Handout # 1

Modeling of Mechanical (Lumped Parameter) Elements

The fundamental components of a **mechanical system** are: **masses or inertias**, **springs** (stiffnesses), and **dampers**.

Lumped elements lead to ordinary differential equations of motion describing the system dynamical behavior.

Each mechanical element has a particular function within the mechanical system:

All systems are just an ensemble of **inertias**, **stiffnesses,** and **damping** elements with definite relationships between its components:

Inertia elements are conservative, store kinetic energy, and relate momentum to velocity (linear momentum to translational velocity, and angular momentum to angular velocity);

Stiffness (compliant or elastic) elements are conservative store potential or strain energy, and relate the element force (torque) to a translational (angular) displacement; and,

Damping elements are non-conservative and dissipate energy from the system. They convert the energy into another form of energy (usually heat). Dampers relate the element force (torque) to a translational (angular) velocity.

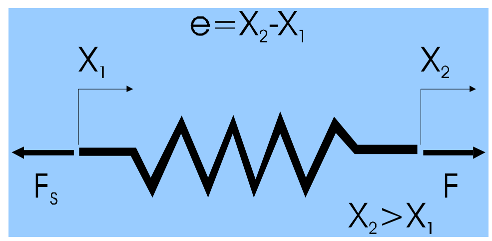
Our objective:

to determine **equivalent system elements** as those capable of reproducing an identical action as all the elements of the same class, and combined by virtue of rendering the same energy or dissipated power. This can be achieved once a particular coordinate or generalized displacement is selected to represent the system or element behavior.

Stiffnesses (Springs) in Mechanical Systems

Translational Springs: relate **force to displacements (deflections)**. Springs are commonly assumed as MASSLESS, so that a force F at one end must be balanced by a force F_s (reaction) acting on the other end. Due to this force applied at one end, the spring undergoes an **elongation** (deformation) equal to the difference between its two end displacements **($e = X_2 - X_1$)**. For small values of (**e**), the spring constant or **stiffness** is:

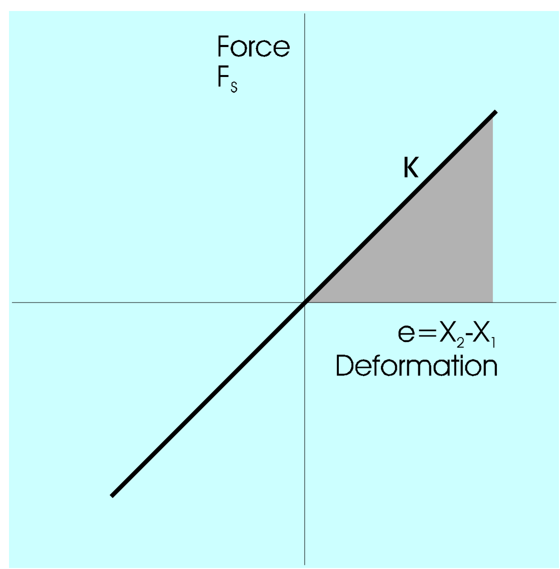
$$K = F/e \qquad \text{[N/m, lb/in]}$$



A spring element is an **energy storage** device. This energy (V_s) is of **strain** (potential) type. In the linear range this energy is:

$$V_s = \int F_s\, dx = \tfrac{1}{2} K e^2$$

and indicated by the area under the F_s *vs.* e (force vs. deformation) curve.



Special cases of non-linear springs are denoted as **softening** if the slope of the curve F_s *vs.* e curve decreases as the elongation increases; and as **hardening** if the slope increases as the deformation e also increases.

In general, for small amplitude motions about an equilibrium point, a local or **linearized stiffness** is defined as:

$$K = -\partial F_s/\partial X\,\big|_o$$

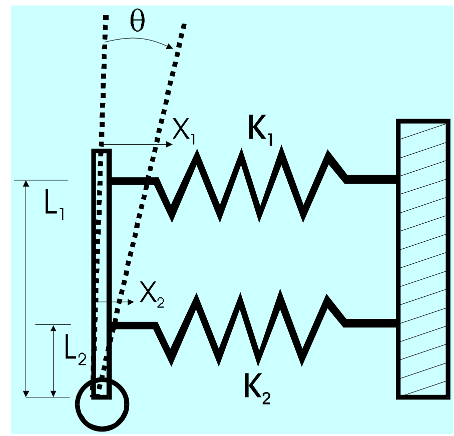
where the sub index **o** denotes a point of static equilibrium.

Equivalent Spring Coefficients:

Select an equivalent displacement, find the total strain energy and equate to that of the equivalent element.

