The Syntax and Semantics of Propositional Logic

Inductive Definition of WWFs

We start by defining our primitive vocabulary—the symbols we expect to see in our language. That is to say, we start with the set of sentence symbols:

SS = { $P_1, P_2, ...$ }.

We assume that this set is infinitely large. In addition, we have the set of connectives and parentheses:

CP = {$\neg, \wedge, \vee, \rightarrow, \leftrightarrow, (,)$}.

The set of primitive vocabulary is the union of these two sets:

PV = $SS \cup CP$.

All well-formed formulae (wffs) are composed of this primitive vocabulary. Let us call the set of all combinations of these symbols EX.

Simply being composed of these things does not suffice for being a well-formed formula. For example, we do not want to treat the following as sentences:

a) $(P \wedge \vee \neg Q($

b) $()P \wedge Q$

c) $P \wedge Q \vee R \leftrightarrow P$

So, intuitively, the set of well-formed formulae is a subset of EX.

Recall how we inductively defined the set of natural numbers. Our goal is to similarly construct the set of well-formed formulae. Our basic set will be the set of sentence symbols, and set of functions will all involve the logical connectives. Let $\boldsymbol{EX}^2 = \boldsymbol{EX} \times \boldsymbol{EX}$.

Let us define the following functions:

$\epsilon_\neg$: $\boldsymbol{EX} \rightarrow \boldsymbol{EX}$, where $\epsilon_\neg(\alpha) = (\neg\alpha)$

$\epsilon_\wedge$: $\boldsymbol{EX}^2 \rightarrow \boldsymbol{EX}$, where $\epsilon_\wedge(<\alpha, \beta>) = (\alpha \wedge \beta)$

$\epsilon_\vee$: $\boldsymbol{EX}^2 \rightarrow \boldsymbol{EX}$, where $\epsilon_\vee(<\alpha, \beta>) = (\alpha \vee \beta)$

$\epsilon_\rightarrow$: $\boldsymbol{EX}^2 \rightarrow \boldsymbol{EX}$, where $\epsilon_\rightarrow(<\alpha, \beta>) = (\alpha \rightarrow \beta)$

$\epsilon_{\leftrightarrow}$: $\boldsymbol{EX}^2 \rightarrow \boldsymbol{EX}$, where $\epsilon_{\leftrightarrow}(<\alpha,\beta>) = (\alpha \leftrightarrow \beta)$

Let set $\epsilon = \{\epsilon_{\neg}, \epsilon_{\wedge}, \epsilon_{\vee}, \epsilon_{\rightarrow}, \epsilon_{\leftrightarrow}\}$—i.e., the set of the previous five functions. We can then provide the following definition:

$\boldsymbol{WFF} = \cap\{A : A \text{ is } (SS, \epsilon)\text{-inductive}\}$

That is to say, the set of well-formed formulae is the smallest set that contains every sentence symbol and is closed under the five functions in set ϵ. It is generated from set SS by the functions $\epsilon_{\neg}, \epsilon_{\wedge}, \epsilon_{\vee}, \epsilon_{\rightarrow}, \epsilon_{\leftrightarrow}$.

Proofs about WFFs

Defining ***WFF*** inductively allows us to prove things about all well-formed formulae by induction. The following form might be helpful for proofs by induction on ***WFF***. It is highly recommended that you follow this form in your proofs.

Prove: Every well-formed formulae has a property Φ

Basis case: [Show that every sentence symbol has property Φ.]

Inductive cases: IH: Assume α and β have property Φ.

i) Inductive ($\neg$): [Show $(\neg\alpha)$ has property Φ].

ii) Inductive ($\wedge$): [Show $(\alpha \wedge \beta)$ has property Φ].

iii) Inductive ($\vee$): [Show $(\alpha \vee \beta)$ has property Φ].

iv) Inductive ($\rightarrow$): [Show $(\alpha \rightarrow \beta)$ has property Φ].

v) Inductive ($\leftrightarrow$): [Show $(\alpha \leftrightarrow \beta)$ has property Φ].

Proof: Prove that every *wff* has the same number of left and right parentheses.

Basis case: Every sentence symbol has 0 left and 0 right parentheses. Therefore, every sentence symbol has the same number of left and right parentheses.

