theorem', Journ. Symb. Logic, vol. 27 (1962), pp. 380 - 381.

[3] Continuous functions in this sense are equivalent to Bishop's weakly continuous functions. (Cf. E. Bishop, Foundations of Constructive Analysis, New York - London, 1967, p. 70.)

A classically discontinuous function, such as f defined by $f(x) = 0$ for $0 \leq x < \frac{1}{2}$, $f(x) = 1$ for $\frac{1}{2} \leq x \leq 1$, would simply not qualify as a function defined on the unit continuum.

The Brouwer - Hilbert Controversy

Towards 1927, support for Brouwer's intuitionist cause had dwindled, Brouwer's attempt to replace classical mathematics by an intuitive, clear constructive system had failed and became a struggle for survival.

The simple alternative presented by the formalist programme proved more attractive to most mathematicians. Hilbert had succeeded in re-establishing Göttingen as the centre of the mathematical world, gathering round him such distinguished mathematicians as Courant, von Neumann, Nordheim, Landau, Noether, Bernays, van der Waerden, Alexandroff and Artin.

Brouwer's earlier overtures (as early as 1909[1]) had failed to make any impression on Hilbert, who claimed never to have read any of Brouwer's papers. After Weyl's 'defection', Hilbert could no longer ignore intuitionism; irritated, he left the field of mathematical physics 'to solve the problem of foundations once and for all'. While Brouwer with the missionary zeal of a prophet of doom, preached the hard and unpleasant truth, Hilbert's programme was pragmatic, inspired by the determination not to give up any of the 'treasures acquired': 'The desires and attitudes which will show us what direction to take are these: wherever there is any hope of salvage, we will carefully investigate frutiful definitions and deductive methods. We will nurse them, strengthen them and make them useful. No one shall drive us out of the paradise which Cantor has created for us.'[2]

The controversy between formalism and intuitionism in this period turned into a bitter personal feud with undignified recriminations on both sides. Hilbert was extremely irritated by 'that man full of temperament and inventiveness', and regarded Brouwer's criticisms as a personal affront: 'Not even the sketch of my proof of Cantor's continuum hypothesis has remained uncriticized.'[3] Brouwer's resentment

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[1] 1928A, p. 375 refers to a meeting with Hilbert 'in the autumn of 1909', where the point of mathematics and metamathematics was raised by Brouwer.

[2] 'Über das Unedliche', MA vol. 95 (1926), reprinted in Benacerraf and Putnam, Philosophy of Mathematics, pp. 134 - 151. An address on the occasion of the celebration in honour of Weierstrass, Munster, 4th June, 1925. The reference here is to Benacerraf and Putnam, p. 141.

[3] D. Hilbert, 'Grundlagen der Mathematik', as reprinted in van Heijenoort, From Frege to Gödel, H.U.P. 1967, p. 476.

was the result of frustration and non-recognition; Brouwer was well aware of the restrictiveness and awkwardness of his intuitionist mathematics ('unfortunately mathematics had to lose much of its elegant character and had to assume much stiffer (Du. stroever), awkward and complicated forms.'[1]) Mainly because of this, supporters such as Weyl had left him and joined 'the formalist camp': 'That from this point of view only a part, perhaps only a wretched part, of classical mathematics is tenable, is a bitter but inevitable fact.'[2] Above all, the highly prized recognition by Hilbert had not been forthcoming.

Brouwer's visit to the Göttingen Mathematics Club at the time is symbolic: Hilbert listened in silence to Brouwer's exposition of the doctrine of the PEM and the ensuing debate. When he finally stood up, he closed the debate with the remark: 'With your methods most of modern mathematics would have to be abandoned and to me the important thing is not to get fewer results, but to get more results.'[3] The enthusiastic applause that followed these remarks by Hilbert sounded a humiliating defeat of Brouwer's great ambition. Disillusioned and embittered, he went back to Holland.

The petty incidents of these years, Brouwer's polemical contributions and his ultimate tragic withdrawal from public life, reveal the weaker sides of a proud and ambitious character; they also show his passionate involvement in his cause, almost to the point of pathological obsession. Göttingen became the 'enemy camp', Hilbert's name so much anathema that Brouwer stalked out of a social gathering in Amsterdam when van der Waerden mentioned Hilbert and Courant as his friends.

On the editorial board of the Mathematische Annalen, Brouwer demanded sole responsibility for all topological contributions and for all papers submitted by Dutch mathematicians; he vetoed contributions in which use was made of the PEM. At the suggestion of Caratheodory, Hilbert dismissed the whole of the editorial board; Einstein unkindly spoke of 'this frog and mouse battle' and resigned his post as principal editor in disgust.[4] The Dutch government so resented this slight on

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[1] 1933, p. 60.

[2] Weyl's comment on Hilbert's second lecture (1927, 'Grundlagen der Mathematik', as printed in J. van Heijenoort, op. cit. p. 483.).

[3] C. Reid, Hilbert, Berlin - Heidelberg - New York 1970, p. 184.

[4] Cf. the covers of MA, vols 100 and 101, which mention only the names of Hilbert, Hecke and Blumenthal.

their leading mathematician that they founded a rival mathematical journal with Brouwer in charge.[1] (Brouwer's campaign to prevent Hilbert and the whole of the German delegation from attending the first international congress after the first world war (Bologna 1928) was more politically inspired). The last phase in the unedifying spectacle of this personal feud between two great but intolerant and arrogant mathematicians came in two contributions.

In Grundlagen der Mathematik (in its original version of 1927[2]), Hilbert presented the revised version of his proof theory and angrily defended: 'Existence theorems ... which to him (Brouwer) are worthless scrip, their use causing mathematics to degenerate into a game ...' 'The formula game that Brouwer so deprecates ...'
and the Principle of the Excluded Middle: 'Taking the PEM from the mathematician would be the same, say, as proscribing the telescope to the astronomer or to the boxer the use of his fists. To prohibit existence statements and the Principle of the Excluded Middle, is tantamount to relinquishing the science of mathematics altogether. For, compared with the immense expanse of modern mathematics, what would the wretched remnants mean, the few isolated results, incomplete and unrelated, that the intuitionists have obtained?'[3]

A similar polemical flavour can be found in Brouwer's Intuitionistische Betrachtungen über den Formalismus (1928A) (Intuitionistic Reflections on Formalism), an address given at the Royal Dutch Academy on 17th December, 1927.[4]

In the opening lines, Brouwer expresses his conviction that 'the disagreement over which is correct, the formalistic way of founding mathematics or the intuitionistic way of reconstructing it, will vanish and the choice between the two activities be reduced to a matter of taste'. The difference between the two schools are listed in four 'insights':

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[1] R.L. Goodstein, Essays in the Philosophy of Mathematics, Leicester 1965, p. 5.

[2] Presented at the Hamburg Mathematical Seminar in July 1927. In its original version given in J. van Heijenoort, op. cit. p. 464 ff.

[3] Ibidem, p. 476.

[4] Also delivered, in a similar form, at the Berlin Academy of Sciences on 16th February, 1928.

1° The formalistic differentiation between the construction of 'the inventory of mathematical formulas' and an 'intuitive (contentual) theory of laws of this construction for which the natural numbers are indispensable;

2° The limited validity of the PEM;

3° The identification of the PEM with the principle of the solvability of every mathematical problem;

4° Recognition of the fact that the (contentual) justification of formalistic mathematics by means of a proof of its consistency contains a vicious circle.

The remainder of the article is concerned with establishing authenticity of these intuitionistic insights by reference to earlier works and with Brouwer's claim that some of these insights have been adopted by formalists without mentioning authorship: '... formalism has received nothing but benefactions from intuitionism and may expect further benefactions. The formalistic school should, therefore, accord some recognition to intuitionism, instead of polemicizing against it in sneering tones, while not even observing proper mention of authorship. Moreover, the formalist school should ponder the fact that in the framework of formalism, nothing of mathematics proper has been secured up to now (since, after all, the metamathematical proof of the consistency of the axiom system is still lacking, now as before), whereas intuitionism, on the basis of its constructive definition of sets and the fundamental property it has exhibited for finite sets, has already erected anew several of the theories of mathematics proper in unshakeable certainty. If, therefore, the formalistic school (according to its utterances in Hilbert, Über das Unendliche p.180) has detected modesty on the part of intuitionism, it should seize the opportunity not to lag behind intuitionism with respect to this virtue.'[1]

Brouwer's Vienna lectures, Mathematik, Wissenschaft und Sprache, (1929A, Mathematics, Science and Language, 10th March, 1928) and Die Struktur des Kontinuums, (1928C, The Structure of the Continuum, 14th March, 1928), are both statements on the two major issues of

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[1] 1928A, p. 377.

Brouwer's intuitionism, the role of language and logic in mathematics and the nature of the continuum. They contain little of the elaborate detail of Brouwer's systematization of intuitionist mathematics.

Mathematik, Wissenschaft und Sprache is chiefly philosophical and reiterates Brouwer's earlier views of the limited role of language. In keeping with Brouwer's mood at the time, the influence of Schopenhauer's pessimistic view of the human will is more marked than anywhere else[1] in Brouwer's work; mathematics, science and language are described as the primary functions of the human will.[2]

Die Struktur des Kontinuums summarizes Brouwer's criticism of the classical and 'old-intuitionist' treatment of the continuum and briefly describes the 'intuitionist continuum'. The classical properties are refuted one by one and replaced by intuitionist analogues where applicable.

Having lost the general support of the mathematical world, Brouwer in these Vienna lectures could still rouse the enthusiasm of philosophers for his grand visions. They inspired Wittgenstein to return to philosophical activity.

At the time when Hilbert retired, Brouwer (in his late forties) left the public (mathematical) scene, leaving the intuitionist lead in the hands of his student, Arend Heyting, who with Brouwer's help[3], was about to publish his formalization of intuitionist logic and mathematics.

There was a temporary return to public mathematical activity after the second world war. Most of Brouwer's published contributions of these years, however, are short and contain little new. Interesting is his rejection (1948A) of van Dantzig and Griss's radical constructive attempt to eliminate negation altogether from intuitionist mathematics, which would obviate Brouwer's theory of absurdities.[4]

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1 The only direct mention of Schopenhauer is made in the other Vienna lecture; see further p. 65.

2 1933 is largely a translation of 1929A.

3 Although Brouwer had his doubts about the value of such a formalization, he helped Heyting to have it published in Sitzungsberichte der preuszischen Akademie, following his ejection from the board of the MA.

4 G.F.C. Griss, 'Negatielooze intuitionistiche Wiskunde' KAW versl. vol, 53 (1944), p. 261; 'Negationless intuitionistic Mathematics', KAW proc. vol. 49 (1946), p. 1127.
P. van Dantzig, 'On the principles of intuitionistic and affirmative mathematics', KAW proc. vol. 50 (1947), p. 918 and p. 1092.
Cf. E. Beth, The Foundations of Mathematics, Amsterdam 1968, pp. 436 - 442.

At the age of seventy, Brouwer retired as professor of mathematics (1951). The first few years of retirement were devoted to visits and lectures at other universities. These lectures helped a revival of interest in intuitionism; like his earlier work on foundations, they concentrated on the broader philosophical issues of language and logic in mathematics (1948C) (where there is a return to the mystical tendencies of 1905), or summarized his views on the PEM, sequences and the continuum (1952B, 1953).

Following the death of his wife in 1961, Brouwer's time was largely taken up with legal problems; his housekeeper writes in 1964, 'His personal circumstances are so intricate and muddled ... he still does very much to try to get his difficulties (financial and inheritance difficulties) setlled ...'[1]

Honours had been bestowed on him earlier in life; he was elected a member of the Royal Dutch Academy (Koninklijke Akademie van Wetenschappen) in 1912, and was knighted in 1932 (Ridder in de Nederlandse Leeuw). In 1948, he was elected a member of the Royal Society.

On 2nd December, 1966, while crossing a road near his home, he was knocked down by a car and killed. He was 85 years old.

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[1] Letter A.W. Vermey to A. Hill, 6th September, 1964.

Bibliography

This bibliography was presented at a mathematics seminar at Bedford College in May 1969. It is included in this work since no complete bibliography of Brouwer's work is yet available[1], and as a guide to Brouwer's activities and interests.

All Brouwer's works are given in the chronological order of their publication.

No attempt is made to classify them into mathematical and non-mathematical, or to distinguish between books and papers; either distinction is in many cases difficutl to make. Inclusion of reference to reprints and republications seems appropriate as the complete works of Brouwer have not as yet been published.

Republications and translations during Brouwer's lifetime are given separately, as some contain additions or alterations (e.g. 1912A and 1913A; 1913C and 1923D, 1923E; 1923C and 1925B). Individual contributions of a series are also given separately, as some extend over a considerable period and, as their title indicates, are not always continued in the same language (e.g. 1910C, 1910D, 1911D, 1911E: Dutch - English, but 1920C: German).

Literal translations of previous publications are indicated by an asterisk.

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[1] A comprehensive bibliography of Brouwer's work was recently published by the Royal Society, London, (Biographical Memoirs, G. Kreisel and M.H.A. Newman).

B I B L I O G R A P H Y

L u i t z e n   E g b e r t u s   J a n   B r o u w e r

Abbreviations:

KAW versl. Koninklijke Akademie van Wetenschappen te Amsterdam (Koninklijke Nederlandse Akademie van Wetenschappen), Verslag der gewone vergaderingen der wis-en natuurkundige afdeelingen.

KAW verh. Koninklijke Akademie van Wetenschappen te Amsterdam, Verhandelingen.

KAW proc. Koninklijke Akademie van Wetenschappen te Amsterdam, Proceedings of the Section of Sciences.

MA Mathematische Annalen.

* Translation of

1904 A Over de splitsing van de continue beweging om een vast punt O van R_4 in twee continue bewingen om O van R_3's, KAW versl ., vol. 12 (1904), pp. 819 - 838.

1904 B On a decomposition of a continuous motion about a fixed point O of S_4 into two continuous motions about O of S_3's, KAW proc., vol. 6 (1904), pp. 716 - 735. (* 1904 A)

1904 C Over symmetrische transformatie van R_4 in verband met R_r en R_1, KAW versl., vol. 12 (1904), pp. 926 - 928.

1904 D On symmetric transformation of S_4 in connection with S_r and S_1, KAW proc., vol. 6 (1904), pp. 785 - 787. (* 1904 C).

1904 E Algebraische afleiding van de splits baarheid der continue beweging om een vast punt van R_4 in die van twee R_3's, KAW versl., vol. 12 (1904), pp. 941 - 947.

1904 F Algebraic deduction of the decomposability of the continuous motion about a fixed point of S_4 into those of two S_3's, KAW proc., vol. 6 (1904), pp. 832 - 838. (* 1904 E).

1905 Leven, Kunst en Mystiek, Waltman, Delft 1905, 99 pages (Life, Art and Mysticism).

1906 A Meerimensionale Vectordistributies, KAW versl., vol. 15 (1906), pp. 14 - 26.

1906 B Polydimensional Vector-distributions, KAW proc., vol. 9 (1906), pp. 66 - 78. (* 1906 A)

1906 C Het krachtveld der niet-Euclidische, negatief gekromde ruimten, KAW versl., vol. 15 (1906), pp. 75 - 94.

1906 D The force field of the non-Euclidean spaces with negative curvature, KAW proc., vol. 9 (1906), pp. 116 - 133. (* 1906 C)

1906 E Het krachtveld der niet-Euclidische positief gekromde ruimten, KAW versl., vol. 15 (1906), pp. 293 - 310.

1906 F The force field of the non-Euclidean spaces with positive curvature, KAW proc., vol. 9 (1906), pp. 250 - 266. (* 1906 E)

1907 Over de Grondslagen der Wiskunde, Maas en van Suchtelen, Amsterdam - Leipzig; Noordhoff, Groningen 1907, 182 pages. (doctoral thesis) (The Foundations of Mathematics)

1908 A Die mögliche Mächtigkeiten, Atti del IV Congresso Internazionale de Matematici, vol. 3, pp. 569 - 571.

1908 B De onbetrouwbaarheid der logische principes, Tijdschrift voor Wijsbegeerte, vol. 2 (1908), pp. 152 - 158. (The untrustworthiness of the principles of logic)

1908 C Over de grondslagen der Wiskunde, Nieuw Archief voor Wiskunde, vol. 2 (1908), pp. 326 - 328.

1908 D Over differentiequotienten en differentiaalquotienten, KAW versl., vol. 17 (1908), pp. 38 - 45.

1908 E About difference quotients and differential quotients, KAW proc., vol. 11 (1908), pp. 59 - 66. (* 1908 D)

1909 A Het wezen der meetkunde, Clausen, Amsterdam 1909, 20 pages. (The essence of geometry, public lecture, 12 October, 1909)

1909 B Die Theorie der endlichen kontinuierlichen Gruppen, unabhängig von den Axiomen von Lie, Part I, MA Vol. 67 (1909), pp. 246 - 267; Atti IV Congr. Intern. Mat.; vol. 2, pp. 296 - 303. Berichtigung hierzu, MA vol. 69, p. 180.

1909 C Over één-éénduidige, continue transformaties van oppervlakken in zich zelf, Eerste bijdrage, KAW versl., vol 17 (1909), pp. 741 - 752.

1909 D Continuous one-one transformations of surfaces in themselves, 1st communication, KAW proc., vol. 11 (1909), pp. 788 - 798. (* 1909 C)

1909 E Over de niet-Euclidische meetkunde, Nieuw Archief voor Wiskunde, series 2, vol. 9 (1909), pp. 72 - 74.

1909 F Over continue vectordistributies op oppervlakken, eerste bijdrage, KAW versl., vol. 17 (1909), pp. 896 - 904.

1909 G On continuous vector-distributions on surfaces, first communication, KAW proc., vol. 11 (1909), pp. 850 - 858. (* 1909 F)

1909 H Over één-éénduidige, continue transformaties van oppervlakken in zich zelf, tweede bijdrage, KAW versl., vol.18 (1909), pp. 106 - 117.

1909 I Continuous one-one transformations of surfaces in themselves, KAW proc., vol. 12 (1909), pp. 286 - 297. (* 1909 H)

1909 J Karakteriseering der Euclidische en niet-Euclidische bewegingsgroepen, Handelingen van het Nederlands Natuurkundig en Geneeskundig Congres, vol. 12, pp. 189 - 199.

1910 A Over continue vector-distributies op oppervlakken, tweede mededeeling, KAW versl., vol. 18 (1910), pp. 702 - 721.

1910 B On continuous vector distributions on surfaces, 2nd communication, KAW proc., vol. 12 (1910), pp. 716 - 734. (* 1910 A)

1910 C Over de structuur der perfekte puntverzamelingen, eerste mededeeling, KAW versl., vol. 18 (1910), pp. 833 - 842.

1910 D On the structure of perfect sets of points, first communication, KAW proc., vol. 12 (1910), pp. 785 - 794. (* 1910 C)

1910 E Zur Analysis Situs, MA vol. 68 (1910), pp. 422 - 434.

1910 F Beweis des Jordanschen Kurvensatzes, MA, vol. 69 (1910), pp. 169 - 175.

1910 G Über ein-eindeutige, stetige Transformationen von Flachen in sich, MA, vol. 69 (1910), pp. 176 - 180. Berichtigung hierzu MA, vol. 69, p. 592 and MA, vol. 79 p. 403.

1910 H Die Theorie der endlichen kontinuierlichen Gruppen unabhängig von den Axiomen von Lie, Part 2, MA, vol. 69, pp. 181 - 203.

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1910 L Beoordeling van Mannoury, 'Methodologisches und Philosophisches zur Elementarmathematik', Nieuw Archief voor Wiskunde, vol. 2 (1910)

1911 A Beweis der Invarianz der Dimensionenzahl, MA, vol. 70 (1911) pp. 161 - 165.

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1911 E On the structure of perfect sets of points, 2nd communication, KAW proc., vol. 14 (1911), pp. 137 - 147 (* 1911 D)

1911 F Over één-éénduidige, continue transformaties van oppervlakken in zich zelf, vierde mededeeling, KAW versl., vol. 20 (1911), pp. 24 - 34.

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1911 J Beweis der Invarianz des n-dimensionalen Gebiets, MA, vol. 71 (1911) pp. 305 - 313.

1911 K Beweis des Jordanschen Satzes für den n-dimensionalen Raum, MA, vol. 71 (1911), pp. 314 - 319.

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1912 B Sur la notion de "classe" de transformations d'une multiplicité, Proceedings of the International Congress of Mathematics, Cambridge 1912, vol. 2, pp. 9 - 10.

1912 C Over omloopscoefficienten, KAW versl., vol. 20 (1912), pp. 1049 - 1057.

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1912 E Over één-éénduidige transformaties van oppervlakken in zich zelf, vijfde mededeeling, KAW versl., vol. 21 (1912), pp. 300 - 309.

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1913 C Über den natürlichen Dimensionsbegriff, Journal für die reine und angewandte Mathematik, vol. 142 (1913), pp. 146 - 152.

1913 D Enige opmerkingen over het samenhangstype η KAW versl., vol. 21 (1913), pp. 1412 - 1419.

1913 E Some remarks on the coherence type η KAW proc., vol. 15 (1913), pp. 1256 - 1263. (* 1913 D)

1915 A Opmerkingen over inwendige grensverzamelingen, KAW versl. vol. 23 (1915), pp. 1325 - 1326.

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1915 C Over de loodrechte trajectorien der baankrommen eener vlakke eenledige projectieve groep, Nieuw Archief voor Wiskunde, vol. 11 (1915), pp. 265 - 290. (on the orthogonal trajectories of curves of the plane projective group).

1917 A Addenda en corrigenda over de grondslagen der wiskunde, KAW versl., vol. 25 (1917), pp. 1418 - 1423; also Nieuw Archief voor Wiskunde, 2nd series, vol. 12 (1918), pp. 439 - 445.

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1917 C Over linear inner, limiting sets, KAW proc., vol. 20 (1917), pp. 1192 - 1194. (* 1917 B)

1918 A Begründung der Mengenlehre unabhängig vom logischen Satz vom ausgeschlossenen Dritten, Erster Teil, Allgemeine Mengenlehre, KAW verh., 1st section, no. 5, pp. 1 - 45.

1918 B Beweging van een materieel punt op den bodem eener draaiende vaas onder den invloed der zwaartekracht, Nieuw Archief voor Wiskunde, 2nd series, vol. 12 (1918), pp. 407 - 417. (Motion under the influence of gravity of a point mass at the bottom of a spinning vessel).

1918 C Opmerking over de vlakke translatiestelling, KAW versl., vol. 27 (1918), pp. 840 - 841.

1918 D Remark on the plane translation theorem, KAW proc., vol. 21 (1918), pp. 935 - 936. (* 1918 C)

1918 E Über eineindeutige, stetige Transformationen von Flächen in sich, sechste Mitteilung, KAW versl., vol. 27 (1918), pp. 609 - 612; KAW proc., vol. 21 (1919), pp. 707 - 710.[1]

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[1] Although different in content, both 1918 E and 1920 D are given by Brouwer as the 6th communication in this series, resumed after 6 years

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1919 E Opsomming der periodische tranformaties van de torus, KAW versl., vol. 27 (1919), pp. 1363 - 1367.

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1919 G Énumeration des surfaces de Riemann régulières de genre un, Comptes rendus hebdomadaires de l'Académie des Sciences, Paris, vol. 168 (1919), pp. 677 - 678.

1919 H Énumeration des groupes finis de tranformations topologiques du tore, Comptes rendus hebdomadaires de l'Académie des Sciences, Paris, vol. 168 (1919), pp. 845 - 848; 1168.

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1923 B Over de rol van het principium tertii exclusi in de wiskunde, in het bijzonder in de functietheorie, Wis-en Natuurkundig Tijdschrift, vol. 2 (1923), pp. 1 - 7. (On the role of the Principle of the excluded third in Mathematics, especially in the theory of functions, reprinted in J. van Heyenoort, From Frege to Gödel, H.U.P. 1967, pp. 334 - 441)

1923 C Intuitionistische splitsing van mathematische grondbegrippen, KAW versl., vol. 32 (1923), pp. 877 - 880. (Intuitionistic analysis of the fundamental concepts of mathematics)

1923 D Over het natuurlijke dimensiebegrip, KAW versl., vol. 32 (1923), pp. 881 - 886.

1923 E Über den natürlichen Dimensionsbegriff, KAW proc., vol. 26 (1923), pp. 795 - 800. (* 1923 D)

1924 A Über die Bedeutung des Satzes vom ausgeschlossenen Dritten in der Mathematik, insbesondere in der Funktionentheorie, Journal für die reine und angewandte Mathematik, vol. 154 (1924), pp. 1 - 7. (* 1923 B)

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1924 E Perfect sets of points with positively irrational distances, KAW proc., vol. 27 (1924), p. 487. (* 1924 D)

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1924 K Intuitionistische Ergänzung des Fundamentalsatzes der Algebra, KAW proc., vol. 27 (1924), pp. 631 - 634. (* 1924 J).

1924 L Opmerkingen over het natuurlijk dimensiebegrip, KAW versl., vol. 33 (1924), pp. 476 - 478.

1924 M Bemerkungen zum natürlichen Dimensionsbegriff, KAW proc., vol. 27 (1924), pp. 635 - 638. (* 1924 L)

1924 N Bewijs van de onafhankelijkeid der onttrekkingsrelatie van de versmeltingsrelatie, KAW versl., vol. 33 (1924), pp. 479 - 480. (Proof of the independence of the relation of disjointness from confluence.)

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1924 P Bemerkungen zum Beweis der gleichmäszigen Stetigkeit voller Funktionen, KAW proc., vol. 27 (1924), pp. 644 - 646. (* 1924 O)

1924 Q On the n-dimensional simplex star in R_n, KAW proc., vol. 27 (1924), pp. 778 - 780; (short note and announcement KAW versl., vol. 33 (1924), p. 1008)

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1925 C Intuitionistischer Beweis des Jordanschen Kurvensatzes, KAW proc., vol. 28 (1925), pp. 503 - 508; abstract and announcement, KAW versl., vol. 34 (1925), p. 657.

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1) The publication date is given on the front page as 1933. It is, however, still possible that it was published in 1944 since it mentions 'prior publication in Euclides in 1944'. This discrepancy may be due to deliberate falsification of the date of publication during the German occupation of Holland, especially in view of the critical content of some Signific Articles.

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1952 C Over accumulatiekernen van oneindige kernsoorten, KAW proc., series A, vol. 55 (1952), pp. 439 - 441; also Indagationes Mathematicae, vol. 14 (1952), pp. 439 - 441. (Accumulation points of infinite species of points).

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CHAPTER II

BROUWER'S PHILOSOPHY

1.1 If philosophy is defined as the sustained process of reflection directed towards ultimate understanding, Brouwer can claim to be a philosopher. He was, however, not an academic, systematic philosopher who left in his writings a coherent body of theories concerning all important issues of human life and its relations to the exterior world. Apart from Leven, Kunst en Mystiek, his philosophic thoughts have only been fragmentarily reported in his mathematical writings such as:

Over de Grondslagen der Wiskunde (1907)
'Die mögliche Mächtigkeiten' (1908A)
'De onbetrouwbaarheid der logische principes' (1908B)
Het Wezen der meetkunde (1909A)
Intuitionisme en Formalisme (1912A and 1913A)
'Intuitionistische Betrachtungen über den Formalismus' (1928A)
Die Struktur des Kontinuums (1928C)
'Mathematik, Wissenschaft und Sprache' (1929A)
'Willen, weten en spreken' (1933)
'Consciousness, Philosophy and mathematics' (1948C)
'Historical background, principles and methods of intuitionism' (1952B)
'Points and spaces' (1953)
'The effect of intuitionism on classical algebra of logic'(1955)

The fact that Brouwer consistently brings his philosophic speculation into his mathematical writings indicates his concern with the deeper questions of the nature of mathematics, its justification and purpose.

We have already remarked[1] how Brouwer's pre-occupation with the foundations of mathematics is shown in his preference at the most important occasions of his life and career and in his emotional and passionate style of writing in marked contrast with the sober, almost clinical approach in his purely mathematical writing.

Leven, Kunst en Mystiek (1905) - and to some extent also his Vienna Lecture (1929A) and 1948C - reveals not only an obsession with with the moral justification of scientific activity but also a nagging

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[1] See above, p. 15.

suspicion of its truth. In his further philosophical writings an attempt is made to justify and redeem mathematical activity and to find a basis for mathematical truth in accordance with his speculative beliefs.

How very much these speculative beliefs have entered into his thinking and writing is clearly shown in the introductory paragraphs of his well-known 'De onbetrouwbaarheid der logische principes' (The untrustworthiness of the principles of logic) (1908B), in which he launched his attack on the Principle of the Excluded Middle. It also illustrates Brouwer's extraordinary use of language (which also in Dutch is difficult to grasp and even more difficult to translate):

'Science considers repetition in time of sequence, qualitatively different but supposably equal. This reduction (Du. vereenzamen, literally isolation) of idea to perceptibility and as such to repeatability appears after the irreligious separation of subject and non-realized realizability which has become something else (a faculty which originates in the primordial sins of fear and desire but which also returns without living fear or desire).

'The urge to realize these realizabilities is guided in the intellect via immediate realizations according to a mathematical system of supposable suppositions born from abstraction of repetition and repeatabilities.

'Everything that can appear as unrealized realizability can be conceived in systems of suppositions, therefore also religion; but then religious science also becomes irreligious: soothing one's conscience or idle play, or has only significance as means to an end.

'And like all irreligiousness, science has no religious reliability, nor any reliability in itself. Least of all, can a mathematical system of suppositions, free from the perceptions received and continued indefinitely, remain reliable in guiding according to these perceptions. Etc.[1]'

Reading Brouwer, one meets striking resemblances to Fichte's extreme solipsistic interpretation of Kant's subjectivism, to Schopenhauer's pessimism, and to Bergson's intuitionism. Yet Brouwer's philosophy is

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[1] 1908B, pp. 5 - 6. The quoted passage takes up more than one page of this article which is only seven pages.

unique. Not only by assimilating certain parts of various forms of idealism but also by pressing for a predominant role for mathematics in the general process of human thinking, he has created a philosophy which is strictly his own. It is this philosophy which Brouwer's intuitionist supporters have refused to adopt but which is the origin and inspiration of his particular brand of mathematics. Without it, intuitionism has only historical significance because of its useful but negative contribution in its criticism of classical mathematics, or it becomes a mathematical school with certain unquestioned traditions and dogmatic practices and abstentions.

We shall not attempt to reconstruct a complete and coherent philosophy from the fragments left in Brouwer's writings. We shall deal with some main doctrines which dominate his writing and are directly relevant to his philosophy of mathematics, and we shall comment on the influence of Kant, Schopenhauer and Bergson at the appropriate places.

1.2 Kant

Brouwer likes to trace the lineage of intuitionism right back to Kant.[1] In his inaugural address he states that with some adjustments Kant's philosophy of the a priori and of synthetic judgments could be accepted as the ultimate basis for the validity of mathematics: 'However weak the position of intuitionism seemed to be after this period of mathematical development, it has recovered by abandoning Kant's a priority of space but adhering the more resolutely to the a priority of time.'[2] The greater part of Chapter II of the Grondslagen is devoted to an examination of the adjustments needed in Kant's philosophy to accommodate the non-Euclidean geometries.[3] Commentators have not failed to characterize Brouwer's philosophy as essentially Kantian', although Heyting qualifies this by saying that 'this interpretation of Kant's theory of knowledge was rather special'.[4]

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[1] 'In Kant we find an old form of intuitionism', 1912A, p. 8; cf. also 1907, p. 94; 1909A, p. 5; 1912A, pp. 10 ff ; 1928C, p.6.

[2] 1912A, pp. 11 - 12.

[3] See above pp. 17 - 18.

[4] S. Körner, The Philosophy of Mathematics, pp. 119 ff. A. Heyting, 'Intuitionism in mathematics', La philosophie au milieu du vingtième siècle, Firenze 1958, p. 10.

The Romantic revolt against the Age of Reason had - to some extent - been initiated by Kant; he had placed the basis of the validity of mathematics firmly in the human mind itself which he saw as the active, sythesizing agent in the formation of concepts. In this respect Brouwer's philosophy can be called Kantian, as can indeed that of most nineteenth-century philosophers. But apart from this most general basis of agreement, and a more detailed account of time as the a priori form of intuition fundamental to mathematics, Brouwer's philosophy can only be related to that of Kant as the extreme consequence of <u>some</u> of Kant's doctrines, such as the subjective tendencies in the process of human cognition and the primacy of the human will. Here Brouwer followed a course more in line with the extreme views of Fichte and Schopenhauer than of the views of their master Kant.

Kant presupposed the a priori character of mathematics and its constructive, i.e. synthetic, character, using it to illustrate his more general metaphysical thesis for which he needed an elaborate and complex system of concepts and distinctions. Brouwer avoided the general problem of metaphysics and that of knowledge of the exterior world, and could, therefore, present a more straightforward consistent and simplified philosophy of life. His prime concern was with mathematics and mathematical objects which, because of their special abstract nature, can be more easily accommodated within the solipsistic isolation of the individual mind. To this restrictive conception of human knowledge Brouwer was already inclined on account of his mystic beliefs; these ultimately determined the central theme and point of departure for the whole of Brouwer's thinking. (Cf. Fichte's maxim that philosophical commitments in the last analysis are made on temperamental rather than evidential grounds).

1.3 Brouwer's egoism

Brouwer's philosophy is more than just idealism (Beth), or even metaphysical solipsism. Descartes' question whether our knowledge of the external world is just an invention of our mind[1] had been answered positively by Locke[2] and Fichte[3], negatively by Kant, who in his

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1 Meditations.

2 Essay concerning human understanding, Ch. IV.

3 Vocation of man, Ch. V.

Critique of Pure Reason set out to remove solipsism, 'this scandal to philosophy'[1]. These solipsists and their modern counterparts, such as Santayana and Schiller, were primarily concerned with the distortion of our knowledge of the exterior world through sense-perceptions.

Brouwer in his mathematical works, confines himself almost completely to mathematical entities, and here he takes a much stronger solipsistic position than any of the metaphysical solipsists, by denying the possibility of any real communication. With more right than Schopenhauer he could say: 'The world is my idea'[2].

Moreover, Brouwer's solipsism is more than a Humean scepticism which simply questions the validity of knowledge as an accurate mirror of exterior reality; Brouwer views external sense perception with moral suspicion, rejects it and shuns it. There is an element of 'self-seeking' in Brouwer's solipsism which can therefore be better described as 'egoism'. 'Turning into oneself' in search of reality and salvation is a dominating theme throughout Leven, Kunst en Mystiek.[3] It is also apparent in Grondslagen and other works on foundations[4] where 'introspection' and 'insight' become the valid and ultimate criteria for mathematical truth.

1.4 Consciousness and the 'self'

To Descartes, knowledge of the 'self' was superior to knowledge of material objects and the origin of all knowledge, the 'self' being apprehended by an act of 'introspection'. Kant introduced an 'inner sense', thereby obviating space as an a priori form of intuition for all knowledge[5]. Avoiding the complications of Fichte's transcendental Ego,

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1 Preface to 2nd edition.

2 The very first words of Schopenhauer's The World as Will and Idea.

3 Especially Chapter II which is named 'Turning into oneself'.

4 e.g. 1928A; 1948C; 1953.

5 Kant asserted that knowledge of the 'self,' like all knowledge, is given by affection of sensibility. He then (B 152) introduces the inner sense through which we ourselves are represented to consciousness as we appear to ourselves not as we are in ourselves.

Brouwer admits simply: 'What this 'self' is we cannot further say, we cannot even reason about it, since - as we know - all speaking and reasoning is an attitude at a great distance from the 'self'; we cannot get near it by reasoning or by words but only by "turning into the self" as is given to us'.[1] Later, the 'self' becomes almost identified with 'consciousness in its deepest home', the 'subject' or 'life in the mind'.[2]

This consciousness is a strange romantic mixture of the spirituality of the Christian, individual soul in the strongly popularized version of the mediaeval mystics, Jakob Böhme and Meister Eckhart, and the universal Hindu soul of the Bhagavad Gîta, in its exodus and return to Nirwana.[3]

Leven, Kunst en Mystiek describes life as the journey of the exiled, spiritual soul through this vale of tears; purely spiritual by nature, it is imprisoned in a body and its peace, 'stillness', is disturbed by sensations. Its own vision in accordance with its spiritual nature is direct, timeless contemplation, such as theologians ascribe to the godhead, the angels and the soul in life hereafter; intuition (Anschauung) in its purest form. In this mystic vision, all is seen directly, not interconnected through causality.[4] Our bodily senses force their

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1 1905, p. 13.

2 1948C, p. 1235.

3 Jakob Böhme: A German shoemaker who lived in the 16th and first quarter of the 17th century.
Brouwer shows appreciation of Christian ascetics (1905, p. 76), but the less dynamic character of Hinduism and Buddhism, its passive stoic acceptance and its awaiting deliverance from this bodily existence is more akin to Brouwer's philosophy of life; more so than the Christian outlook of salvation and elevation of human bodily life through God's incarnation (cf. also Schopenhauer, op. cit., I, p. 494).
In spite of frequent reference to 'God', 'Providence', 'the Saviour', mainly in quotations, Brouwer's religion is pantheistic. Trying to identify the 'self', he quotes Meister Eckhart (1905, p. 25): 'When all images have departed from the soul, and the soul alone contemplates the only One, the naked being of the soul will find the form-less being of divine unity which is present in a transcendental being', and Jakob Böhme: 'When you are silent, you are what God was before creation.'
Organized religion Brouwer regarded as a 'morphine industry'. (1905, p. 25) Usually when he speaks of religion, he means the whole area of (his) philosophy concerning human nature and life.

4 For the mystic there is no logic: 'Nowhere in mysticism is there a thread or a suitable sequence, every sentence stands by itself and does not need another sentence as its predecessor', (1905, p. 76); and 'To the intellect mysticism may sound incoherent, oracular, sometimes bombastic, even full of contradictions' (p. 77).

impressions upon the soul, blur this immediate vision, they impose 'the chains of plurality, separation, time, space .... and causality'.[1] 'The phenomena follow each other in time, bound through causality because you yourself want to see these phenomena clouded in this regularity; but for the free and enlightened, "miracles" constantly penetrate the walls of causality and flow through'.

This immediate vision is sometimes achieved through contemplation of works of art.[2]

Even if with difficulty ('In turning into one elf one will expereince difficulty; it seems as if some inertia has to be surmounted' [3]) man can through his will transcend the world of perceptions: 'You recognise your 'Free Will', in so far as it is free, to withdraw from this world, in which there is causality.'[4] The whole of Brouwer's ethics is an attempt to free oneself (Chapter VIII, The freed life)[5] from the effects the phenomenal world has on the soul. These effects cannot be avoided, they are 'irreversible',[6] and fatalistically Brouwer accepts this 'sad' and 'tragic' world.[7] Like Schopenhauer's, so Brouwer's pessimism is metaphysical; it does not just consist in empirical observation of pain and suffering, but it is a metaphysical belief based on the nature of human reality.[8] Life in this world, mathematical and perceptual cognition and social life are seen by Brouwer as consciousness gradually

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1 1905, p. 15.

2 ibid., p. 48.

3 ibid., p. 14.

4 ibid., p. 15.

5 Also in Chapters IV 'Appeasement', VI 'Immanent truth', VII 'Transcendental truth', and IX 'Economy'.

6 1948C, p. 1235.

7 e.g. 1905, pp. 11, 16, etc.

8 SCHOPENHAUER. Pre-occupation with mysticism, fatalism, anti-rationalism, back-to-the-simple life, are certainly general features of the continental romantic movement towards the end of the 19th century. However, the similarities between Schopenhauer's and Brouwer's philosophy are too many and too striking to be merely accidental. Schopenhauer's appeal, especially to amateur philosophers, was common, e.g. Wittgenstein's Tractatus (Wittgenstein was trained as an engineer), also E. von Hartmann, Philosophy of the Unconscious, and Freud.

Brouwer only twice mentions Schopenhauer by name, 1905, p. 61, and 1928C, p. 1, in both cases referring approvingly to his intuitive approach to knowledge.

moving farther away from its spiritual nature, 'the exodus of consciousness from its deepest home'.[1] In 'Consciousness, philosophy and mathematics' (1948C) Brouwer describes this transition of consciousness from its deepest home to the exterior world[2] as taking place in three phases: 1) The naive phase: the creation of the world of sensations; 2) the isolated causal phase: causal activity setting in; 3) the social phase: involvement in cooperation with other individuals.

From the original, pure state of stillness, of being unaffected by sensations, without a will, consciousness passes to the initial stage of simply having sensations. At this stage awareness of time is born, the primordial phenomenon, the move of time[3] to which Brouwer constantly refers: 'Consciousness in its deepest home seems to oscillate slowly, will-lessly, and reversibly between stillness and sensation. And it seems that only the status of sensation allows the initial phenomenon of the said transition. This initial phenomenon is a move of time.'[4]

1.5 The ability to see sequences in time is the basis of all human knowledge. 'Things' are nothing but 'iterative complexes of sensations whose elements are permutable in point of time' and so are 'individuals, i.e. human bodies, the home body of the subject included'.[5]

There is a strong resemblance on main issues such as: the dominant role of the human will (in Brouwer esp. 1913A, p. 82; 1929A, pp. 153 ff; 1948C, pp. 1235 ff.), a fatalistic acceptance and disapproval of causality, a rejection of any philosophy based on logical systems, an emphasis on intuition; but also in a common dislike of intellectualism, of the power of the state and of organizations (cf. Brouwer 1948C, p. 1241 - 'therefore one will only reluctantly join a clique or a union'), of women (cf. above pp. 8 - 9), a dislike Brouwer shared with his artist friend Piet Mondriaan, an incapacity for love, only sympathy; an emphasis on aesthetic contemplation in appreciation of the arts where there is again a remarkable agreement in personal tastes, and finally an enthusiasm for Eastern philosophies and mediaeval mystics, and a special preference for Meister Eckhart, shared by Goethe, who was at one time Schopenhauer's friend.

[1] 1948C, p. 1235.

[2] ibidem.

[3] see further p. 81.

[4] 1948C, p.1235.

[5] ibidem.

Up to this point Brouwer follows Schopenhauer, who distinguishes between 1° 'Verstand' (mind) - the faculty of perceiving objects, common to man and animal[1] and 2° 'Wissen' (reasoning) - rational knowledge, 'abstract consciousness'[2], the faculty of conceptual, reflective thought, distinctive to man. That Brouwer does not want to ascribe mind to animals follows only from his solipsistic refusal to accept even other human minds[3], 'a plurality of minds',: 'so that as a consequence of the plurality of mind, a mind would have to be assigned to animals as well'[4].

The main difference between Schopenhauer and Brouwer concerns causality. Whereas Schopenhauer in ascribing mind to animals assigns to them also 'knowledge of causality'[5] causality according to Brouwer originates in the human mind, is a human creation, super-imposed on the world by the human mind. Both Schopenhauer's and Brouwer's pessimism is based on the view of mankind possessed by delusion of causality, sliding away in a deteriorative process. But while Schopenhauer's will is not free, but is unconscious instinct, identified with the universal forces and energies of nature and inevitable: ('the force that germinates and vegetates in the plants... that by which the magnet turns to the north'[6], Brouwer's causality is essentially human, the evil effect of

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1 Schopenhauer, op. cit. Chapter I, p. 26. In the Supplement, Schopenhauer explains that by 'Verstand' he means what Kant had called 'pure sensibility'.

2 Op. cit. Chapter I, p. 66.

3 '... the subject in its scientific thinking is induced to place in each individual a mind with free will dependent on this individual, thus elevating itself to a mind of second order experiencing incognizable alien consciousnesses as sensations' (1948C, p. 1239).

4 1948C, p. 1240.

5 Schopenhauer, however, did accept 'degrees' of causal knowledge: 'the degree of its acuteness, and the extension of the sphere of its knowledge, varies enormously with innumerable gradations from the lowest form, which is only conscious of the causal connection between the immediate object and objects affecting it, to the higher grades of knowledge of the causal connection among objects known indirectly.... (op. cit. Chapter I, p. 26).

6 Op. cit., p. 142.

the human will'. 'This causal connection in the world is a dark force of human thought in the service of a dark function of will of mankind, who through it - as by enclouding of a nerve gas - tries to make the intuitive world powerless and ripe for her desires.'[1] 'Causal attention is a free will phenomenon '[2] and '...this (causal) attitude is not necessity but a phenomenon of life subject to the free will; this everyone can ascertain through inner experience'.[3] It goes one step further (away from the intuitive deepest home) than the purely mathematical awareness of time, it is the faculty ('originating in the primordial sins of fear and desire') of 'seeing repetition of sequences, qualitatively different but supposably equal'[4]. The relation of cause and effect is not one that exists in the exterior world, it is reduced to purely a matter of temporal order between elements of sequences, willed by man: 'An iterative complex of sensations, whose elements have an invariable order of succession in time, whilst if one of its elements occurs, all following elements are expected to occur likewise, in the right order of succession, is called a causal sequence.'[5]

Cunning act

It is the most powerful tool in the human armoury, it gives man the power to predict the future.[6] Brouwer's continued moral disapproval of causality is evident from his reference to 'the cunning act', defined in 1948C as: 'The cunning act consists in this, that in a causal sequence of eventualities, a later element not conatively attainable in a spontaneous way but nevertheless desired, (the aim), is realized indirectly by bringing about an in itself perhaps non-desirable but conatively attainable earlier element of the sequence (the means), and in its wake obtaining the desired element as its consequence'. [7] Similar

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1 1933, p. 47.

2 1948C, p. 1235.

3 1929A, pp. 153 - 154; also 1933, p. 46.

4 1908B, p. 5; for full quotations see p. 2.