(a) springs connected by levers

$$V_s = \tfrac{1}{2}\left\{K_1 X_1^2 + K_2 X_2^2\right\} = \tfrac{1}{2} K_{eq} X_{eq}^2 \quad \text{[N.m=J]}$$



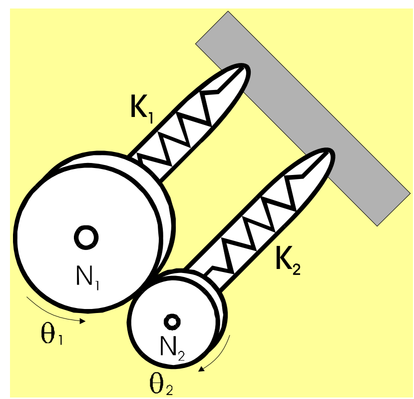
Let $X_{eq} = X_1$, then $X_2 = \theta L_2$, $X_1 = \theta L_1$

so then $X_2 = X_1 (L_2 / L_1)$ and

$K_{eq} = K_1 + K_2 (L_2 / L_1)^2$ [N/m]

(b) torsional springs on geared shafts

$$V_s = \tfrac{1}{2}\left\{K_1 \theta_1^2 + K_2 \theta_2^2\right\} = \tfrac{1}{2} K_{eq} \theta_{eq}^2 \quad \text{[N.m=J]}$$



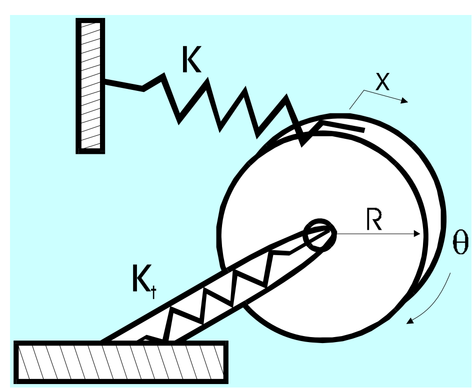
since $\theta_1 N_1 = \theta_2 N_2$, where **N** is the number of teeth

and select $\theta_{eq} = \theta_1$, then

$K_{eq} = K_1 + K_2 (N_1 / N_2)^2$ [N.m/rad]

(c) coupled torsional and translational springs

$$V_s = \tfrac{1}{2}\left\{K x^2 + K_t \theta^2\right\} = \tfrac{1}{2} K_{eq} X_{eq}^2$$



Let $X_{eq} = X$, and since $X = R\,\theta$, then

$K_{eq} = K + K_t / R^2$ [N/m], or

Let $X_{eq} = \theta$, then the equivalent **torsional** spring is

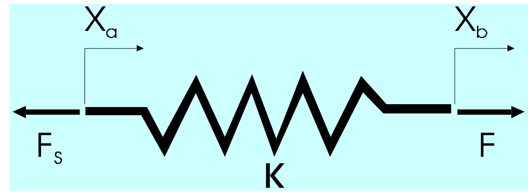
$K_{teq} = K R^2 + K_t$ [N.m/rad]

Note: in (a) thru (c) one of the spring ends is connected to ground.

Equivalent Spring Coefficients: (continued)

(d) springs with two end displacements

$F = K\,(X_b - X_a)$ [N or lb]; $X_b > X_a$



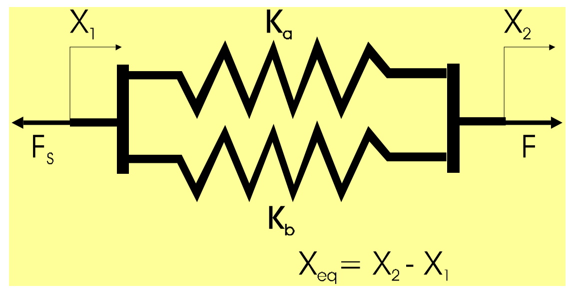
the spring deflection is $e = (X_b - X_a)$

(e) springs connected in parallel

Both of the spring ends share common boundaries.

$F_s = F_a + F_b$, let $X_2 > X_1$

$F_a = K_a\,(X_2 - X_1)$, $F_b = K_b\,(X_2 - X_1)$



hence $F_s = K_a\,(X_2 - X_1) + K_b\,(X_2 - X_1)$

$F_s = (K_a + K_b)\,(X_2 - X_1) = K_{eq}\,X_{eq}$;

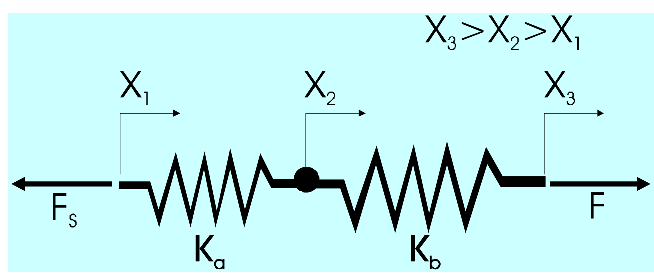
Let, $X_{eq} = X_2 - X_1$

Hence, $K_{eq} = K_a + K_b$ [N/m]

(f) springs connected in series

If end of one spring is fastened to the end of the other springs, then both springs **transmit** the same force.

