# 9 Independence and Random Variables

Computer Science Mentors 70

Apr 5 - Apr 9
Spring 2021

Amogh Gupta, Sylvia Jin, Aekus Bhathal, Abinav Routhu, Debayan Bandyopadhyay

Roast us here: `https://tinyurl.com/csm70-feedback20`

## 1 Independence

1. For each of the following examples, decide whether the listed events are mutually independent, pairwise independent, or neither (if the events are mutually independent, there is no need to also select pairwise independent). Recall the definition of independence–is the probability of both events equal to the product of the probabilities of each event? Should one event influence the probability of the other?

   (a) The event of drawing a jack of hearts from the deck and the event of drawing a jack of clubs from the same deck.

   - Mutually Independent
   - Only Pairwise Independent
   - Neither

   (b) The event of drawing a jack of hearts from the deck and the event of drawing an ace of diamonds from the same deck.

   - Mutually Independent
   - Only Pairwise Independent
   - Neither

   (c) The outcomes of three consecutive coinflips.

   - Mutually Independent
   - Only Pairwise Independent
   - Neither

   (d) Given 2 random integers $x, y$, the event that $x = 5 \mod n$, the event that $y = 7 \mod n$, and the event that $x + y = 20 \mod n$.

   - Mutually Independent
   - Only Pairwise Independent
   - Neither

   **Solution:** A) Neither, the events affect each other. B) Neither, the events affect each other. C) Mutually, all 3 pairs are independent. D) Pairwise. For example, knowing the first two events are true means the third is false.

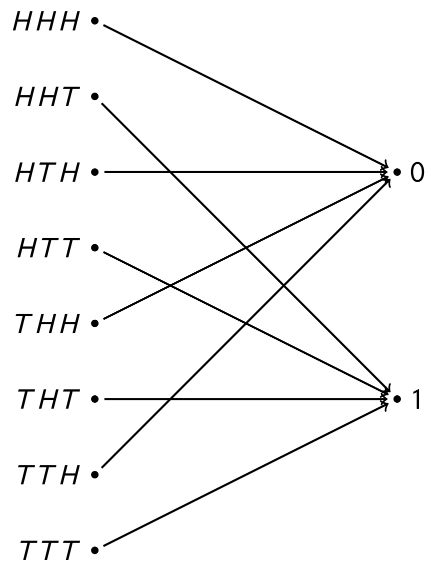
## 2 Random Variables

1. **Intro to Random Variables**
   Suppose that we are flipping 3 coins in a row. Let's try to define some different random variables relating to this process!

   (a) Define the random variable $R_1$ to represent whether the last coinflip was a head or a tail. Draw a mapping from the sample space to the value of $R_1$ that each event corresponds with.

**Solution:**

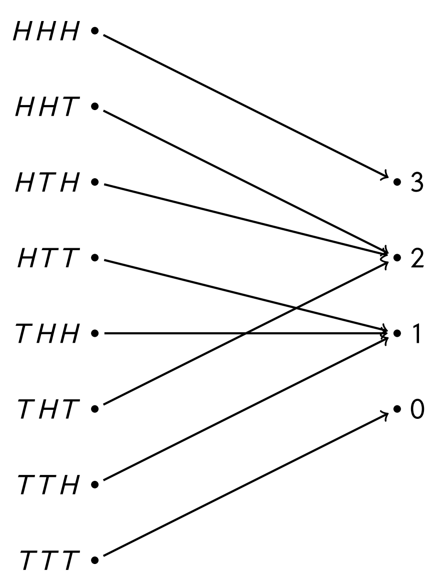
We want to create a function that maps each event in the sample space to one value in $\mathbb{R}$. We do this in this case by assigning Heads to 0, and Tails to 1 (the reverse will also work). The following mapping has the sample space on the left and the corresponding value of $R_1$ on the right. This is an example of a Bernoulli random variable, which takes on a value of 0 or 1 depending on the success or failure of a single event.




(b) Now, define the random variable $R_2$ to represent the number of heads seen in our series of flips. What are possible values that $R_2$ can take on? Draw a mapping from the sample space to the values of $R_2$ that each event corresponds with.

**Solution:**

$R_2$ can be either 0, 1, 2, or 3 since these are the possible number of heads we can see in 3 flips. Now we map each possible outcome to one of these possible values. The following mapping again has the sample space on the left and the corresponding value of $R_2$ on the right. This is an example of a binomial distribution!




2. **Dice Division** (practice.eecs.org Set 11 Problem 1)
Consider the following game: you roll two standard 6-sided dice, one after the other. If the number on the first dice divides

the number on the second dice, you get 1 point. You get 1 additional point for each prime number you roll.