5 1948C, p. 1235.

6 1905, p. 21; cf. 1948C, p. 1236:'Causal attention allows the development of the conative activity of the subject from spontaneous effort to forethinking enterprise'; also 1907, p. 82.

7 1948C, p. 1236.

definitions or descriptions of the cunning act are given in 1929A, p. 154, 1933, P.46, and also in Grondslagen (p.82). Strongest is Brouwer's disapproval in Leven, Kunst en Mystiek: 'The Intellect in the Life of Desire has done mankind a devil's service by linking two phantasies as means and end. Chained in the desires for one thing it causes the intellect to strive after one thing as the means to this end'.[1]

At the root of all science lies this 'desire to predict'[2], the presumed necessity of the causal sequence extended by isolating disturbing irregularities'[3] and by 'complementation'[4].

1.6 Social phase

Furthest away from the intuitive world of the deepest home is consciousness in the social phase: 'there is sadness when in a receding distance naivety vanishes for ever'.[5] At this stage man completely loses his concentration within his inner soul and acts on the assumption of the existence of other minds.[6] Apart from the moral doubt about human intentions in 'cooperative acting' , there is total dependence on experience and the physical senses in all aspects of this world of cooperation.[7]

It is only at this stage that language begins to develop, merely as a means of communication in the process of 'cooperative causal acting'; 'a wire-netting of will-transmission'[8] the third phase of 1929A and 1933 : 'imposition of will through sounds or symbols' .[9]

The emphasis on language as 'will-transmission', especially in 1929A, 1933 and 1948C is indicative of an increased interest in 'Significs' during this period, and its social concern for man at the receiving end of such imposition of will, very much under the influence of his colleague and friend G. Mannoury[10] whose signific views were strongly social.

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1 1905, p. 19.

2 1905, p. 21.

3 1907, p. 82.

4 1907, p. 83.

5 1948C, p. 1242.

6 'The subject in its scientific thinking is induced to place in each individual a mind with free will dependent on this individual thus elevating itself to a mind of second order experiencing incognizable alien consciousnesses as sensations' (1948C, p. 1239).

7 Chapter II of Grondslagen (1907).

8 1948C, p. 1237.

9 1929A, p. 153; 1933, p. 45.

10 Cf. above, p. 15.

Brouwer's signific tendencies, however, were not socially inspired; they originated in a low esteem of language itself and a denial of its truth. Ultimately it is not motivation in the use of language that is questioned and condemned; the metaphysical belief in the primacy and the spiritual nature of the individual consciousness, leading to the impossibility of real, i.e. direct communication, made Brouwer reject language as second-hand, while his solipsistic refusal to accept plurality of mind undermined the meaningfullness of and need for any communication whatsoever. This belief remained with Brouwer throughout his life; it is the main theme of Chapter V of Leven, Kunst en Mystiek, it can be found in 1907, 1908B, and also in 1929A, 1933 and 1948C, e.g. p. 1240: 'By so-called exchange of thought with another being, the subject only touches the outer wall of an automaton'. Page 1242 (1948C) describes the possibility of throwing off the shackles of cooperative causality: 'Recognition of a cooperative world captured in the delusion of causality as a reflex of man's guilt, does not imply eternal bondage to that world... if the delusion of causality could be thrown off, nature gradually resuming her rights would be generous and forgiving to a mankind decausalized and subsiding to more modest and more harmonious proportions... and perhaps at the end of the journey the deepest home vaguely beckons'.[1]

It is this degree of being removed from the deepest home that for Brouwer established the hierarchy: mathematics, science, language,[2] and so the a priority of mathematics.

1.7 Mathematical 'viewing'

It was Schopenhauer's personal tragedy that the will, which he had raised to such a prominent place in his philosophy, became identified with causality and ultimately responsible for all human misery. There is a similar irony in Brouwer's identification of mathematical thinking and causal thinking.

In 1928 Brouwer seems to come very close to Schopenhauer's views by almost identifying mathematical activity and will. The opening lines of his Vienna Lecture (1929A) analyse the 'individual's will to live' into:

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[1] 1948C, p. 1242; cf. also 1905, p. 15 as quoted above p. 65.

[2] The title and first words of 1929A.

1) mathematical viewing (German: Betrachtung), 2) mathematical abstraction, and 3) imposition of will through sounds.[1] In 'Willen, weten, spreken' (1933), which is largely a translation of 1929A, Brouwer corrects the impression that mathematics and will should be identified; he adds to the first paragraph: 'All these three are subject to the free will in extent as well as modality'.[2] Even if mathematics and causality are acts of the free will and in their three aspects constitute 'the individual's (and mankind's (1933)) will to live', the human free will can withdraw from the causal world and transcend it.

The parallel with Schopenhauer lies in their common moral disapproval of causality and Brouwer's identification of causal thinking and mathematical thinking.

Especially against those who made language and logic the starting point for mathematics, Brouwer wanted to show independence of mathematics from language and logic[3], a priority of mathematics to all science, by placing 'mathematical thinking' at the root and beginning of all human, mental activity.

Ability to see sequences in time is the mathematical ability. Chapter II of Grondslagen (1907) begins: 'Man has the faculty - accompanying all his interactions with nature - to view his life mathematically, to see in the world repetition of sequences,

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[1] The opening lines: 'Mathematics, science and language form the main functions of human activity by means of which man dominates nature and preserves order. These functions find their origin in the three forms in which the will to live operates in individual man....

[2] Identification of mathematics and will seems implied in the titles of 1929A and 1933: Mathematik, Wissenschaft und Sprache (Mathematics, Science and Language) becomes Willen, weten, spreken (lit. Wanting, knowing, speaking).

Close comparison of 1929A and 1933, however, shows that exactly those passages or expressions of 1929A which could be interpreted as identification of mathematics and will have been changed in 1933. Apart from the addition mentioned, the opening lines of 1933 leave out mathematics: 'Knowing and speaking are forms of activity by means of which individual man and mankind hold their own and impose their will. They find their origin in the three phenomena of human life, (1929A: in the will to live).

While the second paragraph of 1929A starts: 'Mathematical viewing comes into being as an act of the will, (Germ. Willensakt)', the corresponding paragraph of 1933 simply states 'Mathematical viewing is an attitude of life'.

[3] 1907, pp. 125 - 133; see further Chapter III especially 2.4.

'i.e. causal systems'.[1] Similar views are expressed in 1908B[2], 1912A[3], and especially 1929A. By reducing causality to mere repetition of sequences, Brouwer succeeded in establishing dependence of all forms of science on mathematics; by identifying logic with regularity in language, he placed logic in the sphere of 'social acting' and removed it altogether from the fundamental area of human thinking, this, however, at the cost of associating causality, the root of all human evil, with mathematics almost to the point of complete identification.

Brouwer never succeeded in completely dissociating mathematics from causality, but he moved to a solution whereby there is a branching off in the 'mathematical attitude' at a point as near as possible to the intuitive 'deepest homeof consciousness' into: 1) mere awareness of time, the primordial intuition of time which - without further reference to experience - through a process of pure mental construction leads to pure mathematics; 2) the causal attitude, based on the fundamental but simple mathematical concept of sequence, and leading to the accepted existence of a world of phenomenal objects, to individual and social causal acting, and to science and language.[4]

By making the distinction between 'mathematical viewing' (mathematical attitude, mathematical thinking) on the one hand, and pure mathematics on the other, Brouwer could uphold the a priority of mathematics and at the same time redeem mathematics to such an extent that to mathematics he could attribute beauty, wisdom and truth, 'transcendental' values exclusively found in the intuitive world of the deepest home: 'The fullest constructional beauty is the introspective beauty of mathematics, where instead of elements of playful causal acting, the basic intuition of mathematics is left to free unfolding. This unfolding is not bound to the exterior world, and thereby to finiteness and responsibility; consequently its introspective harmonies can attain any degree of richness and clearness.'[5] But: 'Searching for

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[1] 1907, p. 81.

[2] 1908B, p. 5.

[3] 1912A, p. 5.

[4] 'The mathematical 'Betrachtung' (viewing) comes into being in two phases: 1) temporal attitude (zeitliche Einstellung), 2) causal attitude.' (1929A, p. 153).

[5] 1948C, p. 1239.

'wisdom, we may find it in knowing that causal thinking and acting is non-beautiful and hard to justify...'[1] (Cf. to 1908B: 'In wisdom there is no logic'.- p. 12).

As to truth: 'Truth is only in reality'; observing Brouwer's restrictions on mathematical existence, truth can be found in intuitionist mathematics.[2]

2.1 Intuition

In his inaugural address, Brouwer describes the controversy concerning the philosophic foundations of mathematics as between the French Intuitionist School and the predominantly German Formalist School.[3] In spite of many disagreements on detail especially on the role of language and logi [4], he leaves no doubt about his fundamental loyalty to the cause of the intuitionist revival, to which he refers as neo-intuitionism.[5] When later Brouwer had established himself as 'the father of intuitionism' he used in his historical reviews to refer to those neo-intuitionists as the 'pre-intuitionist school, mainly led by Poincaré, Borel and Lebesgue.'[6]

French philosophic intuitionism

Preceding and underlying the revival in French mathematical intuitionism there is a long and steady tradition of philosophic intuitionism - going back to Pascal - leading, through Maine de Biran, Lachelier, Ravaisson and Boutroux, to the philosophic intuitionism of its most eminent representative Henri Bergson, a biologist as well as a philosopher and mystic. His views on consciousness, intuition and time are expounded in: Essai sur les données immédiates de la conscience (1889),[7] his famous article 'Indroduction a la métaphysique', Revue de Metaphysique et de Morale[8], vol. II (1903), and L'Évolution Créatrice (1907).[9]

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[1] 1948C, p. 1240.

[2] Op. cit., p. 1243

[3] 1912A, p. 7; cf. 1913A, p. 82.

[4] See further Chapter III .

[5] 1912A, p. 12; 1913A, p. 85.

[6] 1952B, p. 140; also 1953, p. 1.

[7] In an authorized translation, 'Time and Free Will, London- New York 1910.

[8] Translated by T.E. Hulme: An Introduction to Metaphysics, London-New York 1912.

[9] In an authorized translation, Creative Evolution, New York, 1911

While Kant starts with phenomenal knowledge and ultimately decides that knowledge of things-in-themselves is impossible, Bergson's starting point is the conviction of our knowledge of reality (i.e. life movement and continuity) and the impossibility of grasping this reality as a compound of representations through sense-perceptions. He distinguishes in human intelligence:

1) Intuition, derived from instinct, the intellect entering into its object, grasping reality without reference to experience or learning, producing knowledge which is simple, absolute, and language-less.
2) The analytical mind, which gathers information from sense perception and reasoning, acquires knowledge which is discursive, compound, relative and expressible in symbols.

Analytical knowledge only touches on the outside, 'it moves round the object, is relative;[1] it breaks up the outer shell of the object from which reality can never be reconstructed. In Bergson's terminology the analytic includes Kant's synthetic. Drawing on his experience as a biologist, Bergson sees the impossibility of grasping life starting from the external symptoms or rebuilding reality from components.

Any attempt at describing reality in words or symbols is analytic in this sense and, therefore, any form of science is analytic: 'It is easy to see that the ordinary function of positive science is analysis. Positive science works above all with symbols'[2]. Science adopts these symbols for reasons of economy and convenience but runs the danger of replacing in our minds the original by its symbols[3]. The inadequacy of language to describe 'reality', anything live, was a favourite topic of Bergson's.[4]

Intuition transcends all these imperfections of analysis; through the 'veils' of phenomena it sees reality directly. It yields knowledge of the self 'through an immersion in the indivisible flow of consciousness'.

Very much under the influence of Darwin's theory of evolution,

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[1] Introduction to Metaphysics, p. 1.

[2] Op. cit. p. 8.

[3] Op. cit. pp. 11 - 17.

[4] Concours General, Distribution des prix 1895; Delalan Bros, University Printers, pp. 6 - 12.

Bergson sees intuition as derived from and akin to instinct; instinct which is natural knowledge, not acquired through experience, prompting to direct action without any time for deliberation or choice.

In man, instinct is raised to intuition. Like instinct, intuition is direct insight, a sympathetic attitude to reality by which one places oneself inside an object, participates with it and identifies oneself with it. But unlike instinct, intuition interacts with the analytic intelligence; it is intellect: the faculty of 'viewing the thing from within' (intueri) or 'reading inside it' (intelligence)[1], or 'intellectual sympathy'[2].

Bergson is sometimes called anti-scientific because of his attitude to analytical knowledge. He does not, however, condemn the discursive methods of analytical science, only its claims of sufficiency in understanding reality. He does not dispense with reasoning, but subordinates analysis and discursive thought to intuition.[3] The totality of reasoning and proving does not constitute 'real' knowledge; reasoning may often be a necessary preliminary but ultimately it is intuition which through it all and beyond it sees reality.

Later, especially in his Creative Evolution (1907), Bergson in his concern with the dynamic continuum of life, time and movement, developed his theory of 'becoming'. As in cinamatography, the optical illusion of movement may be given through a succession of stills, but movement itself can never be analyzed. Movement itself is not produced by the juxtaposition of pause and pause but a continuous process of becoming, only grasped by intuition.[4]

2.2 French mathematical intuitionism

The French mathematical intuitionists show far less concern with the philosophical problem of how the human mind can grasp mathematical truth or even with the nature of mathematical reality.

Against attempts to base the whole of mathematics on logic, or on the consistency of arithmetic, they maintain that mathematics has more than formal significance, has what Heyting would call: 'inhaltliche Bedeuting'.[5] They all - Poincaré, Borel, Baire, Lebesgue and the

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[1] Introduction to Metaphysics, p. 6.

[2] Ibidem, p. 7.

[3] Creative Evolution, p. 177.

[4] See further pp. 183 ff.

[5] Heyting, Mathematische Grundlagenforschung, Intuitionismus, Beweistheorie; Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. III, part 4, p. 3, (Berlin 1934).

Russian Lusin - agree that mathematics is a description of some mathematical reality.[1] For this reason, they are sometimes referred to as realists; Brouwer calls it 'a modified observational standpoint'[2]. Equally common to them all is a resort to intuition in some form, as is indeed a vagueness as to what constitutes this intuition and this reality.

Poincaré is almost as ambiguous on intuition as Kant himself. Apart from intuition in the sense of Kant's empirical intuition - in the senses or the imagination[3] - or as a simple mental generalization through some kind of induction[4], he usually speaks of intuition in the sense of a purely mental grasping of fundamental principles or of reasoning as a whole. More than Russell's or Couturat's intuition, neede for the initial principles and axioms of logic but dispensed with at later stages, Poincaré insists on the need for intuition, as a guide at all stages in selecting axioms as well as the sequences in a logical argument: 'Which of all these possible routes is the best way to our final object? Who will tell us which to choose? We need a faculty that will show us from afar the final goal and that faculty is intuition. It is necessary to the discoverer in helping him to choose the way.' [5] Sometimes intuition becomes identified with synthetic a priori judgments basic to mathematics [6], (fundamental mathematical truths, immediately clear to the mind, which cannot be proved analytically, i.e. reduced to logic, or be ascertained from experience). Such an a priori synthetic judgment is the principle of complete induction [7], but also the axiom of choice. [8]

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[1] The linguistic expression is an essential part of mathematics, quite in contrast with Brouwer; see further p. 118.

[2] 1952B, p. 140; 'observational' to Brouwer is identical with 'empiric'; 'the observational standpoint' is the early classical philosophy of mathematics which deduced axioms from physical reality, (1952B, p. 139; also 1953, p. 1).

[3] Poincaré, La valeur de la Science, Paris 1905, Ch. I; cf. also 'Du role de l'intuition et de la logique en mathématique', Compte rendu du 2e Congres Internationale des Mathématiciens, Paris, 1900.

[4] E.g. by extending the visual image of a simple geometric figure in the imagination to a more complex one, or generalization.

[5] La valeur de la Science, p. 26.

[6] La Science et L'Hypothèse, Paris 1902.

[7] Ibidem,

[8] 'Les mathématiques et la logique', Revue de Métaphysique et de Morale, vol. 14 (1906), p. 313.

Mathematical reality as 'the deeper geometry' is also identified with intuition itself.[1]

Borel - There are many references in Borel's work to intuition, but none that explains the exact nature of intuition. He stresses, as did Poincaré, the necessity of intuition in mathematics, and seems to identify intuition with 'the deeper mathematical reality': 'the negative "freedom from contradiction" is not sufficient for mathematics which must be founded on the 'deeper mathematical reality'[2], and 'purely logical arguments, such as Hilbert's, have only verbal value; the constructed edifice does not for the moment bear any relation to reality'[3]; these logical arguments are 'symbols which do not correspond to any intuition'[4].

In his later works the demand for reality becomes the condition that the mathematical object can be thought in the concrete human mind[5] and can be expressed in a finite number of words.[6] (Also Lebesgue: nommable).

Sometimes intuition is linked to so-called "notions claires"[7] which, again, are not further clarified.

2.3 Kant's intuition

The starting point and only authority of Brouwer's central thesis concerning the intuition of time in his Grondslagen and elsewhere is Kant's theory of time and space as a priori intuitions.

Kant uses the word 'Anschauung', which covers a range of meanings different from that covered by its usual translation 'intuition'. (Brouwer uses in Dutch both 'aanschouwing' and 'intuitie'). While the primary meaning of 'intuition' is: immediate apprehension, and implies instinctive knowledge, the German 'Anschauung' means primarily: contemplation, view, direct inner vision.

Kant's Kritik der reinen Vernunft is an attempt to prove the possibility of real knowledge. Analyzing the process of cognition, Kant recognizes at some stage the need for some direct contact with the object known. The opening lines of the Transcendental Aesthetic refer

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1 Poincaré, Du Role de l'Intuition et de la Logique.

2 Borel, Leçons sur la Théorie des Fonctions, ed. 1928, p. 222.

3 Op. cit. p. 176.

4 Op. cit. p. 181.

5 'Sur l'illusion des définitions numériques', Comptes rendus hebdomadaires des Séances de l'Académie des Sciences, Paris (1947) t. 224, p. 765.

6 'Les paradoxes de l'infini', L'Avenir de la Science, nr. 25, Paris 1946.

7 Leçons sur la Théorie des Fonctions, p. 160.

to this direct link as 'Anschauung', a generalization of what is primarily visual: 'In whatever manner and by whatever means a mode of knowledge may relate to objects, intuition is that through which it is in immediate relation to them, and from which all thought gains its material'[1]. Most commentators agree that there is hardly any other term about which Kant makes more conflicting statements.[2]

He uses the word 'Anschauung' mainly in two meanings:

1) Empirical intuition, also perception or sensible intuition.[3] The process of human cognition starts with (not from) sense impressions. The specific, direct way in which human beings are affected by outside objects and receive representations he calls 'sensibility', a kind of passive receptiveness, an openness to the exterior world, a capacity for gathering in material for real knowledge.[4] 'Objects come to us through this sensibility, and sensibility alone provides us with intuitions.' [5]

This empirical intuition is not yet knowledge: by synthesis, the unifying act of the mind, 'many different intuitions are united into one consciousness'.[6] Kant here (B 104), refers indiscriminately to the plurality or manifold of sense-impressions and the manifold of intuition.

Within this empirical intuition Kant distinguishes:

α. a material element: the sensations;

β. a formal element: the spatio-temporal ordering of sensations. [7]

2) Pure intuition. This has little in common with empirical intuition save the term. It is 'the pure form of sensible intuition'.[8]

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1 A19, B 33. Translation Kemp-Smith. 'that through which' must here refer to 'mode' and 'manner', cf. the original: Auf welche Art und durch welche Mittel auch immer eine Erkenntnis auf Gegenstände beziehen mag, so ist doch diejenige wodurch sie sich .... etc.

2 R.P. Wolff, Kant's Theory of Mental Activity, p. 218; N. Kemp-Smith, A Commentary to Kant's Critique of Pure Reason, pp. 88 - 98.

3 Critique of Pure Reason, B 33.

4 'The capacity (receptivity) for receiving representations through the mode in which we are affected by objects, is entitled sensibility', (A 19, B 33).

5 Ibidem.

6 B 104.

7 B 34.

8 B 35.

Kant arrives at this pure form through abstaction from its matter sensation. Abstraction here is not to be taken in the sense of Aristotle, as providing us with a general notion inherent in the objects themselves, it is more a laying bare of the contribution of the human mind, a priori to sensation: 'Pure intuition which even without any actual object of the senses or of sensation, exists in the mind a priori, as a mere form of sensibility'[1].

It is exactly on this point - most relevant to Brouwer's intuition of time - that Kant is most confusing and self-contradictory. In his Metaphysical Deduction[2] space and time are represented not merely as forms of sensible intuitions, but as themselves intuitions which contain a manifold.

According to Kant, no perceptions at all are required for a system of pure geometry, and therefore pure intuition must exist as an actuality prior to experience.[3]

In none of these meanings, however, does Kant's intuition represent the ultimate, clear, direct, intellectual vision as interpreted by Bergson, rather it represents the initial stages of the complex process of cognition. The 'pure concepts of the understanding' - which include unity, plurality, totality - are the result of <u>synthesis</u>: 'What must first be given - with a view to a priori knowledge of all objects - is the manifold of pure intuition, the second factor involved is the <u>synthesis</u> of this manifold... and even that does not yet yield knowledge.'[4]

Kant maintained that there are 'two pure forms of sensible intuition' serving as principles of a priori knowledge, namely time and space;[5] they are contributions of the human mind since they are not found in objects themselves; they are a priori, not in the sense that they exist independently of a knowing subject, but as a necessary condition for all experience: all our knowledge is spatio-temporal.

<u>Space</u>. The discovery of non-Euclidean geometries certainly did disprove Kant's claim of the a priority of space as set out in A 23 - 29 and B 37 - 45; it does not disprove his general argument that there is an

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[1] B 35.

[2] B 102; cf. B 160; cf. also Wolff's comment, op. cit, p. 74.

[3] B 120; cf. also B 147: 'Through the determination of pure intuition we can acquire <u>a priori</u> knowledge of objects, as in mathematics, but only in regard to their form, as appearances; whether there can be things which must be intuited in this form, is still left undecided. Mathematical concepts are not, therefore, by themselves knowledge...'

[4] B 104, A 79.

[5] A 22, B 36.

a priori mathematics[1]; he simply uses Euclidean geometry as an example to prove this more general proposition.[2] When he speaks of knowledge of the self and the inner sense, Kant even admits the possibility of knowledge not depending on the a priori intuition of space.[3]

Pure intuition of time. While space is 'the pure form of all outer intuition', time is an a priori condition for each and every representation.[4] Time is a contribution of the human mind, a pure form of sensibility: 'Time is not something which exists of itself or which inheres in things - and does not, therefore, remain when abstraction is made of all subjective conditions of intuition'[5], and 'we deny to time all claim to absolute reality', for 'properties that belong to things in themselves can never be given to us through the senses. This is what constitutes the transcendental ideality of time'.[6]

Neither is time the object of knowledge: 'Time itself cannot be perceived'. (This is emphasized in many places, e.g. B 219, B 225, B 233.) Also, time must be distinguished from alteration or change, which requires synthesis: 'I perceive that appearances follow one another... Thus I am really connecting two perceptions in time. Now connection is not the work of mere sense and intuition, but is here the product of a synthetic faculty of imagination'.[7] The category substance is the principle of permanence in change, (unlike the view of Brouwer, who identifies permanence in change 'with the primordial intuition of time').[8] Finally, also the concept of number requires more than time as a pure intuition; it arises through synthesis: 'Number is therefore simply the unity of the synthesis of the manifold of a homogeneous intuition in general'.[9]

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1 A 34.

2 Cf. Wolff, op. cit. p. 293, p. 231.

3 Cf. above p.63.
Critique of Pure Reason, B 52.

4 A 34, B 51.

5 B 49.

6 B 52.

7 B 233.

8 Kant: See whole of First Analogy: 'Substance, Principle of Permanence', A 182 - 189, B 225 - 233.
Brouwer: 'Mathematics develops from one single a priori primordial intuition which can be called permanence in change, or unity in plurality', 1907, p. 179.

9 A 143.

2.4 Brouwer's intuition

The general feeling of confusion as regards the meaning of intuition in the writings of Kant, as well as of Poincaré and Borel, remains with us when we proceed to consider intuition as understood by Brouwer. Even if Brouwer in his definition of the 'primordial intuition of time' remains remarkably consistent, his appeal to intuition at different stages of his system of mathematical foundations and his further remarks on 'intuition' and 'intuitive' indicate a concept of intuition well beyond the simple definition of 'awareness of a move of time' and imply elements of the kind of intuition as understood by the French intuitionist philosophers. Neither can Brouwer's intuitionism be wholly understood as a system of constructive mathematics, differing from that of Kronecker or of E. Bishop only in a further attempt to explain the natural numbers.

2.5 Definition of the primordial intuition of time. Brouwer's account of intuition in his Grondslagen is rather vague and obscure. This could also be said of some of his technical explanations of his alternative analysis such as the continuum, the Principle of the Excluded Middle (PEM), and his choice sequences. But whereas in later works he develops his ideas on the latter and shows more clearly his original intention, his thinking on intuition has remained static. From the language used we can still detect Brouwer's emotional involvement, but he anxiously uses the same wording in defining this primordial intuition without giving further explanation.

In 1907: 'Man has the faculty - which accompanies all his interactions with nature - of viewing his life mathematically, the ability to see in the world repetitions of sequences or causal sequences in time. The primordial phenomenon here is the intuition of time through which repetition of "something in time and again something" is made possible. Because of this, life's moments fall apart as sequences of qualitatively different things, concentrated in the mind as mathematical sequences, not sensed (Du. gevoeld), but perceived (Du. waargenomen, i.e. made aware of);'[1] also: 'the primordial intuition is identical with awareness of time as nothing but change'[2].

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[1] 1907, p. 83.

[2] 1907, p. 98.

In 1912: 'This neo-intuitionism considers the falling apart of moments of life into qualitatively different parts, to be reunited only while remaining separated by time, as the fundamental phenomenon of mathematical thinking, the intuition of bare two-oneness'.[1]

In 1928 (and 1933): 'The becoming aware of time is the primordial happening of the intellect: the falling apart of a life's moment into two qualitatively different things, of which the one makes room for the other, but in spite of this, is retained by memory...'[2].

In 1948: 'By a move of time a present sensation gives way to another present sensation in such a way that consciousness retains the former one as a past sensation....'[3].

In 1952: '... a move of time, i.e. the falling apart of a life moment into two distinct things, one of which gives way to another, but is retained by memory. If the two-ity thus born is divested of all quality, there remains the empty form of the common substratum of all two-ities. It is this common substratum, this empty form, which is the basic intuition of mathematics.'[4]

2.6 In spite of Brouwer's claim that all intuition did, or had to do, was to strengthen Kant's a priori intuition of time[5] - (1909A, p.5 refers to it even as 'the form of intuition') - Brouwer's intuition of time differs essentially and in all respects from Kant's a priori form of intuition. Chapter II of Grondslagen, apart from refuting Kant's a priority of space, lists differences of opinion as to the link between experience and the intuition of time: nothing is said about the intuition of time itself.[6]

Whereas Kant's a priori form of intuition is a passive and highly abstract element in a metaphysical analysis of the process of cognition, very much intertwined with the more active schemata and categories of the synthesizing mind, Brouwer's intuition is a psychological first step, a simple awareness of time: 'the primordial intuition is identical with the awareness (consciousness) of time as nothing but change'.[7]

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1 1912A, p. 12; 1913A, p. 85.

2 1929A, p. 153; 1933, p. 45.

3 1948C, p. 1235.

4 1952B, p. 140; also in 1909A, p. 5; 1928C, p. 6; 1947B, p. 339; 1953, p. 2.

5 1912A, p. 12; 1913A, p. 85.

6 For a summary see above p. 18.

7 1907, p. 98.

Reflecting on individual human thinking, searching within the individual mind, Brouwer discovers what happened at the origin of all human thinking. Significantly, he speaks of the 'primordial happening' (Du. oer-gebeuren). Consciousness in its state of stillness - as described earlier - becomes affected by an exterior sensation and - what is essential for the intuition of time - by another sensation. (Kant's consciousness does not exist prior to any sensation[1] and his pure intuition of time is a priori to each single representation, is 'a form of our inner state', (see above p. 80).

In every account, Brouwer stresses the need for some Aristotelian arithmetical abstraction ('divested of all quality') from the content of sensation.

The 'move of time' consists in the consciousness linking a past sensation - retained by memory - to a present sensation; a simultaneous presence of two sensations, one of which is recognized as past: ' ... in such a way that consciousness retains the former one as a past sensation, and moreover, through this distinction between present and past, recedes from both and from stillness, and becomes mind. As mind it takes the function of a subject experiencing the present as well as the past sensation as object.'[2] There seems to be here some recognition of the synthesizing role of the mind - which Kant anxiously reserves for the 'understanding' (Verstand). But Brouwer is at great pains to point out that this element of linking, this synthesis, is an essential part of the intuition, as 'primordial' as (=' coequal with') the element of discreteness. In this way, he brings the continuous within the primordial intuition of time. '... the primordial intuition ... which is a unity of continuous and discrete, the faculty of thinking together different entities, linked by a "between" that never exhausts itself. Where, therefore, in this primordial intuition continuous and discrete appear as inseparable complements, both coequal and equally clear, it is impossible to treat either of them as the only original and to construct the other from it'[3]. In the summing up of his Grondslagen, Brouwer

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[1] Preface to second edition, footnote α to B xl; time as the form of inner sense: A 33.

[2] 1948C, p. 1235.

[3] 1907, p. 8.

again identifies the linking element with the primordial intuition itself in the form of 'permanence': 'Mathematics is developed from one single a priori primordial intuition which can be called permanence in change or unity in plurality'.[1]

How far removed from Kant's a priori intuition of time such a linking of past and present sensation is may be seen from this passage of Kant's Critique: 'Intellectual synthesis is carried out by the understanding alone without the aid of the imagination. In so far as imagination is spontaneity, I sometimes also entitle it productive imagination to distinguish it from the reproductive imagination whose synthesis is entirely subject to empirical laws, the laws namely of association, and which therefore contributes nothing to the explanation of the possibility of a priori knowledge. The reproductive synthesis falls within the domain not of transcendental philosophy, but of psychology'.[2]

Brouwer's a priori is the a priori of mathematics with respect to logic and all other sciences, a conditio - sine - qua - non.[3] By reducing causality to mere repetition of sequences[4], and so to number, he showed dependence of all sciences on mathematics;[5] by revealing the use of 'again' in the logicist atomic concepts and identifying this 'again' with the conception of number[6] he succeeded in showing 'independence of mathematics from logic and dependence of logic on mathematics'. Brouwer's intuitionist mathematics is an attempt to prove the sufficiency of the intuition of time in the construction of the whole of mathematics. But no attempt is made to prove that the intuition of time, as understood by

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[1] 1907, p. 179; cf. what has been said on change and permanence according to Kant, p. 80.

[2] B 152.

[3] 1907, p. 98.

[4] 'Repetition of sequences involves the primordial intuition of time twice, independently' (1907 p. 106).
Again, causality in Kant is more than repetition of sequences; apart from mentioning simultaeity of many instances of cause and effect, (B 248) Kant stresses 'necessity and universality' (B 233 - 256; A 190 - 211).

[5] 1907, pp. 81 ff; 1908B, p. 5; 1912A, p. 5; 1929A, p. 153; cf. also p. 71.

[6] 1907, p. 168; Brouwer's more powerful argument for dependence of logic rests on the purely linguistic nature of logic.

Brouwer, is a priori i.e. a necessary condition for the conception of number.

In Grondslagen awareness of the discrete in time is simply given as the only and the necessary mental process leading to the number concept, or simply identified with it: 'We cannot call anything a priori except that which is on the one hand common to all mathematics and on the other hand is sufficient to construct the whole of mathematics: the intuition of multiplicity - unity, the primordial intuition of mathematics. And since this intuition is identical with (Du. samenvalt, i.e. coincides) the consciousness of time as nothing but change, we can also say that the only a priori element in science is time.'[1] In further accounts[2] this identification is again stated, not proved.

On the basis of such varying interpretations of the intuition of time by Kant and Brouwer, we can hardly use Kant's positive proof of the intuition of time as an a priori form of intuition of all knowledge, including mathematics, to supplement Brouwer's proof of the a priority of his intuition of time in mathematics. Brouwer did indeed show that attempts to date, purely on the basis of axioms or language, had failed and must fail to provide an adequate foundation for mathematics; he also gave a plausible analysis of a possible way that one may arrive at the concept of number, but he did not prove that this is the only way, and that any other process of conception of number necessarily involves awareness of time.

In a footnote to the above quotation[1], Brouwer refers to a distinction between 'intuitive' time and 'scientific' time: 'Of course I mean here intuitive time, to be distinguished from scientific time, which is indeed a posteriori'[3]. This distinction ha been developed in Time and Free Will (1889)[4] by Bergson, who also refers to this distinction as between 'real' time (or 'time as change' or 'time as it appears to consciousness'), and 'measurable' time (or fictitious time' or 'mechanistic time'). Analyzing intuitive time in

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1 1907, pp. 98 - 99.

2 E.g. 1929A, p. 153; cf. also references given above p. 82.

3 1907, p. 99.

4 'Essai sur les données immediates de la conscience' (1889); see also above p. 73.

'Durée et simultanéité'[1] and 'Matiere et mémoire'[2] (1897), Bergson stresses the fundamental dichotomy of memory and present consciousness. But whereas this becomes the basis for Brouwer's concept of number, Bergson maintains that number is static and necessarily spatial: 'In space alone is a juxtaposition of this kind possible, because space alone, being homogeneous, permits units to be differentiated and identified at the same time, that is, to be added together'[3]. Also, Bergson sees number connected with measurable time rather than with intuitive time.

But against the background of French intuitionism - especially Bergson's identification of subjective time with life and reality - Brouwer's identification of intuitive time and number and his failure to give adequate proof for the a priority of time with respect to number can be understood. His aim was - counter to the claim of logicists and formalists - to find a basis for mathematics anchored in reality, which to him was the inner life of the mind, the reality of consciousness. His primordial intuition 'provided this anchor in reality'; also, if we accept the Bergsonian one and only reality of subjective time, this primordial intuition is a priori.

Finally, Brouwer's claim of a priority is not for time itself, but for awareness of time, the single act of consciousness at the root of 'all intuitive thinking'.

2.7 Brouwer's general concept of intuition

As a generalization of the primordial intuition, Brouwer's intuition in the wider sense is: consciousness acting without reference to any external perception.

Although there are some references in Brouwer's earlier work to 'instinct' and 'instinctive'[4], intuition can in no way be identified with instinct in the sense of knowledge collectively inherent in the

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1 pp. 61 ff.

2 pp. 177 ff.

3 Time and Free Will, p. 77; cf. whole passage pp. 73 - 85.

4 E.g. 1907, p. 81; also such expressions as 'it is intuitively clear', as in 1907, p. 3, might be read in this way.

human race, or individual knowledge which dispenses with all mental activity, a kind of hunch, a knowing 'off-hand'. Brouwer's 'intuitive' is not to be identified with 'self-evident', as is suggested by S. Körner[1], if self-evident implies complete elimination of effort on the part of the subject, the evidence being supplied by and intrinsic in the known object itself. Apart from the fact that Brouwer never uses the word self-evident, he did not recognize 'truths-in-themselves'.[2] Any analytic statements which could claim such self-evidence, Brouwer dismisses as non-statements or as 'meaningless'. As to the role of human consciousness in intuition, this is not one of passive, effortless undergoing of inescapable truths; even the primordial intuition is a creative act of the human mind. Self-evident in this sense certainly does not apply to Brouwer's 'intuitive' mathematics, which requires constructions more complex and rigorous than classical mathematics. Neither has the 'intuitive' anything to do with common-sensical thinking, which Brouwer treated with great suspicion.[3]

It is the restriction imposed on the conscious mind acting intuitively which provides the real key to understanding Brouwer's intuition: the complete independence from experience, except for 'the fact of having a sense impression'. Intuition is 'introspective'[4], inward-looking, (turning-into-the self), completely abstracting from experience through sense perception. Intuitive acting becomes opposed to 'causal acting', which is mental activity on the supposition or 'delusion' of an external world. Brouwer's intuition is the activity of the human mind, restricted exclusively to the data of consciousness itself and the tools provided by it. In the primordial intuition, which is independent of experience, the fundamental data and tools can be found: 'they are the elements of construction which can be read from the primordial intuition, concepts such as continuous, unity, again, "and-so-forth", ...'[5] and include the principle of

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1 The Philosophy of Mathematics, pp. 136 - 139.

2 Cf. 1907, p. 177; 1948C, p. 1243; 1955, p. 113; see further Chapter III, 'Language and logic'.

3 1955, p. 113.

4 Frequent references in 1948C.

5 1907, p. 180.

complete induction.[1] Brouwer claims that all elements of a priori intuitive thinking can ultimately be found in the primordial intuition, indeed that the whole of mathematics can be constructed from the elements of the primordial intuition: 'From the primordial intuition all mathematical systems, including spaces and their geometries, can be constructed independently of experience';[2] 'mathematics is a free creation of the mind, independent of experience; it develops from one single a priori primordial intuition';[3] and, 'Mathematics comes into being when ... the basic intuition of mathematics is left to an unlimited unfolding, creating new mathematical entities'[4]. This restriction, the independence of experience, or acting within the a priori data and tools of consciousness, is the ultimate criterion for intuitive mathematics (which for Brouwer was mathematics); 'Mathematics rigorously treated from this point of view, and deducing theorems exclusively by means of introspective construction, is called intuitionistic mathematics'[5] Intuition in this wider sense: the exclusive demand for introspection throughout the whole construction of mathematics, a complete and exclusive reliance on the elements found in consciousness, characterizes and delimits Brouwer's intuitive mathematics. The question whether the primordial intuition of time contains all the elements to be found in human consciousness, or whether Brouwer had in effect listed all those elements, is of secondary importance. Mathematics defined in terms of intuition in this wider sense is on the one hand restricted and debarred from using 'extraneous' elements such as language and logic, on the other hand it has complete freedom to act within this domain of the 'a priori' consciousness, free from the restrictions imposed by such extraneous elements, free also from the arbitrary restrictions self-imposed by formalization; the complete freedom which Brouwer claims for mathematics and denies to logic. But an obligation is imposed on the mathematician, especially when working within the media of language

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1 E.g. Thesis II, 'Complete induction is an act of mathematical construction already justified by the primordial intuition'.

2 1909A, p. 5.

3 1907, p.179.

4 1948C, p. 1237; cf. 1908A, p. 8; 1912A, p. 13; 1929A, p. 154; 1947B, p. 339.

5 1948C, p. 1243.

and symbols, to be constantly aware of the 'intuitive' nature of mathematics and guard himself against intrusion of extraneous elements: '... the only possible foundation of mathematics is to be found in this (intuitive) construction, or in the obligation to observe carefully what the intuition allows and what it does not'[1].

Intuition does not, therefore, serve us only in the initial stages of the conception of the natural numbers, but is a constant guide and determines the structure of mathematics at every stage. It made possible 'a second insight of intuitionism' at a later stage in Brouwer's career[2] which was not immediately clear from the primordial intuition. This intuition is <u>immediate</u>, like Bergson's intuition, but the immediacy of Brouwer's intuition is nothing else but the exclusion of external experience, the absence of any intermediaries in the process of intuition.[3] Both Bergson and Brouwer are equslly adamant in their insistence on the languageless nature of intuition: Brouwer, because he regards language as one of those extraneous elements, belonging to the external, causal world of cooperation, Bergson, because of the discursive, analytic character of language. Their interpretations of intuition vary in emphasis and depend largely on their different orientation. Bergson as a biologist needed the physical observation; his problem was the role of experience and discursive thought in the process of understanding and grasping the reality of life, a necessary preliminary but not a constituent element of the final, intellectual intuition. The moment a man 'intuits' he abandons the 'veil' of physical perception and discursive thought, and through it sees simply and directly.

Brouwer, a mathematician, could ignore the 'a posteriori' intuition (i.e. post experience); concentrating on the a priori intuition, he saw the possibility of restricting oneself to the 'immediate data of

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1 <u>1907</u>, p. 77.

2 See Chapter IV,'The Continuum after 1917'.

3 Heyting defines intuition as immediate knowledge and sees 'independence from experience' as flowing from it: 'The objects of mathematical knowledge are immediately grasped by the mind; hence mathematical knowledge is independent from experience'. (<u>Mathematische Grundlagenforschung Intuitionismus, Beweistheorie. Ergebnisse der Mathematik und ihrer Grenzgebiete</u>, Berlin, vol. III, part 4, p. 3.).

consciousness' and constructing the whole complex of mathematics with them. In this sense his intuition is synthetic and creative, but it lacks the simplicity of Bergson's intuition. Indeed, its stringent restrictions demand an almost superhuman power of memory and made Brouwer despair of the exactness of mathematics: 'Pure mathematics would indeed be exact provided it were practised in solitude, without the use of linguistic symbols by a human mind endowed with unlimited power of memory'.[1]

These restrictions and the freedom of intuition completely delimit the domain of Brouwer's mathematics and its methods; they dictate Brouwer's views on mathematical existence, language and logic, as well as his attitude towards formalism. Intuitionism is therefore an apt description of his mathematics and his philosophy of mathematics.

3.1 Mathematics

By the turn of the century, Kronecker's constructive demand for the existence of mathematical objects had been silenced. Hilbert's existence-proof and the formal axiomatic method of his 'Grundlagen der Geometrie' had been generally recognized and accepted, even by Poincaré.[2] The antinomies in set theory immediately called into question the logicist attempt to base the whole of mathematics on logic, but their implications for the formal axiomatic method were not fully realized. Hilbert was not to concern himself seriously with the problem of the nature of mathematics until 1917. But the central issue of Brouwer's work on foundations, as early as 1907, was the nature of mathematics and mathematical existence, the role of axiomatics and formalization in mathematics. He showed, indeed, great courage in defending in his dissertation a position partly in line with the discredited

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[1] 1933, p. 58. This is an addition not found in 1929A. Brouwer only gradually came to realize that intuitionist demands did not simplify mathematics. In his earlier works, the simplicity of direct vision is more often emphasized, e.g. 1907, p. 126: 'That in more complex situations a theorem is not directly clear, but only after a sequence of tautologies, only proves that we have made our constructions too complicated to be grasped at once.'

[2] Although with some reservations as to the foundations of these axioms themselves; cf. also p. 119.

constructive demands of Kronecker, and completely against the currently accepted trends of logicism and formalism, in defiance of the unshaken authority of Hilbert. It is not surprising that in the mathematical circles of Amsterdam University 'one did not know what to make of Brouwer's dissertation.[1] Brouwer claimed in 1927[2] that he discussed his views on the nature of mathematics with Hilbert in 1909, but his challenge became widely known and appreciated only after the first world war, when it started what Weyl would call 'the second crisis in the foundations of mathematics'[3]. It then roused Hilbert to renewed activity in the field of foundations, and caused him to make considerable adjustments.[4]

At the time of his dissertation, Brouwer was well aware of current mathematical trends. Frequent references reveal that he had read practically all that Hilbert had written to date. Notably absent is any reference to Kronecker.

The starting point in Brouwer's dissertation and his further foundational work is a simple philosophically-based conception of the nature of mathematics, of language and logic; these are carried to their ultimate conclusions with a typical disregard for current practice.

3.2 The nature of mathematics

> 'Mathematics is a free creation, independent of experience, which develops from one single a priori primordial intuition'.[5]

The problem of the relation between knowledge within a particular sphere, actually in the human mind, and the collective body of knowledge to date within that sphere, systematized and recorded, is not peculiar nor wholly confined to mathematics. This double aspect

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[1] A. Heyting, and H. Freudenthal, 'Levensbericht L.E.J. Brouwer' p. 1.

[2] 1928A, footnote p. 375: 'An oral discussion of the first insight took place in several conversations I had with Hilbert in the autumn of 1909'.

[3] H. Weyl, 'Über die neue Grundlagen krise der Mathematik'.

[4] Cf. H. Weyl, 'David Hilbert and his mathematical work', Bulletin of the American Mathematical Society, vol. 50 (1944), pp. 612 - 654; reprinted in Constance Reid, Hilbert, pp. 245-284.

[5] 1907, p. 179.

of any knowledge is generally understood and does not enter usually into the definition of the discipline concerned.

Brouwer's definition of mathematics is not just the mathematical case of a more general solipsistic claim for the subjectivity of all knowledge. He claims a place for mathematics completely outside the main division of the sciences (in the more general sense of any systematized knowledge); a human activity more akin to the artistic endeavours of a musical composer or of any other artist.

The whole of chapter II of Grondslagen is devoted to the nature of mathematics and its relation to experience. It deals with the necessary, a priori role of mathematics in all sciences, and clarifies the nature of mathematics and mathematical objects by comparing and contrasting them to science.

1° Definitions of branches of science confine themselves to their specific objects and methods. They all concern the exterior world as seen under one particular aspect; their objects are abstractions from the phenomenal world. Even logic comes into this category, since it concerns itself with statements. Mathematical objects are not abstractions in this sense: in all their aspects and at every stage of their construction they are creations of the human mind, only existing in the mind. At the root of mathematics there is no exterior object; in the primordial happening the mind creates its pure mathematical object, the concept of two-ity, and with the objects and tools of its own creation the mind can go on creating new mathematical objects.

2° The first phase of scientific activity, physical observation, is followed by 'causal activity', and finally (within the social phase) by recording. This analysis of scientific activity is typical of Brouwer's low regard for science, so eloquently expressed in 1905. The causal activity of science has all the aspects of immorality (cunning act)[1] and arbitrariness: 'In order to maintain the observed regularity as long as possible, one tries to isolate systems, i.e. to remove that which interferes with the regularity; in this way, man makes far more regularity in nature than occurs in it originally and spontaneously; he wants this regularity because it strengthens him in his struggle for existence by enabling him to predict and act accordingly'.[2] The essence of causal acting 'is a regularity which

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[1] See above, p. 68.