For equilibrium: $F_a = F_b = F$



since $(X_2 - X_1) = F_a / K_a$

$(X_3 - X_2) = F_b / K_b$

and $(X_3 - X_1) = F / K_{eq}$;

let $X_{eq} = X_3 - X_1$

then $F / K_{eq} = F_a / K_a + F_b / K_b$;

$1 / K_{eq} = 1 / K_a + 1 / K_b$. Thus,

$K_{eq} = (K_a \cdot K_b) / (K_a + K_b)$ [N/m]

Equivalent mass (inertia) elements

The mass of a body is a fundamental material property and thought as the amount of matter within a body. The mass (***M***) is a constant (at velocities well below the speed of light) and not to be confused with its weight (***W = Mg***).

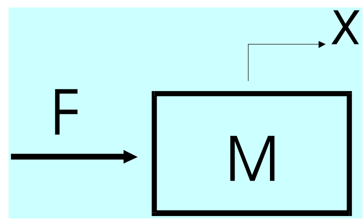
Mass enters the system dynamics through the fundamental laws of motion (linear and angular momentum conservation),

In **translational systems**:

$$F = M\,\ddot{X}$$ [N]

where

$$\ddot{\mathbf{X}} = d^2\mathbf{X}/dt^2$$: acceleration [m/s^2]

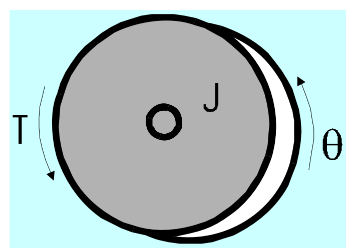


In **rotational systems**:

$$M = I\,\ddot{\theta}$$ [N.m]

where

$$\ddot{\theta} = d^2\theta/dt^2$$: angular acceleration [rad/s^2]



With *I* [N.m.s^2/rad=kg.m^2] as the mass moment of inertia.

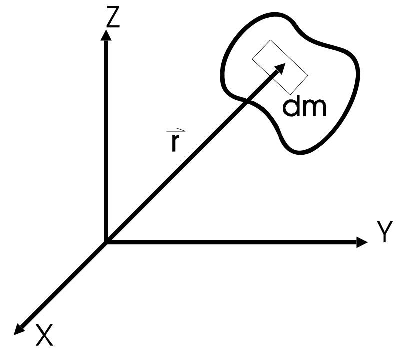
Recall some definitions:

	US system	SI system
displacement	inch =0.0254 m	meter (m) = 100 cm
mass	lb-sec^2/inch=snail	kilogram (kg) = 2.2 lb$_m$
time	second (s)	second (s)
force	lb = 4.448 N	Newton (N)

US: A force of 1 lb applied to a mass of 1 lb$_m$ produces an acceleration of 386 in/s^2

SI: A force of 1 N applied to a mass of 1 kg produces an acceleration of 1 m/s^2

If the body is rigid (not deformable), a lumped mass may be condensed at the center of mass of the body, and hence all material points translate or rotate together, i.e.



$$M = \int \rho \, dV$$

$$I = \int \rho \, r^2 \, dV$$

where ρ: mass density, V: body volume, r: distance from reference axis to $dm = \rho \, dV$

The **kinetic energy** (due to motion) is associated to masses and moments of inertia. **For translation,**

$$T = \frac{1}{2} M \, \dot{X}^2$$

while for **rotation about a fixed axis**,

$$T = \frac{1}{2} I \, \dot{\theta}^2$$

Newton's 2nd law of motion in terms of the body **linear momentum** ($p = M\,v$) as

$$F = \frac{d\,p}{d\,t} = M \frac{d\,V}{d\,t}; \text{ where } V = \frac{d\,X}{d\,t}$$

From this definition, $p = \int F \, dt$ is also known as the **impulse**.

The mass element is energy-conservative since,

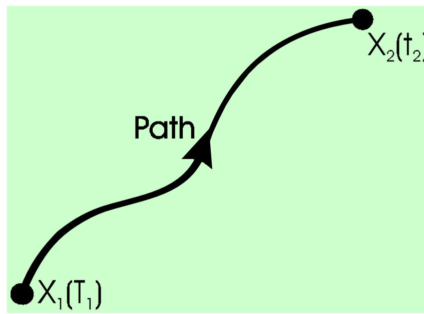
work performed = change in kinetic energy

$$W_{1\to2}=\int_{X_1}^{X_2} F\,dX=\int_{X_1}^{X_2} M\,\frac{dV}{dt}\,dX=\int_{X_1}^{X_2} M\,\frac{dV}{dt}\frac{dX}{dt}\,dt=$$