Define the random variable $R_1$ to be the result of the first roll, and define $R_2$ to be the result of the second roll. Define the random variable $X = R_1 + R_2$ to be the sum of the numbers that come up on both dice, define the random variable $Y = R_1 \cdot R_2$ to be the product of the numbers that come up on both dice, and define the random variable $Z$ to be the number of points you win in the game.

(a) What values can the random variable $Z$ take on (with nonzero probability)? Give examples of an event that would cause $Z$ to take on that value.

> **Solution:** 0 (e.g. (4, 6)),
> 1 (e.g. (6, 2)),
> 2 (e.g. (3, 5) or (2, 4)),
> 3 (e.g. (2,2)).

(b) Say that your first roll is a 3 and your second roll is a 6. What is the value of $Z$?

> **Solution:** $Z = 2$. You get one point because 3 | 6 and a second point because 3 is prime.

(c) Say that your first roll is a 4 and your second roll is a 1. What is the value of $X^2 + Y + Z$?

> **Solution:** Note that $X = R_1 + R_2 = 4 + 1 = 5$, $Y = R_1 \cdot R_2 = 4 \cdot 1 = 4$, and $Z = 0$ because neither rolls are prime and the first number does not divide the second number. Thus, $X^2 + Y + Z = 5^2 + 4 + 0 = 29$.

(d) Conditioned on the fact that your second roll is a 1, what is the probability that $Z = 1$?

> **Solution:** For the score to be 1 when the second roll is 1 means the first roll is a prime number (2, 3, 5) or 1 (because $1|1$). Since there are 6 events with the second roll being 1 and 4 successes, $P[Z = 1|R_2 = 1] = 4/6 = 2/3$.

(e) Conditioned on the fact that your second roll is a 1, what is the probability that $Z = 2$?

> **Solution:** With $R_2 = 1$, there is no possible roll that can garner 2 points. When $R_1 = 2, 4, 6$, $Z = 0$ and when $R_1 = 1, 3, 5$, $Z = 1$. This covers all the possible events, so $P[Z = 2|R_2 = 1] = 0/6 = 0$.

3. **The Binomial to the Multinomial**
A binomial is often a useful distribution when modelling the counts of independent and identically distributed (i.i.d) bernoulli trials. But how can we model trials with more than two possible outcomes?

The content mentors were split on a very important issue: how to eat corn. Some of them ate the corn horizontally (think typewriter-style), while others ate the corn in spirals from left to right. To get to the bottom of who was right, they decided to hold a survey of the CS70 student population. But because they were too lazy to create a google form, they replaced each student with a biased coin with probability $p$ of landing heads and flipped it. They then made the arbitrary rule that heads meant the student ate the corn horizontally and tails meant the student ate the corn in spirals.

(a) Suppose there are $n$ students, find the probability that $k$ of them ate the corn horizontally.

> **Solution:** Let $X$ denote the number of students that eat corn horizontally. This is just the binomial distribution $\mathbb{P}[X = k] = \binom{n}{k} p^k (1 - p)^{n-k}$

(b) A rumour was recently spread that some people ate corn in a double-helix pattern (like a DNA strand). In the spirit of the scientific method, the content mentors decided they should add this options to their test. But since they couldn't find an object with three faces, they had to code up a simulation in python with probability $p$ of the student eating the corn horizontally, probability $q$ of the student eating the corn in spirals, and probability $r$ of the student eating the corn in a double-helix pattern such that $p + q + r = 1$. Find the probability that $k$ student ate the corn in a double-helix pattern.

**Solution:** Let $Z$ denote the number of students that eat corn in double helices. This is once again a binomial distribution, but this time the probability of success is $r$ and the probability of failure is $1 - r = p + q$. Hence, $\mathbb{P}[Z = k] = \binom{n}{k} r^k (p + q)^{n-k}$

(c) But the content mentors are hungry for data, so they want to know something more specific: what is the probability that $k$ students ate the corn in a double-helix pattern **and** $m$ student ate the corn in spirals. Simplify your expression.