[2] 1907, p. 82; also 1912A, p. 6.

is willed, superimposed by man. Even if mathematics is a prerequisite of causal acting, to which Brouwer sometimes refers as 'mathematical observation' or 'mathematical viewing', it is the scientific systematization (i.e. causal acting) which Brouwer wanted to eliminate completely from pure mathematics, together with the other two aspects of scientific activity, observation and recording. He went even as far as to say: 'The edifice of intuitive mathematics is an act not a science'[1]. Causal activity can also be applied to this pure mathematics, subsequent systematization can lead to a 'mathematical science', but this is not Brouwer's pure mathematics. He continues: 'It (mathematics) only becomes a science, i.e. a summary of causal sequences repeatable in time, as mathematics of the second order which is the mathematical consideration of mathematics or of the language of mathematics; only here do we find causal relations in which mathematical systems or words and symbols follow one another; but, as in the case of theoretical logic, this is an application of mathematics, an experience-based science'.[2] Similarly, when referring to synthetic a priori judgments (i.e. 'possibilities of mathematical construction on the basis of the primordial intuition'), Brouwer warns: 'One should, however, not try to make these judgments the foundations of mathematics or of experience, they are a result of mathematical observation of the primordial intuition, therefore presuppose the primordial intuition, in the observation as well as the observed; they belong to what we call mathematics of the second order'.[3] In his analysis of formalization of mathematics,[4] Brouwer will again make this distinction and stress the non-pure-mathematical aspect of this second order mathematics. Later, in his debate with Hilbert, he will refer back to the above quoted passages of his Grondslagen, and identify the second order mathematics with metamathematics: 'The first insight first appeared in the literature in Brouwer 1907, where on pp. 173 - 174 the terms mathematical language and mathematics of the second order are used to distinguish between the parts of formalistic

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[1] 1907, p. 98.

[2] Ibidem.

[3] 1907, p. 119.

[4] 1907, p. 173; this analysis will be further discussed on p. 157.

mathematics mentioned above. This insight penetrated into the formalistic literature with Hilbert 1922[1]; (see in particular p. 165 and p. 174, where mathematics of the second order was given the name 'metamathematics'[2]). Granting the intuitive character of the second order mathematics or metamathematics, or as Brouwer put it: 'Recognizing that for the latter theory the intuitionistic mathematics of the set of natural number is indispensable'[3], does not amount to raising Hilbert's metamathematics to the level of Brouwer's pure mathematics as has been suggested.

As to the definition of pure mathematics as proposed by the formalist school: 'Mathematics is the science of formal systems'[4], this falls outside the scope of Brouwer's pure mathematics on every account. Brouwer's pure mathematics or intuitive mathematics is nothing else but intuition in the wider sense as discussed above: the whole of synthetic, constructive activity of the mind within the a priori limits of complete independence from the content of experience. To this concept of mathematics and his definition given in his Grondslagen Brouwer remained true for the rest of his life: e.g. 'Intuitionist mathematics is an essentially languageless mental structure which comes into being by the self-unfolding of the abstraction of two-ity as the mathematical primordial intuition'.[5] 'Mathematics comes into being, when the two-ity created by a move of time is divested of all quality by the subject, and when the remaining empty form of the common substratum of all two-ities, as basic intuition of mathematics, is left to an unlimited unfolding'.[6] 'Mathematics, deducing theorems exclusively by means of introspective construction, is called intuitionistic mathematics'.[7] 'Mathematics is inner architecture ...'[8]

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[1] 'Neubegründung der Mathematik', Hamburger math. Seminarabhandlungen I (1922)

[2] 1928A, p. 375.

[3] Ibidem.

[4] Hakkell B. Curry, Outlines of a Formalist Philosophy, p. 56.

[5] 1947B, p. 339.

[6] 1948C, p. 1237.

[7] Ibidem, p. 1243.

[8] Ibidem, p. 1249.

The domain of pure mathematics is delimited by the a priori restriction and well-distinct from that of any other branch of knowledge, even if intuition does play a role there.[1] Mathematical objects - elements as well as 'tools', are all concepts, thought, not proceeding from experience but as being synthesized from elements within the a priori intuition: 'By mathematical we always understand: lying within the domain of what can be thought in intuition'(Du. intuitieve denkbaarheden)[2].

3.3 Constructiveness

Referring to this purely mental activity, Brouwer frequently uses the Dutch word 'bouwen' or 'opbouwen' (resp. 'to build' and 'to build up').[3] This metaphor perhaps expresses more adequately the 'upward', synthetic and dynamic character of the mathematical activity than the usual translation 'construction', or the even more static 'structure'. Especially with the present proliferation of notions of constructiveness, even within the intuitionist camp[4], Brouwer's naive conception of construction cannot be overemphasized. They are nothing else but the mathematical activity itself in the strictest sense: intuitive constructions, taking place in the human mind within the a priori limits, a building upwards, step by step, with the self-created elements and tools, so creating ever more new mathematical entities and tools as one progresses in an 'unlimited self-unfolding'. These intuitive constructions are not necessarily simple or immediately clear: '... some constructions that we build are so complex that we cannot grasp them at once.' [5]

Any symbolic or linguistic recording of this mental, constructive process will - though imperfectly - reflect its progressive coherence

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[1] Cf. e.g. Bergson's a posteriori intuition as mentioned on p. 89.

[2] 1907, p. 151.

[3] There are a great many references: e.g. in Grondslagen: The title of Chapter I ('De opbouw der wiskunde), also pp. 37, 62, 63, 64, 67, 98, 119, 126, 128, 141, 147, 148, 149, 151, 152, 165, 167, 170, 173, 175, 177, 180, etc.

[4] See Constructivity in Mathematics, ed. A. Heyting, Amsterdam, 1959; cf also Kleene and Vesley, The Foundations of Intuitionistic Mathematics, Amsterdam, 1965.

[5] 1907, p. 126.

but, being outside the domain of first-order, pure mathematics, no active role can be ascribed to it in the actual mathematical construction itself. 'Mathematics is an autonomic interior constructional activity, which although it has found extremely useful linguistic expression and can be applied to an exterior world, nevertheless neither in its origin nor in the essence of its method has anything to do with language or an exterior world.'[1] Brouwer does use the word 'bouwen' and 'gebouw' (i.e. to build, construct, and edifice, construction) with reference to logic, but only to emphasize how logicists and formalists - unaware of the role of language as a post factum recording of a proceeding mathematical construction - are working completely within a non-mathematical medium, are 'building with words'.[2] Brouwer's mathematical constructiveness excludes any reference to logic, it is a demand for an intuitive construction, a complete intuitive ancestry for every mathematical object and operation; it can, therefore, not be equated with 'derivability', nor does it seem possible to refer to Hilbert and Brouwer as 'constructivists' in the same sense.[3] (Beth's identification of the constructive element in Brouwer's mathematics with the causal 'cunning act'[4] seems altogether out of line with Brouwer's thinking, and is not supported by any evidence from his writings.)

The stringency of this constructive demand became apparent to Brouwer very early on in his ambitious programme of constructing a pure mathematics, free from the pollutions of classical mathematics. Even if Brouwer was prepared to abandon a great part of classical mathematics, he never wanted to part completely with the 'negative'.[5]

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1 1955, p. 113.

2 1907, p. 132; also pp. 141 and 173.

3 Kreisel, op. cit. p. 43; Körner, The Philosophy of Mathematics.

4 E. Beth, The Foundations of Mathematics, p. 636. Beth here links constructivity with a concept of intuition of the kind proposed by Poincaré (see p. 76). Cf. Poincaré: 'Mathematical creation', published in The Foundations of Science, p. 387: 'In fact what is mathematical creation? It does not consist in making new combinations with mathematical entities already known. Anyone could do that ... To create consists precisely in not making useless combinations and in making those which are useful and which are only a small minority. Invention is discernment, choice.'

5 'Essentieel negatieve eigenschappen' (1948A); also 1948B, 1949A.

Searching for an intuitive construction of the negative, he made a valuable and widely acclaimed distinction between pure absence of a construction and 'absurdity' of a construction. However, even this strong Brouwer negation contained elements of hypothesis and could not avoid all appeal to logic.[1] In its simplest but most extreme form, the constructive demand is for a mathematical object to 'have been constructed'[2]. In such a completed construction there is no room for hypothesis, for contradiction, an infinite in any sense, nor for negation. Already in 1907, Brouwer accepted a seemingly weaker form of constructivity: the possibility of a construction; possibility, however, not in the sense of a vague probability or untested hypothesis, not even as an extra-mathematical pronouncement of its non-contradictority. In order to be acceptable to Brouwer as a tool within mathematics, constructibility had to be based on intuition, must have a constructive meaning itself. He regarded intuitive operations - the intuitive tools - as much as mathematical entities as concepts such as number. The intuitive existence of these operations, finite strings of operations founded in intuition, provided the constructive basis for Brouwer's constructibility. Possibility of a construction is only a 'manner of speech', a statement that a method has been constructed, an algorithm. Speaking of his 'denumerably unfinished sets', Brouwer says: 'we could introduce these sets but we then refer to a method, not a set',[3] and: 'we can introduce these words as an arbitrary expression for a known intention'.[4] Constructibility of an object in this sense refers primarily to the completed construction of a method, which is therefore an explicit and finite procedure. It forms the basis of Brouwer's theory of sets and functions, which will be discussed in Chapter IV .

Brouwer does not present a complete theory of constructions nor claim to do so; we quoted above (p.93) his warning that any theory

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[1] See further 'Truth and non-contradictority', p.136.

[2] 1907, p. 177.

[3] 1908A, p. 571.

[4] 1907, p. 148; cf. also Brouwer's division of statements quoted on p.154: 'α has not been proved to be true nor to be absurd but an algorithm is known ...' 1955, p. 114.

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of constructibilites (Du. 'bouwmogelijkheden', i.e. possibilities of building), which he calls 'a priori judgments', belongs to mathematics of the second order. Such possibilities of construction to be read off imediately from the primary intuition are:

1) The possibility of mathematical synthesis - the 'thinking together' in the conception of two-ity and its repetition in further plurality-unities.

2) The possibility of insertion (continuum).

3) The possibility of proceeding ad infinitum (axiom of complete induction).[1]

In the closing lines of his Grondslagen, he names some fundamental elements of mathematical construction, atomic in the sense that they are not themselves constructions of more fundamental constructions and 'therefore in any communication can only be expressed in a single word, sound or symbol: they are the constructional elements that can be read from the primordial intuition or from the intuition of the continuum, concepts such as continuous, unity, again, and-so-on'[2]. Moreover, completeness would be out of the question in view of the freedom of mathematics within intuition to construct ever more new mathematical entities (which include constructions): 'Mathematics develops itself in a self-unfolding (Du. zelfvermenigvuldiging, i.e. self-multiplication), guided by a free arbitrariness (Du. willekeur)[3]; an "arbitrariness" or freedom on the part of mathematics itself, which sets its own rules and is therefore completely autonomous, independent of any element outside the first-order pure mathematics; free of the non-mathematical restrictions of logic and free from the arbitrary restrictions of any formal system.[4] Because of this freedom, mathematics can never be exhausted in any one system.[5]

3.4 Intersubjectivity

Intuitive mathematics is in every respect non-objective, if objective is defined as 'belonging not to the consciousness or the

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[1] 1907, p. 119.

[2] 1907, p. 180.

[3] 1907, p. 119.

[4] 1952B, p. 140.

[5] Cf. also Carnap, Die Antinomien und die Unvollständigkeit der Mathematik, Monatshefte für Mathematik und Physik, 1934.

perceiving or thinking subject but to what is presented to this, or the non-ego, external to the mind' (COD). Indeed, no objectivity is claimed for mathematics by Brouwer. The objectivity he is prepared to ascribe to physical sciences is only 'invariance in our image of nature relative to an important group of phenomena'.[1] (Cf. thesis VII submitted for public defence on the occasion of his doctoral graduation: 'Attribution of objectivity to physical entities such as mass and number originates in a certain invariance relative to an important group of phenomena in the mathematical image of nature'). Objectivity is a 'gradual property', which 'physical time and space possess to perhaps a higher degree than any other physical entity'[2].

Körner insists that 'intuitionists do regard mathematical constructions as intersubjective experience'[3]. This Brouwer has never stated; there is in fact little room for intersubjectivity in Brouwer's general solipsist philosophy, and in particular in his concept of mathematics. Absolute intersubjectivity would require a philosophy of human nature which 1) accepts the existence of other minds, and 2) guarantees a certain sameness of the primordial intuition in all minds; further 3) a perfect medium of communication which assured that the same construction were effected by different subjects. In all its aspects intersubjectivity concerns the 'social sphere', and on every one of these aspects Brouwer expresses his grave doubts.

While 'other individuals, i.e. human bodies, "are only" iterative complexes of sensations whose elements are permutable in time'[4], the existence of other minds is a pure hypothesis. Brouwer strongly rejects what he calls 'plurality of minds': 'There is no plurality of mind, so much the less is there a science of the plural mind ;'[5] and 'in presupposing other minds, the mind elevates itself to a mind of second order, experiencing incognizable alien consciousnesses as sensations; quod non est.'[6]

The a priority (i.e. the necessity) of the primordial intuition for all thinking applies individually. In causal activity the

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[1] 1907, p. 95.

[2] 1907, p. 96.

[3] S. Körner, The Philosophy of Mathematics, London 1960, p. 136.

[4] 1948C, p. 1235.

[5] Op. cit. p. 1240.

[6] Op. cit. p. 1239.

supposition of sameness of the primordial intuition is very important, even essential, but remains a hypothesis: 'A very essential hypothesis in the mathematical viewing of fellow-men is, for instance, the supposition of the presence in each of them of a mathematical-scientific mechanism of intuition, acting, and reflection similar to one's own.'[1] Brouwer's views on the impossibility of real communication between men and the inadequacy of language will be dealt with in Chapter III on Language and logic. One of the reasons given in 1907 for the need for post-factum verification of 'verbal results' is that words may reflect different mathematical systems in different individuals: 'This verification can in different individuals lead to different results, because they relate the words of these conclusions to different mathematical systems in their minds which correspond to these words.'[2] In Brouwer's philosophy of the solitary mathematical mind there is no place for intersubjectivity (although he did not always live up to these beliefs in practice, as is shown in his deep resentment at being misinterpreted [3]).

3.5 Exactness and the 'creating subject'

Denial of intersubjectivity does not call into doubt the 'exactness' of pure mathematics, only of its social practice on the basis of language. Pure first-order mathematics, defined as mental constructive activity within its own rules, excludes all error. As is implied by the Dutch word 'wiskunde' (wis = certain, sure)[4] exactness is an essential element of mathematics since it avoids all reference to an exterior world. Like mathematics itself, its exactness is in the mind: 'The question where mathematical exactness does exist is answered differently: the intuitionist says: in the human intellect.'[5] It was certainly part of Brouwer's programme 'to formulate intuitionist mathematics as adequately as possible'.[6] Brouwer's intuitive

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1 1933, p. 48.

2 1907, p. 136.

3 See further p. 117; also above p. 39.

4 Brouwer comments on the trust expressed by the word 'wiskunde' — 'that the exactness of arithmetic and geometry will never be disturbed by experiment.' (1912A, p. 7).

5 1912A, p. 7; 1913A, p. 83.

6 1947B, p. 339.

mathematics, however, can hardly 'be identified with exact and precise formulation of thought', as Larguier maintains.[1] 'The languageless constructions originating in the self-unfolding of the primordial intuition are exact and true as far as they are present in the human memory.'[2] Brouwer, however, recognised human fallibility even in the individual practice of mathematics through the limits of memory: 'The power of human memory which has to take in these constructions is limited and fallible. For a human mind with an unlimited power of memory, pure mathematics, practised in solitude and without the use of linguistic symbols would be exact.'[2] This recognition leads to the ultimate consequence of Brouwer's conception of pure mathematics as mental activity: the hypothetical nature not only of mathematical exactness but also that of pure mathematics itself. This conclusion was clearly drawn in 1933 where Brouwer speaks of 'hypothetical human beings empowered with unlimited memory '[2]. In his subsequent writings, Brouwer refers to this idealized mathematician as 'the subject' or 'the creating subject'.[3]

Kleene claims that with the introduction of the 'creating subject' Brouwer 'developed a new method after 1948'.[4] In using the term 'creative subject', I believe, Brouwer has not developed a new method or theory, but simply clarified the nature of pure mathematics, which is not only languageless but also free from the limitations of memory in a particular individual. In any formulation of intuitive mathematics - which has exactness only to a degree, ('as adequate as possible') - attempts should be made to eliminate as much as possible both the deficiency on account of language and that of the individual memory; Brouwer's creating subject is 'a hypothetical human being with unlimited power of memory who uses words only as invariant signs for definite

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1 Everett H. Larguier in 'Brouwerian Philosophy', Scripta Mathematica, vol. VII (1940), pp. 69 - 78, states on p. 72: 'According to Brouwer, mathematics is identified with exact and precise formulation'. He does not give any reference, nor have I found any such statement in Brouwer's writings; moreover, it seems completely out of character with Brouwer's views on language.

2 1933, p. 58.

3 'Subject', e.g. 1948C, p 1235 and 1236.
'The creative subject', e.g. 1948A, p. 963; 1948C, p. 1246.

4 S.C. Kleene and R.E. Vesley, The Foundations of Intuitionistic Mathematics, Amsterdam 1965, p. 174. A formalization of Brouwer's theory of 'the creative subject' was also developed by A.S. Troelstra, Principles of Intuitionism, Springer Verlag, Berlin-Heidelberg-New York, pre-publication version 16.1.

elements and definite relations between these elements of pure-mathematical systems created by him'.[1] No new element is introduced with 'the creating subject': awareness is expressed of the distinction between pure mathematics - which, although exact and perfect, is still subjective - and the mathematics practised by the particular individual with all the limitations of memory, time, etc.

A new element had indeed been introduced, but this as early as 1918 with the extension of sequences to 'choice sequences', wholly dependent in their constitution on the free will of the acting mathematician. Because of the arbitrariness of this choice, the active role of the mathematician may seem more prominent; also, the limitations of the particular individual were here strongly felt, especially in relation to the 'proceeding ad infitum'.[2] The supposition, however, of the subject creating all mathematics, even the natural numbers, is absolute and forms the basis and starting point of Brouwer's philosophy of mathematics as early as 1907. The phrase 'the creating subject' could, without change of meaning, replace the words 'I' or 'we' or a passive voice construction, found in earlier attempts at constructive mathematics.[3]

3.6 Applied mathematics

The essential requirement of complete independence from experience leaves no room for an applied mathematics as mathematics in the strict Brouwer sense. Unlike Kant, who regards 'pure mathematics as knowledge only in so far as it can be applied to objects of experience'[4] and

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[1] 1933, p. 58.

[2] 1918A, p. 3, (corrections ad 1919A, p. 34): einer unbegrenzten Wahlfolge; cf. 1925D, footnote p. 245, also 1952B, p. 142.

[3] Compare e.g. the very beginning of Grondslagen: 'If I break off this sequence at 23, and I write down this same sequence again under the previous one, there then exists a one-to-one correspondence between the two rows. If I interchange two of the numbers in the top line, the one-to-one correspondence still remains. I can make sure ...' (1907, p. 3). On page 4 (on the construction of addition): 'By 3 + 4 I understand: I count to 3 first and then continue counting, but make the subsequent elements correspond to the sequence of ordinal numbers 1...4'. A passive voice sentence e.g. in 1918A, p. 3: 'A set (Germ. Menge) is a rule, on the basis of which, whenever an arbitrary numeral-complex of the sequence is chosen, every one of these choices produces a definite symbol or nothing...'

[4] Critique, B. 147.

otherwise 'a playing with a mere figment of the brain'[1], Brouwer treats applied mathematics as second-order mathematics, 'causal acting', science, with all the implications of immorality, arbitrariness and inaccuracy. One of the four conclusions of his dissertation concerns applied mathematics and sums up his views: 'This projection of mathematical systems on experience is a free act, which appears to be effective in our struggle for existence; one mathematical system may appear more practical or economic than another, at least in relation to a certain category of aims we hope to achieve by means of these systems; absolutely effecient none of them are.'[2] In its various forms, applied mathematics presupposes the existence of an exterior world: the physical world, language, and even a completed mathematical system.[3]

The process of application is seen as an arbitrary superimposition of a pure mathematical system on a phenomenal, non-mathematical medium for man's convenience, a 'projection'. Applied mathematics is 'anthropomorphization of nature[4], causal acting in its various stages, analyzed by Brouwer in Grondslagen as:

1° 'The seeing of regularity into nature' (p. 81);

2° Isolating this regularity by 'removing what disturbs regularity' (p. 82);

3° 'Complementing the phenomena with what man himself has brought about' (p. 83), and 'extending reality to possibility through inductive generalization' (p. 84).[5] In this process, mathematics is reduced to 'providing a supply of ready-made "unreal" causal sequences which are waiting for an opportunity to be projected upon reality'.[6]

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[1] Critique, A. 157.

[2] 1907, pp. 179 - 180.

[3] See above, p. 93, quotation [2].

[4] 1907, p. 85.

[5] Cf. 1933, p. 48; cf. also 1948C, p. 1237: 'The significance of mathematics with regard to scientific thinking mainly consists in this, that a group of observed causal sequences can often be manipulated more easily by extending its of-quality-divested mathematical substratum to a hypothesis, i.e. a more comprehensive and more surveyable mathematical system.'

[6] 1907, p. 83.

The projection of a pure-mathematical system on a phenomenal medium is purely arbitrary, the regularity observed and extended is not inherent in the medium in question[1], and no particular mathematical system is necessary: 'Which mathematical systems are chosen is largely a matter of convenience, taste and habit'[2].

Although the application of mathematics is 'a source of power for man[3], and usually effective, failure to distinguish between the pure-mathematical system and its projection can lead to confusion of properties and conclusions either way: the necessity and exactness, proper to pure mathematics is transferred to the 'mathematical image[4] and becomes 'law' in the hypothesis of the so-called scientific theories'.[5] On the other hand - and this applies especially to logic - conclusions within the medium of projection are applied to the mathematical system which underlies it. This is Brouwer's main argument for independence of mathematics from logic against logicists 'who make this jump once and then move alternately in both domains'.[6]

4.1 The foundation of mathematics

> 'Research in foundations of mathematics is inner inquiry with revealing and liberating consequences, also in non-mathematical domains of thought'.[7]

In the study of the foundations of mathematics two main branches can be considered:

1° The actual mathematical activity in its initial and fundamental stages of construction;

2° reflection on the nature of mathematics, its origin, its fundamental principles and concepts, and its relation to other human activities.

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[1] 'Causal attention is an imagining (Du. fantazeering) of the identification of different sequences of phenomena...' (1933, p. 46).

[2] 1909A, p. 13; cf. also 1907, p. 120: 'Experience a posteriori cannot teach us anything about the necessity of certain mathematical systems in experimental science.' See also p. 18.

[3] See above p. 68.

[4] Examples given in 1907, p. 85 are the differentiability and continuity of the 'physical functions'.

[5] 1933, p. 48.

[6] 1907, p. 172.

[7] 1948C, p. 1249.

Much of Brouwer's work in the foundations of mathematics falls in the first category, and can therefore rightly be called intuitive or intuitionistic mathematics. As such, we must consider his construction of number systems, the measurable continuum, theory of sets and functions, mainly found in: Chapter I of Grondslagen (1907), 'The foundations of set theory independent from the logical principle of the excluded middle', parts I and II (1918A and 1919A) and 'The foundations of intuitionist mathematics' (1925D, 1926A and 1927A). Some of his work is of the second kind, e.g. Chapters II and III of 1907, The Untrustworthiness of the Principles of Logic (1908B), The Nature of Geometry (1909A), his inaugural address Intuitionisme en Formalisme (1912A), his Vienna lectures (and 1933), and most of his lectures abroad after the second world war.

More often the distinction is not so clear; many of his contributions on the foundation of mathematics are a mixture of constructive mathematics and statements about his own mathematics and that of others.

4.2 Reflection on the nature of mathematics

In this present chapter we have tried to give a coherent survey and analysis of Brouwer's views on mathematics, starting from his speculative beliefs on the nature of man, consciousness and intuition, summarizing this in the title 'Brouwer's philosophy'.

According to Evert Beth, Brouwer's colleague and friend, the fundamental philosophical problems of mathematics can be approached from without or from within: 'They are approached from without when starting from philosophical, psychological, sociological or psycholinguistic principles, we try to penetrate the more specialized subject matter of extant mathematical theories. They are approached from within when we start from the study of contemporary mathematics and progress towards those problems which, through their general and profound nature and their importance for other fields of science, assume a philosophical character; this second line of approach is currently adopted by those workers in research on foundations who are known as cantorists, logicists, formalists or intuitionists'.[1]

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[1] E.W. Beth, The Foundations of Mathematics, Amsterdam 1968, p. X.

In his Mathematical Thought,[1] the pragmatic character of this distinction appears more clearly when he expresses concern about the alienation between philosophers and mathematicians investigating the problem of foundations of mathematics. Elsewhere[2], he made a similar distinction: between the 'philosophy of mathematics' and the 'philosophical practice of mathematics' (i.e. 'critical investigation of mathematics without leaving the domain of mathematics').

Differentiation between professional mathematicians and philosophers is fairly modern (i.e. post 1800) and has arisen from the need for specialization and division of labour. While there will clearly be a difference in emphasis in their respective treatments, reflecting on the nature of mathematics, is primarily a philosophical activity which is essentially the same for both mathematicians and philosophers.

It requires a certain comprehensive knowledge of mathematics, a grasp of the essential and fundamental mathematical activities; indeed, a formidable demand which disqualifies many a gifted philosopher. As reflection, it requires an outside vantage point from which mathematical activity and its fundamental concepts and principles may be seen in relation to other areas of human life and activity and be delimited. Such fundamental questions as the nature of mathematics and mathematical existence, the criteria for mathematical truth, and the relation of mathematics to a phenomenal world cannot find an answer within the domain of mathematics itself and ultimately require a set of convictions to which we earlier referred as speculative beliefs. How the answers to these questions have profound repercussions on the practice of mathematics is especially clear from Brouwer's programme, which is inspired, not so much by mathematical considerations as by a much wider philosophy covering man and the world.

Since philosophy has no distinctive subject matter, it cannot be placed side by side with mathematics or any physicial science. A clear-cut distinction between philosophical and mathematical can only be made in so far as mathematics, its subject matter, principles and methods are defined. Research into the foundations of mathematics is part of a more general inquiry into the ultimate nature of things,

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[1] E.W. Beth, Mathematical Thought, Dordrecht 1965, p. 2 ff.

[2] E.W. Beth, Inleiding tot de Wijsbegeerte der Wiskunde, Antwerpen-Brussel 1940, p. 6.

which constitutes philosophy. Inside knowledge of mathematics, and training in the use of its methods, are a distinct advantage to the mathematician-researcher into the foundations of mathematics; possible dangers are: overconfidence in the self-sufficiency of its principles and methods, and unwillingness to cross the boundaries of mathematics in his investigation of its fundamental concepts and principles, with a consequent risk of circularity and self-reference.

Brouwer was well aware that in his investigation of the foundations of mathematics he was going beyond what he himself regarded as pure mathematics, and he recognized the philosophical character of much of his work. He distinguished between his work on the foundations of mathematics and 'contributions free of philosophy' (Du. filosofievrije)[1]; some of his major contributions - such as 1908B, 1933 and 1948C - appeared in pure philosophical journals.

Nowhere in his writings is there a trace of fear that the autonomy of mathematics is threatened by philosophical considerations, although no one was more anxious than was Brouwer to safeguard the independence and autonomy of mathematics against the intrusion of alien principles of such phenomenal sciences as logic. The vague reference to 'inner inquiry' (introspection) in defining the study of foundations of mathematics is an indication of Brouwer's criterion for admitting other, supra-mathematical spheres into the inquiry.

Not only has Brouwer's ideological conception of consciousness determined the nature of his intuitive mathematics, in this inquiry the mathematical activity as a whole is subjected to a probing and judging on aesthetical and ethical grounds. Concern for its moral justification is very much in evidence in 1905, 1907, 1929A and 1948C. Especially, 1948C ('Consciousness, Philosophy and Mathematics') is an introspective search for the 'transcendental values'[2] of 'beauty, wisdom and truth' in mathematics[3].

It furthermore affected the basic principles and concepts, content and methods of intuitive mathematics. Dissociation of mathematics from science (applied mathematics) is very much inspired by moral consideration[4] and the lack of beauty and wisdom: 'In causal thinking and

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[1] 1921A, p. 798.

[2] 1905.

[3] 1948C, p. 1238.

[4] see above p. 72.

' acting beauty will hardly be found'[1] and 'Searching for wisdom we may find it in knowing that causal thinking is non-beautiful and hard to justify'[2]. Logic is rejected because of its 'lack in wisdom'., ('In wisdom there is no logic'[3]) and its dependence on language ('where transcendental truth is not found either'.[4]).

There is beauty in constructive mathematics: 'The fullest constructional beauty'.[5]

Whereas Brouwer's friends and followers tried to eliminate philosophical elements from intuitionism, Brouwer himself saw the deeper issues inextricably linked with his mathematical work and ridiculed 'those who wear philosophy like a Sunday suit' [6].

Beth's distinction in approach to the fundamental problems of mathematics between 'from the inside' and 'from the outside' does not coincide with a distinction between 1° the philosophical reflection on the nature of mathematics and 2° Brouwer's 'mathematical consideration of mathematics', as discussed above pp 93 ff. This second-order mathematics or metamathematics is not what Beth vaguely describes as 'starting from the study of contemporary mathematics and progressing towards those problems which, through their general and profound nature and their importance for other fields of science, assume a philosophical character' , but causal acting on the constructed system of pure mathematics, a systematization of mathematics into a science.[7]

5. Brouwer's programme

Preoccupation with the philosophical aspects of the foundations of mathematics marks Brouwer's work before 1917. During that time, he shaped his ideas on the nature of mathematics and its relation to experience and to science in accordance with his mystical and philosophical beliefs. Complete divergence of these ideas from the current trends of logicism and formalism, coupled with Brouwer's passionate

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[1] 1948C, p. 1238.
[2] 1948C, p. 1240.
[3] 1908B, p. 12.
[4] 1905, p. 66.
[5] 1948C, p. 1239.
[6] 1905, p. 41.
[7] See above p. 102.

conviction that his was not just another view but the only right one, accounts for the predominantly critical character of most of Brouwer's earlier work.

In the development of pure intuitive mathematics, language and logic do not play any part. Concentration on these non-mathematical activities was necessitated by the dominant role they played in logicist and formalist practice of mathematics and by Brouwer's desire to convince others, mainly by searching for and exposing inadequacies in their systems as symptoms of a much deeper lying disease, namely the introduction of such extraneous elements as language, logic and formal systems into mathematics itself. This critical, negative treatment dominates the whole of chapter III of Grondslagen, where successively 1° axiomatization of mathematics (pp. 133 - 142), 2° Cantor's naive set theory and transfinite numbers (pp. 142 - 159), 3° Peano's and Russell's logicism (pp. 159 - 169), 4° Hilbert's formalization (pp. 169 - 175), are exposed and critized for their complete reliance on language and logic.

Further steps in this programme of 'liberating mathematics' were the rejection of the Principle of the Excluded Middle and existence theorems. The more positive side of a constructive, intuitive mathematics was obscured by the dramatic impact made by Brouwer's negative and critical claims, which did not have the intended effect of drawing attention away from the linguistic domain. Many commentators, while dismissing his positive philosophy of mathematics, acclaim his negative contributions: Errett Bishop, who has little time for Brouwer's 'preoccupation with the philosophical aspects of constructivism'[1], states: 'Brouwer's great contribution was to analyse intensively the inadequacies of classical mathematics'[2].

Brouwer himself, in retrospect, characterizes his activity during this period - the first act of intuitionism - as purging mathematics from the non-mathematical elements of language and logic: 'The first act of intuitionism completely separates mathematics from mathematical language, in particular from the phenomena of language which are

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[1] E. Bishop, Foundations of Constructive Analysis, New York 1967, p. 6.

[2] E. Bishop, Mathematics as a numerical language (unpublished); cf. also: E. Beth, Constructivity in Mathematics, and Foundations of Mathematics.

described by theoretical logic, and recognizes that intuitionist mathematics is an essentially languageless activity of the mind'[1], and accepts elsewhere[2] that this negative programme 'seems necessarily to lead to destructive and sterilizing consequences'.

As the full implications of rejection of language and logic and Cantor's naive set theory became gradually clearer,[3] Brouwer realized that the construction of intuitive mathematics was not as simple an affair as his first chapter of Grondslagen might suggest. Under the above restrictions, and 'with the basic operation of mathematical construction, the mental creation of two-ity', it was possible 'to generate each natural number, the infinitely proceeding sequence of the natural numbers, arbitrary finite sequences, and infinitely proceeding sequences of mathematical systems previously acquired, finally a continually extending stock of mathematical systems corresponding to the "separable" systems of classical mathematics'[4].

Elsewhere, however, Brouwer had already credited the 'pre-intuitionist school' with establishing autonomy and a priority for the 'separable parts' of arithmetic and algebra: 'For these parts of mathematics they postulated an existence and exactness independent of language and logic, and regarded their non-contradictority as certain, even without logical proof. For the continuum, however, they seem not to have sought an origin extraneous to language and logic'.[5] Brouwer recognized the reliance of the pre-intuitionist school on logic and set-theoretical methods in their treatment of measure-theory and the continuum. He also realized the inadequacy of his own treatment of the continuum in 1907.

Both his objections to logic and axiomatic set-theory had become crystallized in the rejection of the logical Principle of the Excluded Middle and the axiom of comprehension. He now set himself the ambitious task of reconstructing the whole of mathematics on the basis of avoidance of the PEM, existence theorems, as well as the discipline of a formal system.

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[1] 1952B, p. 140.

[2] 1953, p. 2.

[3] Cf. Brouwer's remark in a footnote of 1921A: 'in my earlier publications the implications of intuitionism were not fully clear to me.' (p. 801).

[4] 1953, p. 2.

[5] 1952B, p. 140.

He completed the first part of 'a systematic construction of intuitionist set theory'[1] in 1917: 'Foundations of set theory independent of the logical principle of the excluded middle, (Part I, General Theory of Sets', 1918A), followed a year later by an intuitionist 're-working' of the theory of plane point sets (Part II, Theory of Point Sets, 1919A). In 1923 a start was made with an intuitionist treatment of function theory ('Foundations of the theory of functions independent of the logical principle of the excluded middle, Part I, Continuity, Measurability, Derivability' (1923A); Part II never appeared).

The express mention of avoidance of the PEM in these titles typifies Brouwer's programme at this stage, which had become completely centred round the problem of the PEM and its logical implications. Liberalization of mathematics from the shackles of language and logic proved itself to be a heavier burden, and earlier enthusiastic support for the programme began to wane. Hopes that with the 'second act of intuitionism', i.e. recognition of the possibility of generating new mathematical entities in the form of infinitely proceeding sequences and mathematical species, a 'much wider field of development was opened up, which includes analysis and in several places far exceeds the frontiers of classical mathematics'[2], were not immediately fulfilled. Brouwer had to admit: 'Intuitionism in general submits mathematics to a programme of complete reconstruction; unfortunately, mathematics must in this process often lose its smooth and elegant character and assume much more rigid, awkward and complicated forms. But there it is, the spheres of truth are, regrettably, less permeable than those of illusion'.[3]

A revised version of the first part of the 'Foundations of set theory' (1918A) was published in 1925, 1926 and 1927,[4] and some attempts were made at reconstructing parts of classical mathematics on intuitionist lines; e.g. 'Intuitionist proof of the theorem of the Jordan curve'[5], 'Intuitionist proof of the fundamental theorem of algebra'[6], 'The intuitionist form of the Heine-Borel theorem'[7]. The programme was, however, effectively abandoned by Brouwer in 1927.

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1 Cf. 1921A, p. 798.

2 1952B, p. 142.

3 1933, p. 60.

4 1925D, 1926A and 1927A.

5 1925C.

6 1924F and G; cf. also 1924J and K

7 1926E and F.

CHAPTER III

LANGUAGE AND LOGIC

1.1 Language

The radical rejection of language as in any way contributory to mathematics, the most fundamental thesis of Brouwer's intuitionism, seems to follow naturally from his notion of mathematics as intuitive constructive activity; in the primordial intuition and subsequent intuitive constructions, the question of language simply does not arise.

Brouwer's views on the relation between mathematics and language, however, were undoubtedly influenced by his attitude towards language, his moral disapproval of cooperation and communication. This attitude originated in his solipsistic view of solitary man, isolated within the universe of his own consciousness, unable to communicate with other minds. In this way, it can be argued, Brouwer's notion of mathematics stems from, or is at least inspired by, his views on language.

Dislike of axiomatization and formalization is evident in Leven, Kunst en Mystiek (1905) and in Grondslagen (1907). The increasing emphasis in current mathematical practice on these 'linguistic aspects' provoked Brouwer's reaction and caused him to concentrate on the problem of the nature of language and its relation to mathematics.

At the time that Brouwer was writing his dissertation and generally constructing his intuitionistic mathematics, the phenomenon of language had become the centre of interest of philosophers of different kinds, of mathematicians, and of sociologists. To all, the issue was the role and nature of language, and the relation of language to what it describes. In Chapter II, we mentioned Bergson's Introduction to Metaphysics (1903),[1] where Bergson distinguished between intuitive knowledge, which is languageless, and analytic science, which is expressible in symbols, and warned against the danger of 'replacing in our minds the original by its symbols'.

Mauthner had published his Beiträge zu einer Kritik der Sprache (1901), in which he defended a similar sceptical view of language.[2]

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[1] See above, p. 72; cf. also our footnote on LeRoy and Bergson, p. 183; LeRoy's publication 'Sur la logique de l'invention', Revue de Métaphysique et de Morale, vol. 13 (1905), pp. 193 - 223.

[2] E.g. p. 54: 'By means of language men made it forever impossible to get to know one another.'

Two of Brentano's students, Edmund Husserl and Alexius Meinong, published their major work on language at about the same time[1], taking intentionality as their point of departure.

On the periphery of philosophy, more on the sociological plane, Lady Victoria Welby's What is Meaning? published in 1903, saw language as a source of insight into human nature and started what was later called the 'Signific Movement'.[2] Similar views were expounded by G. Mannoury at the University of Amsterdam; he denied a completely objective meaning to all forms of language including the symbolic language of mathematics.

It is not without significance that many of those researchers into language were mathematicians by profession, such as Frege, Husserl and Mannoury.

1.2 Communication and the nature of language

Brouwer's philosophy of language centres round the impossibility of real communication and the nature of language as a sign.

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[1] E. Husserl, Logische Untersuchungen, (1901 - 1902, especially the first essay, 'Expression and Meaning'); A. Meinong, Über Annahmen, 1902.

[2] Cf. also her later publications, Significs and Language, 1911, and C.K. Ogden and I.A. Richards, The Meaning of Meaning; C.L. Stevenson, 'Persuasive Definitions', Mind, 1938.

The Signific Movement was introduced in Holland by Frederik van Eeden, a national poet, who greatly influenced Brouwer's thinking in his earlier years and with whom Brouwer later collaborated in the Signific Circle. Significs claims to be a science whose object is the study of human communication in all its aspects. Great emphasis is laid on analysis of meaning into indicative, emotional and volitional elements. The mathematicians Mannoury and Brouwer, shared a common interest in the phenomenon of language. Mannoury was mainly interested in the social aspects of language and he determined the general course of the Signific Movement. Even though Brouwer in many of his ideas was influenced by Mannoury, and the general direction of the movement between the two world wars became more 'social' due to the possible catastrophic effects of propaganda, Brouwer was basically an individualist, and his ultimate motivation was a strong suspicion of language as an adequate means of communication.

The 'International Institute for Philosophy' was founded in 1917 by Brouwer, Mannoury, Frederik van Eeden and Jacob Israel de Haan, and changed into 'The Signific Circle' in 1922.

It has been maintained that Brouwer tended to a solipsistic philosophy towards the end of his life.[1] There is a certain increase of non-mathematical activity and involvement in the Signific Movement in the period of 'silence' and after; there is, however, a remarkable consistency and very little development in Brouwer's views on language. These views were most forcibly expressed in Leven, Kunst en Mystiek (1905) and in more moderate terms, but essentially the same, in his later work on the foundations of mathematics, 1907, 1908B, 1912A, 1929A and 1948C.

His solipsistic view of man made him emphasize the generally accepted impossibility of direct communication between minds: 'From life in the mind follows the impossibility to communicate directly or instinctively by gesture or looks or, even more spiritually (Du. materieloozer), through all separation of distance. People then try and train their offspring in some form of mutual understanding by means of crude sounds, laboriously and helplessly, for never has anyone been able to communicate his soul by means of language'.[2] The most effective means of real communication is a work of art, where 'real truth' is to be found.[3] In 1948: '.... there is no exchange of thought .... by so called exchange with another being the subject only touches the outer walls of an automaton.'[4] The impossibility of any real communication also follows from Brouwer's conception of the nature of language.

In the process of indirect communication, i.e. through language, Brouwer seems to distinguish four elements:

1° mental activity;

2° the act of expressing or recording this mental activity in language, spoken, written or otherwise;

3° the resulting spoken or written record;

4° the hearing or reading of this record, which evokes a related mental activity in another mind or in the self-same mind at a different point in time.[5]

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[1] H. Freudenthal and A. Heyting, 'Levensbericht van L.E.J. Brouwer', Jaarboek der Nederlandse Akademie van Wetenschappen, 1966 - 1967, p. 335; A. Heyting, 'In Memoriam L.E.J. Brouwer', Algemeen Nederlands Tijdschrift voor Wijsbegeerte en Psychologie, vol.59 (1967). Letter A. Heyting to W.P. Van Stigt, 29th October, 1969.

[2] 1905, p. 37.

[3] 1905, pp. 47 ff. Similar views were expressed by Bergson in 'Le rire: de quoi rit-on, pourquoi rit-on?', 1884

[4] 1948C, p. 1240.

[5] This distinction is frequently implied in Brouwer's reference to language as 'a means of communication' and 'an aid to memory'.

Brouwer's mental activity is mathematical activity, identified in Chapter II with intuition. There is in his work no systematically graded hierarchy of mental objects and judgements, not even in the manner of Meinong's Objects of Thought or Investigations into the Theory of Objects and Psychology, 1904. Neither is Brouwer concerned with epistemological analysis of the act of perception and the relation between objects and mental acts, as were Brentano, Husserl, and Meinong, even though he does express a subjectivist opinion similar to that of Husserl.[1] In restricting himself in most of his writing to mathematical mental activity, Brouwer does not necessarily imply that all mental activity is mathematical, even if his domain of mathematics stretches further than is commonly accepted. The general term 'mental activity' covers the basic intuitive concepts and the mental construction resulting from them.

Brouwer's interest in semantics was centred round the nature of language as a sign. Unlike manufactured objects that by their very structure denote their function and meaning, or signs that resemble what they signify, such as photographs, onomatopoeia, etc., language is associated with what it signifies by convention only. This type of sign, whether it is a spoken or written word or a symbol of some other form, is an object that conveys a meaning which cannot be found by close study of the object itself. By meaning here is understood an association of the sign and that which it signifies, an association which is totally arbitrary. The actual diversity of human languages illustrates their arbitrariness. This association of the sign with what it signifies comes into being by an act of the will[2], and therefore even when a word relates to a physical object, it can only do so via the human mind. Purely as an object it has no meaning: 'Language by itself has no

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[1] E.g. 'About such concepts as "triangle" there will be little misunderstanding, but even here two different people will never think of exactly the same thing.' (1905, p. 37.)

[2] Brouwer sees the whole process of communication through language within the sphere of the will, even to the extent of claiming that the sole purpose of communication, and language in particular, is the imposition of the speaker's will, a favourite theme in the Signific Movement and also of Schopenhauer. This again is as strongly expressed in 1905 as in 1929A and 1948C: 'The purpose of language is to keep the will (Du. wilsbeweeg) on one path' (1905, p. 38); cf. 1929A, 'Willensauferlegung durch Laute', p. 153; also pp. 157-158.

meaning'[1], it becomes a sign only when a meaning has been given to it. A priori existence of the signified follows from the nature of the sign itself.

As to the nature of the 'signified', Brouwer avoids such philosophical questions as whether, in the case of physical objects, the association is with the object itself, or with the object as it has been perceived by the speaker. He usually restricts himself to mathematical language, i.e. the language to communicate mathematics.[2] The basic mathematical constructs are intuitive concepts, and so are all further parts of mathematics constructed from these.

Brouwer's main argument concerning the inadequacy of mathematical language is based on the general nature of language as a sign. Even if some form of intersubjectivity of the primordial intuition of two-ity were acceptable[3], mutual agreement on the connection of this concept (and further concepts) and the symbol signifying it is brought about by explanation or instruction using common language. In language, insistence on a priori existence of meaning leads ultimately to the impossibility of communication through language or signs, or at least of perfect accuracy of such communication. Brouwer's reservations on mathematical truth are not the result of epistemological scepticism or subjectivism; he fully maintains validity and truth of mathematics based on the primordial intuition and construction in each individual mind: 'Pure mathematics can be exact when practised in solitude in a human mind endowed with unlimited power of memory ... this exactness, however, would be lost in mathematical communication between people'.[4] He doubts the possibility of producing agreement purely by means of language. Indeed, the whole of Chapter IV of 1905, 'On Language', is

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[1] 1905, p. 40.