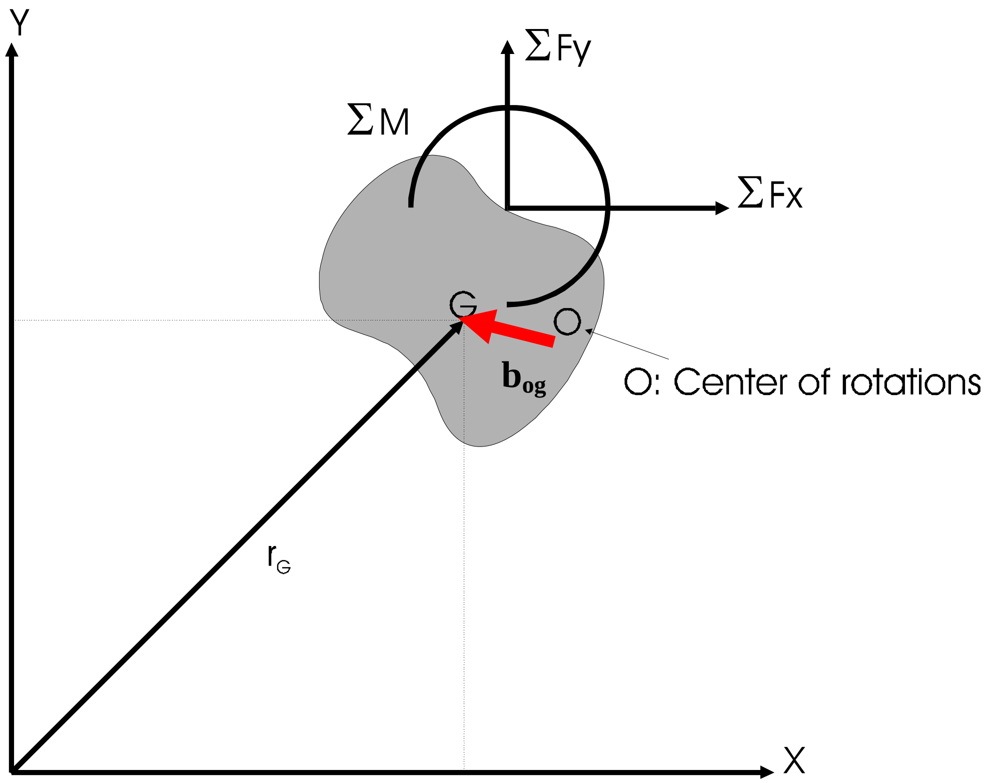
$$=\int_{t_1}^{t_2} M\,V\,\frac{dV}{dt}\,dt=\int_{t_1}^{t_2} M\,\frac{d}{dt}\left(\frac{1}{2}V^2\right)dt=$$

$$=\int_{V_1}^{V_2} M\,d\left(\frac{1}{2}V^2\right)=\frac{1}{2}M\left(V_2^2-V_1^2\right)=T_2-T_1$$

Note: Kinetic Energy (T) is independent of the path followed; it is a function of the end and beginning states.



Equations of motion for a rigid body on plane XY



Summation of Forces: $$\sum \; = \sum \vec{F} = M\,\ddot{\vec{r}}_{G}$$

Summation of Moments: let

$\alpha = \ddot{\theta}$ as the angular acceleration of the rigid body

$$\sum M_G = I_G\,\ddot{\theta}$$ **About center of mass,**

$$\sum M_o = I_o\,\ddot{\theta}$$ **About a fixed axis of rotation, or**

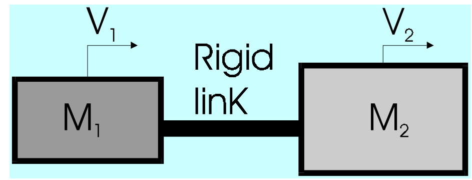
$$\sum M_o = I_o\,\ddot{\theta} + m\left(\vec{b}_{og}\,x\,\ddot{\vec{R}}_o\right)_z$$

About point *o* moving with acceleration $\ddot{\vec{R}}_o$

Equivalent Inertia elements: rendering same kinetic energy

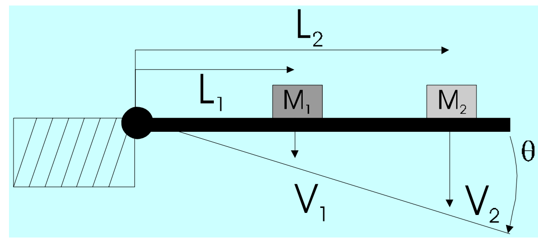
(a) Rigidly connected masses

have identical velocities, and hence
$V_{eq} = V_1 = V_2$
$M_{eq} = M_1 + M_2$



(b) Masses connected by a lever

for small amplitude angular motions. Let $V_{eq} = V_1$, then since the lever is RIGID, $V_2 = V_1 (L_2/L_1)$



and the total kinetic energy is equal to

$$T = \tfrac{1}{2}\{M_1 V_1^2 + M_2 V_2^2\} = \tfrac{1}{2} M_{eq} V_{eq}^2 \text{ [N.m=J]}$$