**Solution:** let $Z$ have the same definition as last part and $Y$ denote the number of students that each corn in spirals. Then, we care to find $\mathbb{P}[Z = k \cap Y = m] = \mathbb{P}[Z = k]\mathbb{P}[Y = m|Z = k]$. The first term is known from part $b$, but the second term is also binomial since now there are only $n - k$ unknowns. Since each unknown has to be either a spiral or a a horizontal, the probability of success is $\frac{q}{p+q}$. Hence $\mathbb{P}[Y = m|Z = k] = \binom{n-k}{m}(\frac{q}{p+q})^m(\frac{p}{p+q})^{n-k-m}$ and

$$\mathbb{P}[Z = k \cap Y = m] = \binom{n}{k}\binom{n-k}{m} r^k (p+q)^{n-k} \frac{q^m p^{n-k-m}}{(p+q)^{n-k}} \tag{1}$$

$$= \frac{n!}{k!m!(n-k-m)!} r^k q^m p^{n-k-m} \tag{2}$$

This is the formula for a multinomial $M(n, r, q)$!

## 3 Expectation

1. **Introduction to Expectation**

Imagine that Sylvia has two, 3-sided loaded (non-uniform probability) dice, which are represented by the random variables $X$ and $Y$, respectively. $X$ and $Y$ are distributed as follows:

$$P(X = 1) = \frac{1}{2} \qquad P(Y = 1) = \frac{1}{6}$$

$$P(X = 2) = \frac{1}{4} \qquad P(Y = 2) = \frac{1}{6}$$

$$P(X = 3) = \frac{1}{4} \qquad P(Y = 3) = \frac{2}{3}$$

(a) Sylvia rolls the first die, represented by random variable $X$.What is the expected value of the roll of the first die. What is the probability it will roll the expected value?

**Solution:** We first find the expected value of the first die, random variable $X$

$$E[X] = \frac{1}{2}(1) + \frac{1}{4}(2) + \frac{1}{4}(3) = \frac{7}{4}$$

As the expected value is not a number on the die, the probability that the die will role it is zero

(b) What is expected value of a roll of the second die, represented by random variable $Y$?

**Solution:**

$$E[Y] = \frac{1}{6}(1) + \frac{1}{6}(2) + \frac{2}{3}(3) = \frac{5}{2}$$

(c) What is the expected value of the product of the two dice?

**Solution:** We take the expectation of each possible product:

$$E[XY] = \frac{1}{2}\cdot\frac{1}{6}(1)(1) + \frac{1}{2}\cdot\frac{1}{6}(1)(2) + \frac{1}{2}\cdot\frac{2}{3}(1)(3) + \frac{1}{4}\cdot\frac{1}{6}(2)(1) + \frac{1}{4}\cdot\frac{1}{6}(2)(2)$$
$$+ \frac{1}{4}\cdot\frac{2}{3}(2)(3) + \frac{1}{4}\cdot\frac{1}{6}(3)(1) + \frac{1}{4}\cdot\frac{1}{6}(3)(2) + \frac{1}{4}\cdot\frac{2}{3}(3)(3) = 4.375$$

2. Assume we have 10 coins, each with a different bias towards heads. The first coin has $p = 0.1$ of flipping heads, the second has $p = 0.2$, etc up to the 10th coin which has $p = 1$. What is the expected number of heads from flipping all 10 coins at once?

**Solution:** Since we know that the expectation of a single Bernoulli distribution is $E(X) = p$, and we have 10 coins, we can write the following (by Linearity of Expectation):

$$E(X) = E(X_1 + X_2 + X_3 + ... + X_{10}) = E(X_1) + E(X_2) + E(X_3) + ... + E(X_{10})$$

$$E(X) = \sum_{i=1}^{10} p = \sum_{i=1}^{10} \frac{i}{10} = 5.5$$

3. **Binomial Mean**
Show that mean of a Binomial random variable $X$ with parameters $n$ and $p$ has mean $np$.

**Solution:** We can model $X$ as the number of flips that show up heads if $n$ independent coins are flipped, and each coin has probability $p$ of being heads. Since a Bernoulli random variable with parameter $p$ represents the whether or not a coin with probability $p$ of being heads is heads, we can express $X$ as the sum of $n$ independent Bernoulli random variables $X_1 + X_2 + \cdots + X_n$, each with parameter $p$.
By Linearity of Expectation, we have that

$$E[X] = E[X_1] + E[X_2] + \cdots + E[X_n].$$

Moreover, we know that the mean of each $X_i$ is $p$. Thus, $E[X] = np$.

# 4 Indicator Variables

4. **Introduction to Indicators**
Imagine drawing a poker hand of five cards from a deck of cards. What is the expected number of face cards that we get?