[2] As to mathematical language, Brouwer does not see any essential difference between ordinary language and symbolic language; cf. e.g. 'Just as any mathematical language, so can also this language (of classical logic) without difficulty be condensed to symbols.' (1907, p. 141, footnote). When Brouwer uses the word 'taal'(language), he has in mind a concrete, meaningful expression in sounds or symbols, not a totality of words together with grammatical and syntactical rules. We shall use the word 'language' in the Brouwer sense of 'a verbal description'.

[3] See above p. 98.

[4] 1933, p. 58.

devoted to stressing the necessity of 'pre-existing harmony of will', or even pre-existing understanding before communication.[1] Typical is his sigh of desperation at the beginning of 1948C, where he expresses his fears that this mutual understanding is lacking and his lecture might 'remain a soliloquy'.

Brouwer's concept of language had far-reaching consequences and affected his views on many topics such as logic, existence and mathematics itself. We will here consider three dominant themes that run through his work and which follow immediately from his concept of language:

1° The essential inadequacy of language as a medium of communication; (1.3)
2° The separation between mathematics and its recording; (1.4)
3° Mathematics as an activity of the mind. (1.5)

1.3 The inadequacy of language for communicating mental activity is a theme that can be found in all Brouwer's work on the foundations of mathematics, it is even echoed in the work of his most prominent student Arend Heyting:[2]

In 1907: '... language, which is only a defective means of communicating mathematics and has nothing to do with mathematics itself except as its accompaniment.'[3] (p. 141)

'... the language of mathematics, which itself is not mathematics but only a defective medium of communicating mathematics and an aid to memory.' (p. 169)

In 1912A: '... neither ordinary language nor any symbolic language can have any other role than that of serving as a non-mathematical auxiliary.' (p. 13)

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[1] How much Brouwer still stood by his ideas as expressed in this Chapter is evident from a letter to Mr. A. Hill (6th September, 1964). In order to show that an English publication of Leven, Kunst en Mystiek was a worthwhile exercise, Brouwer singled out this Chapter for translation.

[2] There are many instances; typical is the opening line of his most important work, 'Die intuitionistische Mathematik ist eine Denktätigkeit unde jede Sprache, auch die formalistische, ist für sie nur Hilfsmittel zur Mitteilung'. (Die formalen Regeln der intuitionistischen Logik', Sitzungsberichte der preuszischen Akademie von Wissenschaften, physikalisch-mathematische Klasse, 1930, p. 142.

[3] Brouwer frequently uses the Dutch word 'begeleiding' in this context, in later English publications usually 'accompaniment' which is the literal translation. He also uses the expression 'taalparallel' (linguistic parallel), e.g. 1907, p. 173. In other places he speaks of the projection of mathematics into language.

In 1929A: 'Even for pure mathematics there is no certain language, i.e. a language which excludes all misunderstanding and error in communication, and when used as an aid to memory.' (p. 157)

In 1933: 'Because of the nature of language, errors and mistakes can never be completely prevented. (p. 54)

In 1947B: '... in intuitionistic thinking the role of mathematical language can only be that of an aid to help memorize mathematical constructions or construction methods, or to suggest them to others, sufficient for all practical purposes, but nevertheless never completely safeguarding against error ..., intuitionistic mathematics is therefore an essentially languageless activity.' (p. 339)

1952B, p. 140 and 1953, p. 2, are literal restatements of this last passage. The inadequacy of language in expressing mathematics was the fundamental reason for Brouwer's claim, long before Gödel's proof in 1931, that formal systems are necessarily incomplete.

1.4 It is the complete separation of mathematics and its describing language that Brouwer later saw as his main achievement: 'The first act of intuitionism completely separates mathematics from mathematical language, in particular from the phenomenon of language which are described by theoretical logic, and recognizes that intuitionist mathematics is an essentially languageless activity of the mind...'[1]

The absolute distinction between mathematics and its expression was offered by Brouwer as the cure of the crisis in the foundations of mathematics. In Grondslagen he had already criticized Poincaré for not going far enough in his criticism of logicism and cantorism: 'He does not go to the root of the trouble, which lies much deeper, i.e. the confusion of the act of constructing mathematics and the language of mathematics.'[2]

Here, Brouwer indeed puts his finger on the main difference between himself and the 'pre-intuitionists,[3] Poincaré, Borel and Lebesgue.

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[1] 1952B, p. 140.

[2] 1907, p. 176; in this passage, Brouwer refers to Poincaré's articles in Revue de Métaphysique et de Morale, vol. 13 (1905), pp. 815 - 835 and vol 14 (1906), pp. 17 - 34. Cf 1912A, p. 13 footnote; also p. 14.

[3] A term used by Brouwer in later publications (e.g. 1952B, p. 140). It is interesting to compare this to the earlier term 'neo-intuitionist' (e.g. 1912A, p. 12), used for all those in the intuitionistic movement including himself.

Whereas they are all in agreement on mathematics as a human mental activity, the pre-intuitionists see expression in language or symbols as an essential part of mathematics.[1] Brouwer's point of view, however, was not entirely original. Poincaré's stand on language and mathematics was formulated mainly as the result of a publication by E. LeRoy in the Revue de Métaphysique et de Morale.[2] LeRoy expresses views very similar to those of Brouwer: 'Science contains two elements of very different character, an act by which it is inspired and which forms its origin, and a representation which, as far as possible, expresses the acquired results. Progress is due to the former, the latter provides a language more or less well adapted, but not a true source of verification.'[3]

In his criticism of LeRoy, Poincaré maintained that certainty of a mathematical theorem can only be the result of an argumentative proof, and that a concept is only truly mathematical if it can be defined in a finite number of words and is free from contradiction.[4] In this, Poincaré may seem to come very near Hilbert's original point

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[1] Borel's 'realistic' viewpoint demands of mathematical objects that we can put them into language (cf. Leçons sur les séries à termes positifs, Paris, 1902, p. 169).
Lebesgue also demands of mathematical objects that they are 'Nommables', (cf.'Sur les fonctions représentables analytiquement', Journal de mathématiques pure et appliquées. Cf. also 'Sur les controverses sur la théorie des ensembles et la question des fondements', Les Entretiens de Zurich, p. 118).
In the quoted passage, Brouwer maintains that Poincaré's chief objection to logicism and cantorism was petitio principii and acceptance of the actual infinite. In all fairness to Poincaré, it must be said that he based his opposition to logicism in those articles on:
i. The inadequacy of logic in its choice of axioms, and
ii. in deciding which course to follow through the vast number of logical combinations without the guidance of intuition, (op. cit. 1905, p. 817).
iii. The need of the principle of mathematical complete induction or the intuition of the sequence of the natural numbers (op. cit. 1906, pp. 27 - 30; cf. also La Science et L'Hypothèse, Paris 1902, pp. 20 ff.)
It is typical of Brouwer to mention here only the points of disagreement. On the other hand, he uses iii in Grondslagen (pp.170-171) to prove a priority of mathematics with respect to logic, and claims later (1928A, p. 375) that 'this first appeared in writing in Grondslagen', although he admits here that 'a case had been prepared by Poincaré'.

[2] 'Sur la logique de l'invention', Revue de Métaphysique et de Morale, vol. 13, (1905) pp. 193 - 223.

[3] Ibidem. p. 203. See further our comment on Bergson and LeRoy, p. 183.

[4] Op. cit., p. 226.

of view. However, in the roles ascribed to complete induction there is an essential difference, and also in the definitions of mathematics. The logicist (and formalist) view as expressed by Couturat, 'Pure mathematics is a collection of formal implications independent of content'[1], was strongly attacked by Poincaré, even though he seems rather vague about the existence of mathematical content.[2] Opposition to this radical formalism, and emphasis on the role of the human mind before expression in language, form the basis of a common front of both Brouwer and the pre-intuitionists, while the difference between Brouwer and the pre-intuitionists is mainly Brouwer's more radical view of language.

K. Popper in his article, 'Epistemology without a knowing subject'[3], seems unaware of this, and is essentially on the side of the pre-intuitionists. In this article, which is meant as a tribute to Brouwer,[4] he claims it to be Brouwer's main achievement that 'he showed that mathematics is created by man[5], and that 'mathematical objects must exist before we can talk about them[6], but on the other hand he rejects Brouwer's separation of mathematics and language. As to the 'man-made' character of mathematics, this had already been claimed by Poincaré as early as 1894[7]; it also was the main theme of Borel's anthropological realism.

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[1] 'Les principes de mathematiques', Revue de Métaphysique et de Morale, vol. 12 (1904), p. 21.

[2] Cf. p.76. This intuition sometimes seems to be purely psychological, ('L'invention mathématique', L'Enseignement mathématique, vol. 10 (1908), pp. 357 - 371), sometimes more rational.

[3] Logic, Methodology and Philosophy of Science III, Proceedings of the third international congress for logic, methodology and philosophy of science, Amsterdam 1967, ed. B. van Rootselaar and J.F. Staal, pp. 333 - 373.

[4] Ibidem, p. 353.

[5] Ibidem, p. 358.

[6] 'Sur la nature de raisonnement mathématique', Revue de Métaphysique et de Morale, vol. 2 (1894), pp. 371 - 374.

[7] Cf. 'La logique et l'intuition en mathématiques', Revue de Métaphysique et de Morale, vol. 15 (1907), pp. 273 - 283, further developed especially in Leçons sur la théorie des fonctions, Paris 1898 with additions of 1928 edition (notes IV, V, VI, and VII), and in 'La théorie des ensembles et les progrès recents de la théorie des fonctions', Revue générale des sciences pures et appliquées, vol. 20 (1909), pp. 315 - 324.

Popper's rejection of Brouwer's separation of language and mathematics is based on the conviction that interaction between mathematicians through argumentative language contributes substantially to mathematical knowledge[1], and that language can be 'exosomatic' and constitute a linguistic third world.[2]

Even though Brouwer had his subjectivist suspicions about the value of scientific debate[3], he certainly in his life's practice and in his writings[4] conceded the possibility of cooperation between mathematicians and the value of rigorous criticism. But this would not necessarily alter his main thesis, nor is the distinction between descriptive and argumentative language convincing. In the interaction resulting from argumentative language, as in all mathematical activity, the 'cooperative' creation of mathematics takes place in the individual minds, influencing each other and communicating through a medium which - especially in argument - has its deficiencies. While not excluding interaction and various outside influences or more complex situations, Brouwer reduced the relation between mathematics and language to that of the sign and the signified, two elements of entirely different nature, each with its own set of rules. His main conclusion is that language by itself cannot contribute anything to mathematics.

As to Popper's linguistic 'third world' of 'books in themselves, theories in themselves, problems in themselves' [5], this is not a disagreement on minor points but is diametrically opposed to Brouwer's

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[1] Op. cit., p. 361.

[2] Ibidem, p. 346. Popper claims that Brouwer sometimes admits that objects of mathematics owe their existence partly to language, and he then quotes 1925D where Brouwer says 'Der Mathematik liegt eine unbegrenzte Folge von Zeichen bzw endlichen Zeichenreihe zugrunde...' This, Popper claims, shows that Brouwer was aware that signs were needed to carry out constructions. In 1947B, p. 339, however, Brouwer points at some identification of sign and signified for practical purposes: 'Because of the languagelessness of mathematics, the word sign (Zeichen) and especially the word numerical compound (Ziffernkomplex) must be thought of as thought-signs (Du. gedachtenteekens), consisting in mathematical entities (Du. denkbaarheden, i.e. literally: thinkabilities) already constructed.

[3] Cf. 1905, Chapter III.

[4] Cf. 1928A, pp. 374, 375.

[5] Op. cit. p. 342.

fundamental thinking on language.[1] Popper's case against language as purely a medium of communication is not very strong, neither does his attempt to undermine the essential relation of language and the human mind sound very convincing. The case of the book that remains a book even 'without the reader', like 'the deserted wasp's nest even after it has been deserted'[2], seems immaterial in this context, especially when Popper admits that a book in order to belong to the world of objective knowledge, in principle, or virtually should be capable of being grasped by somebody.[3] Popper claims that, like Brouwer and unlike Plato, he wants this third world to be man-made, but in order to stress the autonomy of his third world he cannot help quoting the logarithm book entirely written and produced by machine.

Brouwer's simple case remains: that a human mind has attached - quite arbitrarily - a meaning to a physical object, which can then communicate this meaning to a mind which knows the association; that without 'the reader' it would never be a sign but simply remain the physical object it was. The hypothetical case of the human race destroyed and books surviving, subsequently deciphered and read by visitors of outer space[4], does not alter this simple fact nor disprove Brouwer's insistence on the separation of mathematics and its linguistic description.

1.5 Mathematics as mental activity

In his philosophical analysis of the nature of mathematics, 'intuitive mental activity' (see above Chapter II), Brouwer starts with the human mind, a complete blank, and arrives at the concept of

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[1] Cf. the whole of Chapter III, 1905, typical of which is:'Language by itself has no meaning and all philosophy which looked for certainty in that direction came to grief ... a language which tries to live on independently in the so-called pure concept is an absurdity, similar to a frog's heart that according to physiologists is 'kept alive even when separated from its organs'.(p. 40)
Cf. also 1907, p. 177: 'one should not regard mathematical formulae as truths leading an independent existence...'
In 1955 Brouwer clearly condemned the conviction 'that after the extinction of humanity, mathematical truths, just as laws of nature, will survive.' (p. 113)
Cf. also 1948C, p. 1245.

[2] Op. cit. p. 341.

[3] Ibidem, p. 342.

[4] Ibidem, p. 342.

two-ity through awareness of past and present sensations and mathematical abstraction.

On the other hand, he was enough of a realist to know that this is not the usual process by which people learn about the natural numbers. In fact the very first line of Grondslagen starts: 'One, two, three, ..., the sequence of sounds we have all learned by heart ...'[1] Language is accepted as 'a means of communicating mathematics', even if it is a defective one; it is, however, stressed that mathematical concepts, even the simplest, are not grasped at first sight or hearing, but that in the final analysis they are only fully assimilated as the result of an active process of reconstruction. If original mathematics is a construction in the mind, mathematics communicated through language is the same process repeated in another mind, not a passive undergoing of a mathematical programme with necessarily following understanding: 'Mathematical language originates in the human will to evoke (Du. te doen oprijzen) in others by means of sounds and symbols a copy of their own mathematical construction.'[2] It is this emphasis on active reconstruction that made Brouwer in his analysis think of sets, real numbers and functions as procedures.

There is, however, no language that can guarantee absolute identity of the construction and its 'copy', reconstructed in another mind. (See 'Intersubjectivity'[3]).

2.1 Logic

The general interest in language towards the end of the nineteenth century was closely allied to a revival of interest in formal logic. This revival was largely due to the publication in 1847 of de Morgan's Formal Logic and especially Boole's Mathematical Analysis of Logic; his discovery of the algebraic structure of logic and his attempt at a symbolic notation started a fresh line of research (mainly in England)[4],

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[1] 1907, p. 3.

[2] 1907, p. 128.

[3] See above, p. 98.

[4] First in England, W.S. Jevons, Pure Logic, 1864 (but see footnote [1], p. 124), J. Venn, Symbolic Logic, 1881; J. N. Keynes, Studies and Exercises in Formal Logic, 1884; W.E. Johnson, 'The logical calculus', Mind, 1892; later in America (C.S. Peirce), Germany (Frege and Schröder) and Italy (Peano).

in which mathematics and logic were closely linked. On the Continent, traditional Aristotelian logic retained its authority much longer, partly due to the great influence of Kant's philosophy and his complacent view of logic 'as having to all apearances reached its completion', and partly, especially in Germany and Holland, because of the traditional character of the universities.

The mathematician Boole concentrated his efforts on the mathematical aspects of traditional logic and showed successfully the benefits of symbolic methods. In no way did he attempt to establish a more fundamental relation between logic and mathematics; mathematics and logic were still seen as two autonomous disciplines.[1]

In this respect, Boole's approach differs essentially from Leibniz's earlier effort to create 'a mathesis universalis', which was to provide the fundamental principles of every science together with a symbolic notation and calculus. The advances in symbolic logic, however, led to renewed attempts by the end of the nineteenth century to seek the ultimate foundations of mathematics in logic itself. Peirce, Dedekind and Cantor were the first to base their number concept on logical analysis, and in this way placed the foundations of mathematics effectively within the domain of logic. Gottlob Frege went a step further. In his Begriffsschrift (1879) he invented a new system of symbolic logic; he then used this rigorously to base the whole of arithmetic on logic and its security on the laws of logic. A less fundamental attitude had been adopted by Peano, who set out to show that arithmetic and algebra can be constructed on the basis of a few undefined primitive concepts and undemonstrated primitive propositions. Brouwer in particular directed his main criticism against Peano, Dedekind and Cantor, and nowhere mentions the name of Frege.

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[1] Anxiety on the part of logicians not to be dominated by mathematics is well expressed by Jevons (Pure Logic), who wanted to show the 'real structure of Boole's logic by divesting his system of a mathematical dress, which, to say the least, is not essential to it'. It is interesting to compare this quotation with one by Boutroux, who in l'Idéal Scientifique des Mathématiciens, (p. 203), expresses a reverse anxiety, very much in line with Brouwer's ideas: 'The mathematical fact is independent of the logic or algebraic dress in which we seek to represent it.'
Brouwer showed his appreciation of both Boole and Jevons in 1955, p. 113.

Edmund Husserl, a mathematician of great repute on the Continental scene, rejected formal logic but derived the necessity of mathematical principles from the laws of logic, first a kind of logic based on psychology[1] but later a 'pure logic' concerned with the concepts common to all sciences.[2]

In 1903, Russell published his Principles of Mathematics, in which he set himself a twofold task: 'The present work has to fulfil two objects; first, to show that all mathematics follows from symbolic logic, and secondly, to discover, as far as possible, what are the principles of symbolic logic itself'. Since the International Congress of Philosophy and Mathematics in Paris in 1900, the role of logic in mathematics had become the subject of an international debate, the Russell - Poincaré debate, mainly conducted in the Revue de Métaphysique et de Morale. The discovery of Russell's antinomy[3] had shaken the security of set theory and logic as the ultimate foundations of mathematics.

2.2 It was at this time that Brouwer was writing his doctoral dissertation at the University of Amsterdam. His supervisor, Professor D.J. Korteweg, strongly encouraged an interest in the topical controversy of Foundation study in Holland.

Brouwer saw the debate in a much wider context, the central issue being the basis of the validity of mathematics. On page 94 of Grondslagen he says: 'The classical problems of space and time, which were for some time neglected by those mathematicians engaged in foundation study, have been brought to the fore again by B.A.W. Russell in his work Essay on the Foundations of Geometry.'[4] It is this work,

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[1] Philosophie der Arithmetik, 1891.

[2] Logische Untersuchungen, 1901 and Phänomenogie des Geistes, 1910. There is no direct evidence that Brouwer at the time of his Grondslagen was influenced by Husserl, or even was familiar with his writings, even if this is most likely in view of Husserl's fame, similarity of interests and certain striking similarities in definitions of logic, etc. Interesting to note is Brouwer's later friendship with Husserl at a time when Husserl tended more and more towards a solipsistic kind of idealism, abandoning his earlier demands for objectivity, so characteristic of his Logische Untersuchungen.

[3] Published in Principles of Mathematics, London 1903, par. 78 and Chapter X. Earlier discovery of antinomies (Cantor 1895, Burali-Forti 1897) was not widely known, nor taken seriously. Cf. A. Fraenkel, Y. Bar-Hillel - Foundations of Set Theory, pp. 1 - 18.

[4] 1907, p. 94.

according to Brouwer, that initiated the debate '... and led to continued discussions in the Revue de Métaphysique et de Morale between Couturat, Poincaré, Lechalas and the author (Russell) himself'. Later[1], Brouwer describes the debate as simply between the German-Italian school (logicists and formalists) on the one hand, and the French (intuitionist) tradition on the other. In playing down the role of logic as the central issue of the Russell-Poincaré debate, Brouwer was certainly influenced by his own views on language and logic, and the minor role he ascribes to logic in mathematics.

Even if opinions on Brouwer's positive contribution to mathematics vary[2], he is generally given credit for his consistency and radicalism. Brouwer's consistency in rejecting Dedekind's and Russell's reduction of mathematics[3] to logic, Pasch's and Hilbert's axiomatic formalization, Cantor's and Zermelo's theory of sets and transfinite numbers, as well as Poincaré's acceptance of non-contradictority, originates in his complete denial of any contributory role of language in mathematics. In spite of all their differences, Brouwer brackets all these mathematicians as apostles of the word: 'The question as to where mathematical exactness exists is answered differently by both sides; the intuitionist says: in the intellect, the formalist says: on paper.'[4] In Grondslagen he carefully distinguishes between axiomaticists, cantorians, logicists and formalists, and the whole of Chapter III, ('Mathematics and Logic'), is devoted to refuting one system after

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[1] 1912A, p. 7.

[2] Cf. e.g. Weyl, Philosophy of Mathematics and Natural Science, Princeton 1949, p. 54: 'Mathematics with Brouwer gains its highest intuitive clarity.' E. Bishop: 'Mathematics as a numerical language (unpublished), p. 1: 'Brouwer's great contribution was to analyze intensively the inadequacies of classical mathematics.' Cf. also E. Beth, op. cit. p. 70.

[3] Brouwer is more outspoken in his attacks on Dedekind, Peano and Cantor (and later on Hilbert) than on Russell. Even though Russell showed the logicist intention of reducing mathematics to logic equally clearly, if not more clearly, Brouwer singled out Dedekind for a personal attack on this score (1907, p. 138). As is clear from Chapter II of Grondslagen, Brouwer identified Russell with the attempt to revive the 'more fundamental' issues of time and space. After condemning those mathematicians who ignored these questions, he continues: 'Only recently a book has been published that in the light of the latest results in mathematics has again posed these philosophical questions, and this work by B.A.W. Russell, An Essay on the Foundations of Geometry, has received general attention'. (1907, p. 94)

[4] 1912A, p. 7.

another. In most of his later publications he refers to them all as the 'Formalist School' or 'the Old Formalist School'[1], ' ... which merged logic and mathematics into a single linguistic science, operating on meaningless words or symbols by means of logical rules, thus divesting logic and mathematics of their difference in character as well as their autonomy.'[2]

Poincaré had already shown that logicists did accept some fundamental undefined concepts and principles, such as implication, logical sum and product, negation, and the principle of syllogism, a point conceded by Russell and Couturat.[3] The formalist solution, and in particular Hilbert's attempt to use words without reference to meaning, to reduce mathematics to purely formal systems, represented a more extreme challenge to Brouwer's conception of mathematics, language and logic. Brouwer continued to attack in his later work the misuse of logic and logical principles, especially the Principle of the Excluded Middle, but the main target of his continued and sometimes violent attack was formalism, not logicism.

2.3 Brouwer did not attack logic as such; he recognized logic as an autonomous science and later in life expressed appreciation for advances made in symbolic logic: 'Classical algebra of logic [4] has its merits quite apart from the question of its applicability to mathematics.

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[1] In his inaugural address (1912A), he simply refers to 'the formalists for whom these meaningless series of relations, to which mathematics has been reduced, have mathematical existence only when they have been represented in spoken or written language, together with the mathematical-logical laws upon which their development depends, thus forming what is called symbolic logic'. (p. 9)
In 1952B, p. 139: '... The Old Formalist School, (Dedekind, Cantor, Peano, Hilbert, Russell, Zermelo, Couturat) finally for the purpose of a rigorous treatment of mathematics and logic, rejected any element extraneous to language and logic'.

[2] 1953, p. 1.

[3] See p. 119, footnote 2.

[4] i.e. as 'founded by Boole, developed by de Morgan, Jevons, Peirce, and perfected by Schröder' (1955, p. 139).

'Not only as a formal image of the technique of common-sensical thinking has it reached a high degree of perfection, but also in itself, as an edifice of thought, it is a thing of exceptional harmony and beauty. Indeed, its successor, the sumptuous symbolic logic of the twentieth century, which at present is continually raising the most captivating problems and making the most surprising and penetrating discoveries, likewise is for a great part cultivated for its own sake.'[1]

In Grondslagen, Brouwer distinguishes between:

1° Intuitive logical deduction;

2° Theoretical logic, later extended to include the logic of relations;

3° Logistics.

Even when Brouwer sometimes conforms to contemporary usage and speaks of logic as 'life in the mind'[2], or of logical laws as 'laws of thought'[3], he certainly does not wish to imply that logic deals with the mental process of thinking.[4] Neither must 'Brouwer's logic' (see further page 95) be seen as an analysis of the process of mental constructions. Indeed, the complete separation between mathematics as a mental process and logic is based on his firm belief that logic only deals with the verbal expression of a narrow field of human thought.

1° When Brouwer speaks of intuitive logical reasoning he does refer to the mental process of mathematical thinking but he stresses that this only applies to the form of some of our mathematical thinking, namely that part of our thinking that concerns itself with the relation of 'whole' and 'part', with the question, how mathematical objects relate to elements or parts of a mathematical structure:

' ... intuitive logical reasoning is that special kind of mathematical reasoning that remains when one, in considering mathematical systems, restricts oneself exclusively to the relations of whole and part'.[5]

Brouwer emphasizes that there is nothing fundamental about this relation and that this pattern is not necessary to mathematical thinking.

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[1] 1955, p. 116.

[2] 1905, p. 37.

[3] 1907, p. 125.

[4] Frege, almost simultaneously, expressed his concern about the confusion of thought, judgement and assertion, ('The thought: a logical inquiry', translated and reprinted in Mind, vol. 65 (1956), pp. 289 - 311).

[5] 1907, p. 127.

Indeed, this form is often superimposed on mathematical thinking and its verbal description; '... even in mathematics where relations of whole and part do not enter into the mental construction, the relations that were in the mind are often - for the purpose of communication to others - transformed into relations of whole and part, so that the usual language of general mathematics is full of the modes of expression of logical reasoning.'[1]

Any verbal description of a mathematical construction in this deductive form, Brouwer calls the language of logical reasoning.[2]

Both intuitive logical thinking and the language of logical reasoning are cultural conventions rather than something universally and essentially human. Brouwer even leaves open the possibility ' ... that with the same organization of the human intellect - and therefore with the same mathematics - a different language of understanding had emerged in which there was no room for our language of logical reasoning. Probably there are people living outside our cultural sphere where this would be the case. Neither can we preclude the possibility that at a further stage of development, logical reasoning will lose its position in the language of our culture.'[3]

2° Theoretical logic is the mathematical study and analysis of the language of logical reasoning, the classical Aristotelian and scholastic logic. In Grondslagen Brouwer emphasizes the mathematical simplicity of the structure of logical reasoning, 'a transition of relations through tautology'[4] and of the mathematical structure of the verbal rendering of this logical reasoning, i.e. the theory of syllogism.

He expresses here already some doubt as to the usefulness of the Principle of the Excluded Middle, which, he says, is meaningless: 'A theorem such as, "a function is either differentiable or not differentiable", says nothing, expresses the same as, "if a function is not differentiable, it is not differentiable". [5] With some reservation as

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[1] 1907, p. 128 footnote.

[2] See p. 116, footnote 2.

[3] 1907, p. 129.

[4] 1907, p. 131.

[5] 1907, p. 131. Soon after, Brouwer took up a much more aggressive position. In fact, this is one of the few passages that Brouwer has ever withdrawn - 1917A, p. 442.

to its mathematical interest and the use of the Principle of the Excluded Middle, Brouwer accepted classical theoretical logic, but the regularity observed and the validity of the so-called logical principles were due to the mathematics it described. As long as mathematical systems, already constructed, were described, the 'linguistic phenomenon' of a contradiction could not occur and 'we can safely introduce logical sum, logical product and complementary set ... and safely apply the principles of identity, syllogism, distribution, contradiction, and tertium non datur'.[1]

3° Theoretical logic had never claimed to be the ultimate foundation of mathematics. It is the logicists - Leibniz, Peano, Russell and others, who take these logical principles to be rules for mathematics, applying to the domain 'of a chimerical all', who are the real target of Brouwer's criticism. He defines logistics as the study of the mathematical structure of any verbal description of a mathematical construction[2], extending beyond the 'logical' relation of whole and part, supplementing classical logic with the so-called logic of relations 'which studies the language of mathematics in general, starting from the words "follow on", ... (i.e. to be successor of), the most fundamental act of mathematical building'.[3] In identifying logic with the relation whole and part or with Aristotelian logic, Brouwer in some way followed a line taken by Poincaré. Both agreed concerning classical logic 'that this pure logic can only lead to tautologies and create nothing new; no science can originate from it ...'[4] Poincare attacked the logicists for extending the domain of classical logic[5] to include principles that would effectively lead to the principle of complete

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[1] 1907, p. 160.

[2] 1907, p. 129.

[3] 1907, p. 165.

[4] H. Poincaré, La valeur de la Science, Paris 1905, p. 20.

[5] 'Les mathematiques et la logique' (H. Poincaré, Revue de Métaphysique et de Morale, vol. 13 (1905), p. 129), was taken by Couturat to show that Poincaré saw logic as Aristotelian logic by definition: 'This would imply that by definition only those principles formulated by Aristotle and the scholastici are logical principles'. (L. Couturat, 'Pour la Logistique', Revue de Métaphysique et de Morale, vol. 14, (1906), p. 217.

induction. The controversy between Poincaré and Couturat is concerned with those principles themselves in so far as they do or do not belong to the domain of logic.[1]

2.4 Brouwer's main concern is not with these principles , nor with logic itself, but with the role of logic and the relation between logic and mathematics and especially with the logicist claim of priority of logic over mathematics. His views can be briefly summed up as follows:

1° A priority of mathematics with respect to logic, based on the view that logic is mathematical analysis of the verbal description of mathematics; independence of mathematics.

2° Therefore logic is not a source of new mathematical knowledge.

3° The so-called logical principles only apply to the mathematical system from which they have been derived by retrospection.

1° The object of classical logic, with its restriction to the relation whole and part, but also of the 'extended logic' of logicists and formalists is ultimately the verbal expression of a mathematical construction. This view of the essentially linguistic nature of logic is apparent from Brouwer's frequent use of the derogatory 'taalgebouw' (i.e. verbal edifice).

Mathematical analysis of this language may reveal regularity and a certain, though simple, mathematical structure; it may even be possible to express this in so-called logical principles. These principles, however, are not what guides the mathematical construction described, but are expressions of regularity in the accompanying language, observed a posteriori and originating in the mathematics it describes.[2] 'That in the language accompanying mathematics the

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[1] Poincaré: 'Sur la nature de raisonnement mathématique', <u>Revue de Métaphysique et de Morale</u>, vol. 2 (1894), pp. 371 - 384. 'Les mathématiques et la logique', <u>Revue de Métaphysique et de Morale</u>, vol. 13 (1905), pp. 815 - 835; vol. 14 (1906), pp. 17 - 34.
L. Couturat: 'Les principes de mathématiques', <u>Revue de Métaphysique et de Morale</u>, vol. 12 (1904), pp. 19 - 50, pp. 211 - 240, pp. 664 - 698, pp. 810 - 844; vol. 13 (1905), pp. 224 - 256.
'Logique et Philosophie des Sciences', <u>Revue de Métaphysique et de Morale</u>, vol. 12 (1904), pp. 1037 - 1077.
'Pour la logistique', (Réponse à M. Poincaré), <u>Revue de Métaphysique et de Morale</u>, vol. 14 (1906), pp. 208 - 250.

[2] Cf. <u>1908B</u>, p. 7.

'sequence of words follows certain rules is obvious; the fault lies in regarding these laws as a guide in the construction of mathematics.'[1] Brouwer therefore speaks of logic and logistics as 'empirical sciences and applications of mathematics' and continues: 'It is as absurd to call "the language of logical reasoning" an application of theoretical logic as it is to say that the human body is an application of anatomy.'[2]

As mentioned earlier, there are already signs in Grondslagen of his doubts as to whether these so-called logical laws do indeed reflect the structure of a truly mathematical construction. These doubts are more apparent in 1908B (The Untrustworthiness of the Principles of Logic), when he launched his attack on the Principle of the Excluded Middle. But even when Brouwer became more and more involved in logic, even to the extent of developing his own logic[3], he continued to see the role of logic as an essentially a posteriori corroboration, and this view is in fact the ultimate reason for his rejection of the Principle of the Excluded Middle.[4]

2° This post factum character of logic and logical principles is also the reason for Brouwer's claim of 'independence of mathematics from logic', (1907, p. 125), 'dependence of logic on mathematics', (p. 127), and his rejection of logic as a source of new mathematical knowledge. Poincaré had based his rejection on the tautological character of classical logic and was unwilling to accept new extensions to logic. Brouwer rejects logic as an instrument 'to discover new

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[1] 1907, p. 165. In this context, it is interesting to see Brouwer's views on Euclid. He did accept logical imperfections in Euclid's geometry but leaves open the possibility that Euclid truly mathematically constructed his geometry, and that his reasonings as they have come down to us are only a logical, verbal 'accompaniment of an exploration of this structure constructed before ...' He continues: '... It is of course also possible that Euclid did not see it this way and fell into the same trap as many others who thought that it was possible to reason logically about things, different from the mathematical system they had constructed in advance.' (1907, p. 135)

[2] 1907, p. 130.

[3] 1953, p. 3.

[4] E.g. 1923B, 1929A, 1952B, 1953; cf. also A. Heyting, 'Die formalen Regeln der intuitionistischen Logik', Sitzungsberichte der preuszischen Akademie von Wissenschaften, physikalisch-mathematische Klasse, 1930, p. 42.

Cf. 1907, stelling III.

truths' out of hand[1], because of its essentially linguistic nature. This is the main theme of the whole of Chapter III of Grondslagen and of most of his work on the foundations of mathematics. In 1953 he writes: 'The wording of a mathematical theorem has no sense unless it indicates the construction either of an actual mathematical entity or of an incompatibility ... so that mathematical language, in particular logic, can never by itself create new mathematical entities, nor deduce a mathematical state of things.'[2]

3° If Brouwer accepted that logical principles may in some sense be said to apply to the mathematical system from which they had been derived by retrospection, he wholly rejects the logicist attempt to take them out of their mathematical context and apply them to other parts of mathematics or to a different mathematical system[3], or more generally to use them as a point of departure in the construction of the whole of mathematics. The root of all deviations of logicists and cantorians is, 'that they create nothing but verbal edifices which can never be transformed into the mathematics proper ... ...we must limit ourselves to what can be constructed in intuition and not try and extend this by logical combinations which cannot be realized', (i.e. in mathematics proper)[4]. The mathematics of logicists becomes a purely verbal structure in which there is no guarantee of non-contradictority. 'Since Russell's logic is nothing but a verbal system (Du. woordsysteem), without a presupposed mathematical system to which it relates, there is no reason why contradictions should not occur.'[5] This is, in fact, Brouwer's explanation of the paradoxes in 1907. Referring to these paradoxes in 1908 he says: 'These paradoxes originate in the same error as the paradoxes of Epimenides: regularity in language accompanying mathematics has been extended to a language which does not accompany mathematics; logistics concerns itself with mathematical language

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[1] 1907, p. 177; cf. also 1907, p. 135 and 1933, p. 59.

[2] 1953, p. 3; cf. also 1928A, p. 3.

[3] Or any other domain: 'Logical principles are only valid for the language of mathematics, for other verbal systems, however closely connected with mathematics, they need not be valid.' (1907, p. 164).

[4] 1907, p. 176. Recognition of the essential difference between the logic of language and the laws of mathematical construction came later also from outside the Intuitionist School e.g. G. Gentzen (1935): 'The formalization of logical inference, as developed in particular by Frege, Russell and Hilbert, deviates considerably from the way conclusions are made in mathematical proofs.'

[5] 1907, p. 163.

'rather than mathematics itself ...'[1]

Summing up his views on logistics: 'The conclusions concerning logistics must therefore be: it cannot teach anything about the foundations of mathematics since it remains irrevocably separated from mathematics; on the contrary, in order to guarantee its survival as a science in its own right and secure itself against contradictions, it will have to reject all its own special principles and restrict itself to being a faithful, mechanical, stenographic copy of the language of mathematics, which is not mathematics itself, but only a defective means of communication...'[2] The final sentence of his Grondslagen starts: 'A purely logical construction independent of the mathematical intuition is impossible, since in this way we obtain nothing but a verbal system that remains irrevocably separated from mathematics proper ...'[3]

2.5 Brouwer did consider the possibility of some form of compromise between these two extremes of mathematics all constructed in the mind and the logical principles being just a post factum analysis on the one hand, and the logicist reduction of mathematics to logic on the other. In 'The Untrustworthiness of the Principles of Logic he asks himself the question whether one can leave the real mathematical structure and work within its projection - the verbal edifice - along the logical principles established for that mathematical system, in other words, work as most mathematicians do, 'There still remains the more specific question: can one in pure mathematical constructions and transformations temporarily neglect the constructed mathematical system and move within the accompanying language structure guided by the principles of syllogism, contradiction and tertium exclusum, confident that whenever one returns to the mathematical construction, every part of the reasoned argument will be confirmed?'[4] Almost fifty years later he poses the same question, but here the emphasis is on the existence of a mathematical construction parallel to the continued logical argument: 'Suppose that an intuitionist mathematical construction has been carefully

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[1] 1908B, p. 8.

[2] 1907, p. 169.

[3] 1907, p. 180.

[4] 1908B, p. 8.

'described by means of words, and then, the introspective character of the mathematical construction being ignored for a moment, its linguistic description is considered by itself and submitted to a linguistic application of a principle of classical logic. Is it then always possible to perform a languageless mathematical construction, finding its expression in the logico-linguistic figure in question?'[1] Brouwer's answer to both questions is 'yes', as far as the principle of contradiction and syllogism are concerned, although even here he cannot help warning for possible errors due to 'the inevitable inadequacy of language as a mode of description'.[2] As far as the Principle of the Excluded Middle is concerned, his answer was simply 'no' in 1952; in 1908 he slightly qualifies this and admits that for 'finite discrete systems' this principle does apply, 'since this kind of investigation can be performed even by machines or an animal suitably trained'.[3]

The question of the existence of a mathematical system corresponding to a purely logical system had been raised by Brouwer in 1907: 'Suppose we have somehow proved - without thinking of any mathematical interpretation - that a logical system built up from some verbal axioms is non-contradictory, i.e. that at no point of the development of the system two contradictory statements occur; if we then find a mathematical interpretation for the axioms, does it then follow from the non-contradictority of the logical system that such a mathematical construction does indeed exist? But this has never been proved by axiomaticists, not even for the case that the axioms include the condition that the system is mathematically constructible; thus it has never been proved that if a finite number must satisfy a set of conditions which can be shown, to be non-contradictory, this number then actually exists.'[4] As an example Brouwer then quotes Cantor's second class of transfinite numbers, which, according to him, is non-contradictory but does not exist.

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[1] 1952B, p. 141.

[2] Ibidem.

[3] 1908B, p. 10.

[4] 1907, p. 141.

3.1 Truth and non-contradictority

Complete rejection of Platonic or realistic existence of mathematical objects, independent of human thought, excludes the notion of truth as conformity of thought to object. Equally unacceptable to Brouwer was the formalist notion of truth as derivability within a formal system and completely free 'acceptability' of the system as a whole, as for example proposed by Haskell B. Curry.[1]

Even if Brouwer is rather vague about truth in his philosophical writings, ('truth is only in reality, i.e. in the present and past experiences of consciousness'), and this vagueness is freely admitted by Heyting[3], his frequent distinctions between true and meaningless statements, and between true and false statements, leave no doubt about his simple identification of truth and existence in the Brouwer sense, their 'having been constructed intuitively'. In a mathematical construction, ultimately with the true atomic concepts of intuition, the truth of a theorem is the existence of a mathematical construction supporting, or rather, creating it. Unknown truths, neither given in intuition nor proved by mathematical construction, do not exist: 'There are no non-experienced truths ... expected experiences, and experiences attributed to others are only true as anticipations and hypotheses; in their contents there is no truth.'[4] A theorem which is true is a theorem together with its complete construction. As to the truth of the proof of this theorem: the construction at any of its stages, each of the elements constituting the construction, including the dynamic, i.e. the constructive operations, is a mathematical object, a mathematical truth, originating by construction ultimately in the primordial intuition.

Strictly within the domain of mathematics, i.e. constructive mental activity, there is only room for truth. The recording of

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[1] Outlines of a Formalist Philosophy of Mathematics, Amsterdam 1951, p. 59.

[2] 1948C, p. 1243.

[3] Haskell B. Curry, Op. cit. p. 5; cf. also Heyting, 'Intuitionism in Mathematics', Philosophy in the Mid-century. A Survey, Firenze 1958, p. 107, where Heyting says: '... a theorem does not express that a theorem is true; it expresses the success of a certain mathematical construction.'

[4] 1948C, p. 1243.

a mathematical construction is a true statement, which is of a completely different nature from a non-contradictory statement or even a false statement in the Brouwer sense, which are both rooted in language.

3.2 To Brouwer, a contradiction is 'a purely linguistic phenomenon',[1] 'a linguistic figure' (Du. het taalfiguur der contradictie)[2]. In a verbal description of a truey mathematical construction, a contradiction cannot arise: 'two contradictory theorems cannot be true of a mathematical construction.'[3] Non-contradictority is, therefore, accepted as a necessary condition for mathematical existence. There is, however, a certain inconsistency in Brouwer's strict demands for mathematical true statements and his use of the contradiction as the basis for his negation. Contradictority is described as a kind of incompatibility of two mathematical constructions, or to use Brouwer's phrase, 'two systems do not fit into each other'.[4] The verbal contradiction reports an end of the road of a mathematical construction; addressing the logician, Brouwer says: 'the words of your mathematical demonstration are only the accompaniment of a wordless mathematical building, and where you pronounce a contradiction, I simply observe that the construction cannot go further, that in the given construction there is no room for the posited structure.[5] The word 'posited' here indicates that Brouwer indeed had in mind a verbal hypothesis as the constructive part of a false statement. Whereas the construction leading to a positive theorem or statement is a building upwards, firmly grounded in the atomic concepts of intuition, or other constructed mathematical entities, the start of the constructive false statement is a hypothesis, a mathematical non-entity which leads to an absurdity such as $1 = 0$. Such an unrealizable supposition is the start of Brouwer's and Heyting's proof of the theorem, 'A square circle cannot exist'.[6] This was clearly stated in the famous passage of 1908B: 'This principle (i.e. the PEM) requires that every supposition

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[1] 1907, p. 132.

[2] 1933, pp. 57 - 58; cf. 1929A, p. 160.

[3] 1907, p. 137.

[4] 1908B, p. 10.

[5] 1907, p. 127.

[6] Heyting, Intuitionism. An Introduction, Amsterdam 1956, p. 120.

(Du. onderstelling) is either true or false.'[1] If the truth or validity of the constructive proof is its mathematical existence at every stage, then the validity of the constructive proof leading to absurdity seems in doubt because of the not only doubtful but unrealizable character of its basic ingredient. Brouwer's insistence on the completely mathematical, i.e. languageless, nature of the Brouwer negation does not seem justified.

3.3 In rejecting non-contradictority as a sufficient condition for mathematical existence, Brouwer not only repudiated contemporary logicist views of Frege and Russell and the earlier views of Hilbert, but even the view of Poincaré, who insisted that mathematical existence is nothing but freedom from contradiction.[2] Indeed, his main attack on non-contradictority in Grondslagen (and in 1912A) is directed against Poincaré: 'Logicism and cantorism have already been strongly criticized by Poincaré in the Revue de Métaphysique et de Morale 1905, No. 6, 1906, No.1, 3; he chiefly attacks logicism for petitio principii and cantorism for accepting the actual infinite. He therefore does not go to the heart of the matter which lies much deeper, i.e. the confusion of the act of constructing mathematics and the language of mathematics.'[3]

Brouwer's main and simple argument against Poincaré is that contradiction lies within the medium of language; within this medium even a petitio principii is acceptable.[4] 'Freedom from contradiction is of little importance to the mathematician and should only be of interest to the logician.'[5] A similar distinction between 'mathematical'

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[1] 1908B, p. 9.

[2] Brouwer quotes Poincaré as saying: 'En mathématiques le mot exister ne peut pas avoir qu'un sens, il signifie exempt de contradiction'; (can be found in 'Les Mathématiques et la Logique', Revue de Métaphysique et de Morale, vol. 13 (1905), p. 819.

[3] 1907, p. 176.

[4] Ibidem. 'Petitio principii is in some sense permissible, because when applied in the act of constructing the verbal system, it does not interfere with the completeness (Du. volkomenheid, i.e. completeness, perfection) of that verbal system as such ... The mistake of the logicians consists in creating nothing but a verbal edifice, that can never be transferred to mathematics proper.'

[5] 1907.

and 'logical' importance can be found on p. 139: 'Mathematical importance can therefore not be afforded to Dedekind's system; to give it any logical meaning an independent proof of its non-contradictority would have been necessary and this Dedekind did not give either.'[1] The fundamental disagreement between Poincaré and Brouwer is on the nature of mathematical existence and the role of language. Poincaré accepted definition of mathematical objects by axioms[2], although restricted by the demand that these axioms should not lead to contradictions; in so far, he seems very close to the earlier views of Hilbert. He differs from Hilbert in using the principle of complete induction to prove non-contradictority of axioms and claiming that the principle of complete induction is not a logical but a mathematical principle. He demanded definability in a finite number of words as a condition for mathematical existence: 'I am of the opinion - and I am not alone in this - that it is of the utmost importance to introduce only objects which can be completely defined in a finite number of words.'[3] In this he agreed with Borel and Lebesgue[4], although both disagreed with Poincaré on non-contradictority: 'Mathematical reality cannot be the result of the purely negative property of being free of contradiction.'[5] Both these views were clearly denounced by Brouwer on the grounds of their purely linguistic nature: 'Neither the ordinary language nor any symbolic language can have any other role than that of serving as a non-mathematical auxiliary, ... For this reason, the intuitionist can never feel assured of the exactness of a mathematical theory by such guarantees as the proof of its being non-contradictory, the possibility of defining its concepts by a finite number of words, (see, however, Poincaré in Scientia, No. XXIV, p. 6), or the practical certainty

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[1] 1907, p. 139.