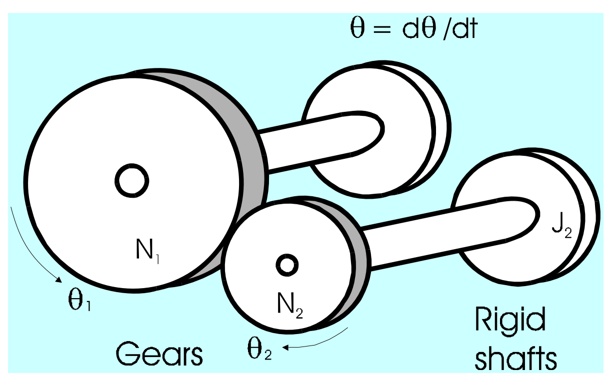
$$T = \tfrac{1}{2}\{M_1 + M_2 (L_2/L_1)^2\} V_1, \quad \text{let } V_1 = V_{eq}$$

hence $M_{eq} = \{M_1 + M_2 (L_2/L_1)^2\}$ [kg]

(c) Inertias on geared shafts (rotation)

Consider two shafts with mass moments of inertias, I_1 and I_2, connected by massless gears.
Let the number of teeth on each gear be N_1 and N_2, respectively.

Since $\dot{\Theta}_2 N_2 = \dot{\Theta}_1 N_1$,
because the contact speed is the same.
Select $\dot{\Theta}_{eq} = \dot{\Theta}_1$



and using the equivalence of kinetic energies find:

$$I_{eq} = I_1 + I_2 \, (N_1 / N_2)^2$$ [N.m/(rad/s²) = kg.m²]

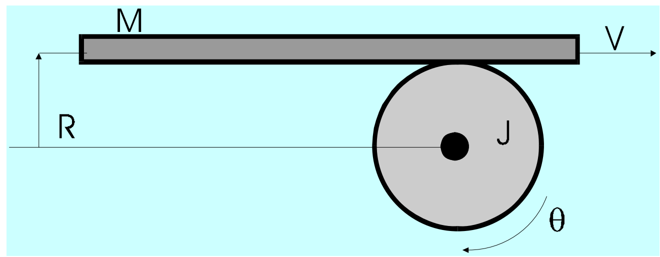
(d) Masses in series

Masses do not have two ends (terminals or ports), and thus can not be connected in series.

(e) Coupled rotation and translation

T = ½ M_{eq} V_{eq}^2 = ½ M V^2 + ½ I $\dot{\theta}^2$ [N.m]

since $V = R\dot{\theta}$ at the contact point without slipping, then

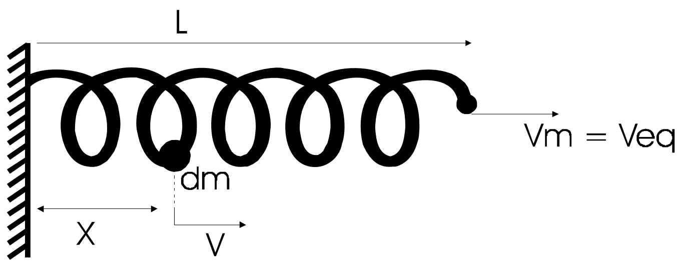


for **V_{eq} = V,** $\rightarrow$ **$M_{eq} = M + I/R^2$**, [kg]

or

for **V_{eq} =** $\dot{\theta}$**,** $\rightarrow$ **I_{eq} = I + M R^2**, [kg·m^2]

LUMPED MASS FOR SPRING ELEMENT:



$$T = \frac{1}{2} M_{eq} V_{eq}^2 = \frac{1}{2} \int_0^M v_{(x)}^2 \, dm. \quad \text{Let } V_{eq} = V_m, \text{ the spring tip speed,}$$

since $v_{(x)} = V_m \dfrac{x}{L}$ and $dm = M \dfrac{dx}{L}$ for uniform density

Then

$$M_{eq} = \frac{\frac{1}{2}\int_0^M v^2 \, dm}{\frac{1}{2} V_{eq}^2} = \frac{\int_0^L V_m^2 \left(\frac{x^2}{L^2}\right) \frac{M}{L} \, dx}{V_m^2} = \int_0^L \left(\frac{M}{L^3}\right) x^2 \, dx$$

$$M_{eq} = \frac{M \, x^3}{3 \, L^3} \Big]_0^L = \frac{M}{3}$$: Not all of the element mass moves with same speed

Dissipation elements in mechanical systems

These are mechanical components which dissipate power (remove energy from system) converting it usually into heat. The process is irreversible!

Energy dissipation elements are known as **dampers or dashpots**, and typically used as:
ISOLATORS: to reduce the amount of transmitted forces (moments) to other system components, and/or
VIBRATION ABSORVERS: to dissipate energy of undesirable vibrations and to decrease the amplitude of vibrations in a mechanical system.