**Solution:** Let $X$ denote the number of face cards. We will denote the indicator $X_i$ that takes on the value 1 when we have a face card, and 0 else. The probability that we have a face card is $\frac{3}{13}$ as there are three face cards out of the 13. (jack, queen, king). Thus, we have:

$$E[X] = E[\sum_{i=1}^{5} X_1]$$

By the linearity of expectation:

$$E[X] = \sum_{i=1}^{5} E[X_i] = 5E[X_1] = 5 * \frac{3}{13} = \frac{15}{13}$$

5. **Random Homework Party**
N for $N \geq 3$ people join a Homework party. After joining each person is assigned to a breakout rooms randomly. Each assignment was equally likely to be one the $m \geq 2$ breakout rooms and independent of all other assignments. This helps the course staff and students in the way that in each breakout room, every pair of people could help each other. Course staff would like to find the expected number of pairs who are in the the same room.

(a) Let the random variable X be the number of pairs of people who are in the same breakout room. Lets model this using indicator variables. Come up with a model for indicator variable for the pair of people without over counting them. Are the random variables mutually independent ?

**Solution:** For each pair of people i, j with $i < j$, let the indicator random variable $X_{i,j}$ be 1 if i and j are in the same breakout room, and 0 otherwise. Then we have $X = \sum_{i<j} X_{i,j}$

The random variables $X_{i,j}$ are not mutually independent. We notice that if 1 and 2 are in the same breakout room and 1 and 3 are in the same breakout room then 2 and 3 must be in the same room too. That is $P(X_{2,3} = 1 | X_{1,2} = 1 \cap X_{1,3} = 1) = 1$ while the unconditioned probability $P(X_{2,3}) = \frac{1}{m}$

(b) Let the random variable X be the number of pairs of people who are in the same breakout room. What is the expectation E[$X$]

**Solution:** Using our previous indicator variables model we have $X = \sum_{i<j} X_{i,j}$ Also since each $Xi, j$ is an indicator r.v., we have $E[X_{i,j}] = Pr(X_{i,j} = 1) = Pr(\text{i, j get the same color hat}) = \frac{1}{m}$. By linearity of expectation, we have

$$E[X] \sum_{i<j} E[X_{i,j}] = \binom{n}{2} \frac{1}{m}$$

since there are $\binom{n}{2}$ pairs of i,j

# 5 Joint Distribution

1. You have a waste-disposal chute that trash is falling through. At the end of the chute, trash either falls into the right bin with probability 3/4, or the left bin with probability 1/4. The right bin can hold 3 pieces of trash, while the left bin can only hold 1 piece. Being the sanitary student that you are, if you notice that a bin is filled, you block off the chute so that no more trash can fall through the chute.
For this problem, let $R$ be the pieces that fall into the right bin, and $L$ be the number of pieces that fall into the left bin.

1. Determine the sample space (all possible outcomes of the falling trash), along with the probability of each sample point.

**Solution:** The sample space is the set of all sequences the trash can fall in. $\omega = \{l, rl, rrl, rrr\}$

$$P(l) = \frac{1}{4}$$
$$P(rl) = \frac{3}{4} \cdot \frac{1}{4} = \frac{3}{16}$$
$$P(rrl) = \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{1}{4} = \frac{9}{64}$$
$$P(rrr) = \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} = \frac{27}{64}$$

2. Compute the Joint Distribution of $L$ and $R$. Fill in the table:

| | R=0 | R=1 | R=2 | R=3 |
|---|---|---|---|---|
| L=0 | | | | |
| L=1 | | | | |

**Solution:**

| | R=0 | R=1 | R=2 | R=3 |
|---|---|---|---|---|
| L=0 | 0 | 0 | 0 | 27/64 |
| L=1 | 1/4 | 3/16 | 9/64 | 0 |

3. Use the Joint Distribution to compute the Marginal Distribution of $L$ and $R$. Fill out the tables.

| P(L=0) | P(L=1) |
|---|---|
| | |

| P(R=0) | P(R=1) | P(R=2) | P(R=3) |
|---|---|---|---|
| | | | |

**Solution:**

| P(L=0) | P(L=1) |
|---|---|
| 27/64 | 37/64 |

| P(R=0) | P(R=1) | P(R=2) | P(R=3) |
|---|---|---|---|
| 1/4 | 3/16 | 9/64 | 27/64 |

4. Are $L$ and $R$ independent?

**Solution:** No. we just have to find some $l, r$ such that $P(L = l, R = r_) \neq P(L = l) \cdot P(R = r)$. For $l = 0$, $r = 0$, We find that this is the case.

5. What is the total expected number of pieces of trash that will fall before you have to empty the bins?

**Solution:**

$$E[Total] = 1 \cdot \frac{1}{4} + 2 \cdot \frac{3}{16} + 3 \cdot (\frac{9}{64} + \frac{27}{64}) = 2.3125$$