[2] Science et Méthode, Paris 1908, pp 161 - 162.

[3] Op. cit. p. 41.

[4] Borel, Leçons sur la Théorie des Fonctions, Paris 1898, ed. 1928, pp. 204 - 205. Cf. also Lebesgue, 'Sur les fonctions représentables analytiquement', Journal de Mathématiques pures et appliquées, vol. 10 (1905), p. 205.

[5] Borel, op. cit. p. 222; cf. Lebesgue, 'Sur les controverses sur la théorie des ensembles et la question des fondements', Les Entretiens de Zurich, p. 118.

'that it will never lead to a misunderstanding in human relations. (See, however, Borel in Revue du Mois, No. 80, p. 221).'[1]

Poincaré never withdrew the demands for definability and non-contradictority, even though in his discussion with Couturat and Russell on natural numbers he provided a concrete example of the inadequacy of logic in the construction of mathematics. Brouwer with his restrictive views on language and logic could take full advantage of these arguments and present a much more coherent view, basing both the natural numbers, the principle of complete induction and the whole of mathematics firmly on intuition and banning definitions in axioms and non-contradictority to the mathematically irrelevant domain of language and logic.

In 1907, Brouwer reproached Poincaré for 'looking very much like his opponent Russell'[2], in his demand for non-contradictority and 'not seeing the confusion of the act of constructing mathematics and the language of mathematics'.[3] His later, more mature, view recognizes the true spirit and merit of Poincaré: 'For these "separable" parts of mathematics, (i.e. natural numbers, principle of complete induction and all mathematical entities and theories springing from this source), they, (Poincaré, Borel and Lebesgue), postulated an existence and exactness independent of language and logic ...'[4]

3.4 This change of heart is partly due to Brouwer's different views on non-contradictority after 1923, when he published his 'Calculus of absurdity', (1923C). Non-contradictority, previously often referred to as freedom from contradiction and dismissed as mathematically irrelevant, is given the constructive meaning of 'absurdity of absurdity'. Realization of the equivalence of non-contradictority and absurdity of absurdity may have been gradual. In 1948C he frequently equates the two, e.g. '... the non-contradictority, i.e. the absurdity of the absurdity'[5] and 'by non-contradictority we understand the absurdity of

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[1] 1912A, p. 13.

[2] 1907, p. 177.

[3] 1907, p. 176.

[4] 1952B, p. 140.

[5] 1948C, p. 1245.

contradictority.'[1] There is no doubt about Brouwer's views on the constructive character of 'absurdity': in 1924N he speaks of the assertion of absurdity as an operator; in point of fact, the whole calculus of absurdity in 1923C and 1925B forms a short introduction (less than half a page) in an article on relations, all generated by application of the operators of absurdity and of absurdity of absurdity, leading him dangerously into the field of logic itself.

Even if in 1927[2] he still speaks disapprovingly of those who seem to make freedom from contradiction the norm for mathematical truth and continues to refer scathingly to its linguistic nature, non-contradictority is now taken seriously. Whole sections of 1928A, 1929A, 1933, 1948C, 1953 and 1955 are devoted to analyzing the PEM and to examining in how far to results obtained by the use of this principle, non-contradictority can be ascribed.

Non-contradictority then stands for derivability of a single result within a consistent system, as well as for the consistency of the system itself.

4.1 The Principle of the Excluded Middle (PEM)

Brouwer's intuitionism is probably best known for its rejection of the universal validity of the Principle of the Excluded Middle. This rejection is also Brouwer's most original contribution to the study of the foundations of mathematics. Even if the rejection of the PEM is to be found in some of Borel's work[3], Borel admits that he does not regard acceptance of 'empirical logic' as of great mathematical importance.[4] Also, his rejection dates from well after 1908.

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[1] 1948C, p. 1248. But already in 1923B non-contradictority is seen as impossibility of contradictority:'...an incorrect theory that cannot be inhibited by any contradiction'. (p.2)

[2] 1929A, p. 160; also 1933, pp. 57-58: 'These investigations concern words, the world of mathematical ideas supposedly expressed by these words is kept out of consideration. Their main aim, not by far yet achieved, consists in the construction of a linguistic mechanism which - apart from some drastic amputations - is capable of producing a linguistic projection of the whole of mathematics, known to the present day, and in whose functioning the appearance of the linguistic figure of the contradiction is excluded.'

[3] Leçons sur la Théorie de Fonctions, ed. 1928 (note VII), p. 219. Cf. also:'Sur l'illusion des définitions numériques', Comptes rendus hebdomadaires des Séances de l'Académie des Sciences, vol. 224 (1947), pp. 765, 767; also 'Sur les difficultés des définitions asymptotiques', Ibid. pp. 1597 - 1599.

[4] 'A propos de la récente discussion entre MM. R. Wavre et M.P. Levy', Revue de Métaphysique et de Morale, vol. 34 (1927), pp. 271 - 276.

Because of this originality, it is important to trace Brouwer's rejection of the PEM back to its earliest forms and development.

In 1928A, page 375, Brouwer states that 'the second insight of intuitionism', (i.e. the rejection of the thoughtless use of the logical principle of excluded middle), 'was found in literature for the first time in Brouwer in 1908B.' What he calls the third insight, the identification of the PEM with the principle of solvability of all mathematical problems, is indeed found for the first time in 1908A, pages 9 and 10, but as we stated earlier[1], already in 1907 Brouwer expresses his reservations as to the value of the PEM. In the passage there quoted, (1907, pp. 131 - 132), he plays down the value of the PEM by reducing it to a mere tautology, in support of his general thesis that language cannot be a source of new mathematical knowledge, banning the PEM to the domain of language: 'Considering the words of the sentence (i.e. a function is either differentiable or not differentiable) and discovering regular behaviour in the sequence of the words of this sentence and similar sentences, the logician projects even here a mathematical system and calls such a sentence an application of the principle tertium non datur.'[2] On page 160 (1907), he accepts the PEM together with other logical principles, provided they occur in a description of a previously constructed mathematical system and are not used as a starting-point, applying to a 'chimerical all '. Although both passages are withdrawn in 1917A (respectively p. 442 and p. 444), the second passage (p. 160) seems to agree with Brouwer's modified view and acceptance of the PEM as restricted to finite cases.

Similarly, Brouwer already in 1907, page 142, shows his doubts about the solvability of all mathematical problems. This follows his thesis that non-contradictority does not imply mathematical existence. In a footnote, he remarks: 'It is therefore a fortiori not certain that of every mathematical problem either a solution can be given, or it can be logically shown that it is unsolvable, of which, according to Hilbert in 'Mathematische Probleme', every mathematician is most deeply convinced. But of course, it is not certain that this question itself can ever be dealt with[3], i.e. that it can be shown either to have been solved or to be unsolvable.'

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[1] See above, p. 130.

[2] 1907, pp. 131 - 132.

[3] Brouwer later uses the word 'test' (e.g. 1923B, p. 1 and 1948C, p. 1245), or 'judge' (e.g. 1952B, p. 141; cf. also 1955, p. 114).

The full realization that the unlimited application of the PEM in mathematics is also a problem of existence and presupposes the solvability of every mathematical probelm came in 1908B. The actual passage in which Brouwer rejects the PEM is short and we quote it in full: 'Now the Principle of the Excluded Middle; this asserts that every supposition is either true or false; mathematically, that for every hypothesis that two systems fit into one another in a certain way, we can always either complete the construction or will meet with impossibility. (Literally, 'either its completion can be constructed, or its collision with impossiblity'). The question of the validity of the PEM is, therefore, equivalent with the question of the possibility of unsolvable mathematical problems. For the sometimes (ilbert) expressed conviction that unsolvable mathematical problems do not exist, there is not even a hint of proof. As long as only finite discrete systems are considered, the investigation into the possibility or impossibility can be terminated and lead to an answer and then the PEM is a reliable principle.'[1]

Most of Brouwer's work after the first world war consists of reconstructions of mathematics expressly without the use of the PEM. When in some of his work, 1921A, footnote p. 798, 1923B, pp. 1 - 2, 1929A, p. 158, 1933, p. 55, 1948C, p. 1247, 1952B, p. 142, 1953, p. 5, 1955, pp. 113 - 114, he discusses the PEM itself, he gives an historical interpretation, invariably expressed in the same terms. This is basically a summing up of his views on language and logic and their relation to mathematics. He describes 'the dogma of the PEM' as a phenomenon of the history of civilization of the same kind as the old-time belief in the rationality of π or the rotation of the universe on an axis through the earth. First classical logic was abstracted from the language of the mathematics of 'part sets' of a certain finite system. A priori existence independent of this mathematics was then attributed to this logic, and on the basis of this a priority the rules of logic were then applied to the mathematics of infinite sets. The reasons Brouwer gave for the persistence of this dogma were: the obvious non-contradictority of the principle for an arbitrary single assertion and the practical validity of the whole of classical logic for an extensive

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[1] 1908B, pp. 9 - 10.

group of simple, every-day phenomena. The first reason accounts for the clumsiness of the counterexamples given by Brouwer to refute the PEM.[1] Apart from these counterexamples and the above historical interpretation, Brouwer did not elaborate on the grounds for his rejection of the PEM. It may be going too far to trace it down to Schopenhauerian aversion to Hegel's dialectic; Brouwer was clearly suspicious of any device that would seem to produce mathematical knowledge from a purely verbal hypothesis. It is in this context and on these grounds that he rejects the PEM out of hand, in 1907 as meaningless, and from 1908 as unjustified.

Most of the ingredients of this rejection can already be found in 1907: the identification of 'mathematically existent' with 'having been constructed',[2] Brouwer's constructive views of a positive mathematical statement 'recording the fitting of a constructed mathematical entity within the structure of a mathematical system', reflected by a logical inference of the simple pattern of 'whole and part'. This demand for a constructive proof for every mathematical statement, implies the necessity of a constructive proof for the negative statement and leaves open the possibility of a third alternative: the pure absence of a proof either way. Even though Brouwer shows on page 131, 1907, that he was not quite sure about the constructive proof for a negative statement, he did in one instance draw this conclusion and demanded a positive proof for the unsolvability of all mathematical problems (see above-quoted footnote).

What is new in 1908B is a clear, constructive definition of a negative statement in terms of a contradiction.[3] 1908B demanded for the negative statement $\sim p$ a constructive proof of the contradictority of p.

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[1] Brouwer was well aware of this and, seemingly in desperation, sighs: 'That classical mathematics cannot be silenced in a simple way is because of the favourable fact that - if one limits oneself to finite groups of properties - the PEM in spite of being incorrect is non-contradictory, so that intuitionism, in its fight against the errors of classical mathematics, is deprived of the most current means of repression, i.e. reductio ad absurdum, and has to rely exclusively on admonition to use reason (Du. redelijke bezinning).' 1933, p. 63.

[2] 1907, p. 177.

[3] A recording of an end of the road of a construction, 'there being no room for the posited structure within the given construction'. See above, p. 137.

It is this definition of negation that is the key to most of Brouwer's divergent views.[1]

The consistent demand of 1907 for an actual constructive proof for every mathematical statement implies the complete rejection of the PEM. Acceptance of the PEM for the finite case implies a qualification of this requirement that every mathematical statement is a record of a previous construction. After 1907, Brouwer dropped this requirement and was satisfied with the promise and possibility of such a construction. Indeed, the whole question of the valdity of the PEM in the period 1908 - 1923 was a question of the possibility of corroboration by investigation and identified with the question of the solvability of every mathematical problem.[2] Because of its finite character, Brouwer claimed in 1908B that 'such an investigation can be terminated and lead to an answer and the PEM is a reliable principle';[3] and in 1923B: 'For properties derived within a specific finite main system by means of the PEM, it is always certain that we can arrive at their empirical corroboration if we have a sufficient amount of time at our disposal.'[4] There is no doubt that by 'reliable' Brouwer meant, 'leading to true statements'. As late as 1923, he states: '... for properties conceived within a specific finite main system, the Principle of the Excluded Middle holds, i.e. the principle that for every system every property is either correct or impossible, and in particular the principle of the reciprocity of the complementary species, i.e. the principle that for every system the correctness of a property follows from the impossibility of the impossibility of this property.'[5]

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[1] See above pp. 137 ff.

[2] See quotation on p.142 above; also 1921A, p. 797, and 1928C, p. 375.

[3] 1908B, p. 10.

[4] 1923B, pp. 1 - 2. Interesting are the examples Brouwer gives of the valid use of the PEM: 'If, for example, the union (p,q) of two mathematical species p and q contains at least eleven elements, it follows on the basis of the PEM (which in this case also appears as "the principle of disjunction") that either p or q contains at least six elements. Likewise, if we have proved in elementary arithmetic that, whenever none of the positive integers $a_1, a_2, \ldots a_n$ is dividisle by a prime number c, the product $a_1 . a_2 . \ldots a_n$ is not divisible by c either; it follows on the basis of the reciprocity of the complementary species that, if the product $a_1 . a_2 \ldots a_n$ is divisible by the prime number c, at least one of the factors of the product is divisible by c.'

[5] 1923B, p. 1.

Brouwer's stand on the PEM in the period 1908 - 1923 can be summed up as follows:

1° The question of the PEM i to be identified with the question of the possibility of actually carrying out the investigation.

2° Within a finite main mathematical system, such an investigation can be terminated, and the PEM is a valid but trivial[1] principle, leading to true mathematical statements.

3° In an infinite mathematical system, the use of the PEM is not justified.

4° Even the unjustified use of the PEM will never lead to contradiction.[2]

5° Freedom from contradiction is, therefore, not a guarantee of the validity of a mathematical argument.

4.2 The PEM after 1923 and the calculus of absurdities

On the 23rd November, 1923, Brouwer presented a paper to the Royal Dutch Academy entitled, 'Intuitionistische splitsing van mathematische grondbegrippen', (lit., 'Intuitionistic splitting of some fundamental concepts of mathematics'), 1923C, in which he developed his well-known calculus of absurdity. It marks an important shift in his views on the PEM, on non-contradictority[3] and truth, and a temporary change in his attitude towards logic.

The opening lines of the Dutch version seem to indicate a complete rejection of the PEM: 'The intuitionistic conception of mathematics not only rejects the principium tertii exclusi completely, (Du. 'in zijn geheel', i.e. in its entirety), but also the special case which is rooted in the principle of reciprocity of the complementary species, i.e. the principle that for an arbitrary system the correctness of a property may be inferred from the absurdity of the absurdity of that property.'[4] The reference to 'arbitrary systems' could indicate agreement with Brouwer's previous general statements which allowed an exception for the finite case, especially since in its German version the above passage was changed into: 'This article deals with some

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[1] Cf. Brouwer's remarks on machines and suitably trained animals in 1908B, p. 10, quoted on p.135 above.

[2] Cf. 1908B p. 11.

[3] Cf. p.136 above.

[4] 1923C, p. 877.

'conclusions from the intuitionistic thesis which maintains that the logical Principle of the Excluded Middle only applies with unlimited validity to parts of mathematics within a definite finite system, and therefore only to those parts of natural sciences on which a definite finite mathematical system can be projected.'[1] The stronger statement in the Dutch version may well be the better reflection of Brouwer's real thoughts at the time, while the German version reflects a more cautious line at the time of a heated controversy with Hilbert.[2] Moreover, rejection of the classical double negation, as set out in 1923C, undermines the validity of the PEM even for the finite case: 'Classical mathematics postulates for every property the alternatives as either correct or absurd, and, therefore, equivalence of correctness and absurdity of absurdity. For the intuitionist, absurdity of absurdity is not equivalent to correctness but follows from it.'[3] Absurdity, and absurdity of absurdity, are then universally applied to relations without any distinction being made between finite and infinite systems.[4] (The 'logical nature' of Brouwer's argument in 1923C will be further discussed in Chapter IV, 'The Principle of Reciprocity of Complementarity').

The complete rejection of the PEM as a logical principle producing *true* mathematical statements can be found in 1928C, 1933, 1948C, 1953 and 1955. Apart from true statements, Brouwer here distinguished between those that are false or absurd, and those that are non-contradictory. Already in 1908B Brouwer had hinted at such a distinction: 'Therefore, in mathematics, one should in all the theorems that are usually accepted as having been proved, distinguish between those that are *correct* and those that are *non-contradictory*.' [5]

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[1] 1925B, p. 251.

[2] The added reference to natural sciences might also point to this. 1928A, mainly directed at Hilbert, clearly shows Brouwer's anxiety to prove that his views in practice had been adopted by Hilbert. Cf. p.39.

[3] 1923C, p. 877.

[4] See further, p. 217.

[5] 1908B, p. 12 footnote.

But apart from the newly found more constructive meaning of 'absurdity of absurdity', and it greater respectability, non-contradictority is now ascribed within finite systems: 'Finally, we observe that the PEM in intuitionistic mathematics, even if not correct, is free from contradictions if one limits onself exclusively to finite species of properties, which first explains why errors in classical mathematics could for so long survive, and secondly, has encouraged formalistic tendencies. On the basis of intuitionistic insights, therefore, there are, apart from accurate theories which can be developed independently of the PEM, non-contradictory theories, which, with the above restriction, can be derived with the help of the PEM, and which cover a much larger part of classical mathematics than the correct theories. An appropriate mechanization of the language of such intuitionistic non-contradictory mathematics will in this way provide what the formalist school is aiming at.'[1]

Within an infinite main mathematical system, there is no such guarantee for non-contradictority. This Brouwer tried to prove in his counterexamples.

4.3 Counterexamples refuting the PEM

In keeping with his views at the time, Brouwer refutes the PEM in 1908 by simply quoting mathematical problems that have not yet been solved, such as: 'Is there in the decimal expansion of π a number that is found more frequently than others?' And: 'Are there in the decimal expansion of π infinitely many pairs of the same subsequent digits?'[2]

Fermat's problem is a favourite example of an unsolved problem refuting the PEM in his later work.[3] It is quite clear here - as well as from Brouwer's general views on mathematical existence and a priori pronouncements on mathematics - that Brouwer did not have in mind problems that are essentially unsolvable or undecidable in Gödel's sense: 'Furthermore, we remark that an assertion which is in the fourth case[4], may at some time pass into one of the other cases, not only

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[1] Vienna Lecture, 10th March, 1928; 1929A, p. 163.

[2] 1908B, p. 12.

[3] 1948C, p. 963 footnote; 1955, p. 114.

[4] Cf. further p. 154.

'because further thinking may generate a construction accomplishing this passage, but also because in intuitionist mathematics, a mathematical entity is not necessarily predeterminate, and may in its state of free growth, at some time acquire a property which it did not possess before. An example of the fourth case is the assertion: "Three natural numbers, a, b, n. cannot satisfy the equation $a^n + b^n = c^n$ for $n > 2$". Each mathematical assertion which is in the fourth case yields a refutation of the principle of the excluded third.'[1]

Preoccupation with the rationality of π is evident in his later counterexamples, but after 1923 it is used to show up contradiction due to the use of the PEM, an aim he expressed in 1928: 'The simultaneous enunciation of the PEM for arbitrary species of properties can indeed be contradictory.[2] Indeed, the contradictority of the principle that "all real numbers are either rational or irrational", can be proved.'[3] The proof was given by Brouwer in 1923C[4], where he constructed a real number r which was not rational but whose rationality could not possibly be absurd. In his definition of the terms of the sequence converging to r, Brouwer made the value of the terms and of r dependent on the occurrence of the sequence 0123456789 in the decimal expansion of π :

'Let d_ν be the ν-th digit after the decimal point in the decimal expansion of π and let $m=k_n$ if, as the decimal expansion of π is progressively written, it happens at d_m for the nth time that the segment $d_m, d_{m+1}, \dots d_{m+9}$ of this decimal expansion forms the sequence 0123456789. Further, let $c = (-\frac{1}{2})^{k_1}$ if $\nu \geqslant k_1$, otherwise let $c_\nu = (-\frac{1}{2})^\nu$. Then the sequence $c_1, c_2, c_3, \dots$ converges to a real number r.

If we call a real number g rational, if two integers p and q can be calculated whose quotient equals g, the r canot be called rational, but on the other hand, the rationality of r cannot possibly be absurd for in that case k_1 could not possibly exist, which would imply that $r = 0$ and therefore is rational.'[5]

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[1] 1955, p. 114.

[2] Italics are mine.

[3] 1929A, p. 164.

[4] Cf. also 1928A, p. 378.

[5] 1923C, p. 877.

The same example had been given in 1923B, but here Brouwer concentrated on the fact that for r none of the conditions $r = 0$, $r > 0$, $r < 0$ holds, and on these grounds rejected the view that the points of the continuum form an ordered point species, as well as the notion of the L-integral.[1] The species of the positive integers k_ν, defined above, provided Brouwer with a counterexample refuting the maxim that every mathematical species is either finite or infinite; this resulted in the rejection of the Bolzano-Weierstrasz theorem as well as the Heine-Borel covering theorem.[2]

The occurrence of the sequence 0123456789 in the decimal expansion of π is just an example of a more general property x, whose existence or impossibility can be derived for every definite positive integer, although neither can any positive integer be determined which possesses x, nor can the impossibility of x be proved for all positive integers. Later, (1929A, 1933, 1948C, 1952B, 1953 and 1955), Brouwer would refer to this property as a fleeing property, (Du. vluchtende eigenschap; Germ. fliehende Eigenschaft) and to the sequence as a drift.[3] On the basis of this generalized property, he then refutes the PEM and such classical theorems as: for two straight lines of the projective plane at least one common point can be determined, Bolzano's theorem, and Rolle's theorem.[4]

4.4 The controversy concerning 'the Brouwer-logic'

Brouwer's search for proof of contradictority of the PEM in an infinite system was suddenly halted by a publication of Glivenko in 1928, which followed a series of controversial articles on Brouwer's

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[1] 'This notion of integral is bound to the notion of 'measurable function' and - as indicated in the above example - not even a constant function satisfies the condition of measurability.'

[2] 1923B, pp. 337 - 339.

[3] This fleeing property is generally described by Brouwer as a property f of natural numbers satisfying the following conditions:
i) For each natural number n it can be decided whether or not n possesses the property f;
ii) No way is known to calculate a natural number possessing f;
iii) The assumption that at least one natural number possesses f is not known to be contradictory.

[4] 1933, p. 61 et al.

calculus of absurdity and the PEM.[1]

R. Wavre was the first to proclaim Brouwer's calculus of absurdity as the basis for a 'logique brouwerienne'. His contribution: 'Sur le principe du tiers exclu'[2], publicized Brouwer's calculus of absurdity and started a controversy on the Brouwer logic, mainly conducted in the Bulletin de l'Académie Royale de Belgique,

M. Barzin and A. Errera in 'Sur la logique de M. Brouwer', showed complete lack of understanding of Brouwer's views on negation and the PEM by introducing a third truth value 'tierce', and using classical negation side by side with the Brouwer negation, even to the extent of adopting the 'principle of the excluded fourth'. They claimed to prove

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[1] R. Wavre, 'Y-a-t-il une crise des mathématiques? A propos de la notion d'existence et d'une application suspecte du principe du tiers exclu.' Revue de Métaphysique et de Morale, vol.31 (1924), pp. 435 - 470.

R. Wavre, 'Logique formelle et Logique empiriste', Revue de Métaphysique et de Morale, vol. 33 (1926), No. 1, pp. 65 - 75.

R. Wavre, 'Sur le principe du tiers exclu', Revue de Métaphysique et de Morale, vol. 33 (1926), pp. 425 - 430.

M. Barzin et A. Errera, 'Sur la logique de M. Brouwer', Académie Royale de Belgique, Bulletin de la Classe des Sciences, vol. 13 (1927), pp. 56 - 71.

Paul Levy, 'Sur le principe du tiers exclu et sur les theorèmes non susceptibles de démonstration', Revue de Métaphysique et de Morale, vol. 33 (1926), pp. 253 - 258.

Paul Levy, 'Critique de la logique empirique, Réponse à M. Robin Wavre', Revue de Métaphysique et de Morale, vol. 33 (1926), pp. 545 - 551.

Paul Levy, 'Logique classique, Logique brouwerienne et Logique mixte', Académie Royale de Belgique, Bulletin, vol. 13 (1927), pp. 256 - 266.

E. Borel, 'A propos de la recente discussion entre MM R. Wavre et M. P. Levy', Revue de Métaphysique et de Morale, vol. 34 (1927), pp. 271 - 276.

S. Avsitidysky, 'Note relative au travail de MM Barzin et Errera, "Sur la logique de M. Brouwer"', Ac. R. de Belg., Bulletin, vol. 17 (1927), pp. 724 - 730.

A. Khintchine, 'Objection à une note de MM Barzin et Errera', Ac. R. de Belg., Bulletin, vol 14 (1928), pp. 223 - 224.

V. Glivenko, 'Sur la logique de M. Brouwer', Ac. R. de Belg. Bulletin, vol. 14 (1928), pp. 225 - 228.

V. Glivenko, 'Sur quelques points de la logique de M. Brouwer', Ac. R. de Belgique, Bulletin, vol. 15 (1929), pp. 183 - 188.

[2] See footnote [1] above.

that within the Brouwer logic, this third truth value leads to contradiction.

Subsequent articles by Paul Levy and Serge Avsitidysky are attempts to improve Barzin and Errera, both based on the conviction that the third truth value 'tierce' is an essential element in Brouwer's logic.

Khintchine, in 'Objection à une note de MM. Barzin et Errera', does not express such a position but simply points at some logical flaws in Barzin's and Errera's argument.[1]

Brouwer himself did not directly take part in this debate. A man so deeply concerned with public recognition - as is shown in his controversy with Hilbert - must certainly have been flattered being the centre of this public controversy and being heralded by some as the revolutionary innovator of the century in the field of logic. On the other hand, he showed a sense of discomfort in the field of symbolic logic where he professed himself to be an outsider[2], and his deep-rooted suspicion of logic remained with him for the whole of his life.

V. Glivenko's 'Sur la logique de M. Brouwer' remained more true to the spirit of Brouwer. He showed that the introduction of a third truth value ('tierce') is as illegitimate in Brouwer's logic as it is in classical logic. He made the first attempt at formalizing intuitionistic logic by stating and listing the axioms acceptable to Brouwer and proving some of Brouwer's results.[3]

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[1] Khintchine shows that the contradiction Barzin and Errera claimed to have found is based on the false assumption of some 'generally accepted logical rules', which, however, will equally lead to contradiction in classical logic.

[2] <u>1955</u>.

[3] Much of Heyting's formalization of logic (1930) was based on Glivenko, (Cf. references in Heyting, 1930, pp. 43, 44, 49, 52). Glivenko listed the logical axioms acceptable to Brouwer as:

I $p \supset p$
II $p \supset q :\supset: q \supset r . \supset . p \supset r$
III $pq \supset p$ IV $pq \supset q$
V $r \supset p :\supset: r \supset q . \supset . r \supset pq$
VI $p \supset p \lor q$ VII $q \supset p \lor q$
VIII $p \supset r :\supset: q \supset r . \supset . p \lor q \supset r$
IX $p \supset q :\supset: p \supset \sim q . \supset . \sim p$

Glivenko also observed that adding $p \lor \sim p$ to these axioms produces the classical propositional calculus (p. 226). (Cf. also Glivenko 1929, p. 183, where he proved completeness of the system).
To the axioms, Glivenko added the principles derivable from I - IX:
principle of contradiction $\sim(\sim p . p)$
XI $p \supset \sim q . \supset . q \supset \sim p$
XII $p \supset q . \supset . \sim q \supset \sim p$
Brouwer accepted $p \supset \sim(\sim p)$ as an axiom. Heyting admits in 1930 (p. 49) that he changed his mind because of the work of Glivenko.

A comparison between the typically informal proof of Brouwer and the corresponding proof of Glivenko shows the essential agreement between the two, and the elegance and superiority of the latter:

Brouwer: Absurdity of absurdity of absurdity is equivalent with absurdity.
Proof: a) When the property y follows from the property x the absurdity of x follows from the absurdity of y. Therefore, since from correctness follows absurdity of absurdity, absurdity must follow from the absurdity of the absurdity of the absurdity.
b) Since absurdity of absurdity of a property follows from its correctness, in particular absurdity of the absurdity of the absurdity must follow from the correctness of absurdity.[1]

Glivenko: In the Brouwer logic, the falsity of the falsity of the falsity of a proposition q implies the falsity of the proposition q:

$$\sim(\sim(\sim q)) \supset \sim q$$

Proof: $\sim q \supset \sim q$ (I)

$$\frac{\sim q \supset \sim q \;.\supset.\; q \supset \sim(\sim q) \qquad \text{(XI)}}{q \supset \sim(\sim q)}$$

$$\frac{\begin{array}{c} q \supset \sim(\sim q) \\ q \supset \sim(\sim q) \;.\supset.\; \sim(\sim(\sim q)) \supset \sim q \qquad \text{(XII)} \end{array}}{\sim(\sim(\sim q)) \supset \sim q}$$

Most disturbing, however, to Brouwer must have been Glivenko's first theorem: 'In the Brouwer logic, the proposition, "the proposition $\sim p \vee p$ is absurd", is absurd in the Brouwer sense, or $\sim(\sim(\sim p \vee p))$'[2] This amounts to accepting the universal non-contradictority of the PEM. In the proof, Glivenko makes use of VI, VII, IX and XII, all acceptable to Brouwer.

Brouwer's immediate reaction is difficult to measure from his writings. Both 1928A and 1929A were publications of lectures given before Glivenko's article appeared (respectively 17th December, 1927, and 10th March, 1928). With the exception of 1933 - which is for the main parts an almost literal translation of 1929A - Brouwer does not

[1] 1923C, p. 878 and 1925B, p. 253.

[2] Op. cit. p. 226.

mention the PEM until 1948. This almost complete silence, starting immediately after Glivenko's publication, could by itself be an indication of Brouwer's reaction. According to G. Kreisel, this silence is due to intellectual shock due to the publication of papers by Heyting (1930) and Gödel (1931).[1] If there has been such a shock, it seems more likely to have been started by Glivenko's publication, which coincides with the beginning of this period of silence, rather than by the publication of Gödel's work three years later. Moreover, Gödel's work on the incompleteness of formal systems confirmed Brouwer's views even if Gödel's methods of proof were different. Brouwer later, in an interview[2], showed how little he was impressed by Gödel's proof and shrugged it off. As far as Heyting's formalization of intuitionistic logic is concerned, this came as no surprise to Brouwer, who was to some extent involved in its publication.

4.5 In his post-war publications, Brouwer remains adamant in his refusal to accept the PEM as 'an instrument to discover new mathematical truth'.[3] In mathematics proper, only four situations may arise; accordingly, in the language of mathematics, distinction can be made between statements which are:

1° True, i.e. have been proved;
2° False, i.e. the supposition of which has led to a contradiction, such as $1 = 0$;
3° Neither true nor false, but an algorithm is known leading to a decision that they are true or false;
4° Neither true nor false, nor do we know an algorithm leading to the decision that they are true or false.[4]

The PEM, here also referred to as the principle of judgeability, is refuted by the very existence of statements of the fourth kind, 'each mathematical assertion which is in the fourth case yields a refutation of the principle of the excluded third'.[5] Granting the PEM in the finite case, ('an assertion of possibility of some construction of bounded finite character in some finite mathematical species is

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[1] Biographical Memoir of Luitzen Egbertus Jan Brouwer, Royal Society, (1970), p. 45.

[2] Interview with A. Hill.

[3] 1948C, p. 1244; cf. also 1953 and 1955.

[4] 1955, p. 114.

[5] Ibidem.

necessarily judgeable')[1], does not amount to more than accepting that 'such a construction can be attempted only in a finite number of ways, each of which will succeed or fail after a finite number of steps'.[1]

The PEM itself, and statements about the PEM, its truth or contradictority, are 'assertions about assertions', and belong to the 'introspective theory of mathematical assertions, a theory which with some right may be called intuitionist mathematical logic.[2]

5.1 Symbolic notation and formalization

In 1907 Brouwer accepted that there is no essential difference between the use of language and that of symbolic notation in mathematics or in logic: 'Just as all mathematical language, so can also this language, (i.e. of classical logic), without difficulty, be condensed to symbols.'[3] (He then, approvingly, quotes Whitehead's A Treatise of Universal Algebra.) Yet there is throughout Brouwer's work a reluctance to make use of widely accepted symbolic notation. Long phrases are preferred to concise symbols, not only in his 'logical' contributions,(e.g. 'The absurdity of the absurdity of the absurdity follows from the correctness of the absurdity'[4]), but also in his mathematical work; set-theoretical symbols $\cup$ $\cap$ $\subset$ $\in$ are never used, the only concession made here is the occasional use of $\mathfrak{S}$ for union and $\mathfrak{D}$ for intersection. Much of the alleged incomprehensibility of Brouwer's writings is due to his refusal to conform to common practice and to make use of a clear and concise symbolic notation.

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[1] 1955, p. 114.

[2] 1953, p. 3.
Even within this domain of intuitionist logic, Brouwer is equally insistent that truth only follows 'realization'. Extended principles, i.e. principles concerning an infinite domain of assertions, can never be true since it is impossible to complete verification of all individual assertions: 'In intuitionism, of course, all these principles, being assertions about assertions, are only then "realized", i.e. only then convey truths, when these truths have been experienced. On this basis, it can be proved that the extended principles are not only not true, but even contradictory.' (1953, p. 3.)

[3] 1907, p. 159.

[4] 1923C, p. 878; 1925B, p. 253.

Brouwer, himself, explained his opposition to symbolic usage by stressing that expression of a mental construction would be further complicated, if first an explanation had to be given of an agreed symbol; moreover, that this explanation had to be given in ordinary language.[1]

Association of symbolic notation with logicism and formalism has certainly been a contributing factor to Brouwer's dislike of symbolism. By the time that most of the symbols had become generally established, and did not need prior explanation, Brouwer's attitude had hardened and became more like a stubborn refusal to 'receive benefactions from formalism'.[2] Besides, formalist emphasis on symbols to Brouwer presented the real danger of drawing attention away from the real nature of mathematical objects; symbols, more than language, could be used as primitive constructs without any prior meaning, a formalist practice already condemned in 1907: 'Hilbert even declares explicitly that with words such as "Punkt", "Gerade", "zwischen", one should not think of any mathematical interpretation.'[3]

However, the most fundamental reason for Brouwer's reluctance to use symbols is probably an a priori unwillingness to accept the possibility of a precise formulation of a mental construction. In contrast with a living language, which is not precise, where words have various connotations and shades of meaning, a formal symbol claims to convey a definite and precise meaning. As instruments conveying mathematical truth mechanically, necessarily and completely, both language and formal symbols had been rejected because of both the 'spiritual' nature of mathematics as mental, 'living' activity, and the nature of language as a sign. Paradoxically, because of the very vagueness of living language, Brouwer claims, truth can be found through language. In Leven, Kunst en Mystiek, Brouwer ridicules precision in language, 'to be able to speak for some time without being caught in contradiction and without making silent assumptions is indeed a great art, to be valued in an acrobat'[4], but claims that truth can

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[1] As quoted by A. Heyting during a private conversation, (Logic Conference, Bedford College, 1970).

[2] 1928A, as quoted above, p. 31.

[3] 1907, p. 137.

[4] 1905, p. 40.

be found through 'works of art' and 'in poetry more so than in prose'.[1] This power of the living language to convey truth transcends the immediate precise meaning of words or symbols.

5.2 This Bergsonian unwillingness to accept the possibility of grasping living mathematical reality completely and exhaustively in definite, predefined words or symbols is the fundamental reason for Brouwer's consistent opposition to formalism and to any formalization of mathematics. 'Post factum' formal analysis of his own intuitionist mathematics and logic he regarded as a futile academic exercise; to any such attempt, he reacted with the irritation of the artist who sees his own work analyzed and explained away by experts. Above all, he recognized the impossibility of reconciling the post factum nature of such analysis with the true spirit of Hilbert's formal axiomatic method, a contradictio in terminis. Poincaré had rejected formal methods as not sufficient; he appealed to intuition as the guide in selecting the significant.[2] Brouwer rejected formalization on the grounds of its arbitrary restriction of the 'freedom of mathematics', which 'can never be exhausted in any one system, and develops in a self-unfolding guided by a free arbitrariness'.[3]

The impossibility of formalization without appeal to intuitive mathematics was repeatedly stressed: 'It presupposes the intuitive mathematics of the set of natural numbers';[4] and in 1907, criticising Hilbert's Grundlagen der Geometrie, 'Hilbert's formalization needs intuitive acts even more than Peano's.'[5]

However, the main reason for Brouwer's rejection of formalism and formalization is the non-pure-mathematical nature of formalized mathematics. This is illustrated in an anlysis of Hilbert's formalization (in Grondslagen) into eight successive stages ('in genetic order'):

1° The pure construction of intuitive mathematical systems;

2° The verbal parallel of mathematics, mathematical language;

3° The mathematical analysis of this language, which reveals the logical structure of this verbal edifice;

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[1] 1905, pp. 48 and 49.

[2] See above p. 76.

[3] 1907, p. 119; see above p. 98.

[4] 1928A, p. 375.

[5] 1907, p. 174.

4° Abstraction from the meaning of the elements of these logical structures, and reconstructing the logical structures in a mathematics of the second order;

5° The verbal expressions of these logicist structures in symbolic systems (Peano, Russell);

6° The mathematical analysis of this language (Hilbert);

7° Abstraction from the meaning of this language, leading to a mathematical system of the third order;

8° The verbal parallel of this mathematical system of the third order, motivating its construction and demonstrating its non-contradictority.[1]

As far as the mathematical relevance of logic and formalization is concerned, Brouwer leaves us in no doubt: mathematics proper is only found in stage 1°: 'In practical every-day life we cannot dispense with language, stage 2°, 'but this stage remains a non-mathematical activity[2]', as do indeed all further stages which depend on it.

5.3 Brouwer's logic

Mathematical logic in the Brouwer sense has been described in this Chapter as, 'post factum analysis of verbal or symbolic recording of intuitive constructions', very much on the lines of a study of grammar of one's mother tongue, which does not prescribe how one ought to speak but merely observes and analyses spoken language. As post factum analysis of an ever-growing, never completely closed domain of a mathematical activity, intuitionistic logic lacks completeness, universality, and a priority with respect to the mathematical activity.

A comprehensive, systematic analysis of the 'language of intuitionist mathematics to date' was never produced by Brouwer himself, even if he claims that 'intuitionist mathematics has its general introspective theory of mathematical assertions, which, with some right, may be called intuitionist logic'.[3] Neither does Brouwer here refer to Heyting's formalization; reference here is to his own fragmentary observations on such topics as the PEM, the Brouwer negative, non-contradictority and mathematical assertions.[4]

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[1] 1907, pp. 173 - 175.

[2] 1907, p. 175.

[3] 1953, p. 3.

[4] See above p. 154.

5.4 Heyting's formalization

The first attempt at a systematic formalization of intuitionistic mathematics and logic was made by Arend Heyting[1], the most loyal and enthusiastic intuitionist supporter among Brouwer's students. Fraenkel claims it to be 'the most decisive step since the establishment of neo-intuitionism in 1907'.[2] In his doctoral thesis, Intuitionistische Axiomatiek der Projectieve Meetkunde, Heyting had already shown a more positive appreciation of formal methods. In reply to a challenge by Mannoury, he produced in 1928 a first formalization of intuitionist logic and mathematics (not published), which was awarded a prize by the Wiskundig Genootschap. This work formed the basis for Heyting's well-known formalization of intuitionistic logic and mathematics, which appeared as:

'Die formalen Regeln der intuitionistischen Logik', (Sitzungsberichte der preussischen Akademie von Wissenschaften, phys. math. Klasse, 1930, pp. 42 - 56;)

'Die formalen Regeln der intuitionistischen Mathematik', II and III, ibidem, 1930, pp. 57 - 71 and pp. 158 - 169.

Brouwer's reaction to Heyting's formalization[3] has been described by Heyting as 'negative'.[4] Brouwer remained unconvinced of the value of the exercise; he agreed to its publication, mainly because of Heyting's expressed support for Brouwer's views on language and logic and their relations to mathematics: 'Intuitionist mathematics is an activity of the mind, and every language, including the formalistic language, only a means of communication. It is in principle impossible to construct from formulae a system which is equivalent to intuitionistic mathematics, since the possibilities of human thinking cannot be reduced

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1 A complete bibliography of Heyting's work is given in A.S. Troelstra, 'The scientific work of A. Heyting', Logic and the Foundations of Mathematics, Groningen 1968, ('dedicated to Prof. A. Heyting on his 70th birthday').

2 A. Fraenkel and Y. Bar-Hillel, Foundations of Set Theory, Amsterdam 1958, p. 228.

3 For an account of Heyting's formalization, see e.g.,
A. Fraenkel and Y. Bar-Hillel, op. cit. pp. 229 ff.
G.T. Kneebone, Mathematical Logic and the Foundations of Mathematics, Van Nostrand 1963, pp. 254 ff.
A. Mostowski, Thirty Years of Foundation Study, Acta Philosophica Fennica, Fasc. XVII, Helsinki 1965, pp. 10ff.

4 In a personal interview at the Logic Conference, Bedford College, 1970.

'to a finite number of rules, laid down in advance'[1], and as for the formalization of intuitionist logic: 'The formulation of generally valid logical rules is not needed for the construction of mathematics; in each individual case, these laws are discovered anew as valid for the mathematical system under consideration.'[2]

There is, however, in Heyting, as indeed in all later intuitionist supporters, a more positive appreciation of language, symbolic language and formalization.

While Brouwer insisted on the languageless nature of mathematics and could only bring himself as far as to admit that 'language in practical everyday practice is difficult to dispense with', for Heyting, 'understanding between mathematicians is essential'.[3]

In his introduction, Heyting also expressed a definite preference for symbolic language over ordinary language: 'The attempt to express the most important parts of mathematics in formal language is justified on the grounds of its greater conciseness and definiteness (grössere Bündigkeit und Bestimmtheit), as compared with ordinary language.'. '... properties which help deeper understanding of the intuitionist concepts and their application in further research.'[4]

Heyting has consistently denied any claims that his formalization should completely reflect the whole of intuitionist mathematics.[5]

He also continued to express his loyalty to Brouwer's fundamental

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1 'Die formalen Regeln der intuitionistischen Logik', p. 42.

2 Ibidem.

3 A. Heyting, 'Intuitionism in Mathematics', Philosophy in the Midcentury, Firenze 1958, p. 103.
Heyting, more than his master Brouwer, was concerned with dialogue between mathematicians, and especially between 'the two warring factions in this ruthless contest', (i.e. the Brouwer-Hilbert controversy). In September 1930, he represented the intuitionist viewpoint at the symposium at Königsberg, where representatives of logicism, formalism, and intuitionism met for the first time. Even if Heyting at this occasion did not go as far as Hahn, who remarked that he considered intuitionism and formalism as important investigations inside mathematics, he expressed the opinion that, 'the problems yet to be solved were more of a technical than a fundamental nature.' (Cf. 'After thirty years', Logic, Methodology and Philosophy of Science, Proc. 1960 International Congress, Stanford 1962.)

4 'Die formalen Regeln der intuitionistischen Logik', p. 42.

5 Cf. e.g. 'Logique et intuitionisme', Actes du 2e colloque international de logique mathematique, Paris 1954, pp. 75 - 82. Also, Intuitionism. An Introduction, Amsterdam 1956.

maxim that 'every mathematical or logical theorem must express the result of a mathematical construction.'[1] ('the principle of positivity', Heyting.)

But in spite of these assurances of loyalty to Brouwer's intuitionist philosophy of mathematics, Heyting's efforts to combine the best of formalism and intuitionism led to a compromise in which some of Brouwer's major principles became untenable, and were surrendered.

Brouwer's outright rejection of formalization was based on the conviction that formalization was incompatible with what he regarded to be the only true conception of mathematics.[2] In logic as a post factum analysis, there is no place for principles applying universally to an infinite mathematical domain.[3] The value of such principles lies in the guarantee of correctness in their application. This was to some extent also claimed by Heyting: 'The relation between this system, (i.e. propositional calculus), and mathematics, is this, that with a particular interpretation of constants, and with certain restrictions as to the substitution of variables, every formula represents a correct mathematical statement.'[4] His added warning that the validity of a logical principle must be ascertained every time it is applied in a mathematical proof,[5] not only implies suspicion of the validity of such principles, but destroys their usefulness as safe working principles in mathematics. Heyting recently stated[6] that he would not completely endorse his earlier views on the role of logical principles.

As for the need or necessity of formalization, Heyting expresses in 1930 his complete agreement with Brouwer: 'For the construction of mathematics, the formulation of universally valid logical rules is not necessary.'[7] However, in 1969 he writes, 'Brouwer has always maintained

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1 'Intuitionism in mathematics', <u>Philosophy in the Midcentury</u>, Firenze 1958, p. 108.

2 Heyting took a much more liberal view: 'No direction of research has any longer the pretension to represent the only true mathematics.' ('After thirty years', p. 194.)

3 In the light of Brouwer's notion of post factum analysis, it seems significant that Heyting interpreted his formalization in 1931 as a calculus of <u>intended</u> constructions. ('Die intuitionistische Grundlegung der Mathematik', <u>Erkenntnis</u> 2 (1931), pp. 121 - 122).

4 'Die formalen Regeln der intuitionistischen Logik', p. 42.