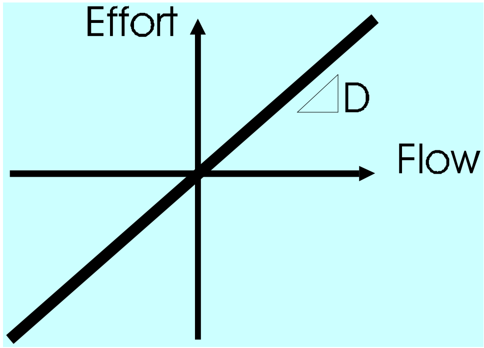
(Note: **effort**=force or moment, **flow**: velocity or angular speed)

TYPES of DAMPING:

VISCOUS (Linear)

effort = *D* flow

Typically found at low speeds and with viscous fluids

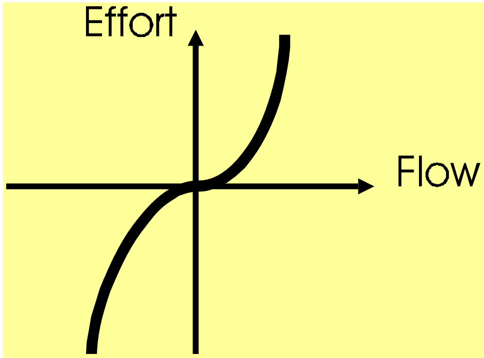


NONLINEAR DAMPING:

Aerodynamic:

effort = *C* |flow| flow

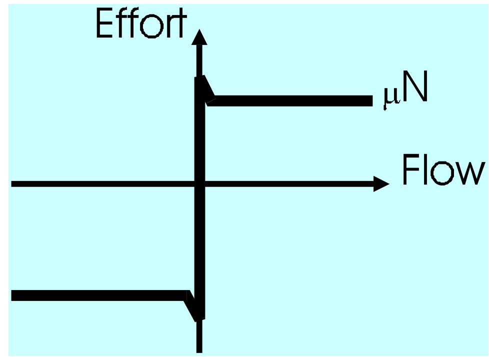
Typically found at high speeds and with non-viscous fluids



Dry (Coulomb) friction:

Effort = {|flow|/flow} μN

Typically found between dry surfaces in relative motion. μ is the coefficient of friction (static and dynamic),



Histeretical or structural damping:
Material damping with structural components

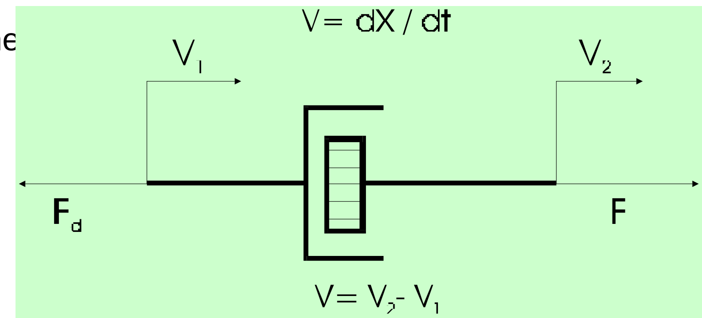
Except for linear (viscous) damping, all other forms of damping are difficult to analyze (and to include) in dynamic models, even for simple cases.

Viscous damping elements

Viscous (Linear) Damper: is the mechanical dissipative element which relates **force to velocity**. Dampers or DASHPOTS are commonly assumed as MASSLESS, so that a force **F** at one end must be balanced by a force $\mathbf{F_D}$ (reaction) acting on its other end. Due to the force applied at one end, the damper follows a motion equal to the difference between its two end velocities $\mathbf{(V=V_2-V_1)}$. For small values of **V**, a linear relationship appears so that the damping constant is defined as:

D = F/V [N-sec/m, lb-sec/in]

A linear damper denotes a viscous she mechanism for its action. A damper **dissipates energy** lost to the surroundings (outside of system) or transferred to other systems usually in the form of **heat**. The energy dissipated is given as:



$$E_d = \int_{x_1}^{x_2} F_D \, dX = \int_{t_1}^{t_2} F_D V \, dt$$

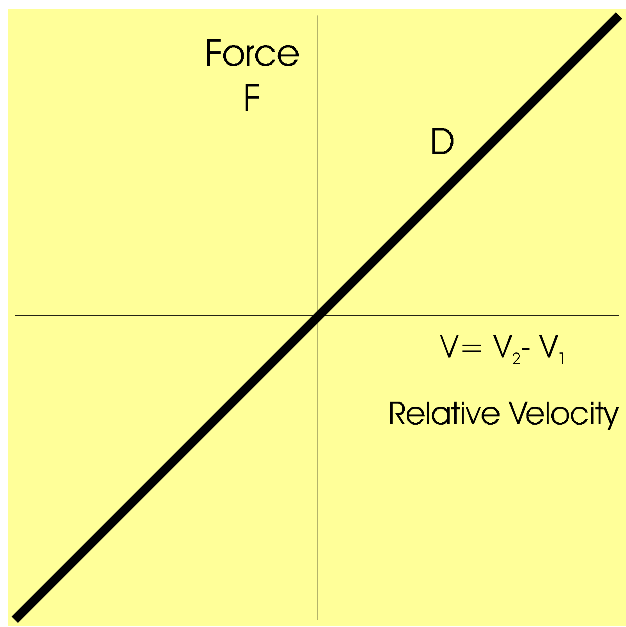
$$E_d = \int_{t_1}^{t_2} P_v \, dt = D \int_{t_1}^{t_2} V^2 \, dt \quad > 0$$

if ***D*** **> 0**

In general, for small amplitude motions about an equilibrium point, a local or **linearized damping coefficient** can be defined as:

$$D = -\partial F/\partial V \big|_o$$

where the sub index o denotes a point of equilibrium.