5 Cf. Beth, op. cit. p. 434.

6 Private conversation, Logic Conference, Bedford College, 1970.

7 'Die formalen Regeln der intuitionistischen Logik', p. 42.

'that formalizing mathematics is a fruitless exercise, as mathematics is a construction in the mind, of which language, and therefore also any formal system, can only give an inadequate representation. I have become more and more convinced that, at least in the communication of mathematics, formalization has its great advantages. From recent research into the notion of choice sequences, it has even appeared that formalization is necessary for any sufficiently clear representation. I rerer here to the outline given by A.S. Troelstra in his Principles of Intuitionism.'[1]

Heyting's formalization led to a great many investigations by Kolmogorov, Gödel, Jaskowski, Gentzen, Tarski, Stone, McKinsey, Mostowski, Fitch, Henkin and others.[2]

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[1] Letter A. Heyting to W.P. Van Stigt, 29th October, 1969.

[2] A.N. Kolmogorov, 'Zur Deutung der intuitionistischen Logik', Mathematische Zeitschrift, vol. 35 (1932), pp. 58 - 65.

K. Gödel, 'Zur intuitionistischen Arithmetik und Zahlentheorie', Ergebnisse eines mathematischen Kolloquiums, Heft 2, 1933, pp. 34 - 38.

S. Jaskowski, 'Reserches sur le système de la logique intuitioniste', Actes du Congrès International de Philosophie Scientifique, vol. 6, Paris 1936, pp. 58 - 61.

G. Gentzen, 'Die Widerspruchsfreiheit der reinen Zahlentheorie', MA, vol. 12, pp. 493 - 565; also, 'Untersuchungen über das logische Schliessen', Mathematische Zeitschrift, vol. 39 (1934), pp. 176 - 210, 405 - 431.

A. Tarski, 'Der Assagenkalkül und die Topologie', Fundamenta Mathematica, vol. 30, (1938), pp. 103 - 104.

M.H. Stone, 'Topological representations of distributive lattices and Brouwerian logics', Časopis pro pěstovani mathmatiky a fysiky, vol. 67 (1938), pp. 1 - 25.

J.C.C. McKinsey, 'Proof of the independence of the primitive symbols of Heyting's calculus of propositions', Journ. Symb. Logic, vol. 4, (1939), pp. 155 - 158.

'A solution of the decision problem for the Lewis Systems S2 and S4, with an application to topology', Journ. Symb. Logic, vol. 6 (1941)

A. Mostowski, 'Proofs of non-deducibility in intuitionistic functional calculus', Journ. Symb. Logic, vol. 13 (1948).

F. B. Fitch, 'Intuitionistic modal logic with quantifiers', Portugalice Math., vol. 7 (1949), pp. 113 - 118.

L. Henkin, 'An algebraic charachterization of quantifiers', Fundamenta Math., vol. 37 (1951), pp. 63 - 74.

For comment on these contributions, see A. Fraenkel and Y. Bar-Hillel, Foundations of Set Theory, Amsterdam 1958, pp. 229 ff.; cf. also E. Beth, The Foundations of Mathematics, Amsterdam 1968, pp. 435 ff.

5.5 More recently, new formalizations of intuitionistic mathematics have been attempted by:

C. Kleene and R.E. Vesley, <u>The Foundations of Intuitionistic Mathematics</u>, Amsterdam 1965;

G. Kreisel, 'Mathematical Logic', <u>Lectures in Modern Mathematics</u>, London 1963, pp. 95 - 195;

G. Kreisel et al., <u>Report of the Seminar on the Foundations of Analysis</u>, (Mimeographed), Stanford 1963;

J. Myhill, 'Notes towards an Axiomatization of Intuitionistic Analysis', <u>Logique et Analyse</u>, vol. 33 (1967), pp. 280 - 297;

also, 'Formal Systems of Intuitionistic Analysis', <u>Logic, Methodology and Philosophy of Science</u>, vol. III (ed. B. van Rootselaar and J.F. Staal), Amsterdam 1968.

A.S. Troelstra, 'The Theory of Choice Sequences', <u>Logic, Methodology and Philosophy of Science</u>, vol. III, Amsterdam 1968, pp. 201 - 223;

also, <u>Principles of Intuitionism</u>, Lecture Notes in Mathematics 95, Springer Verlag 1969.

CHAPTER IV

THEORY OF SETS AND THE CONTINUUM

Brouwer's first systematic treatment of the theory of sets and the continuum is found in 'Begründung der Mengenlehre unabhängig vom logischen Satz vom ausgeschlossenen Dritten' (1918A and 1919A). In this and a whole series of contributions during the following ten years, an intuitionist theory of sets and the continuum is developed which not only diverges from the classical theory but also introduces many notions which are completely new and different from his views on set theory as found in his earlier writings.

1.1 Theory of sets and the continuum before 1918

Brouwer's views on set theory and the continuum in this period can be found in: De Grondslagen der Wiskunde (1907),
'Die mögliche Mächtigkeiten' (1908A),
Intuitionisme en Formalisme (1912A; also 1913A), and
'Review of Schoenflies'(Die Entwickelung der Mengenlehre und ihrer Anwendungen) (1913B).

De Grondslagen der Wiskunde reflects the attitude of the neo-intuitionist camp at the time when Cantor's naive set-theory was dominating the mathematical scene, an attittude which was critical of Cantorism and mainly negative. It searches for the weaker aspects in Cantor's set-theory and directs its attack on its idealistic and logical basis. A great part of 1907 is devoted to pointing out that Cantor's set-theory is purely logical and therefore mathematically irrelevant.

Brouwer's worst suspicions were confirmed when Zermelo, in 'Grundlagen der Mengenlehre'[1], made a first attempt at a rigorous axiomatization of set theory. The logical and formalistic nature of non-intuitionist set theory, crystallized in the axiom of choice and comprehension, became the main target of Brouwer's attack on formalism in his inaugural address (1912A), and helped him to clarify his own thinking on certain aspects of set theory and to formulate his position more clearly.

1.2 Definition of set

Typical of the negative attitude towards set-theory is the complete absence of any definition of set in Brouwer's earlier writings.

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[1] MA, vol. 65 (1908), pp. 261 - 281.

There is no attempt to replace Cantor's definition of a set as: 'the comprehension (Zusammenfassung) of distinct objects into a totality'[1]. At this stage Brouwer, together with Poincaré and Borel, seems to be of the opinion that the notion of set is sufficiently primitive[2], and concentrates on the nature of its elements, the formation of sets, and the power of sets.

1.3 The nature of the elements of sets

While Cantor's definition of a set segregates a part of a pre-existing universe of individual objects, Brouwer's elements are either given in the primary intuition or are the result of a mathematical construction. In a subsection of Chapter I (1907) headed, 'Mathematics can only deal with objects that it has constructed itself', he makes this constructive demand for the whole of mathematics. Whole systems can become 'elements' within another system, but only in so far as their construction has been completed (and 'at every stage of the construction'[3]). It is only after the set of natural numbers ('positive ordinals') have been generated by means of the primordial intuition that Brouwer speaks of them as a 'set'. The irrationals are introduced as 'a symbolic agglomerate of numbers previously introduced ... It therefore follows that at any stage of development of the theory the totality of known numbers remains countable'[4]. And in 1912A: 'From the present point of view of intuitionism all sets of mathematical entities deserving of this name can be constructed from the primordial intuition, and this can only be done by a finite combination of the operation of "creating a finite ordinal number" and "creating an infinite ordinal number ω '.' [5]

The cantorian extension of sets of elements beyond the constructive has led to paradoxes and this - on logical grounds alone - should have been sufficient reason for the Cantorians to drop this kind of set

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1 'Beitrage zur Begründung der transfiniten Mengenlehre', MA, vol. 46, (1895), p. 481.

2 This in spite of the fact that Brouwer disagreed with them on the value of definitions in general. Cf. also p. 119 above.

3 1907, p. 77; cf. also 1908A, p. 569.

4 1907, pp. 6 - 7.

5 1912A, p. 13.

like a hot brick: 'The cantorians should have rejected immediately as non-mathematical a concept which leads to contradictions'[1].

1.4 Formation of sets

Closely allied to the non-constructive character of Cantor's elements is the idea of 'comprehension' (Zusammenfassung) - which goes back to Boole's operation of 'election' - and not only presupposes the existence of a universe of individual objects but also that of, as yet, not attributed predicates. Cantor's principle of the formation of sets[2] as 'the allocation of non-contradictory predicates to "ein eigenschaftsloses Ding ... welcher in uns geschlummert" is quoted in full[3]; Brouwer then simply remarks that this must be rejected on the same grounds as axiomatics.[4]

In 1907 - or any other writing before 1918 - Brouwer does not yet distinguish between sets and species. A property is vaguely described as 'the possibility of fitting in new systems within a main system'.[5] Whenever in this period he speaks of sets, he restricts himself to aggregates which are generated by the progressive construction of their elements, and therefore all real[6] sets are denumerable and linearly ordered. The inductive nature of the primordial intuition, which lies at the basis of Brouwer's number system, determines the ordinal character of the natural numbers (to which Brouwer always refers as 'the positive ordinal numbers'). All 'proper' sets to him are therefore, sequences, y distinctions between these linearly ordered sets is on the basis of order: the order type ω (natural order), ω^* (reverse natural order) or the order type η of the rational numbers

1.5 Axiom of comprehension and Axiom of choice

What Brouwer rejects in the axioms of choice and comprehension is not the mental activities of 'choosing' and 'comprehending': these indeed play a significant role in Brouwer's own theory of sets and the

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1 1907, p. 144.

2 Grundlagen einer allgemeinen Mannigfaltigkeitslehre, p. 45.

3 1907, pp. 143 - 144.

4 He had earlier rejected the axiomatic method as being purely verbal or logical, without any mathematical, i.e. constructive, basis.

5 1907, p. 78.

6 1908A, p. 571.

continuum. In his analysis of the primordial intuition,[1] Brouwer repeatedly stresses the ability of the human mind to 'think together' (Du. samendenken), which is ultimately the origin of the concept of number and which is most adequately expressed by the word 'comprehension'. Similarly, the role of human choice in Brouwer's theory of sets and the continuum is larger than in any other school of mathematics.

The main reason for his rejection of both axioms is: ' the presupposition of a world of mathematical entities, existing independently of a thinking individual and governed by the laws of classical logic, which among themselves should possess the "relation of a collection to its elements"'[2], in other words: the complete absence of a human mind which alone is capable of choosing and comprehending. Later[3], Brouwer sees the rejection of the axiom of comprehension as his main contribution to the controversy of set-theory during this period.

Even when in 1918 species are introduced, an intuitionist equivalent to classical sets, Brouwer will insist on the construction of mathematical entities preceding the constitution of a species on the basis of property and independent of it.

In the pre-1918 period Brouwer's sets are formed by the 'binding' or 'linking' aspect of the primordial intuition which creates 'two' and as a binary 'operation' generates the natural numbers; all other sets are formed by a setting up of a one-one correspondence with this sequence or with any other previously constructed set, and here again the 'binding' intuition plays the creative role.

1.6 Power of sets

Two of the theses offered by Brouwer for public (oral) defence at the occasion of the doctoral graduation ceremony[4] concern powers of sets; they summarize his views on powers during this period:

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[1] See above p. 83.

[2] 1912A, p. 14.

[3] 1921A, p. 797.

[4] 19th February, 1907; these theses have been published as a supplement to the dissertation (1907); the two theses in question appear on pp. 4 - 5 of this supplement; similar statements can be found in 1907, p. 149; 1908A, p. 571.

Thesis XII: 'The only possible powers are: the finite,
the denumerably infinite,
the unfinished denumerably infinite,
the continuum.

Thesis XIII: 'Cantor's second number class does not exist'.

Even if Brouwer here, as elsewhere, uses the word power (Du. Machtigheid) when referring to the finite and the various degrees of infinity, there is sufficient evidence that in each instance the word power refers to different concepts as do indeed the words existence and sets.

In the next few paragraphs we will discuss:

1° The finite and the denumerable, the only powers admissible on the basis of a constructive set;

2° Brouwer's rejection of Cantor's transfinite powers;

3° What meaning must be given to the power of 'the denumerably unfinished infinite. The continuum and the power of the continuum will be discussed separately in the following section.

1° Finite

The extreme constructive demand of 1907 which equates mathematical existence to 'having been constructed'[1], only allows finite sets; in this sense Brouwer may be called a 'finitist'[2]. Since the emphasis in 'denumerable' is on the order, Brouwer often speaks of denumerable sets when exclusively finite sets are intended, e.g. 'the system of constructed numbers is denumerable at every stage of the theory'.[3]

A denumerably infinite set is a set to which always new elements can be added. This unrestricted possibility originates in the primordial intuition, the intuitive "and-so-on"[4], and is the basis of induction. In some sense Brouwer's infinite can be called negative as it only refers to the absence of any restriction of this freedom. Later (in 1918), in order to accommodate finite sets within his definition of sets, Brouwer will find it necessary to mention the possibility of positively 'arresting' the process. [5]

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[1] 1907, p. 177.

[2] J.A. Bernadette, Infinity, Oxford 1964.

[3] 1907, p. 10.

[4] 1907, p. 142.

[5] Cf. 1918A, p. 3; 1921A, p. 798; 1925D, p. 244.

The constructive demand for the generation of all sets - by progressive generation of each of their elements and on the basis of equipotency ultimately with the set of natural numbers - can only lead to denumerable sets. This Brouwer states in many instances: discussing 'cuts' of rationals in 1907, p. 7: 'At every stage of the construction the totality of known numbers remains denumerable, for a denumerable number of denumerable quantities is by a simple proof of Cantor also denumerable'.

On page 10: 'Cantor maintains that not all points of the continuum are denumerable, i.e. that besides every denumerable number of such points there are still others, whereas we have seen that the system of constructed numbers is denumerable at every stage of the theory'.

On page 62: 'Only denumerable quantities can be created by the mathematical intuition as individualized', and on page 142: 'we can only create mathematically: finite sequences on the basis of intuition, further the order type ω on the basis of the intuitively clear "and-so-on"'.

In 1908A, page 571: 'There exists only one power for mathematical infinite sets: the denumerable'.

In 1912A, page 23: 'The intuitionist can only construct denumerable sets of mathematical objects', and on page 13: 'Therefore the intuitionist only knows denumerable sets, i.e. sets whose elements can be put into one-one correspondence either with a finite ordinal number or the infinite ordinal number ω'. etc., etc.

This infinite ordinal number ω cannot be thought of in the same way as any finite number, which is the result of a completed process of construction and is associated with the totality of its predecessors. As to the exact nature of the infinite ordinal number ω, Brouwer remains rather vague and confusing. He speaks of: 'the smallest infinite ordinal number ω by considering this process (i.e. constructing the next number) as being repeated indefinitely.'[2] He also accepts Cantor's definition of well-ordered sets and the construction

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[1] The distinctions within the concept of equipotency date from 1918. As to the 'negative' aspects of infinity: Brouwer refers to the various degrees of infinity as 'the negative part of the theory of potencies'. 1912A, p. 26.

[2] 1912A, p. 12.

of ω^ω : 'There is no objection to considering ω as a new ordinal number and to start counting again $\omega, \omega+1, \omega+2, \ldots 2\omega, 2\omega+1, \ldots m\omega+n, \ldots$'[1]

That Brouwer had in mind simply an ordering and in no way intended to include the concept of a consummated construction of the infinite is clear from his uncomprising opposition to Cantor's transfinite sets.

2° Brouwer's views on infinite sets are diametrically opposed to Cantor's as expressed in his 'Essay on Linear Aggregates', in which he launched his theory of the transfinite in 1883:
'It is traditional to regard the infinite as the indefinitely growing or in the closely related form of a convergent sequence, a form it acquired acquired during the seventeenth century. As against this, I conceive the infinite in the definite form of something consummated, something capable not only of mathematical formulation, but of definition by number. This conception of the infinite is opposed to traditions which have grown dear to me, and it is much against my own will that I have been forced to accept this view. But many years of scientific speculation and trial point to these conclusions as to a logical necessity, and for this reason I am confident that no valid objections could be raised which I would not be in a position to meet'.

Because of its purely logical basis, Brouwer rejects the whole of Cantor's theory of the transfinite as meaningless and trivial. Even when concentrating some times his attack on some detail, such as Cantor's continuum hypothesis[2], the Schröder-Bernstein theorem[3], or the well-ordering theorem[4], in the last resort he dismisses them all as self-evident, trivial or meaningless: 'We knew already from the start' (1907, pp. 147, 155, 159), or 'A proposition without any real meaning to the intuitionist' (1912A, pp. 22, 23, etc.). 'From our point of view these arguments of Cantor are meaningless; the only thing we can make of it - with some alterations - is the following triviality: If we introduce the logical entity T (power of the second class) then the axiom $T = A_1$ (A is the power of ω) would in the logical structure lead to a contradiction; so would the introduction of a logical entity I which would fulfill the logical function of a power and

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[1] 1907, p. 144.

[2] 1907, p. 149.

[3] 1907, p. 153.

[4] 1907, p. 152.

' satisfy the axioms $A < I < T$. This is the logical, mathematically worthless result of those proofs of Cantor'.[1]

A year later, he is not so outright in his condemnation and grants that there is some mathematical sense in the method of constructing certain mathematical systems, quoting as examples 'the totality of numbers of the second class, the totality of definable points on the continuum, and the totality of mathematical systems', but he insists that 'it is wrong to call this whole system a set, for it is impossible to construct it as complete from the mathematical primordial intuition'.[2]

For Brouwer, the mathematical actual infinite did not exist, as it did not for Poincaré and Borel.[3] A set cannot be considered as a closed totality unless each of its elements has been individually constructed. Since for infinitely many elements this is impossible, the infinite set as a totality does not exist. In his definition of the second number class, Cantor assumes the existence of an infinite totality: 'Wir definiren daher die zweite Zahlenclasse als den Inbegriff aller mit Hülfe der beiden Erzeugungsprincipe bildbaren in bestimmter Succession fortschreitenden Zahlen α : ω , $\omega + 1, \ldots \omega^{\omega} \ldots \alpha$ ... welche der Bedingungen unterworfen sind, dass alle der Zahl α voraufgehenden Zahlen, von I an, eine Menge von der Machtigkeit der ersten Zahlenclasse bilden'.[4]

Brouwer denies the second number class any mathematical existence on the grounds that the 'Inbegriff aller' can never be constructed: 'Notice the "Inbegriff aller"; he speaks here of something that cannot be thought, i.e. that cannot be constructed mathematically'.[5] Elsewhere in a seemingly circular argument he simply states: 'Such a system as a whole can never be constructed, since it is supposed to be non-denumerable'.[6]

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[1] 1907, p. 147.

[2] 1908A, p. 569.

[3] Poincaré, Reflections sur les deux notes précédentes, Acta Mathematica, vol. 32 (1909), pp. 195 - 200; cf.' Über transfinite Zahlen' (sixth lecture in 'Sechs Vorträge über ausgewählte Gegenstande, Leipzig - Berlin, 1910); Borel, Leçons sur la théorie des fonctions, Paris, 1898, Ch. I.

[4] Grundlagen, p. 35. (Italics are mine WVS).

[5] 1907, p. 145.

[6] 1908A, p. 569

Through some improvement by Hilbert[1] the second number class can be granted purely logical existence (i.e. non-contradictority), but not any mathematical existence as is done by Cantor: 'Cantor speaks of his second number class as if he saw it before his very eyes; the way he expresses himself does not indicate that all he had in mind is a purely logical system'.[2]

3° If Brouwer is prepared to say anything at all about the totality of the set of well-ordered numbers, it is with reservations and, like Gauss[3], only as a manner of speech: 'The power of the totality of well-ordered numbers is countably incomplete; under a countably incomplete set we mean a set of which only a countable group can be given as well-defined and when at the same time from each of such a countable group, by means of some previously defined mathematical process, new elements can be derived which can also be said to belong to the set in question. But strictly mathematically speaking, this set does not exist as a totality, neither does its power; we may, however, introduce these words as an arbitrary expression for a known intention'.[4]

Nothing is said here of the totality as such, but only of a method which at any stage of its use can only produce a denumerable set: '... with such a method - as always in mathematics - only denumerable sets can be constructed.'[5]

The infinite, both the simply denumerable and the denumerable incomplete, is essentially unfinished; both are denumerable (cf. footnote 1907, p. 149: ' ... one can say that the unfinished denumerable and the simply denumerable sets are equipotent, since every unfinished denumerable set can be made to correspond to ω^2.') The 'unfinished' here refers to the correspondence itself.

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[1] Hilbert introduced "das Inbegriff aller" as a logical entity in: Verhandelungen des internationalen Mathematiker Congress in Heidelberg, (1904), pp. 183 - 184.

[2] 1907, p. 146.

[3] Gauss's letter to Schumacher in 1831, where he writes: 'I must protest most vehemently against your use of the infinite as something consummated, as this is never permitted in mathematics. The infinite is but a façon de parler ...'. Cf. also Waisman, Introduction to Mathematical Thinking, London 1951, p. 128.

[4] 1907, p. 148.

[5] 1908A, p. 569.

Brouwer concludes his contribution 'On possible powers' (1908A) by stating: 'Therefore there exists only one power for infinite mathematical sets, i.e. the denumerable. One could add to this:
1° the denumerable unfinished, but then one refers to a method, not a set;
2° the continuous; then, indeed, something complete is referred to but only as a matrix, not as a set.'[1]

The continuum before 1918

2.1 The intuitive continuum

The primordial intuition of two-ity, the intuition of time, is the human mind linking a present sensation to the memory of a past sensation.[2] From this intuition both the 'linking', the comprehension, to which Brouwer refers as 'thinking-into-one' (Du. samendenken), and distinctness emerge simultaneously as equally basic elements. This recognition made Brouwer speak of the continuous and the discrete as one and the same intuition, 'appearing as each other's complement'.[3] It is the linking element of the primordial intuition that Brouwer sees as the continuous: 'Not only is there a new element in the totality of two, but also in that which binds them: that which is not this totality itself, nor the constituting elements'[4]. And: 'In the primordial intuition of two-one-ness the intuitions of the continuous and discrete meet; 'first' and 'second' are being hold together and in this holding-together consists the intuition of the continuous (continere = hold together)'. [5]

He also refers to the continuous as the 'fluid' , 'the "between" that never exhausts itself by insertion of new entities' .[6] Also in 1912: 'Finally, with the primordial intuition of mathematics in which the connected and the separate, the continuous and the discrete are united, the intuition of the linear continuum is immediately present,

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[1] 1908A, p. 571.

[2] 1907, pp. 8 and 119; see also above p. 81.

[3] 1907, p. 8.

[4] 1907, p. 120.

[5] 1908A, p. 569.

[6] 1907, p. 8.

i.e. the intuition of the "between" that will never be exhausted by insertion of new entities and therefore can never be thought of as a collection of units'.[1] This possibility of insertion is 'one of the synthetic a priori judgments'[2], one of the constructions that can be traced to the primordial intuition itself and which prepares the way for Brouwer's later 'medium of free becoming'.

The complementary character of the discrete and the continuous make it impossible to consider the continuum as an aggregate of fixed elements (Leibniz, Cantor) or to attempt a construction of the one in terms of the other: 'When therefore in the primordial intuition both the continuous and the discrete appear as inseparable complements, both coequal (Du. gelijkgerechtigd) and equally clear, it is impossible to limit oneself to one as the original entity and to construct the other from it'[3] ... 'The continuum as a whole is intuitively given; a construction or act which would create "all" its points individually by means of the mathematical intuition is unthinkable and impossible'.[4]

Borel and Lusin also accepted the continuum as intuitively given: 'the geometric continuum'. But unlike Brouwer's intuitive continuum, theirs was a set, the totality of all points of an interval (Borel), the totality of all real numbers (Borel and Lusin). There is a certain inconsistency between 'the effectively given' (i.e. all elements generated through finite procedure) which Borel demands for all sets and this concept of the continuum: 'We accept that the set C of numbers between 0 and 1 is given without investigating how they can be effectively given'. [5]

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[1] 1912A, p. 12.

[2] 1907, p. 119.

[3] 1907, p. 8.

[4] 1907, p. 62.

[5] Borel, Leçons sur la théorie des fonctions, Paris 1898 (ed. 1928 p. 16). This inconsistency was pointed out by A. Fraenkel in 'Discontinu et continu', Travaux du IXe Congrès int. de philosophie (VI. Logique et mathématique), Actualités scientifiques et industrielles, No. 535, pp. 193 - 200). For Brouwer's criticisms see p.179. According to Borel and Poincaré, we can speak of all points of the continuum, since this collection as a whole is given, but not of every arbitrary point since not every point (or real number) is definable in a finite number of words or can be arrived at through a finite procedure. In a different way, (by Richard's proof in 'Sur un paradoxe de la théorie des ensembles et sur l'axiome de Zermelo', L'Enseignement mathématique, vol. 9 (1907), pp. 94 - 98), they arrive at the same conclusion as Brouwer did: all sets of points in the continuum are denumerable (See Poincaré, 'Über transfinite Zahlen, Op. cit.).

Brouwer attempted a definition of the continuum which avoids the idea of a whole to be analysed into points as its parts or the building up of points into a continuum. The 'continuum is the matrix in which points can be thought together'[1] (Du. het matrix van samen te denken punten). In this definition, the continuum appears as more static than Brouwer's later 'medium of free becoming', but there is no totality of atomic points nor an attempt to bridge the chasm between a set of discrete elements and the continuum, which are different in kind.

2.2 The measurable continuum

Even if there is a development of the notion of real number in Grondslagen, this is only done - at this stage - in relation to powers of sets and in answer to Cantor's continuum problem as repeated by Hilbert in 'Mathematische Probleme'[2]. For Brouwer's own treatment of the basic arithmetical operations characterized as group-theoretical transformations[3], a simple extension of the rationals was sufficient, described as 'making the continuum measurable'[4].

(It is typical of Brouwer's rejection of current views on the continuum, even those of the pre-intuitionists, 'who did not seek for the continuum an origin extraneous to language and logic'[5], that in the whole of Chapter I of Grondslagen the words 'real number' are not mentioned).

This 'making the continuum measurable' consists in the projection of an everywhere dense scale of rationals on the intuitive one-dimensional continuum:

First: 'A row of points of the order type of all positive and negative numbers can easily be constructed on it; we then take in

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[1] 1907, p. 9; cf. also 1908, p. 570.

[2] Hilbert, Mathemathische Probleme, problem no. 1 (p. 263): 'To prove Cantor's 'continuum hypothesis', that any set of real numbers can be put into one-to-one correspondence either with the set of natural numbers or with the set of all real numbers (i.e. the continuum)'.

[3] Research in this direction was continued in his more topological work; a detailed analysis of the group-theoretical characterization of operations on the measurable continuum was given in 'Die Theorie der endlichen kontinuierlichen Gruppen, unabhängig von den Axiom von Lie', 1909B.

[4] 1907, p. 11.

[5] 1952B, p. 140.

'each interval another point, and in each of the resulting intervals another point, etc.'[1] Then the scale is made everywhere dense by 'thinking as contracted into one point every segment in which the scale does not penetrate, in other words: we regard two points only then as different if their dual expansions differ after a finite number of figures'.[2]

Finally, 'if we on the constructed scale adopt one point as the zero point, the scale has made the continuum into a measurable continuum.'[3] On the basis of this measurable continuum, operations of addition and multiplication are defined as 'shift-transformations'[4] and geometry is then arithmetized.[5]

2.3 The power of the continuum

The measurable continuum preserves its intuitive character, i.e. that of 'the between that never exhausts itself by insertion of new entities'[6] and therefore cannot have indivisisble points as its atomic parts.

Even if Brouwer distinguishes between 'the denumerably unfinished' and 'the continuous' (see above p.173) 'the totality of definable points on the continuum' is given as an example of the denumerably unfinished.

The power of the continuum as a point-set in Brouwer's views is probably best characterized as purely negative in the sense that it is non-denumerable; in accordance with Cantor's proof, Brouwer merely accepts that 'it can be proved satisfactorily for the intuitionist as well as the formalist that for every denumerably infinite set of real numbers between 0 and 1 immediately a real number between 0 and 1 can be indicated which does not belong to that set'[7]. Brouwer then draws the inevitable conclusion that 'all denumerably unfinished sets

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[1] 1907, p. 9.

[2] Op. cit. p. 10.

[3] Op. cit. p. 11.

[4] Op. cit. pp. 12 ff.

[5] Op. cit. pp. 35 ff.

[6] Op. cit. p. 8 ; 1908A, p. 570. See

[7] 1912A, pp. 22 - 23.

'have the same power'[1] and Cantor's continuum problem, ('are there any sets of real numbers between 0 and 1 whose power is smaller than the continuum but greater than aleph null?), is dismissed as meaningless.[2] In Grondslagen, following the attempt at 'making sets continuous' (to be discussed below), 'every point-set on the measurable continuum which is not denumerable has the power of the continuum' and Brouwer closes the section with: 'In this way, the continuum problem stated by Cantor in 1873 and re-stated by Hilbert as being still topical seems to have been solved, i.e. by strictly observing the views that we cannot speak of a continuum as a 'set' of points except in relation to a scale of order type η'[3].

2.4 'Completing into a continuum

Of greater significance in view of later developments are some of Brouwer's ideas which in 1907 are only expressed in the context of Cantor's continuum hypothesis, but gradually in the course of 1908A, 1912A, 1913B and 1917A become more prominent.

Grondslagen (p. 62) considers the possibility of constructing a continuum on a scale of order type η : 'The mathematical intuition can indeed construct a scale order type η and then place over it a continuum as a whole, which continuum can then again be taken as matrix of the points of the scale'[4].

Brouwer states here three principles of construction, of which one was later withdrawn.[5] Two principles are concerned with what Brouwer calls 'completing into a continuum'[6](Du. completeering tot een continuum) or 'making continuous' (Germ. Operation des Continuierlichmachens)[7] and contain the basic ingredients of Brouwer's later developments of choice sequences:

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[1] 1912A, p. 24, footnote: 'Calling denumerably unfinished all sets of which the elements can be individually realized, and in which for every denumerably infinite subset there exists an element not belonging to this subset, we can say in general, in accordance with the definitions of the text, "All denumerably unfinished sets have the same power'.

[2] Ibidem, p. 23.

[3] 1907, p. 67.

[4] 1907, p. 62.

[5] 1917A, p. 440; cf. also 1913B, p. 79.

[6] 1907, p. 66.

[7] 1908A, p. 570.

A scale of order type η defines a denumerable collection of pairs of points and intervals. Each of these intervals can be made dense with 'irrational' points (i.e. relative to the dual scale in it) by a process of infinite remification, while blotting out the 'rational' points of the scale.

In Grondslagen this density is expressed and illustrated in the following way: 'This density can be characterized in the following way, by approximating to an everywhere dense dual scale constructed on the segment under consideration taken as unit segment: the fixing of each following dual figure is either determined by the previous one or leaves a choice between two; in the latter case for either choice the next dual figure again is either determined or leaves a choice between two, which can be illustrated by a figure of this form:

etc.

If we break off every branch which never remifies again there remains in the end either nothing or a continued self-multiplying bifurcation. In the latter case, the collection is dense in the interval, in the former it is not'.[1]

In 1908A the operation of 'making continuous' is explained in a similar way, but the freedom of choice is more explicitly mentioned: 'we now construct a remified chain in which every branch simply continues if the choice is not free, and splits itself into two when the choice is free, and then destroy every branch that does not split itself again'[2]. 1913B makes a further generalization and considers the splitting into more than two sub-branches[3], leading to Brouwer's later development of

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[1] 1907, pp. 64 - 65 (The only illustration in Grondslagen).

[2] 1908A, p. 570.

[3] 1913B, p. 79.

'Ausfüllungselmente' and 'Ergänzungselmente'.

Brouwer later recognized that in the above process of ramification two tacit assumptions were made:

1° That the point-set can be constructed as 'individualized', i.e. such that two different indefinitely proceeding branches always lead to two different points;

2° That this individualized constructed point-set can be analyzed internally, i.e. that the process of breaking off can indeed be executed.[1]

His justification for making these assumptions seems less convincing: 'One has the right to consider as implicitly contained in the principles of construction such suppositions as are desirable for the sake of viability of the theory'[2]. Indeed, in later developments, he tries to avoid relying on these assumptions.

As observed above, there is not yet at this stage a systematic treatment of the real numbers and the basic operations on the basis of the real numbers in the above sense. That Brouwer already saw early on the possibility of such development is clear from a remark in his inaugural address where he mentions the possibility of admitting 'free choice sequences as an element of construction of real numbers between 0 and 1'.[3]

Nowhere in his earlier work is there yet any definition of real numbers as convergent sequences of nested intervals. Brouwer's own historical reflections on this period, however, emphasize the insufficiency of the classical and pre-intuitionist treatment of the real numbers and the continuum, (including Borel's 'practical continuum'), on the grounds that 'the measure of this "ersatz continuum" is zero'[4]: 'In doing so, they seem to have overlooked that such an ever-unfinished and ever-denumerable system of "real numbers" is incapable of fulfilling the mathematical functions of the continuum, for the simple reason that it cannot have a measure positively differing from zero.'[5]

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[1] 1917A, p. 441.

[2] Ibidem.

[3] 1912A, p. 23.

[4] 1929A, p. 2.

[5] 1952B, p. 140

Early traces of Brouwer's views on the interval as the fundamental constructive element of the continuum can already be found in some of his topological work, especially in 'Über Abbildung von Mannigfaltigkeiten' (1911I) and 'Invarianz des n-dimensionalen Gebiets' (1911J), where simply connected pieces, of which the space is composed are taken as the fundamental constructs.[1]

Perhaps even a remark in Grondslagen could point in this direction, where Brouwer describes the 'making measurable' of the continuum as 'constructing the continuum from individualized measurable pieces[2].

3.1 Theory of sets and the continuum after 1917

Beth claims that 'the central place in intuitionistic mathematics is occupied by the theory of the continuum'[3]. Indeed, Brouwer's interpretation of the continuum and real numbers forms the basis of an 'intuitionist analysis', completely divergent from the classical, and inspired an intuitionist set theory which could accommodate this continuum.

Systematization of this intuitionist set theory and analysis dominates Brouwer's work in the period 1917 - 1927, and the issues involved, apart from the Brouwer logic, have become almost the sole object of research within the intuitionist school.

Need for an intuitionist set theory

Errett Bishop in his excellent book, Foundations of Constructive Analysis, ridicules Brouwer's preoccupation with the continuum: 'A bugaboo of both Brouwer and the logicians has been compulsive speculation about the nature of the continuum', but he diagnoses one of the chief reasons for this preoccupation: 'In Brouwer's case, there seems to have been a nagging suspicion that unless he personally intervened to prevent it, the continuum would turn out to be discrete'.[4] The increasing prominence of measure-theory, and the reliance even of

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[1] Weyl made this the starting point of an investigation into methods of constructing spaces by continued insertion of 'net-points', ('Über die neue Grundlagenkrise der Mathematik', p. 78).

[2] 1907, p. 67.

[3] Beth, The Foundations of Mathematics, p. 422.

[4] E. Bishop, The Foundations of Constructive Analysis, p. 6

such 'neo-intuitionists' as Lebesgue and Borel[1] on logical, Cantor set-theoretical methods in their treatment, highlighted the need for a theory of the continuum which could bridge the gap between the continuum and the 'everywhere dense set of definable numbers of the linear continuum, whose measure is zero'[2]. This need had also inspired Weyl to write Das Kontinuum in 1917, an attempt abandoned by Weyl a few years later in favour of Brouwer's 'continuum of free becoming'[3].

We have already mentioned Brouwer's dissatisfaction with the 'logical treatment of the continuum'[4]. Real numbers as Dedekind cuts of rationals were unacceptable to Brouwer because of the presumption of the axiom of choice; moreover, they only defined a part of what Brouwer would call 'the reduced continuum' whose measure is zero.[5] Cantor's theory of real numbers as sequences, in which a definition of real numbers as sequences of nested intervals can easily be accommodated[6]; had obvious attractions, but apart from objections against Brouwer's concept of sets, this definition was inadequate for all points of the continuum for the same reasons as the Dedekind cuts.

Brouwer's fundamental views concerning the continuum and sets had remained virtually the same. The simple measurable continuum had proved inadequate, and further development of real numbers on the basis of an 'indefinitely proceeding sequence' needed the rigorous framework of a set theory. Brouwer confessed in 1919: 'In my non-philosophical contributions I have made regular use of old methods, always taking care, however, to derive, as far as possible, only such results as could be expected to find a place (may be with some modification, but substantially the same) in an intuitionistic set theory, after this had been systematically constructed.'[7] Since the more general rejection of logic and the demand for construction had become more specific in

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1 Cf. Brouwer's remark on Borel's practical continuum quoted above (p.179).

2 Borel, 'Les Paradoxes de l'infini', L'Avenir de la Science, vol. 25, p. 189.

3 Weyl, 'Über die neue Grundlagenkrise der Mathematik', p. 56: 'Therefore I abandon now my own attempt and join Brouwer'. Weyl's 'atomic' continuum was similar to Borel's practical continuum and failed on the same grounds.

4 See above p. 179.

5 1928C, p. 5.

6 F. Waisman, Introduction to Mathematical Thinking, London 1951, p. 194.

7 1921A, p. 798.

the rejection of the Principle of the Excluded Middle and the axiom of choice, such a systematization had become more feasible.

It is beyond the scope of this present work to give a detailed account of Brouwer's set theory and analysis.[1]

We will concern ourselves here mainly with the Brouwer set as a fundamental tool and consider Brouwer's attempt to introduce this new notion of a set, which on the one hand satisfies his strict constructive demands and replaces the logically-based Cantor set, and which, on the other hand, is wide enough to accommodate 'sets' of classical cardinal number $2^{\aleph_0}$ and sets whose elements are 'essentially unfinished'.

This will be done in the following sections:

1° The Brouwer set inspired by Brouwer's notion of the continuum; the continuum as 'medium of free becoming';

2° Brouwer's set theory: sets and species;

3° The splitting of equipotency;

4° Ordering;

5° The Brouwer continuum and the real numbers.

3.2 For the development of the separable parts of mathematics the primordial intuition of time was sufficient. For the system of natural numbers and its immediate extensions there is no need for any set-theoretical foundations. Moreover, Brouwer's set theory presupposes the existence of the sequence of natural numbers; this is clear from the very first line of Brouwer's systematization of set theory: 'At the basis of set theory lies an unlimited sequence of symbols, which is determined by a first symbol and the rule that from each of these symbols derives its successor. Among the many rules that can be used, the most appropriate appears to be the one which generates the sequence ζ of the numerals 1, 2, 3, 4, 5, ...'[2]

Cantor expressed surprise when, in the course of the development of his set theory, he met such concepts as the 'actual infinite' which

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[1] All studies of intuitionistic mathematics so far contain additions and extensions of Brouwer's version. Among these we may mention: A. Heyting, Intuitionism - An Introduction, Amsterdam 1956; S.C. Kleene and R.E. Vesley, The Foundations of Intuitionistic Mathematics, Amsterdam 1964. (Chapter III is a revised edition of Vesley's Ph.D. thesis on 'The Intuitionist Continuum'; here careful distinction is made between Brouwer's theory and addition by others. A.S. Troelstra, Principles of Intuitionism, (Lecture notes in Mathematics, vol. 35), Berlin-Heidelberg-New York 1969.

[2] 1918A, p. 3; cf. 1925D, p. 244.

he was 'forced to accept much against his will'[1]. Brouwer set out to construct a set theory as a tool, found necessary in his interpretation of the continuum and completely determined by it. In order to emphasize this, Brouwer starts his various contributions on intuitionist set theory (1918A, 1920B, 1921A, 1925D), not with the much simpler notion of species (of which very little is said and which is much more in line with the traditional definition of set), but with the highly complex notion of a set ('Menge') which can only be fully understood in the context of the continuum.

3.3 The Brouwer continuum, 'medium of free becoming'

'To Brouwer we owe the new solution of the problem of the continuum.'[2]

H. Weyl.

With the publication of L'Évolution Créatrice, Bergson had revived Zeno's paradoxes and initiated an international debate on the continuum.[3] He accepted the Dedekind-Weierstrass-Cantor solution to

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[1] G. Cantor, 'On linear aggregates'; for full quotation see above p. 170.

[2] H. Weyl, 'Über die neue Grundlagenkrise der Mathematik', p. 56.

[3] Although there is no direct evidence that Brouwer was influenced by Bergson and there is no reference to Bergson anywhere in Brouwer's writings, (except the general reference to 'the French intuitionist school' of which Bergson was the founder and most prominent representative), there are undoubtedly many links between Brouwer and Bergson. In Chapter II, we pointed at the similarity of their views on intuition. It seems highly unlikely that the writings of such an eminent contemporary intuitionist were unknown to Brouwer. Bergson's famous article on intuition, 'Introduction à la Métaphysique', appeared in the Revue de Métaphysique et de Morale (1903) at the time of the Russell-Poincaré debate, which Brouwer followed closely (1907, p. 94). Again, it is possible that Brouwer's new solution of the continuum as medium of 'free becoming' is completely independent from Bergson's 'becoming' of continuous time; but it is rather unlikely that Brouwer was unaware of the debate on Bergson's time as 'becoming', which involved philosophers and mathematicians (a.o. Whitehead and Poincaré).
Elsewhere, page 119, we mentioned the close similarity of Brouwer's and Leroy's views on language and logic; Brouwer was well acquainted with Leroy's views which were criticized in an article by Poincaré, which in turn was criticized by Brouwer in 1907, p. 176. Edward Leroy was an intuitionist and strong supporter of Bergson; in his book on Bergson, Une philosophie nouvelle: Henri Bergson, Paris 1912, (very much approved of by Bergson himself), the problem of time and the continuum are discussed in great detail and so Bergson's 'becoming'. It is possible that the link Bergson - Brouwer is via Leroy.

the problem of the mathematical continuum; Zeno's paradoxes simply arose because of identification of time and the everywhere-dense mathematical linear continuum. Bergson did not consider time intervals seen retrospecively which indeed - as James and Whitehead would point out[1] - are isomorphic with the mathematical linear continuum. Time to him was time as it is lived, with its own natural temporal order, a continuous 'becoming'[2] which admits of a discreteness of 'before' and 'after'; (time in this sense is the basis of Brouwer's primordial intuition of two-ity). The impossibility of an isomorphism between the perceived order of 'temporal becoming' and the dense order of the mathematical linear continuum has recently been pointed out by A. Grünbaum[3], but then the mathematical continuum is taken as the completed totality in the classical sense.

Bergson's 'becoming' of time and motion is a complete denial of a composition of either through successive states or immobilities; referring to Zeno's paradox of the arrow, he says: 'At bottom, the illusion arises from this that the movement, once effected, has laid along its course a motionless trajectory on which we can count as many immobilities as we will ...' and 'The arrow never is at any point of its course ... To suppose that the moving body is at a 'point' is in effect putting a stop to the movement.'[4]

Brouwer in a similar way had already upheld this complementary character of the discrete and the continuous with respect to the mathematical continuum; he now tried to extend the 'becoming' to the natural dense order of the mathematical linear continuum, the continuum based on the relation of betweenness. It could be said that Brouwer claims the dynamic 'becoming' of Bergson's time continuum for the mathematical continuum, (Aristotle's potential infinite in the real sense: δυνάμει),[5] and returns to the pre-Weierstrass - Cantor stage, (when functions were held 'to move' and 'reach' a limit), in conceiving a 'point of the continuum' as an 'indefinitely proceeding sequence'

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1 A.N. Whitehead, Process and Reality, New York 1929.
W. James, Some Problems of Philosophy, London-New York 1948.

2 Cf. Time and Free Will, pp. 73 - 85.

3 A. Grünbaum, Modern Science and Zeno's Paradoxes, London 1968.

4 Creative Evolution, London-New York 1911, Chapter V.

5 Physics 263[a], 28.

which is never completed, neither as a'set'[1] nor even always as a method. Brouwer had earlier introduced 'free-choice sequences' to show the insufficiency of the definition of points of the continuum in terms of sequences well-defined in the classical sense. Now the free-choice sequence is used to emphasize the 'becoming', growing aspect of all infinite sequences, and the difference between the discrete point and the 'point of the continuum'. In the same way that in Bergson's time as 'becoming' there is no place for an atomistic, discret 'now', there is ultimately in the Brouwer continuum no room for a sicrete point. Brouwer's refusal to accept the continuum as a 'totality of discrete points' becomes more than just a matter of potency, it affects the nature of the point of the continuum. The fact that in some way sequences determined by an algorithm can be thought of as finished, i.e. as a law, has obscured the true nature of the infite mathematical sequence which is the progressive allocation of values by the human mind, according to a predeterminate law or otherwise, and this ad infinitum, i.e. never ending or for ever proceeding. This 'becoming', growing, unfinished character, so well illustrated by the free-choice sequence but also present in the law-like sequence, distinguishes the point of the continuum from the discrete. E. Bishop's comment that 'Brouwer introduced the method of free-choice sequences for constructing the continuum, as a consequence of which the continuum cannot be discrete because it is not well enough defined'[2], might well in some way have applied to Brouwer's earlier 'completing into a continuum', which indeed he introduced to disprove the cantorian discreteness of the continuum. With the new emphasis on 'becoming', the free-choice sequence illustrates the possibility of generalization of the concept of the infinite sequence, in which the manner of its continuation is immaterial and which as a never finished, always growing mathematical entity, characterizes the point of the continuum and the continuum itself. Weyl recognized this: 'When a sequence is generated step by step by means of free acts of choice, it must be seen as a sequence that is 'becoming' (eine werdende Wahlfolge)[3]; it is

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[1] 'Set' here in the sense of Brouwer 1908; see above pp. 172 - 173.

[2] E. Bishop, op. cit. p. 6.

[3] 'Die neue Grundlagenkrise der Mathematik', p. 50.

because of the becoming character of Brouwer's continuum that he was attracted to intuitionism. Referring to his own previous attempt, he abandons it 'fully conscious that it does not catch the intuitive continuum ... but Brouwer's attempt contains the promise to do full justice to the 'becoming' in a valid and lasting manner'.[1]

Set theory

4.1 The Brouwer set

Brouwer's set theory is primarily an attempt to construct a framework for this more generalized notion of the sequence.

Definition

As stated above, the definition of set starts from the existence of a denumerable sequence of 'symbols' (Zeichen, 1918A), 'symbols or finite rows of symbols' (Zeichen bzw. endlichen Zeichenreihen, 1925D), most appropriately represented by the sequence of the numerals 1, 2, 3, 4, 5, ...[2] We give here the definition of 1918A:

> 'A set (Menge) is a rule, on the basis of which - whenever an arbitrary numeral complex (Ziffernkomplex) of the sequence is chosen - every choice generates either a definite symbol or nothing, or causes the arresting (Hemmung) of the process and the definite destruction of its result, while for every $n > 1$ after every non-arrested sequence of $n - 1$ choices, at least one numeral-complex can be indicated which, when chosen as the n-th numeral complex, will not effect the arresting of the process. Every sequence of symbols - generated in this way by an unlimited sequence of choices (and therefore in general not presentable as finished) - is called an element of the set. We will refer to the common mode of generating elements of the set M, also as the set M.'[3]

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1 'Die neue Grundlagenkrise der Mathematik', p. 51.

2 In 1947B (p. 339) Brouwer remarks: 'Because of the languageless nature of mathematics, the word symbol (Zeichen) and especially the word numeral-complex (Ziffernkomplex) in this definition must be understood as thought-signs (Du. gedachtenteekens), consisting in mathematical mental entities (Du. denkbaarheden) previously acquired.'

3 1918A, pp. 3 - 4.

The same definition is given in 1920B, 1921A, and with some change of wording, also in 1925D; ('each of these choices either generates a sequence of symbols - with or without the termination of the process - or the arresting of the process'[1]). In a footnote to the 1925D definition, Brouwer apologizes for the 'inevitable longwindedness', and in quoting the continuum as a simple example of a set, he gives an indication of what inspired his notion of set.[2] Brouwer's post-war interpretations start with the definition of elements of the set (for the first time referred to as a 'spread'[3]), 'the infinitely proceeding sequences', a method generally adopted by other intuitionists.[4]

Even if Brouwer uses the word 'set' (Du. verzameling, Germ. Menge), the Brouwer set from 1917 onwards can in no way be taken as a totality of its elements, unlike his earlier practice (1907, 1908A, and 1913B), when he did use the word set in its traditional sense of a totality, well-distinct from a 'method'.[5]

The Brouwer set is simply defined as a 'rule', a law (ein Gesetz), in 1924H even as 'an algorithm'[6] law, however, must here be taken in the more general sense of a prescribed, effective procedure ('common mode of generating') for generating the special kind of elements that Brouwer had in mind: sequences in the widest sense, including those whose components are determined by such unorthodox methods as free choice, or throwing of dice; a more complex procedure than the classical algorithm which generates denumerable sets, one that could generate sets of classical cardinal number $2^{\aleph_0}$.

The main requirement for this law in the general definition of set is a guarantee of the 'infinite proceedability' of each of the elements of the set: 'for every $n > 1$ after every non-arrested sequence of $n - 1$ choices, at least one numeral can be found that will not effect the arresting of the process'. This is also emphasized

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1 1925D, p. 244.

2 Ibidem.

3 1947B, p. 339 (Du. spreiding).

4 Heyting, Intuitionism, An Introduction, Amsterdam 1956; Kleene and Vesley, The Foundations of Intuitionistic Mathematics, Amsterdam 1965.

5 See above pp. 172 - 173.

6 'Algorithmus', 1924H, p. 189.

by what 1920B (1921A) calls 'the essentially unfinished character of the elements'[1], 1925D: 'its freedom to proceed' (Forsetzbarkeitsfreiheit)[2], 1942A: 'the possibility to proceed' (Forsetzbarkeitsmögligkeit)[3]. When a sequence is 'arrested', it is simply eliminated as an element of the set, it causes 'die definitive Vernichtung seines Resultats'. (Weyl even more dramatically speaks of 'den Abbruch des Prozesses, ihren eigenen Tod (death)[4]).

When a sequence is simply terminated, the process of choosing can still be thought of as continuing but 'after a certain choice, instead of generating a series of symbols, every further choice produces nothing'.[5]

The law regulates the process of generating an element. This in Brouwer's definition is described as:

1° A progressive selection from a pre-defined[6] fundamental sequence of symbols (1925D: symbols or finite series of symbols), represented by the numerals (Nummern) 1, 2, 3, ... [7]

2° A correlation of each of the chosen numerals to 'a definite symbol' (1918A), (1925D: 'a definite sign series'; 1952B: 'a mathematical entity previously acquired), or to nothing, or to the arresting of the process.

This dual aspect of the generating process is illustrated by the very first example of a set in 1925D, where the fundamental sequence is the set ζ and the subsequent correspondence is with the numbers themselves: 'To help understanding, I may here already point at a special case: when there is no termination nor arresting of the process, and

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[1] 1920B, p. 950; 1921A, p. 799.

[2] 1925D, p. 245, footnote [3].

[3] 1942A, p. 323.

[4] Weyl, op. cit. p. 65.

[5] 1925D, p. 244, footnote [2].

[6] 1942A considers the possibility of 'floating sets' (schwebender Mengen) where the fundamental sequence - instead of being fixed in advance - is proceeding indefinitely, or as Brouwer puts it, 'is in free becoming', (befindet sich im "freien Werden"). 1942C provides the proof that such a set M 'of second order' constitutes a sub-species of a set M_1 derivable from M, and that therefore such a set of second order - and similarly those of higher order - are not a fundamental notion in intuitionist mathematics.

[7] See above, p.186.

'when, moreover, every choice of a number generates that number itself. This case produces the set C'.[1] The dual role of the Brouwer set as a law led to Heyting's distinction between the 'spread law', governing the choice from the set ζ, and 'the complementary law' which correlates a particular mathematical object to each of the numerals.[2] This usage was never adopted by Brouwer himself[3], neither was the distinction between the various kinds of infinitely proceeding sequences, i.e. into lawless, choice-sequences and lawlike.

The term 'choice sequences' (Wahlfolgen) is used in the most general sense and covers all infinite sequences, the classical well-defined lawlike as well as those where no restriction is imposed on the freedom of choice; they are best characterized by Brouwer's non-commital 'more or less freely chosen'[4]. The restrictions on freedom of choice were not systematically analyzed by Brouwer; (recent and present research within the intuitionist school, both in Holland and America, is concentrated on the implications of choice sequences.[5]

For Brouwer's immediate purpose of defining the continuum and the real numbers, the broad distinction between complete freedom, progressively restricted freedom of choice and complete pre-determination suffices. Although not explicitly mentioned, progressive restriction of choice is allowable within the definition of 1918A. 1925D states in a footnote vaguely that 'this freedom to proceed can be narrowed arbitrarily after every choice, even up to the point of complete determination (Bestimmtheit), but in every case in accordance with the set-law'.[6] 1942A clarifies this further by stressing that 'indeed the

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1 1925D, p. 244.

2 Heyting, Intuitionism, An Introduction, Amsterdam 1956, pp. 34 - 35.

3 The 'Mengengesetz' (setlaw) mentioned in 1925D (footnote [3]), p. 245, is to be interpreted appositionally, i.e. the law which is the set.

4 1952B, p. 142.

5 Cf. e.g. G. Kreisel, 'Lawless sequences of natural numbers', Compositio Mathematica, vol. 20 (also as Logic and Foundations of Mathematics, Amsterdam 1968, pp. 222 - 248;

J.R. Myhill, 'Notes towards an axiomatization of intuitionistic analysis', Logique et Analyse, vol. 35 (1967), pp. 280 - 297;
A. S. Troelstra, 'The theory of choice sequences', Logic, Methodology and Philosophy of Science III, Amsterdam 1968, pp. 201 - 223;
A. S. Troelstra, Principles of Intuitionism, Berlin-Heidelberg-New York 1969. Research on choice sequences is continued at present by Myhill, Kreisel and Troelstra.

6 1925D, p. 245, footnote [3].

arbitrariness of these restrictive clauses which can be associated with each individual choice - provided proceedability is safeguarded - constitutes an essential characteristic of the 'free becoming' of the elements of the set'[1]. (The further suggestion of a restrictive clause of second order, 'restricting freedom to restrict in future', was later withdrawn: 'This admission is not justified by close intospection and, moreover, would endanger the simplicity and rigour of further development'.[2]

Particular cases of restrictions are the Brouwer finite sets (see below). The most typical example of a set is the continuum. We quote it here as given by Brouwer in 1918A, since it also illustrates the way Brouwer defines a particular set:

'The set C is the set of unlimitedly proceeding sequences of numerals of ζ , we denote its cardinal number by c...
If we let the sequence of positive whole numbers $a_1, a_2, a_3, \ldots$ correspond to the real number:

$$\frac{1}{2^{a_1}} + \frac{1}{2^{a_1 + a_2}} + \frac{1}{2^{a_1 + a_2 + a_3}} + \ldots$$

C appears to be the means of generating the real numbers between 0 and 1, including 1 but excluding 0.
If we associate the fundamental sequence $a_1, a_2, a_3 \ldots$ with the real number:

$$\cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\cdot}}}}$$

the set C appears as the generator of the irrational numbers between 0 and 1'.[3]

4.2 Finiteness

1° 'Endliche Mengen'

The notion of the Brouwer set was primarily introduced to accommodate the superdenumerable set. Sets that are finite in the classical sense

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1 1942A, p. 323.

2 1952B, p. 142, footnote.

3 1918A, p. 9.

do exist, as the above example of the set M illustrates. Brouwer then uses the word 'endlich'. No definition of the finite set in this sense is given, but as a special case of species, the 'endliche Menge' could be defined as 'equipotent with an initial eventually terminating segment of the sequence ζ '.[1]

In accordance with his rejection of the PEM, Brouwer stresses that there is no ground for maintaining that every set (or species) is either finite (endlich) or infinite (unendlich).[2]

2° 'Finite Mengen'

Another Brouwerian divergence from conventional usage is his so-called 'finite Menge', referred to by Brouwer in his post-war English contributions as 'fan'[3] and in present intuitionistic usage as 'finitary spread'.

The finite here does not refer to 'the number of elements' of the sets, nor to the initial segment of each element, but to the fundamental sequence from which the choices are made.

The 'finite set' does not appear in the 1918A version but can find its place in that definition as a particular restriction of the freedom of choice. 1925D introduces it as a restriction which is progressive in the sense that the restricted number is made dependent on the particular choice: 'If for every n in ζ a number k_n is defined, such that, when for the nth choice a number in ζ is chosen lying higher than k_n, the process is arrested, the set is called finite'[4]. The example given here is the set of 'all unlimitedly proceeding sequences of single digit numbers'.

A special case is Brouwer's later 'n-finite Menge', when the number is fixed for the whole set. 1928C shows how the unit continuum can be generated by the 3-finite set [5], the earlier bifurcation of 1907 can be seen as a precursor of the 2-finite set.

The 'finite set' proved its usefulness and became gradually more prominent as a means of restricting intervals in the representation of the unit-continuum.

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1 1918A, p. 5 (with correction from 1919A, p. 34; 1925D, p. 248.).

2 Ibidem.

3 1952B, 1953.

4 1925D, p. 245.

5 1928C, p. 5.

3° Initial segment

The most important finite aspect of the Brouwer set derives from the 'becoming' character of the elements itself; sequences which are finite at every stage of their growth, where only finite initial segments possess the strongest form of existence (i.e. having been constructed).

When considering the continuum as a whole, its power and such topological properties as density, separability, and compactness[1], the emphasis is on the possibility to proceed indefinitely. In the context of the theory of functions, special attention needs to be paid to properties possessed by the finite initial segments of the sequence and the restrictions on further choices in order to make a function on a Brouwer set at all meaningful. Brouwer's strict interpretation of functions demanded that if a function f is to assign a definite object $f(x)$ to an element x of a Brouwer set, $f(x)$ should be determined on the basis of a finite initial segment of the infinite sequence that constitutes x.[2] That Brouwer was well aware of this already in 1917, is shown in a proof of 1918A that the power of the set C is greater than that of the (denumerable) set A: 'A rule that associates an element h of A to every element g of C must have defined the element h completely after a certain initial segment α of the number sequence g. But then with every element of C which has α as its initial segment the same element h is associated. It is therefore impossible to associate with every element of C a different element of A'.[3] The requirement of 'securability' on the basis of a finite initial segment is reinforced by Brouwer's later extension of the notion of function replacing the 'sharp' law of earlier definitions of functions (1923A) by a 'free associating' on the lines of the Brouwer set: 'I have for some time shown in my lectures that an arbitrary continuous function grows 'in free becoming' in exactly the same way as an arbitrary point of the continuum.' ('Note on the free becoming of sets and functions'[4]).

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[1] See further p. 223.

[2] For an account of the resulting 'bar theorem' and 'fan theorem' see: Kleene and Vesley, op. cit. pp. 43 ff; Beth, op. cit. pp. 427 ff; see above p. 35.

[3] 1918A, p. 13; 1925D, p. 253 (slight modification).

[4] 1942A, p. 322.

4.3 Apart from the notion of the Brouwer set itself, there is no detailed systematic treatment in 1918A, nor in 1925D, of the theory of Brouwer sets.

Typical of Brouwer's suspicion of formalization and formalist symbolism is the almost complete absence of a standard (or any other) notation; (not even $\cup$, $\cap$, $\subset$, or $\in$; the only concession made is $\mathfrak{D}$ (M,N) (Germ. Durchschnitt) for intersection and $\mathfrak{S}$ (M,N) (Vereinigung) for union). We will use the symbols $\cap$ and $\cup$ respectively for intersection and union.

Since the Brouwer set is primarily a procedure for constructing each element, the only immediate, natural equality of Brouwer sets would be the trivial equality of procedure. Any equality of sets in terms of their elements (cf. Weyl's 'Umfang' (extent)[1]), can only be on the basis of a property possessed by the elements and is an equality natural to classical sets and the Brouwer 'species'. Because of their common mode of generating, Brouwer sets are species ('sets are special cases of species of the first order'[2]), and as such can be subject to operations on species.

Because of this 'weaker' equality (i.e. relating to it as a species not as a set), the theory of Brouwer sets contains some unexpected results; also, with the emphasis on the elements and their equality, it is quite natural that the 'species' dominates 1918A (and 1925D), in spite of its title, 'Begründung der Mengenlehre'.

Some important definitions and results:

Equality:

'Two elements of a set[3] are equal or identical when one is certain that for every n, the n-th choice for both elements generates the same series of symbols.'

'Two elements of a set are called distinct (verschieden) if the impossibility of their equality has been established, i.e. when one is certain that in the course of their generation their equality can never be proved.'

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[1] Weyl, 'Über die neue Grundlagenkrise der Mathematik', p. 41.

[2] 1918A, p. 4; 1925D, p. 245.

[3] The text has 'Mengenelemente', which means literally 'set elements' and can also be interpreted as two elements of two different sets.

'two sets are said to be equal or identical if for every element of one set an equal element of the other set can be given.'[1]

Individualized:

'A set is said to be individualized (individualiziert) if different non-arrested sequences of choices always lead to different sequences of symbol-series.'[2]

Subset:

'The set M is said to be a subset (Teilmenge, i.e. part-set) of the set N, if for every element of M there exists an element of N equal to it.'[3]

Union and intersection

Neither union nor intersection are specially defined for sets, but implied in the definition of union and intersection of species. All that Brouwer states concerning union and intersection of sets is the following:

'The union of two sets is again a set. The union, however, of two individualized sets need not be an individualized set.'[4]

'The intersection of two sets need not be a set.'[5] 'That the intersection of two sets need not be a set can be seen, when we take M to be the set containing one single infinite dual fraction a, and N the set containing the single infinite dual fraction b, and when, moreover, we can neither prove that a and b are equal nor that they are distinct.[6]

4.4 Species

The notion of 'species' is quietly introduced in 1918A where its definition only occupies one line. After reserving the phrase 'mathematical entity' only for 'sets and elements of sets', 'a species of the first order' is defined as 'a property which only a mathematical entity can possess.'[7] 1925D adds: 'a property defined in a conceptually completed form', (begrifflich fertig definierte Eigenschaft).[8]

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[1] These definitions are only given in 1925D, not in 1918A.

[2] 1918A, p. 3; 1925D, p. 245.

[3] 1925D, p. 245.

[4] 1925D, p. 247.

[5] 1918A, p. 4.

[6] 1925D, p. 247.

[7] 1918A, p. 4.

[8] 1925D, p. 245.

1920B (1921A) simply states: 'On the notion of set rests further the definition of the notion "mathematical species" (Du. soort, i.e. kind), which contains the notion set as a special case.' (page 52)

In Brouwer's later writings it achieves greater prominence and is heralded as 'a new mathematical entity' as the result of the second act of intuitionism: 'A much wider field of development which includes analysis is opened up by the second act of intuitionism which recognizes the possibility of generating new mathematical entities, first in the form of infinitely proceeding sequences ... secondly in the form of mathematical species, i.e. properties supposable for mathematical entities previously acquired and satisfying the condition that if they hold for a certain mathematical entity, they also hold for all mathematical entities which have been defined to be equal to it.'[1]

Like the Brouwer set, the species is not the totality of its elements, but a means of generating elements by selection from 'previously acquired mathematical entities, and so constitutes an intuitionist form of the principle of comprehension, based on the requirement that the elements are selected - not from a cantorian universe - but from elements that have been constructed. In this way, not only will Russell's paradox be avoided, but also the 'circulus vitiosus' of 'Stufenbildung' rejected by Weyl in his 'Das Kontinuum '.[2] Brouwer also develops a hierarchy of species of second order ... nth order: ('By a species of second order we understand a property that only mathematical entities or species of first order can possess ... In a similar way we define species of n-th order.'[3]) But whereas Weyl's 'Stufenbildung' for its first stage resorts to logical properties and relations which are 'original', Brouwer demands existence as 'a mathematical entity' for his species of first order.

Equality is then defined within each order of species: 'Two species of n-th order are equal or identical when for every element of one species an equal element of the other can be specified.'[4]

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1 1952B, p. 142.

2 Weyl, Das Kontinuum, 1917 (reprinted Chelsea Publ. Co.), p. 23.

3 1918A, p. 4; 1925D, p. 246.

4 1925D, p. 246.

4.5 Divergence from the classical theory of sets is caused mainly by the strong Brouwer negation and is more evident in 1925D, (after Brouwer's publication of the calculus of absurdity[1]), than in 1918A. This is clear from the following definitions:

Distinct

'Two species are said to be distinct (verschieden) if the impossibility of their equality has been ascertained.'[2] (Not found in 1918A).

Discrete

'A species of which any two elements can be recognized as either equal or distinct is said to be discrete.'[3]

Subspecies

'A species M is said to be a subspecies of the species N if for every element of M an equal element of N exists. If, moreover, an element of N can be specified that cannot be an element of M, then M is said to be a proper subspecies of N.'[4] (1918A only has: 'an element of N which is not an element of M.')

'Standing out'

'We say that the species M stands out (herausragt) from the species N, if N has an element distinct from every element of M.'[5] (Not in 1918A).

Congruent

'Two given species are said to be congruent if neither can stand out from the other, in other words, when every property, impossible for the elements of one species is also impossible for the elements of the other.'[6]

Half-identical

'If a subspecies M of N is congruent to N, it is said to be half-identical with N.' [7]

Union

'The species which contains those elements that belong either to the species M or to the species N, is called the union of M and N and is denoted by $\cup$ (M,N) .' [8]

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[1] See further: PEM after 1923, p. 146.

[2] 1925D, p. 246.

[3] Ibidem.

[4] Ibidem.

[5] Ibidem

[6] Ibidem

[7] Ibidem; also in 1918A, p. 4, as corrected in 1942A, p. 323.

[8] 1925D, p. 247, note: Entweder ... oder.

Disjoint

'Two species M and N are said to be disjoint (Elementefremd) if they are distinct and it is impossible that an element of M and an element of N exist which are identical.'[1] (1918A: 'No element can exist which belongs to M as well as N.')

4.6 Complementary species

As can be expected, the greatest divergence from classical practice is found in the set (species) theoretical equivalent to negation. The cantorian concept of a set as a totality allows a subsequent partition on the basis of the principle of contradiction into a set and its complement.

The Brouwer species (and the Brouwer subspecies) is defined as the property itself, and only exists as a property. If the Brouwer species in any way is to be seen as an aggregate of its elements, it is the aggregate which results from an effective selection on the basis of a property from mathematical entities previously acquired.

Brouwer does not deny the principle of contradiction; he denies the possibility of effective decision in all cases which alone can generate a species - or subspecies. A single property α and the Brouwer negation of that property, $\sim\alpha$, generate two subspecies, the subspecies of those elements of which it can be decided that they possess the property α, and the subspecies of those elements of which it can be decided that they cannot possibly possess the property α.

This interpretation of subspecies allows a distinction in the manner in which a species can be composed of subspecies. Brouwer distinguishes between 'zusammensetzen' (composition) and 'zerlegen' (splitting)[2], which results in respectively 'Komplementarspezies' (complementary species) and 'konjugierte Zerlegungspezies' (conjugate splitting species).

1° 'If M' and M" are disjoint and subspecies of N, and $\cup$ (M',M") is half identical with N, we say that N is composed of M' and M", and we call M' and M" mutually complementary species in N.'[3]

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1 1925D, p. 247.

2 Cf. also 1929A: 'We say that the species P is composed of its disjoint subspecies Q and R if the existence of an element of P distinct from $\cup$ (Q,R) leads to absurdity. We further say that the species P is split into its disjoint subspecies Q and R if P = $\cup$ (Q,R).' (p. 10)

3 1925D, p. 247; in 1918A where the definition of 'half identical' is given in terms of 'congruence', the condition is: 'If $\cup$ (M',M") and N are congruent' (1918A, p. 4).

2° 'If M' and M" are disjoint and subspecies of N and ∪ (M',M") is identical with N, we say that N is split into M' and M" and we call M' and M" conjugate splitting species of N and both M' and M" separable subspecies of N.'[1]

It must be noted that in both cases 'complementarity' is relative to a given species (N - M) and that, moreover, M must be a subspecies of N.

Brouwer's rejection of the PEM could in species-theoretical terms simply be formulated as: in general a well-defined property will decompose, but not necessarily split a species.

Rejection of the PEM is not specifically mentioned in either 1918A nor in 1925D, (except in the title of 1918A), nor are general rules given as to when a species splits into subspecies on the basis of a property, although the example of complementary species given here is the favourite example refuting the PEM: ('This is the case when N contains whole (sämtliche) numbers, M' those which as indices of the Fermat equation make this equation solvable, and M" those which as indices of the Fermat equation make this equation unsolvable.'[2] Also, the example of splitting subspecies refers to a property which can always be ascertained because of its finiteness: ('The case when N contains whole numbers, M' those which have at most five digits and M" those which have more than five digits.'[3] (All Brouwer's examples refuting the PEM will consist in quoting a species and a property which will not split that species by indicating one or more elements for which the property cannot be decided).

4.7 The principle of reciprocity of complementarity

In his set-theoretical contributions (1918A, 1919A, 1920B, 1921A, 1925D, 1926A and 1927A), Brouwer does not mention the principle of reciprocity of complementary species, the cantorian principle (A') ' = A.

1923C (1925B) defines 'the principle of reciprocity of complementary

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[1] 1925D, p. 247.

[2] Ibidem.

[3] Ibidem; see also p. 148.

species' as a logical principle: 'the principle which concludes the truth of a property from the absurdity of its absurdity'.[1] It is referred to in 1923C as 'a special case of the PEM' and in 1925B as 'a corollary of the PEM'.

Brouwer subsequently recognized that application of the calculus of absurdities ($\sim\sim\alpha = \alpha$) leads to the validity of the principle of reciprocity of a complementarity for statements involving negative properties; with this 'wider field of validity', the principle of reciprocity of complementarity can hardly be called a special case of the PEM: 'The equivalence of the PEM and the principle of reciprocity of complementarity, mentioned there, (i.e. 1923C), in a footnote by way of remark, has been recognized as non-existent. In fact, as was also shown in the present paper, the fields of validity of these principles have turned out to be essentially different.'[2]

This confusion is partly due to an inconsistency in Brouwer's treatment of logic and species.

In the case of the (logical) principle of the reciprocity of complementary species, Brouwer makes a definite distinction between a positive and an (essentially) negative property. For a statement α asserting a positive property: $\sim\sim\alpha \neq \alpha$; but in the case of a statement β asserting a negative property: $\sim\sim\beta = \beta$.

However, in Brouwer's set-theoretical treatment of complementary species, neither of the complementary subspecies are given as a positive nor as a negative property. They are defined as two subspecies, M' and M" of N, which are disjoint in the sense of Brouwer, i.e. mutually exclusive, every element of M' cannot possibly belong to M" and every element of M" cannot possibly belong to M', while moreover M' $\cup$ M" is not presumed identical with N. Within this definition of complementary species there is no room for reciprocity, (where such reciprocity could be proved, i.e. in the case of 'conjugate splitting species', no attempt is made at defining a limited principle of reciprocity).

Brouwer's distinction between positive and negative properties in his logical treatment of the calculus of absurdities can therefore not be based on notion of (mathematical) complementary species, in spite of his usage of 'complementary species'. Neither can the logical calculus of absurdities be called a 'post factum analysis of the language' of Brouwer's mathematical theory of species.

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[1] 1923C, p. 877; 1925B, p. 252.

[2] 1953, p. 14 footnote.

5.1 Splitting of Equipotencies

In spite of Brouwer's claim in 1907 'to have finally solved Cantor's continuum hypothesis', (in agreement with Cantor, but on the basis of a crude distinction between denumerable and non-denumerable)[1], the question of the power of the continuum and of infinite sets continued to occupy him for years to come. In 1912 he confessed, 'this question is still waiting for an answer, and is one of the most difficult and fundamental of mathematical problems'.[2]

In a footnote to 1913A, (not in 1912A), he 'defines' a subset of the continuum whose power is greater than the denumerable but smaller than the continuum: 'If we suppose that the problem concerning the pairs of digits in the decimal fraction development of π cannot be solved, then the question of the text, (i.e. Is it impossible to construct infinite sets of real numbers between 0 and 1, whose power is less than that of the continuum, but greater than aleph-null?), must be answered negatively. For let us denote by Z the set of those infinite binary fractions whose n-th digit is 1 if the n-th pair of digits in the decimal fraction development of π consists of unequal digits; let us further denote by X the set of all finite binary fractions. Then the power of Z + X is greater than aleph-null but less than that of the continuum.'[3] At this stage, however, this kind of 'definition' is still branded as formalistic and not acceptable to Brouwer as a construction;[4] Brouwer still accepts Cantor's continuum hypothesis.

With the introduction of species, sets of this type could be considered; indeed, this same example is used in 1917A to refute Cantor's continuum hypothesis [5], but the earlier crude distinction between denumerable and non-denumerable proves insufficient. Further analysis of the continuum emphasized the need for subtler subdistictions.

Cantorian comparison of powers of two sets A and B allows three possibilities: (using α for Cantor's notation $\bar{A}$):

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[1] 1907, p. 67; full quotation above p. 177.

[2] 1912A, p. 23; 1913A, p. 92.

[3] 1913A, p. 92.

[4] Ibidem: 'If "construct" here were replaced by "define" in the formalist sense...'

[5] 1917A, p. 441.

1° equipotency (equivalence): $\alpha = \beta$ (there is a one-one correspondence between elements of A and of B)

2° of greater power: $\alpha > \beta$ ($\alpha \neq \beta$ and there is a subset B' of B such that $\alpha = \beta'$, or in other words, there is a one-one correspondence between A and a proper subset of B)

3° of smaller power: $\alpha < \beta$ (similarly: $\alpha \neq \beta$ and $\alpha' = \beta$).

Both Cantorian equivalence and non-equivalence are based on a one-one correspomdence; the only possibilities considered are a one-one correspondence between A and B, and a one-one correspondence between A and B', (or vice versa: A' and B).

Brouwer made various distinctions within the concept of equipotency. First, the general notion of equipotency, (simply defined as a one-one correspondence[1]), is distinguished from a stricter relation of equivalence, which is defined as follows: 'Two species M and N (and their corresponding cardinal numbers m and n) are said to be equivalent, if by a law G_1 with each element of M such an element of N is associated, that equal elements of N correspond to equal and only equal elements of M, and when by a law G_2 with every element of N such an element of M is associated, that equal elements of M correspond to equal and only equal elements of N, a property which we also express by means of the formula m = n.'[2]

Brouwer's splitting of the notion of equipotency is based on his distinction between identical and half-identical species. In the process of relating two species M and N, a one-one correspondence can be set up, not only between M and a subspecies N' of N (and vice versa), but also between a subspecies M' of M and a subspecies

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[1] Equipotency (Gleichmächtigkeit): 'If between two species M and N a one-one correspondence can be established, which associates with each element α of M an element β of N, such that each element of N coresponds to one and only one definite element of M, we write $M \sim N$ and say M and N have the same power or cardinal number.' (1918A, p. 5). In this context, Brouwer also defines a notion of semi-equipollent: 'If with each element α of M a different element β of N is associated in such a way that the species of the β is half-identical with N, then M and N are said to be semi-equipollent (halbgleichmächtig).' (The text, 1918A, p. 5, has 'identical with N', which must be a misprint, even if it is not included in the list of errata of 1919A, p. 34. Cf. also 1925D, p. 248).

[2] 1925D, p. 252; cf. 1918A, p. 10.

N' of N, and both M' and N' (or: either M' or N') can be identical or half-identical with respectively M and N.

The process of generating various relations can be generalized in the following way:[1]

The general notion of equipotency (say $\underset{\cdot}{=}$) rests on the relation $\underset{\cdot}{\leq}$. By combinations of this relation and the Brouwer negation, the relations $\underset{\cdot}{=}$ and $\underset{\cdot}{<}$ are generated;

1° $m \underset{\cdot}{=} n$: if both $m \underset{\cdot}{\leq}$ and $n \underset{\cdot}{\leq} m$.

2° $m \underset{\cdot}{<} n$: if $m \underset{\cdot}{\leq} n$ and it is impossible that $n \underset{\cdot}{\leq} m$.

Because of the rejection of the PEM, it follows in general that:

1° $m \underset{\cdot}{\leq} n$ does not necessarily imply that either $m = n$ or $m < n$;

2° the relations $m \underset{\cdot}{=} n$, $m \underset{\cdot}{<} n$, and $m \underset{\cdot}{>} n$ are mutually exclusive;

3° $m \underset{\cdot}{=} n$ and $n \underset{\cdot}{=} p \Rightarrow m \underset{\cdot}{=} p$;

4° $m \underset{\cdot}{>} n$ and $n \underset{\cdot}{>} p \Rightarrow m \underset{\cdot}{>} p$;

5° $m \underset{\cdot}{\geq} n$ and $n \underset{\cdot}{>} p \Rightarrow m \underset{\cdot}{>} p$.

5.2 The various relations and equipotencies as generated by Brouwer are given here in their German names of 1925D; a translation, especially of the corresponding degrees of denumerability, is well-nigh impossible.[2]

i. Übergeordnet ($\underline{\S}$)

'A species M, and its cardinal number m, is übergeordnet (over-associated) with respect to N, and its cardinal number n, and we write m $\underline{\S}$ n, if with every element α of a certain subspecies M_1 of M always an element β of N is associated in such a way that the species N' of the β is half-identical with N.'[3]

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[1] In all cases the notion of equipollent is generated by a combination of $m \underset{\cdot}{\leq} n$ and $n \underset{\cdot}{\leq} m$, except in the case of equivalence, where definitions of $\dot{<}$ (smaller than), and $>$ (greater than), $\leq$ and $\geq$, follow the definition of equivalence. (1918A, p. 10; 1925D, p. 252).

[2] Formulation of 1918A and 1925D vary considerably:
überlagert of 1925D = superponiert of 1918A;
superponiert of 1925D is left out in 1918A;
In the notation of 1918A Brouwer uses $\underline{\S}$ $\underline{>}\cdot$ and $\cdot\underline{>}\cdot$ resp. for übergeordnet, superponiert and überdeckt.

[3] 1925D, p. 253.

von grőszerem Umfang ($\overset{\circ}{>}$)

'If the species M is overassociated with respect to N, but the species N cannot possibly be overassociated with respect to M, then M (or m) is said to be of greater range (Umfang) than N (or n) and we write $m \overset{\circ}{>} n$.'[1]

von gleichen Umfang ($\overset{\circ}{=}$)

' M (or m) and N (or n) are said to be of equal range if both $m \overset{\circ}{\geq} n$ and $n \overset{\circ}{\geq} m$.'[2]

In a similar manner:

ii. überlagert ($\overset{\cdot}{\geq}$) (literally: overlaid)

'Die Spezies M heisst N überlagert' if: $m \overset{\circ}{\geq} n$ and N' is identical with N.

von groszerer Weite ($\overset{\cdot}{>}$) (literally: of greater width)

The species M is of greater width than N if:

$m \overset{\cdot}{\geq} n$ but it is impossible that $n \overset{\cdot}{\geq} m$.

von gleicher Weite ($\overset{\cdot}{=}$) (literally: of equal width)

The species M is of the same width as N if both: $m \overset{\cdot}{\geq} n$ and $n \overset{\cdot}{\geq} m$.

iii. superponiert ($\overset{\cdot\cdot}{\geq}$)

' M heisst N superponiert' if $m \overset{\circ}{\geq} n$ and M_1 is identical with M.

von groszerer Ausdehnung ($\overset{\cdot\cdot}{>}$) (literally: of greater extension)

' M is said to be of greater extension than N, if:

$m \overset{\cdot\cdot}{\geq} n$ but it is impossible that $n \overset{\cdot\cdot}{\geq} m$.

von gleicher Ausdehnung ($\overset{\cdot\cdot}{=}$) (literally: of equal extension)

' M is of the same extension as N if both $m \overset{\cdot\cdot}{\geq} n$ and $n \overset{\cdot\cdot}{\geq} m$.

iv. überdeckt ($\overset{\cdot\cdot\cdot}{\geq}$) (literally: covers)

M is said to cover N, if $m \overset{\circ}{\geq} n$, M_1 is identical with M and N' is identical with N.

von grőszerem Gewicht ($\overset{\cdot\cdot\cdot}{>}$) (literally: of greater weight)

M is of greater weight than N if $m \overset{\cdot\cdot\cdot}{\geq} n$ but it is impossible that $n \overset{\cdot\cdot\cdot}{\geq} m$.

von gleichem Gewicht ($\overset{\cdot\cdot\cdot}{=}$) (literally: of equal weight)

M is said to be of equal weight as N if both $m \overset{\cdot\cdot\cdot}{\geq} n$ and $n \overset{\cdot\cdot\cdot}{\geq} m$

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[1] 1925D, p. 254.

[2] 1925D, p. 254. (Definitions of all other relations are taken or reconstructed from pp. 253, 254, and 255).

Denumerability

The above relations applied to species, and the set A and its cardinal number a (symbols sequences of ζ) produce the following distinctions within the classical denumerable:

A species M is said to be:

abzählbar if $m \leq a$,

zählbar if there is a one-one correspondence between M and a separable subspecies of A,

nachzählbar if $m = a$,

auszählbar if $m \overset{\circ}{\leq} a$,

überzählbar if $m \overset{\cdot}{\leq} a$,

durchzählbar if $m \overset{\cdot\cdot}{\leq} a$,

aufzählbar if $m \overset{\cdot\cdot\cdot}{\leq} a$.[1]

Examples of species of various 'Zählbarkeite' are given in *1925D*, pp. 255 - 256. Very little use has been made by intuitionists, and indeed by Brouwer himself, of these elaborate distinctions; some proved their usefulness, e.g. 'überlagert, (and 'überabzählbar)[2] in 'die Struktur des Kontinuums' (*1928C*, p. 4), 'zählbar' in 'well-ordering'[3], etc. Since these relations concern species of previously constructed elements, the splitting of equipotency does not help in generating ever higher powers.

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[1] In *1918A* the various forms of 'numerability' are given (p. 7) before the general splitting of equipotency (p. 10 ff.) and defined as: (note footnote [2], page 202 above).
'A species is abzählbar if a law exists which associates a distinct numeral complex of ζ with every one of its elements. In particular, it is called zählbar if this law allows a decision for every numeral complex of ζ either with which element of the species it is associated or that it is not associated with any element of the species. A species is called auszählbar if an element β of S is associated with every element α of a certain subspecies A_1 of A in such a way that the subspecies of β is half-identical with S. If, moreover, A_1 is identical with A then S is said to be durchzählbar. Finally, if the species of the β is identical with S then S is called aufzählbar.'

[2] 'Über*ab*zählbar' on pp. 5 and 6 of *1928C* is to be taken in the sense of 'super-denumerable'; (Brouwer speaks here of 'die fertige überabzahlbare Vielfachheit des Kontinuums'), well-distinct from the 'überzählbar'.

[3] E.g. *1918A*, p. 30; *1927A*, p. 455.

Powers higher than the continuum are not considered. The theorem that 'the set F of all real-valued functions has power greater than c', accepted with some reservations in 1907[1], is now rejected and replaced by the theorem, 'The species S of continuous functions of a variable between 0 and 1 is equivalent with the set C. Also: the species S and C, as can easily be seen, are of the same 'Umfang' (range) and 'Ausdehnung' (extension).'[2]

Ordering and Well-Ordering

6.1 Order and virtual order

In 1918A the classical definition of order-relation is still unquestioned and applied to the continuum. Here, an 'ordered species' is defined as 'a species where between every two elements a and b of the species - known as distinct - an asymmetric relation, referred to as an ordering relation, is defined in one sense or abother. (We express this one sense by '$a < b$', or 'a before b', or 'a to the left of b', or $b > a$, or 'b after a', or 'b to the right of a'; the other sense by '$a > b$', or 'a after b', or 'a to the right of b', or '$b < a$', or 'b before a', or 'b to the left of a'), such that the relation $a < c$ follows from the relations $a < b$ and $b < c$.'[3]

This differs from the classical definition in that the elements a and b are required to be 'known as distinct'[4], and the relations $\geq$ and $\leq$ are excluded.

1918A also presumes that the set C can be ordered 'in a natural ordering': 'The ordinal number of the set C, ordered on the basis of the natural ordering in correspondence with the real numbers between 0 and 1 (including 1 but excluding 0) will be denoted by θ_1 and the ordinal $1 + \theta_1$ by θ'.

The ordinal numbers θ and η are compared and a complex characterization of θ is given:

'Every ordered species P which contains a denumerably (abzählbar) infinite subspecies M which is everywhere dense in the strict sense in such a way that:

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[1] 1907, pp. 158 - 159.

[2] 1918A, p. 13.

[3] Ibidem.

[4] Cf. also 1920B (1921A, p. 799: 'which have appeared to be different.') 1918A, p. 20.

1° Between every two elements of P there lie elements of M;

2° the species of elements of M lying before an arbitrary element P are a separable subspecies of M, which is either empty (elementlos), or possesses at least one definable (bestimmbares) element;

3° for every fundamental sequence of relations 'after' or 'not after' joined to elements of M in accordance with the ordering relation, an element of P can be constructed which satisfies these relations,

possesses the ordinal number θ .'[1]

The first change of mind concerning the ordering of the continuum can be found in 1923B ('The role of the PEM in mathematics especially in the theory of functions'). The 'fundamental property', that the points of the continuum form are ordered point species, is quoted as one of the cases where the use of the PEM has led to wrong conclusions in classical mathematics.[2] A real number r is defined (dependent on the occurrence of the sequence 0123456789 in the decimal expansion of π [3]), 'for which none of the conditions $r = 0$, $r > 0$ or $r < 0$ hold'.

'Die Struktur des Kontinuums' (1928C) goes further and maintains that not only such a natural ordering is out of the question, but 'one can also show that any other ordering (eine anderweitige Ordnung) of the intuitionist continuum has no hope (ist hoffnunglos), a result which stands in stark contradiction with the until recently generally held belief, that the continuum could not only be ordered in the most varied ways, but even well-ordered'.[4]

Virtual order

Faced with the impossibility of ordering the continuum, Brouwer, in 1926A, introduces the concept of 'virtual ordering', a weaker order relation which does not order all elements of a species but is only defined for elements of a subspecies:

'A species P is said to be virtually ordered if an asymmetric relation, referred to as the ordering relation (< or >), is defined for

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[1] 1918A, p. 20; I have added the numbers 1°, 2°, 3°, to facilitate reading. WVS.

[2] 1923B, p. 3.

[3] See also above p. 148.

[4] 1928C, p. 8.

'the elements of the subspecies of the species of pairs (a,b) of elements of P ... If then identity of two elements p and q of P is expressed by "p = q", the ordering properties are:

1° The relations r = s, r < s and r > s are mutually exclusive;

2° from r = u, s = v and r < s follows u < v;

3° from the simultaneous absurdity of the relations r > s and r = s follows r < s;

4° from the simultaneous absurdity of the relations r > s and r < s follows r = s;

5° from r < s and s < t follows r < t.'[1]

(In 1926A the 'ordered species P' is then simply defined as: 'one where the above order-relation applies to every pair (a,b) of elements of P'; in particular a discrete ordered species is referred to as completely ordered.[2]

6.2 Virtual ordering of the continuum

Brouwer's definition of real numbers as 'intercalation-elements' (Einschaltungselemente) will be discussed in the next section. For a virtual ordering of the continuum it will be sufficient if we take an intercalation-element e to be the species of coinciding partitions t_r of elements of a denumerable infinite species M (each partition t constitutes two subspecies Left and Right of elements of M).

A virtual ordering (indicated by <, $\leq$ and =) of elements e of the continuum can be established on the basis of the relation $<_o$ defined as follows:

1° e' $<_o$ e" (in 1951: e' "measurably smaller than "e")[3]; if a partition t' of e' can be specified and a partition t" of e" and also elements g_σ and g_τ of M, which respectively belong to the Right subspecies of t' and the Left subspecies of t".

2° e' $\leq_o$ e": if e' $_o>$ e" is impossible.

'The relation e' < e" : if e' $\leq_o$ e" and e $\neq$ e" satisfies the five stipulations of the virtual ordering'[4], - referred to as 'the five

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1 1926A, p. 453.

2 1926A, p. 455.

3 1951, p. 357.

4 1926A, p. 467; Kleene's 'On order in the continuum' is an investigation into the properties between real numbers resulting from these relations. (Kleene and Vesley, op. cit. Chapter IV).

axioms of virtual ordering in 1928C[1] - and establishes a virtual ordering of the continuum, or as Brouwer puts it: 'We call the species of intercalation-elements of M, virtually ordered in this way, the continuum over M, and denote it by K (M).[2]

6.3 Well-ordering

Zermelo's proof[3] that every set can be well-ordered had already been rejected by Brouwer in 1907: 'We know that, apart from denumerable sets where this theorem certainly applies, there is only the continuum for which the theorem certainly does not apply; first, because we must consider the greater part of elements of the continuum as unknown and we cannot, therefore, order them individually, and secondly, because all well-ordered sets are denumerable. Therefore, this question remains illusory.'[4] This conviction was again expressed in 1913B: 'Non-denumerable well-ordered sets cannot exist'[5], and in 1917A: 'A point set of power greater than the denumerable, (Du. aftelbaar, i.e. Germ. abzählbar), can certainly not be well-ordered.'[6]

Even if in 1918A, 1927A and 1927B, and further in 1953, Brouwer develops a well-ordering theory, this has little in common with the classical theory of well-ordering, save the name.

The classical definition of a well-ordered set (a totally ordered set in which every non-empty subset has a first element) is rejected[7] and is replaced by a constructive definition of a well-ordered species on the basis of Cantor's first and second principle of construction.

1918A defines a well-ordered species in the following way:

'1° A species without an element or with one single element is a well-ordered species and is referred to as the 'basic' species[8] (Urspezies);

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[1] 1928C, p. 9.

[2] 1926A, p. 467.

[3] MA, vol. 59, pp. 514 - 516, (following Cantor's problem in Grundlagen, pp6); also rejected the following year by Borel in MA vol. 60 (1905), pp. 194 - 195.

[4] 1907, p. 153.

[5] 1913B, p. 81.

[6] 1917A, pp. 443 - 444.

[7] 1920B; 1921A, p. 800.

[8] 'An arbitrary element of a well-ordered species is either an element of the first kind, called a full element, or an element of the second kind, called a null-element.' (1927A, p. 451.)

'2° from known well-ordered species further well-ordered species can be derived by two generating operations which consist in addition of two known well-ordered species or a fundamental sequence of known well-ordered species.'[1]

Ordinal addition had been discussed earlier (respectively 1918A, p. 15, and 1926A, pp. 456 - 457; cf. also 1953, pp. 9 - 10).