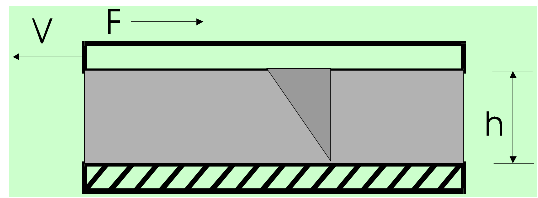
MORE on **VISCOUS DAMPER ELEMENTS**

A fluid is a material that can not withstand tension. A fluid flows as soon as a shear stress τ is applied to it. In general, it is sufficient to know (for now) that the shear stress τ [N/m^2] in a newtonian fluid is proportional to the spatial rate of change of its velocity $\boldsymbol{V}$ [m/s]. The proportionality constant is the fluid viscosity μ [Pa.s = N.s/m^2], i.e.

$$\tau\ [N/m^2] = \mu\, \partial V / \partial x$$

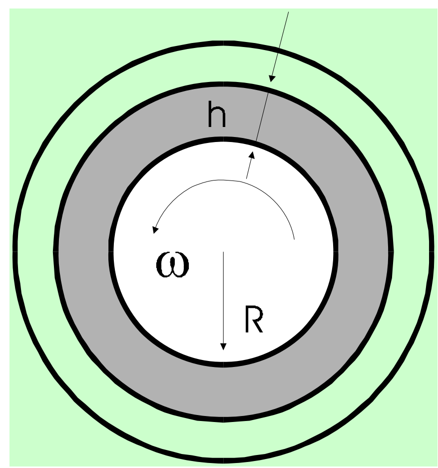
Consider two parallel plates of area $\boldsymbol{A}$ and separated by a gap $\boldsymbol{h}$. A viscous fluid fills the space between the plates. The top plate moves with velocity $\boldsymbol{V}$. Then, the shear force resisting the relative motion is:

$$F = \tau A = \mu A\, V/h = D_{eq}\, V$$



Assumed a linear velocity profile between the stationary plate and the moving plate.
Note that D_{eq} corresponds to a damping coefficient for translation with physical units [N.s/m].

Consider two cylinders of length L and radii R and $R+h$, respectively. The inner cylinder rotates at angular speed ω (rad/s). The gap between the two cylinders is filled with a viscous fluid. Note $h << R$.



For concentric cylinders, the resistive torque T_o opposing the rotation at angular speed (ω) is

$$T_o = F\, R = \mu\, A\, (V/h)\, R = \mu\, (2\,\pi\, R\, L)\, (\omega\, R/h)\, R$$

$$T_o = 2\,\mu\,\pi\, R^3\, (L/h)\,\omega = D_{eq}\,\omega$$

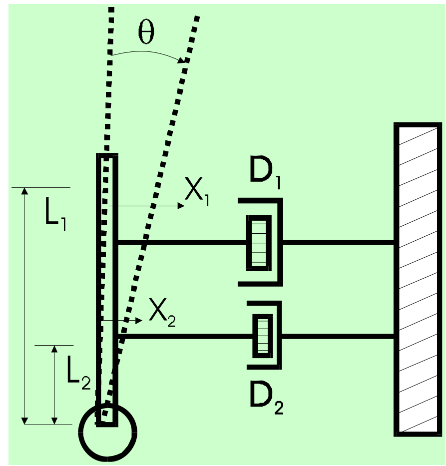
Note that D_{eq} is a damping coefficient for rotational motions with physical units [N.m.s/rad].

Equivalent Damping Coefficients:

Select an equivalent velocity and equate the dissipated powers

$$P_D = F_d \dot{X} = D V^2$$

(a) dampers connected by levers



$$\mathbf{P_D} = \{D_1 V_1^2 + D_2 V_2^2\} = D_{eq} V_{eq}^2 \quad [\text{N.m=J}]$$

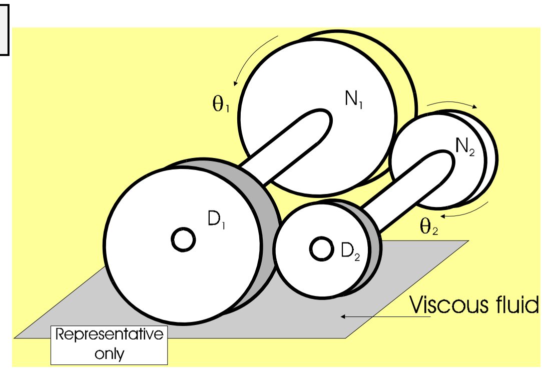
$$\omega = \dot{\theta}$$

Let $\mathbf{V_{eq} = V_1}$ & since $\mathbf{V_2 = \omega L_2}$, $\mathbf{V_1 = \omega L_1}$

then $\mathbf{V_2 = V_1 (L_2 / L_1)}$ and

$$D_{eq} = D_1 + D_2 (L_2 / L_1)^2 \quad [\text{Ns/m}]$$

(b) torsional dampers on geared shafts

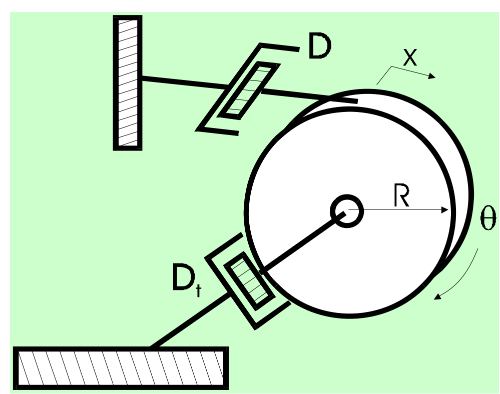


Since $\dot{\theta}_1 N_1 = \dot{\theta}_2 N_2$, where *N* is the number of teeth

and selecting $\dot{\theta}_{eq} = \dot{\theta}_1$, then

$$D_{eq} = D_1 + D_2 (N_1 / N_2)^2 \quad [\text{N.m.s/rad}]$$

(c) coupled torsional and translational dampers



Let $\omega = \dot{\theta}$, $\mathbf{V_{eq} = V}$ & since $\mathbf{V = \omega R}$, then

$$D_{eq} = D + D_t / R^2 \quad [\text{N.s/m}]$$

Or let $\dot{\theta}_{eq} = \dot{\theta}_1$ then the equivalent **torsional** damper is

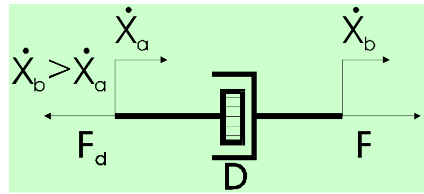
$$D_{eq} = D R^2 + D_t \quad [\text{N.m.s/rad}]$$

Note: in (a) thru (c) one of the dashpot ends is connected to ground.

Equivalent Damping Coefficients: (continued)

(d) damper with two end velocities

$$F_D = D\left(\dot{X}_2 - \dot{X}_1\right)$$ [N or lbs]



(e) dampers connected in parallel

Both damper ends share common boundaries

$F_p = F_a + F_b = F_d$

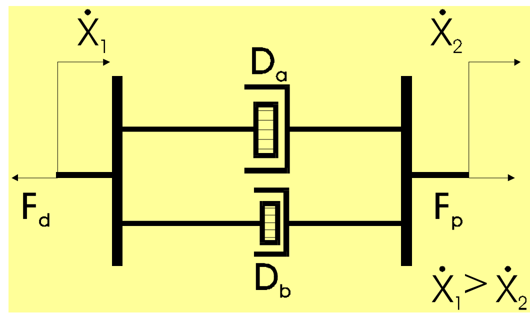
$F_a = D_a (V_2 - V_1)$, $F_b = D_b (V_2 - V_1)$

hence $F_p = (D_a + D_b)(V_2 - V_1)$

select $V_{eq} = (V_2 - V_1)$

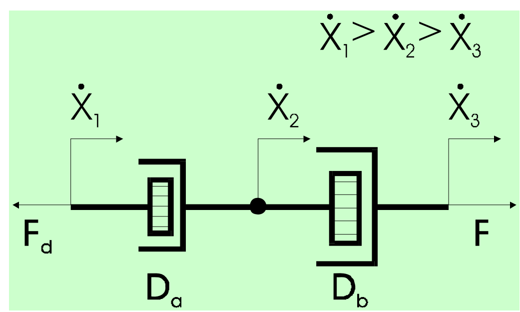
Thus,

$$D_{eq} = D_a + D_b$$ [N.s/m]



(f) dampers connected in series

If the end of one damper is fastened to the end of the other damper. The same force is transmitted.



For equilibrium: $F_a = F_b = F = F_d$

since $(V_2 - V_1) = F_a / D_a$

$(V_3 - V_1) = F_b / D_b$

$(V_3 - V_1) = F / D_{eq}$; and let $V_{eq} = (V_3 - V_1)$

then $F / D_{eq} = F_a / D_a + F_b / D_b$

$$1/D_{eq} = 1/D_a + 1/D_b$$

Thus, $D_{eq} = (D_a D_b) / (D_a + D_b)$ [N.s/m]