The ordinal sum M of species N can be defined as the sum-species $M = \cup\ (N)$, virtually ordered in the following way:

i) Let R be a virtually ordered species of virtually ordered species N. (Order relations: $<_\bullet$, $\leq_\bullet$, $=_\bullet$)

A relation $<_\circ$ is defined on the elements e of M, such that

1° for every e' of N' and e'' of N'', $e' = e''$ implies $N' = N''$;

2° if $N' = N''$ they are virtually ordered in the same way;

3° $e' {}_\circ\!> e''$ if either $N' {}_\bullet\!> N''$, or $N' = N''$ and $e' {}_\bullet\!> e''$;

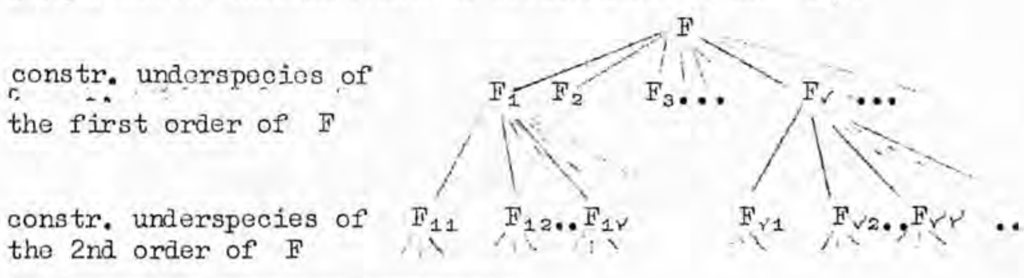
4° $e' {}_\circ\!\geq e''$ if $e' <_\circ e''$ is impossible.

ii) The relation $>$ defined by: $e' > e''$ iff both $e' {}_\circ\!> e''$ and $e' \neq e''$, virtually orders M.

In an attempt to establish some intuitionist form of transfinite induction, Brouwer analyses a constructed well-ordered species working backwards:

'Every well-ordered species which played a role in the construction of a well-ordered species F is called a constructive under-species[2] of F. The constructive under-species which played a role in the last generating operation of F are called constructive underspecies of F of the first order and are distinguished from each other by an index and are therefore denoted by $F_1, F_2, F_3 \ldots F_m$ or $F_1, F_2, F_3, \ldots$'[3]

Each of the F has its constructive underspecies, which are then constructive underspecies of F of the second order. This, and the allocation of indices could be illustrated in this way:



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[1] 1918A, p. 22.

[2] Although Brouwer himself in 1953 uses the word 'subspecies', we have preferred the literal translation 'underspecies' to distinguish it from subspecies (Teilspezies).

[3] 1918A, p. 22; 1927A, p. 451.

In this understructure of F every 'basic species' that has been used must appear as a constructive underspecies of finite order of F_ν and accordingly their indices are finite numeral sequences. Both the species of 'rows of indices (Indizesreihen) of these elements and the constructive underspecies of F form a separable subspecies of the species of finite numeral sequences'.[1]

From these definitions and this analysis, Brouwer derives two fundamental theorems concerning well-ordered species:

1° 'A well-ordered species F is either finite or denumerably (abzählbar) infinite, and the species of its full elements is zählbar.' (1927A)[2]
(1918A:'Every well-ordered species is zählbar'.[3])

2° 'A law which in a well -ordered species F specifies a constructive underspecies F' and with every $F^{(\nu)}$ already specified associates either the arresting of the process or a constructive underspecies $F^{(\nu+1)}$ lying in F before $F^{(\nu)}$, certainly specifies a natural number n and a corresponding underspecies $F^{(n)}$ with which it associates the arresting of the process.'[4]

This last theorem - that every descending chain is finite - replaces the fundamental classical assumption of well-ordered sets, (i.e. every subset of a well-ordered set has a first element), and forms the basis for an intuitionist theory of transfinite induction. The implications of Brouwer's well-ordering for the theory of functions and for proof theory were already apparent in 1927B and have been the subject of various studies:

S.C. Kleene, 'On notation for ordinals', J.f.Symb. Logic, vol. 3 (1938), pp. 150 - 155;
also Introduction to Metamathematics, Amsterdam-Groningen-New York-Toronto, 1952;

C. Spector, 'Recursive well-orderings', J.f.Symb.Logic, vol. 20 (1955), pp. 151 - 163;

A. Heyting, 'Infinitistic methods from a finitist point of view', Proc. the Symposium on the Foundations of Mathematics, Warsaw 1959, pp. 185 - 192;

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[1] 1918A, p. 23; 1927A, p. 452; cf. also 1953, p. 10.

[2] 1927A, p. 455.

[3] 1918A, p. 30.

[4] Literally the same in 1918A, p. 30 and 1927A, p. 455.

Kleene and Vesley, The Foundations of Intuitionistic Mathematics especially in relation to Recursive Functions, Amsterdam 1965;

W.A. Howard and G. Kreisel, 'Transfinite induction and Bar-induction of types zero and one and the role of continuity in intuitionistic analysis', J.f.Symb. Logic, vol. 31 (1966), pp. 325 - 328;

A.S. Troelstra, 'Theory of choice sequences', Logic, Methodology and Philosophy of Science III, Amsterdam 1968;

also Principles of Intuitionism, Lecture notes in Mathematics vol. 95, Berlin-Heidelberg-New York 1969;

(cf. also C. Parsons, 'Introduction to reprint of Brouwer's 1927B (On the domains of definition of functions) in J. van Heijenoort, From Frege to Gödel, H.U.P. 1967).

7.1 The Brouwer continuum and the real numbers

In 'Begründung der Mengenlehre' (1918A and 1919A) references to real numbers are few. In 1919A ('Theory of point sets'), the possibility is pointed out of extending the definition of a point of the continuum to the real line and n-dimensional space[1], but the plane is singled out for treatment throughout. After intervals (squares λ) have been defined, the point of the plane is defined as 'an unlimitedly proceeding sequences of squares λ of which each one is contained in the interior (im Innengebiete) of its predecessor'.[2]

Weyl, in his 'Über die neue Grundlagenkrise der Mathematik' (1921), rightly interpreted Brouwer's characterization of a real number as: 'an infinite sequence of dual intervals i , i' , i" , ... such that every interval of this sequence contains its successor;'[3] and elsewhere: 'Brouwer sees the essential character of the continuum not in the relation between element and set, but in that between part and whole.'[4] Weyl also points out that 'it is an inessential simplification if we instead of sequences of nested intervals consider unlimited sequences of natural numbers'.[5]

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1 1919A, p. 3.

2 Ibidem.

3 Op. cit. p. 49.

4 Philosophy of Mathematics and Natural Science, Princeton 1949, p. 52.

5 Ibidem [3].

However, Weyl's subsequent interpretation of Brouwer's real number seems to miss an important point; in his explanation of the difference between 'a law-like' sequence and a sequence generated by free choice, Weyl characterizes the real number as defined by a certain law and the continuum as 'becoming' and generated by free choice: 'Whereas the law ϕ which defines a sequence ad infinitum represents a single real number, the choice sequence, which is not restricted in the freedom of its development, represents the continuum'[1], and: 'Brouwer has noted something simple but very deep: here a "continuum" is generated in which the real numbers find their place, which, however, does not resolve itself into a set of ready-made (fertig seiender) real numbers, but is far more a medium of free becoming.'[2]

Brouwer's article, 'Does every real number have a decimal expansion?' (1921B), could almost be interpreted as a correction of Weyl's interpretation.[3] Here Brouwer stresses the difference between points of the continuum as fundamental sequences of nested intervals and as unlimitedly proceeding sequences of nested intervals: 'The definition of a point of the continuum suffers a considerable limitation if we read fundamental sequence instead of unlimitedly proceeding sequence.'[4] The examples given in 1918A[5] might indeed support Weyl's interpretation of Brouwer's real number in the restricted sense, although in a footnote in 1919A, Brouwer had already warned that 'the phrase point of the straight line or point of the n-dimensional Cartesian space has already been used in Part I, page 10, but in a different sense from the definition here.'[6]

Borel distinguished[7] between the points of the continuum (geometric continuum) and the set of definable real numbers (practical continuum).

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[1] 'Über die neue Grundlagenkrise der Mathematik', p. 50.

[2] Ibidem.

[3] Although Brouwer's 1921B was published in the same year as Weyl's article, it was read as a paper by Brouwer at the Naturforscherversammlung in Bad-Neuheim on 22nd September, 1920.

[4] 1921B, p. 804.

[5] 1918A, p. 9, quoted above p.190.

[6] 1919A, p. 4.

[7] E. Borel, Leçons sur la théorie des Functions, Paris 1898; ed. 1928, pp. 165 - 166; also p. 210; see also above p.174.

Brouwer makes a similar distinction between the full continuum and the reduced continuum, but he claimed for those elements of the full continuum which do not belong to the reduced continuum the full status of point of the continuum;[1] he remains reluctant to use the word real number.

Both the cantorian characterization of real number[2] and real numbers as Dedekind cuts are rejected on logical grounds (because of their reliance on the existence of an upper bound[3], but mainly because 'they only form a part of the continuum; to this part of the continuum we will refer as the reduced continuum which overlays (überlagert[4]) the system of rational numbers'. [5]

Brouwer was well aware that between characterization in terms of nested intervals and cuts there is no essential difference; 1921B defines points of the continuum as sequences of nested intervals, whereas the intercalation elements of 1927A could well be described as Brouwer cuts. Both are generated by infinite sequences and an isomorphic correspondence can be established between them.[6] In fact, 1928C simply refers to convergent sequences of rational numbers or dual fractions. Brouwer's extension of the real numbers beyond those of the reduced continuum is mainly the extension of sequences beyond the classical notion of sequences by including those less well-defined sequences of his famous counterexamples refuting the PEM, and choice sequences.

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1 In the whole of 1919A the words real number are not used; neither are they used in Die Struktur des Kontinuums (1928C). 1921B uses reelle Zahlen in the title and subtitles, in the text only in the combination algebraic real numbers (p. 808), and in the final sentence: 'As to the examples of real numbers without a decimal expansion, there is always the possibility, with the further development of mathematics, that they will be eliminated, but then they can always be replaced by others which have retained their validity.' (p. 812)

2 Given by Brouwer as, 'species of species of together-belonging convergent fundamental sequences of rational numbers', (1928C, pp. 2 and 4), a treatment which he describes as 'old-intuitionistic' and ascribes to Poincaré and Borel.

3 1928C, p. 3.

4 See above p. 203.

5 1928C, p. 4.

6 Cf. F. Waismann, Introduction to Mathematical Thinking, London 1951, p. 205.

Before discussing the reduced continuum and the full continuum, we will give a brief account of two aspects of Brouwer's treatment of real numbers: the relation of coincidence and inequalities, and real numbers as Brouwer cuts.

7.2 Coincidence

For the various interpretations of real numbers (sequences of nested intervals, Brouwer cuts or simple sequences) Brouwer defines a relation of coincidence (Germ. Zusammenfallen). Post war contributions[1] all describe it as, 'self explanatory'. It is the equivalence-relation to which 1952B refers in the general definition of species: 'Properties supposable for mathematical entities previously acquired, and satisfying the condition that, if they hold for a certain mathematical entity, they also hold for all mathematical entities which have been defined to be equal to it, relations of equality having to be symmetric, reflexive and transitive.'[2]

In terms of nested intervals: 'Two sequences, ${}_1F_1$, ${}_1F_2$, ${}_1F_3$, ... and ${}_2F_1$, ${}_2F_2$, ${}_2F_3$, ... are coincident if every ${}_1F_\nu$ has an element in common with every ${}_2F_\nu$.'[3]

In the case of the Brouwer cut, where various 'arbitrary enumerations' of the set M of rationals are allowed, 'intercalation partitions t coincide if an element of the Left subspecies of one partition never lies to the right of an element of the Right subspecies of another'.[4] The coincidence relation in the treatment of 1928C (convergent sequences of rational numbers) is described as a relation of 'belonging together' (Germ. zusammengehörig).[5] In accordance with Brouwer's requirements for coincidence relations and a formulation of the relation of apartness,[6]

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[1] 1952B, p. 143; 1953, p. 4; 1954A, p. 104.

[2] 1952B, p. 142.

[3] 1921B, p. 805, also 1919A, p. 5 and 1923C, p. 878: 'Two points P_1 and P_2 coincide if every square of P_1 contains a square of P_2 and every square of P_2 contains a square of P_1.' (also 1925B, p. 253).

[4] 1926A, p. 467.

[5] 1928C, pp. 3, 4, 5. Together-belonging (zusammengehörig) was defined in 1918A, p. 16, for two monotonic increasing and for two monotonic decreasing fundamental sequences: 'Two monotonic sequences with resp. elements $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ are said to belong together if for every a_μ a $b_\nu > a_\mu$ can be indicated and for every b_μ an $a_\nu > b_\mu$ can be indicated. Similarly for decreasing fundamental sequences.'

[6] From 1948C, p. 1246.

this could be expressed as follows: If the real numbers a and b are defined respectively by the convergent sequences of rational numbers $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$, a coincides with b if for every natural number n a natural number m can be calculated such that $|b_\nu - a_\nu| < 2^{-n}$ for all $\nu \geq m$.

The species of coincident sequences of nested intervals, or coincident intercalation partitions, or together-belonging convergent sequences of rational numbers are then the equivalence class constituting the real number, a species of second order. Brouwer prefers to call them 'number cores' (1953), 'limiting point cores' (1952B), or simply 'cores' (1954A).

Apartness

Within the concept of inequality, Brouwer makes some unusual distinctions.

The notion of apartness was already introduced in 1919A, page 3 under the name locally distinct (Germ. örtlich verschieden), here applying to two points: 'If a square q_1 of the point P_1 and a square q_2 of P_2 lie outside one another (ausserhalb voneinander liegen), P_1 and P_2 are said to be locally distinct.'[1] 1923C uses both expressions, locally distinct and lying apart, (Du. van elkaar verwijderd; Germ. (1925B) voneinander entfernt), and adds, 'if squares q_1 and q_2 can be indicated'.[2]

1923C also defines the relation of apartness for two species of points: 'Two species of points, Q and R are said to lie apart, if one species contains a point which lies apart from the other species,'[3] i.e. 'if it lies apart from every point of the other species'.[4]

1948C and 1953 suggest the following definition of apartness for real numbers :[5]
Two real numbers a and b defined respectively by the convergent sequences of rational numbers $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$

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1 1919A, p. 3.

2 1923C, p. 878.

3 Ibidem, p. 880.

4 Ibidem, p. 879.

5 1948C, p. 1246 footnote.

are said to lie apart if two natural numbers m and n can be calculated such that $|b_\nu - a_\nu| > 2^{-n}$ for all $\nu \geq m$.

Deviation

A weaker relation than apartness, deviation can simply be described as the absurdity of the identity relation.[1] In 1948C a notation is suggested: 'If a = b is absurd, we write $a \neq b$.'[2]

Deviation (Du. afwijking; Germ. Abweichung) was introduced in 1923C as, 'the absurdity of coincidence of two points'[3]. 'Two species of points are said to deviate from one another if one species contains a point that deviates from the other species.'[4] 1953 uses both the word deviate and distinct: 'If the species M possesses an element which cannot possibly belong to the species N, we shall say that M deviates from N' (page 6), and: 'Two species are said to be equal or identical if for each element of either, an element of the other, equal to it, can be indicated. They are called different if their equality is absurd.' (page 6)[5]

In 1923C Brouwer proceeds to apply his calculus of absurdity to these 'three fundamental relations' of coincidence, apartness and deviation between species of points and generates altogether nine

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1 i.e. The strict identity relation as given for species on p.196 Cf. the definition of distinctness ref.[1] on following page.

2 1948C, p. 1246. This relation is to be distinguished from the 'sharp difference' defined in 1928C; R.E. Vesley has shown (Kleene and Vesley, Foundations of Intuitionistic Mathematics, pp. 133 ff.) that this 'sharp difference' is equivalent with the apartness relation; he uses, however, the notation $\neq$ for 'sharp difference'.

3 1923C, p. 478; 1925B, p. 252; the notion of distinctness (see above p.) was in addition of 1925D, p. 246, not found in 1918A.

4 1923C, p. 880.

5 See above footnote [3].

relations between species of points[1]. In his further writings, however, only the three fundamental relations, and of course the relation of identity, play a major role.[2]

7.3 The Brouwer Cut or Intercalation Element

The notion of intercalation element (Einschaltungselement), to which we have referred as the Brouwer cut, is developed in 1926A.[3] Brouwer defines the intercalation element in general for a species M which is ordered, denumerably (abzählbar) infinite, and everywhere dense in the strict sense.

We shall define intercalation element in the species Q of rational numbers.

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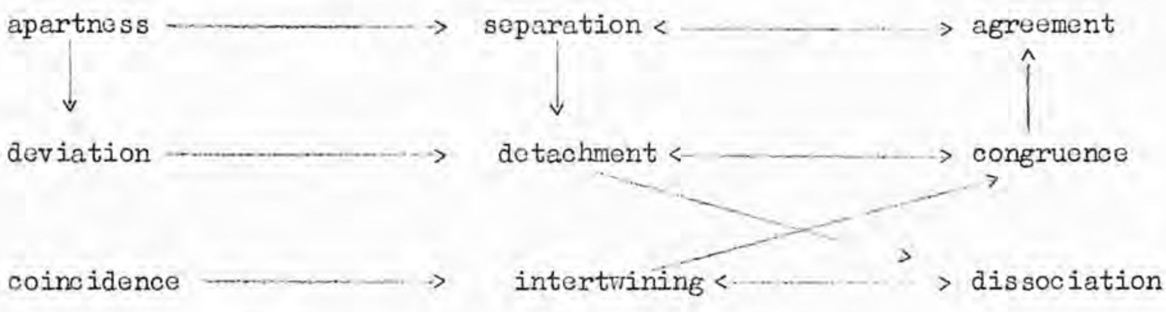
[1] As has been stated elsewhere, the so-called calculus of absurdity only occupies half a page of 1923C; the main part of 1923C is concerned with the logical generation of relations between two points (3 relations), points and species (6 relations), and between species of points (9 relations.)

'Using the symbols: a <—> b (i.e. a and b are contradictory), a -> b (b follows from a), symbols which have the property that from a -> b and b <-> c follows a <-> c, and from a -> b and b -> c follows a -> c', (p. 879). Brouwer produces the following figure by application of absurdity, and absurdity of absurdity:



dissociation (Du. loswikkeling; Germ. Loswindung): absurdity of the absurdity of coincidence;

intertwining (Du. ineenvlechting; Germ. Verflechtung): absurdity of coincidence;

congruence absurdity of deviation;

detachment (Du. loshechting; Germ. loslösung): absurdity of the absurdity of deviation;

agreement (Du. overeenstemming; Germ. Übereinstimmung): absurdity of apartness;

separation absurdity of the absurdity of apartness

[2] But congruent in this sense is used, e.g. 1953, p. 6.

[3] 1926A, pp. 467 - 469.

First, the elements of Q are arbitrarily denumerated[1], i.e. the elements of Q are placed arbitrarily in a sequence: $r_1, r_2, r_3, \ldots$ Let $s_\nu = \cup(r_1, r_2, r_3, \ldots r_\nu)$.

An intercalation partition t in Q is established by generating in Q a Left and a Right subspecies in an unlimited sequences of choices by assigning the predicates Left and Right to each r in Q, such that:

1° The Left and the Right subspecies are determined successively in $s_1, s_2, s_3, \ldots$;

2° An element of the Left subspecies always precedes any element of the Right subspecies; (preceding: in the natural order of Q)

3° At every stage ν only one element r_{α_ν} may be left indeterminate; at any subsequent stage $\nu + 1$ this element can remain indeterminate ($r_{\alpha_{\nu+1}} \equiv r_{\alpha_\nu}$); if the last element $r_{\nu+1}$ in $s_{\nu+1}$ is left indeterminate and is not identical with the previous exception r_{α_ν}, then to r_{α_ν} is assigned the predicate Left or Right according to its natural order relation to $r_{\nu+1}$.[2]

The arbitrariness in the denumeration of the sequence $r_1. r_2, r_3, \ldots$ allows various intercalation partitions, t_1 and t_2 or more: ('according to different distinct denumerations'), and a coincidence relation between such different intercalation partitions: 'Two intercalation partitions t_1 and t_2 are said to coincide if no element of the Left subspecies of t_1 lies right of any element of the Right subspecies of t_2, and vice versa.'

The 'coincidence class' c of intercalation partitions t of Q coinciding with a certain intercalation partition t_1 is called an intercalation element of Q ; each t coinciding with t_1, as well as is called a partition of the intercalation element c.

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[1] 'Arbitrarily denumerated by the fundamental sequence $g_1, g_2, g_3, \ldots$'

[2] 'For every s_ν only one single element r_{α_ν} of s_ν may remain excluded; for every ν the element $r_{\alpha_{\nu+1}}$ if it exists, is either identical with r_{α_ν} or with $r_{\nu+1}$.' (p. 467)
Brouwer only states that at any stage in this process one may not leave more than one element indeterminate; if one leaves the last element in $s_{\nu+1}$ indeterminate, then all other elements in $s_{\nu+1}$ have been assigned either the predicate Left or Right.
It is important to remember that Brouwer is not describing an already existing cut, but gives a procedure by which one progressively generates an intercalation partition.

This characterization of points of the continuum in 1926 was due to a growing realization of the impossibility of ordering the continuum on the basis of Brouwer's earlier[1] interpretation of points as sequences of nested intervals. Apart from the impossibility of accommodating those real numbers whose definition depends on the occurrence of the sequence 0123456789 in the decimal expansion of π, the definition of Ergänzungselement of 1921B allows coincidence of individually different Ergänzungelemente[2]. To overcome this last difficulty, Heyting's dissertation, written in 1925 under the supervision of Brouwer, introduced the concept of pseudo ordering, a natural ordering of Ergänzungselemente which are locally distinct (lie apart)[3]. Virtual ordering, introduced by Brouwer in 1926, includes the requirement that '$x = y$, $x < y$ and $x > y$ are mutually exclusive', and can be applied to all intercalation elements. (see above page 207)

(Characterization of points of the continuum as nested intervals is still used, especially in the treatment of functions; e.g. in 1927B: 'By a point of the continuum we understand an unlimited sequence of intervals[4], (the generating intervals of the point) such that each of them is contained, in the strict sense, in the preceding one and their size, therefore, converges positively to zero.' [5] Brouwer then simply

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1 1919A and 1921B.

2 Cf. A. Heyting, Intuitionistische Axiomatiek der Projectieve Meetkunde, Dissertation. Amsterdam 1925, p. 8.

3 Ibidem; cf. also A. Heyting, Intuitionism. An Introduction, Amsterdam 1956, p. 156, edition 1966, (7 . 3).
Equivalent to Brouwer's $<_o$ (measurably smaller) of 1926 (see above p. 207), the pseudo ordering ($<$ in Heyting's notation) is related to the virtual ordering ($\lessdot$ in Heyting's notation) by $x \lessdot y$ if $x \not> y$ and $x \neq y$.

4 1925D, p. 253, describes the λ interval as, 'an interval in the number continuum with the endpoints $a. 2^{-n}$ and $(a + 2). 2^{-n}$, where a is an arbitrary integer (ganze rationale Zahl) and n an arbitrary natural number'.

5 In 1923B, pp. 5 - 6 the distinction was made between positive and negative convergence. A non-oscillating infinite sum sequence s_n is '1° negatively convergent, if there exists a real number s with the property that for every $\epsilon > 0$, the impossibility has been established of the existence of an infinite sequence of indefinitely increasing positive whole numbers $n_1, n_2, n_3, \ldots$ such that $|s - s_{n_\nu}| > \epsilon$ for all ν;
2° positively convergent, if there exists a real number s with the property that for every $\epsilon > 0$ there exists a whole positive number n_ϵ such that $|s - s_n| < \epsilon$ for all $n > n_\epsilon$.'

states further, 'the point cores of the linear continuum can also be virtually ordered in a way similar to the virtual ordering of the intercalation elements', and establishes a 'similarity correspondence between this continuum and K (M)'.[1]

By allowing the exceptional case (3°), the Brouwer cut distinguishes itself from the Dedekind cut. It includes those special real numbers to which Brouwer refers as 'Pendalzahl' (pendulum number).[2] On the evidence of this particular kind of real number, Brouwer claims that 'the definition by means of Dedekind cuts and as species of together-belonging convergent fundamental sequences on the intuitionistic basis are not equivalent'.[3] Dedekind cuts are therefore a particular case of the Brouwer cut; they are said (1928C) 'to have a degree of precision (Präsisionslage) of order one'[4], or in 1921B: 'The Ergänzungselemente of first order of H correspond to the Dedekind cuts of H.'[5] Moreover, the Dedekind cuts are only elements of the Reduced Continuum with this first degree of precision. This had been stressed before the introduction of the term reduced continuum; 1921A ends: 'It must be noted that the linear continuum - even with the above-mentioned restriction of the notion of point of the continuum (i.e. reading 'fundamental

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Brouwer convergence and divergence was the subject of a dissertation by J.G. Dijkma, Convergentie en Divergentie in de Intuitionistische Wiskunde, 's Gravenhage 1952.

[1] 1927B, p. 60 : the species of intercalation elements over M (see page 208).

[2] See p. 148.

[3] 1928C, p. 3.

[4] 1921B, pp. 804 - 805; 1926A, pp. 467 - 469 and 1928C, p. 4, introduce, with slight variations, a hierarchy of orders of precision. In the 1928C version:

'An element 1 has, in relation to the fundamental sequence $r_1, r_2, r_3, \ldots$ a degree of precision of
1st order, if for any n it can be decided whether $1 \geq r_n$ or $1 < r_n$,
2nd order, 1st kind, if for any n it can be decided whether $1 \geq r_n$ or $1 < r_n$,
2nd order, 2nd kind, (3rd order 1921B) if for any n it can be decided whether $1 \leq r_n$ or $1 > r_n$,
3rd order (not in 1921B) if for any n one of the relations $1 \circ\!> r_n$, $1 <\!\circ\, r_n$, or $1 = r_n$ can be established.

[5] 1921B, p. 806.

sequences' instead of 'unlimitedly proceeding sequences') - contains considerably more than the classical notion of point of the continuum on the basis of a cut.'[1]

7.4 Reduced continuum and full continuum

The distinction between the reduced continuum and the full continuum is based on the distinction between fundamental sequences and free choice sequences. The fundamental sequence was defined in 1918A and 1926A as, 'an ordered species of ordinal number ω '.[2]

The reduced continuum is defined as, 'the species of species of together-belonging convergent fundamental sequences of rational numbers'.[3] It includes the classical real numbers as well as those allowed within Brouwer's definition of the exceptional case.[4]

1928C reiterates the claim of 1921B 'that the elements of the reduced continuum form only a part of the continuum'.[5] This is emphasized again in the definition of the full continuum, 'in the intuitionistic theory, in order to obtain (erhalten) the full unit continuum which overlays (überlagert) the rational numbers within the unit continuum (or the decimal fractions), it is necessary to introduce besides (neben) the finished elements of the reduced continuum also unfinished elements by admitting - besides the convergent fundamental sequences of rational numbers - also convergent sequences of rational numbers generated by free choice.' [6]

It is quite clear from the use of the word elements and besides that Brouwer here claims for these choice sequences, like the fundamental sequences, to generate a point of the continuum.

In Weyl's and Fraenkel's interpretation, the choice sequence is a part of the continuum: 'The free choice sequence does not endeavour to yield a single object (say a real number), but to produce an aggregate which is continuous.' [7]

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1 1921A, p. 802; 1920B, p. 208.

2 1918A, p. 14; 1926A, p. 455.

3 1928C, p. 4.

4 See above p.

5 1928C, p. 4.

6 1928C, p. 5.

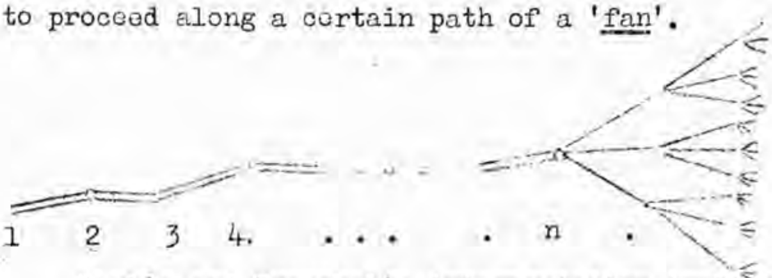
7 A. Fraenkel and Y. Bar-Hillel, Foundations of Set Theory, Amsterdam 1958, p. 249; for Weyl see above p. 183 ff.

At any stage of its construction, at the n-th step, the choice step is:

1° A finite sequence of n mathematical objects ('nodes') which have each been selected at each successive step, (this finite sequence is referred to as a 'rod' in 1953);

2° The possibility of choosing one single entity at the (n + 1)th, (n + 2)th, (n + 3)th, ... step from a Brouwer set. (In the case of a completely free choice this would be a universal set, or a finite set).

The situation at the n-th step of a choice sequence in a 3-finite set can then be represented by a 'rod of order n', and the possibility to proceed along a certain path of a 'fan'.



Weyl's and Fraenkel's interpretation implies that this fan at the n-th node forms part of the choice sequence. However, apart from the difference in existence between the completed rod and fan or Brouwer set, the choice at any stage means the selection of one object and this inductively ad infinitum. 1928C leaves aside the question of the nature of the individual point of the continuum and that of the individual choice sequence. The Brouwer set (here the '3-finite Menge') is introduced at this stage to 'express the super-denumerability of this full continuum'[1] and so to find a basis for a measure different from zero.[2] That Brouwer meant the individual choice sequence to generate a single point is evident from 1953, where a distinction is made between an arrow and a spread. and between the general choice sequence (arrow) and the 'predeterminate' sequence (sharp arrow):

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[1] 1928C, p. 5.

[2] 'Only this degree of multiplicity, which is an essential characteristic of the pure n-finite set and which has extendibility (Ausdehnungscharakter) by virtue of its admission of unfinished elements, allows the introduction of a content different from zero for a subspecies of the continuum.' (1928C, p. 5).
For Brouwer theory of measure, see further 1952B, and B. van Rootselaar, Generalization of the Brouwer Integral, Amsterdam 1954, (dissertation), and C.G. Gibson, The Radon Integral in Intuitionism, Groningen 1967, (dissertation).

' An infinite (not necessarily predeterminate) sequence of nodes consisting of[1] a node p_1 of order 1, an immediate descendant p_2 of p_1, an immediate descendant p_3 of p_2, and so on ad infinitum, will be called an arrow.

Naturally an arrow may grow in complete freedom, i.e. with the passage from p_ν to $p_{\nu+1}$, the choice of a new constituent $p_{\nu+1}$ to be joined to those of p_ν may be completely free for each ν, on the other hand, this freedom in the generation of the arrow may at any stage be completely abolished... From this moment the arrow concerned will be called a sharp arrow. ... Furthermore the freedom in the generation of the arrow, without being completely abolished, may at any p_ν undergo some restriction...'[2]

Although a point of the continuum is defined as a species of coincident convergent sequences, one individual choice sequence is sufficient to generate a point.

7.5 Properties of the continuum

Part II of Die Struktur des Kontinuums (1928C) succesively refutes seven classical properties of the continuum (discreteness[3], order, density in itself, separability in itself, connectedness, everywhere density, compactness), in all cases using the counterexample of the 'Pendelzahl' or 'Lösungszahl' of a fleeing property.[4] For each of these properties, except discreteness, an intuitionistic analogue is established (Brouwer calls this 'a logical transformation of the definition'), which is applicable to the intuitionist continuum or, in some cases, only to the reduced continuum.

These properties have been studied by R.E. Vesley in his dissertation, (S.C. Kleene and R.E. Vesley, The Foundations of Intuitionistic Mathematics, Amsterdam 1965, Chapter III; cf. also A.S. Troelstra, Intuitionistic General Topology, Groningen 1966, (dissertation).

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[1] The text, written in English by Brouwer, has or in several places instead of of a typical Dutch mistake, (Dutch of = English or, Dutch van = English of).

[2] 1953, p. 7.

[3] See above p. 196.

[4] See above p. 150.

CHAPTER V

CONCLUSIONS

1.1 Brouwer's philosphy of life and his mystical tendencies are relevant to his philosophy of mathematics and intuitionist mathematics itself. Not only do they provide a motivation for the course he took, but often clarify his views.

1.2 Leven, Kunst en Mystiek proves that Brouwer's mystic and solipsistic ideas were not an aberration of old age. These views were held by Brouwer at the time he was writing his dissertation, and changed very little during his lifetime.

1.3 Leven, Kunst en Mystiek is a source of information on Brouwer's general views on life, science, language, logic and intuition. In spite of the triviality of some of its content, it should be included in Brouwer's complete works and be preserved for posterity. (Only very few copies of the original Dutch version are still available).

2.1 There are undoubtedly in Brouwer's general philosophy elements of Schopenhauer's pessimism, Bergson's intuitionism, and Eastern philosophies. Brouwer's philosophy is unique in assimilating certain parts of various forms of idealism and in pressing for a predominant role of mathematics in the general process of human thinking.

2.2 Brouwer's philosophy has been the source and inspiration of his particular view of mathematics and the role of language and logic in mathematics. In this way, Brouwer has demonstrated the importance of philosophy for mathematics, how a change in philosophy can have far-reaching results and even affect the content of mathematics.

2.3 It has been maintained[1] that intuitionist mathematics can be practised without subscribing to Brouwer's philosophy.

Many of Brouwer's mystical and philosophical views are irrelevant in intuitionist practice of mathematics, and others may need drastic revision, (such as his low valuation of communication and 'communal mental activity', see p. 381); however, a clear philosophy of mathematics, of its relation to other areas of human activity, and of its fundamental

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[1] Cf. Heyting, 'Intuitionism in Mathematics', Philosophy in the Mid-century, Florence 1958, p. 102; E. Beth, 'Remarks on intuitionistic Logic', Constructivity in Mathematics ; Amsterdam 1959; E. Bishop, Foundations of Constructive Analysis, New York 1967, p. 6.

concepts and principles seems essential for any school of mathematics. Mathematical methods and mathematics itself are not sufficient in deciding these fundamental issues.

2.4 To set aside intuitionist philosophy, and start from a dogmatic acceptance of certain principles and restrictions as the basis of intuitionist mathematics,[1] is contradictory to the spirit of Brouwer, who denied completeness to mathematical systems for which he claimed a freedom always to develop further.

The fundamental reason for Brouwer's opposition to formalization was its claim to fix notions and principles in advance.

Moreover, new developments during Brouwer's lifetime, notably that of choice sequences, were not always an application of previously accepted intuitionistic principles but follow from his particular philosophy of mathematics.

2.5 Brouwer's approach to the foundation of mathematics was philosophical, not 'an approach from within', as claimed by Beth (i.e. a mathematical consideration of mathematics).

3.1 Brouwer's primordial intuition of time 'has little in common with Kant's a priori form of intuition of time, in spite of Brouwer's own claim. There is some resemblance to Bergson's intuition; however, as awareness of time, leading to the notions of discrete and continuous, the primordial intuition of time is an original notion of Brouwer.

3.2 The primordial intuition of time provides an adequate basis for the conception of number and continuity. Brouwer has not proved its a priority, i.e. the necessity of the primordial intuition of time as the only basis for the number concept and for mathematics.

3.3 In the primordial intuition of time, the continuous appears as an even more fundamental concept than the discrete; Brouwer claims that they are 'co-equal'.

3.4 The notion of 'intuitive' is, in Brouwer's philosophy, not to be identified with 'self-evident', which implies elimination of effort on the part of the subject, the evidence being supplied by the object known.

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[1] Cf. J. Myhill, 'By "intuitionistic mathematics" we mean roughly the practice of Brouwer and Heyting'. ('Formal systems of intuitionistic analysis', Logic, Methodology and Philosophy of Science, III, Amsterdam 1968, p. 162.

3.5 Intuition in the general sense is to be identified with human consciousness acting independently of experience (except for the fact of having sensations.)

3.6 The role of intuition in mathematics is not confined to the initial stages of the conception of natural numbers; it determines mathematics at every stage and allows the possibility of 'new insights'. Brouwer's intuitionism is more than a Kroneckerian constructive school of mathematics; not only does it seek to construct the natural numbers and treat the continuous as equally fundamental as the discrete, it accepts intuition as its final arbiter.

3.7 In identifying mathematics with intuitive mental activity, Brouwer has succeeded in providing a comprehensive definition of mathematics which delimits the domain of mathematics and determines its methods.

3.8 Solipsistic unwillingness to accept 'communal mental activity' and recognition of the limitations of each individual human mind led to Brouwer's introduction of the 'creative subject' in 1933. While remaining essentially human (there will still be unsolved mathematical problems, the infinite still remains unfinished, etc.), the creative subject is an 'idealized mathematician' in the sense that this 'hypothetical human being' is empowered with 'unlimited memory'. Removing further limitations inherent in the individual human mind, such as restrictions of powers of concentration, intellect, age, etc., one would be driven dangerously into the realms of phantasy, or at least some Platonic real world.

Brouwer's views on the nature of language may well have been inspired by a low valuation of communication. It seems, however, possible to accept his analysis of language without necessarily subscribing to his extreme solipsistic views. Brouwer's notion of mathematics and his reservations on language do not exclude the possibility of mathematics as communal mental activity. The hypothesis of intersubjectivity of the primordial intuition, necessary for mathematics as communal mental activity, is more acceptable than the hypotheses required by the theory of 'the idealized mathematician'.

With Heyting, intuitionism has recognized the necessity of communication between mathematicians. Accepting the possibility of mathematics as communal mental activity obviates the need for the idealized mathematician.

4.1 It is one of Brouwer's great merits to have drawn attention to the essentially different nature of mathematics and language - that language as a sign can only convey a meaning which has been given to it; that a symbol without a meaning is a contradictio in terminis; that language as such cannot contribute anything to mathematics; and 'that it follows from the nature of language that errors and mistakes can never be completely prevented in the recording of mathematics'.

4.2 Formalism, logicism, and even the 'pre-intuitionism' of Borel and Poincaré, were rejected by Brouwer ultimately because of their reliance on language.

5.1 Logic, described by Brouwer as post factum analysis of a mathematical recording in language, is rejected as contributory to mathematics on the same grounds.

5.2 Like Boole, Brouwer accepts mathematical logic as 'applied mathematics', 'mathematical consideration of the language of mathematics'. Logic is not the foundation of mathematics, nor can logical principles produce mathematical results not obtainable otherwise.

5.3 Unacceptable to Brouwer are 'extended logical principles', i.e. universal statements about an infinite domain of statements.

5.4 Brouwer identifies mathematical truth with the completion of its construction; a theorem which is true is a theorem together with its construction, the proof is the construction itself. A statement is true if it reports the completion of a construction, and only in so far as the construction has been completed.

5.5 Non-contradictority and the Brouwer negative are both rooted in language, and have a weaker form of constructive existence. The Brouwer negative is based on a verbal hypothesis, an unrealizable supposition.

A hierarchy of degrees of mathematical existence can be seen in Brouwer's enumeration of mathematical statements of <u>1955</u>, (see above page 154), into: 1° true statements, i.e. those which have been proved;

2° false statements, i.e. those the supposition of which has led to a contradiction, such as $1 = 0$;

3° judgeable statements, i.e. statements for which an algorithm is known which will lead to a decision that they are true or that they are false;

4° statements which are neither true nor false, and for which there is no algorithm leading to the decision that they are true or that they are false.[1]

5.6 An affirmative intuitionist mathematics on the lines of van Dantzig[2] is more in accordance with Brouwer's strict constructive demands than Brouwer's own intuitionist mathematics on the basis of the Brouwer negation.

5.7 Brouwer's notion of mathematics and of the role he ascribed to language and logic in mathematics led to his rejection of the PEM. The importance of the PEM for intuitionism has been unduly emphasized, as has indeed that of Brouwer's counterexamples. In 'mathematics proper', the question of the PEM does not arise, neither do the 'real numbers' of Brouwer's counterexamples play any role in Brouwer's systematic development of intuitionist mathematics. (During the period 1923 - 1927, Brouwer became involved in the 'logical argument' concerning non-contradictority and the PEM. In his post-war publications, he returns to his original position where non-contradictority and the PEM are treated as mathematically irrelevant).

5.8 Brouwer's logical calculus of absurdities and the rejection of 'the principle of reciprocity of the complementary species', ($\sim\sim\alpha \rightarrow \alpha$), cannot be based on his mathematical theory of complementary species. (see 8.2)

6.1 Brouwer's reluctance to use symbolic notation derives from an unwillingness to accept the possibility of precise and exhaustive expression of mathematical activity.

He recognized that there is no essential difference between language and symbols. It is possible, in accordance with Brouwer's principles, to express a mathematical entity or operation in a symbol, but only <u>after</u> this entity has been mathematically constructed.

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[1] Cf. also Heyting's hierarchy of degrees of self-evidence, ('Intuitionism in mathematics', p. 103, and 'After Thirty Years', p. 195), or Beth's hierarchy of spheres of reality, (*The Foundations of Mathematics*, p. 644).

When Heyting rejects 'truth in mathematics' as 'intuitionistically irrelevant', he means truth as conformity with external reality, ('Over waarheid in de wiskunde', p. 129). Cf. also above p.

[2] Van Dantzig's affirmative, stable mathematics; also Griss's negationless intuitionistic mathematics; cf. above pp. 40 and 137.

6.2 Formalization of intuitionist mathematics, starting from symbols and laying down rules in advance, is contradictory to the spirit of Brouwer's intuitionism.

6.3 In many of its aspects, Brouwer's conception of mathematics can be illustrated by comparison with music: like music, mathematics exists in time; the musical score is not music itself, but a means of reactivating it; the creative artist, the composer, is not guided by the principles discovered by musicologists in other compositions, nor restricted by them, etc.

7.1 The Brouwer set (spread) was not developed as an attempt at reconstructing cantorian set theory while avoiding its antinomies, neither is an intuitionist set theory needed as the foundation of natural numbers and their extensions.

The Brouwer set was specifically intended as a general framework for the continuum and the real numbers.

7.2 In no way can Brouwer sets (or species) be taken as completed totalities of their elements.

7.3 For the Brouwer set, the existence of the sequence of natural numbers is presupposed.

7.4 The Brouwer set is an effective procedure for generating the special kind of elements that Brouwer had in mind: sequences in the widest sense. As a procedure for active generation of an infinite sequence by progressive allocation of values in time, the Brouwer set is a truly constructive tool within the intuitionist conception of mathematics.

7.5 The term 'choice-sequences' (Wahlfolge) in Brouwer's writings is to be taken in its most general sense, and covers all infinite sequences. It emphasizes the human role in the generation of a sequence and the essentially unfinished character of the infinite sequence. It allows a generalization of the notion of an infinite sequence by abstraction from the manner of its continuation.

7.6 The dynamic 'becoming' is an aspect of each infinite sequence ('werdende Wahlfolge'); it characterizes not only the continuum but each point of the continuum.

7.7 There is a close parallel between Bergson's time as 'becoming' and Brouwer's mathematical continuum. It is Brouwer's original idea to extend the dynamic character of Bergson's time continuum to the mathematical continuum, which until then had been treated - also by Bergson - as static.

Just as in Bergson's time as becoming there is no room for an atomistic, discrete 'now', so there is in Brouwer's continuum no room for a discrete point.

7.8 From this conception of sequence, and Brouwer's strict constructive demand for mathematical existence ('having been constructed'), it follows that only finite initial segments can be considered when the actual sequences, points of the continuum, become the basis for further construction. This is the case when points of the continuum form the domain of functions; it leads ultimately to Brouwer's Bar and Fan theorems and the uniform continuity theorem for all functions.

8.1 Even if the intuitionist notion of species may reasonably be referred to as the intuitionist equivalent of the classical set, there are important differences: the species is primarily a property; if the species is in any way to be seen as an aggregate of elements, it is the aggregate resulting from an effective selection on the basis of this property from mathematical entities previously constructed.

8.2 In Brouwer's notion of complementary species, neither of the two complementary species (or of the conjugate splitting species) appears as a positive property.

8.3 In Brouwer's splitting of fundamental mathematical concepts, relations are generated by application of his calculus of absurdities. This is a logical operation in the strict Brouwer sense.

9. Of Brouwer's great qualities as a professional mathematician, his topological work is sufficient proof.

Yet Brouwer will undoubtedly always be remembered, not so much for this work, nor for his adding yet another sect to the schools of mathematics, but for the impact he has made on general mathematical thinking well beyond the tiny minority of professed intuitionists.

Outside influences and circumstances have definitely determined many of Brouwer's views, and made Brouwer's dramatic impact at the time possible, but the real key to its success lies in the person of Brouwer himself, a combination of personal qualities, and in a clear vision of the nature of mathematics which was pursued consistently, regardless of the consequences to which it might lead.

At the root of Brouwer's questioning of current mathematical practice lies the desire to justify his work as a mathematician and to relate it to his deeper philosophical convictions. Brouwer was a revolutionary in the sense that he was not prepared to play his specialized role as a mathematician in conformity with current conventions and just accept the truth of the fundamental assumptions of his discipline.

Having succeeded in finding a philosophy of mathematics in agreement with his mystical beliefs and his philosophy of life, he consistently drew his daring conclusions, prepared to sacrifice some of the content of mathematics in the process. Brouwer had the courage to flout public mathematical opinion; he lacked academic modesty, and passionately believed that his conception of mathematics was the only true one. He saw his task as that of a reformer of general mathematical practice.

Through his persistent and extreme claims, Brouwer alarmed many mathematicians, while others were inspired by his vision of mathematical reality beyond a mechanical handling of signs and symbols, deeper and more human, and yet demanding utmost rigour.

At a critical stage in the history of mathematics, Brouwer halted a general movement in mathematics towards over-reliance on language, logic and formalization. The fact that on many major issues he was later proved right is evidence of his genius and the soundness of his approach to mathematics.

I N D E X

R E F E R E N C E S

A bibliography of all Brouwer's works has been given on pp. 43 - 58; underlined dates constitute references to this bibliography. Selected bibliographies are given on p. 151 (The controversy concerning the Brouwer Logic), pp 162 and 163 (Formalization of intuitionistic mathematics); cf. also pp. 182, 189, 210.

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