Inductive cases: IH: Assume α is a well-formed formula with k right and k left parentheses, and that β is a well-formed formula with l right and l left parentheses.

i) $(\neg\alpha)$: '$(\neg\alpha)$' has $k+1$ left parentheses and $k+1$ right parentheses. Therefore, '$(\neg\alpha)$' has the same number of left and right parentheses.

ii) $(\alpha \wedge \beta)$: '$(\alpha \wedge \beta)$' has $k+l+1$ left parentheses and $k+l+1$ right parentheses. Therefore, '$(\alpha \wedge \beta)$' has the same number of left and right parentheses.

iii) $(\alpha \vee \beta)$: '$(\alpha \vee \beta)$' has $k+l+1$ left parentheses and $k+l+1$ right parentheses.

Therefore, '$(\alpha \vee \beta)$' has the same number of left and right parentheses.

iv) $(\alpha \rightarrow \beta)$: '$(\alpha \rightarrow \beta)$' has $k + l + 1$ left parentheses and $k + l + 1$ right parentheses.
Therefore, '$(\alpha \rightarrow \beta)$' has the same number of left and right parentheses.

ii) $(\alpha \wedge \beta)$: '$(\alpha \leftrightarrow \beta)$' has $k + l + 1$ left parentheses and $k + l + 1$ right parentheses.
Therefore, '$(\alpha \leftrightarrow \beta)$' has the same number of left and right parentheses.

Therefore, every well-formed formulae has the same number of left and right parentheses.

Total Truth Assignment

The goal of our semantics is to assign a truth-value to each sentence in the language of propositional logic in a manner consistent with our previous discussions. In particular, we want a *function* from *WFFs* to *truth-values*. That is, we're looking for the following:

$$\bar{v}\text{: } \mathbf{WFF} \rightarrow \{\mathbf{T}, \mathbf{F}\}$$

Because the set of *WWF*s in propositional logic is defined inductively, we will define $\bar{v}$ recursively. We start out by defining a function for the 'base set' of our sentences; i.e., a function from sentence symbols to truth-values. Luckily, the truth-values of sentence symbols are independent from one another, so any function from sentence-symbols to truth values counts as a truth-assignment. That is:

$$v\text{: } \mathbf{SS} \rightarrow \{\mathbf{T}, \mathbf{F}\}$$

We then require five ways to determine the truth-values of compound sentences that correspond to the five functions under which **WFF** is inductively closed: $\epsilon_{\neg}$, $\epsilon_{\wedge}$, $\epsilon_{\vee}$, $\epsilon_{\rightarrow}$, and $\epsilon_{\leftrightarrow}$:

$\mathscr{F}_{\neg}$: $\{\mathbf{T}, \mathbf{F}\} \rightarrow \{\mathbf{T}, \mathbf{F}\}$ such that $\mathscr{F}_{\neg}(x) = \mathbf{T}$ if $x = \mathbf{F}$
$\mathbf{F}$ if $x = \mathbf{T}$

$\mathscr{F}_{\wedge}$: $\{\mathbf{T}, \mathbf{F}\}^2 \rightarrow \{\mathbf{T}, \mathbf{F}\}$ such that $\mathscr{F}_{\wedge}(x, y) = \mathbf{T}$ if $x = \mathbf{T}$ and $y = \mathbf{T}$
$\mathbf{F}$ otherwise

$\mathscr{F}_{\vee}$: $\{\mathbf{T}, \mathbf{F}\}^2 \rightarrow \{\mathbf{T}, \mathbf{F}\}$ such that $\mathscr{F}_{\wedge}(x, y) = \mathbf{F}$ if $x = \mathbf{F}$ and $y = \mathbf{F}$
$\mathbf{T}$ otherwise

$\mathscr{F}_{\rightarrow}$: $\{\mathbf{T}, \mathbf{F}\}^2 \rightarrow \{\mathbf{T}, \mathbf{F}\}$ such that $\mathscr{F}_{\rightarrow}(x, y) = \mathbf{F}$ if $x = \mathbf{T}$ and $y = \mathbf{F}$
$\mathbf{T}$ otherwise

$\mathscr{F}_{\leftrightarrow}$: $\{\mathbf{T}, \mathbf{F}\}^2 \rightarrow \{\mathbf{T}, \mathbf{F}\}$ such that $\mathscr{F}_{\leftrightarrow}(x, y) = \mathbf{T}$ if $x = y$
$\mathbf{F}$ otherwise

We are now in a position to define $\bar{v}$:

$\bar{v}$: **WFF** $\rightarrow$ **{T, F}**

$\bar{v}(\alpha) = v(\alpha)$ for every $\alpha \in \mathbf{SS}$
$\bar{v}(\epsilon_\neg(\alpha)) = \mathscr{F}_\neg(\bar{v}(\alpha))$
$\bar{v}(\epsilon_\wedge(\alpha, \beta)) = \mathscr{F}_\wedge(\bar{v}(\alpha, \beta))$
$\bar{v}(\epsilon_\vee(\alpha, \beta)) = \mathscr{F}_\vee(\bar{v}(\alpha, \beta))$
$\bar{v}(\epsilon_\rightarrow(\alpha, \beta)) = \mathscr{F}_\rightarrow(\bar{v}(\alpha, \beta))$
$\bar{v}(\epsilon_\leftrightarrow(\alpha, \beta)) = \mathscr{F}_\leftrightarrow(\bar{v}(\alpha, \beta))$

Semantic Concepts:

Definition: Let $\mathbf{S} \subseteq \mathbf{SS}$. Then v is a *truth assignment* for **S** iff $v : \mathbf{S} \rightarrow \{\mathbf{T}, \mathbf{F}\}$, where v satisfies the conditions of $\bar{v}$. This corresponds to a column on a truth table for the sentences in **S**.

Definition: v is a *total truth assignment* (*tta*) iff v is a truth assignment and $\mathbf{S} = \mathbf{SS}$. We can think of a *tta* as a column on an infinitely large truth-table; one that exhausts the truth-values of every sentence whatsoever. For the most part in this class we will focus on total truth assignments.

Definition: A truth assignment v *satisfies* α (which we denote $\models_v \alpha$) iff $v(\alpha) = \mathbf{T}$

Definition: Let Σ be a set of *wffs*. Σ *tautologically entails* α (which we denote $\Sigma \models \alpha$ iff every tta v that satisfies all members of Σ also satisfies α.

Definition: α is *tautologically equivalent* to β iff $\alpha \models \beta$ and $\beta \models \alpha$.

Definition: α is a *tautology* iff $\varnothing \models \alpha$.

Definition: Propositional logic is *sound* iff, if $\Sigma \vdash \alpha$ then $\Sigma \models \alpha$. Note that we have not yet formally defined $\vdash$.

Definition: Propositional logic is *complete* iff, if $\Sigma \models \alpha$ then $\Sigma \vdash \alpha$.

Sample Proof:

Prove: If $\neg\beta \in \Sigma$ and $(\alpha \rightarrow \beta) \in \Sigma$, then $\Sigma \models (\neg\alpha)$.

Suppose that $\neg\beta \in \Sigma$ and $(\alpha \rightarrow \beta) \in \Sigma$. Either Σ is satisfiable or it is not. If it is not satisfiable, then trivially $\Sigma \models (\neg\alpha)$. Suppose that it is satisfiable, and select an arbitrary v that satisfies every member of Σ. Because $\neg\beta$, $\alpha \rightarrow \beta \in \Sigma$, $v \models \neg\beta$ and $v \models \alpha \rightarrow \beta$. Thus:

i) $v((\neg\beta)) = \mathscr{F}_\neg(v(\beta)) = \mathbf{T}$, therefore $v(\beta) = \mathbf{F}$

ii) $v((\alpha \rightarrow \beta)) = \mathscr{F}_\rightarrow(v(\alpha), v(\beta)) = \mathscr{F}_\rightarrow(v(\alpha), \mathbf{F}) = \mathbf{T}$.

Therefore, $v(\alpha) = \mathbf{F}$. So, $v((\neg\alpha)) = \mathscr{F}_\neg$ **(F)** = **T**. So $\models_v \neg\alpha$. And because the selection of v was arbitrary, $\Sigma \models (\neg\alpha)$.

Practice Problems:

Prove each of the following:

1. Every *wff* has a finite length.

2. There is no maximum length to a *wff*.

3. α is a tautology iff for every tta v, $\models_v \alpha$.

4. Neither of the following two formulas tautologically entails the other:

i. $(A \leftrightarrow (B \leftrightarrow C))$

ii. $((A \wedge (B \wedge C)) \vee ((\neg A) \wedge (\neg B) \wedge (\neg C)))$

5. $\Sigma; \alpha \models \beta$ iff $\Sigma \models (\alpha \rightarrow \beta)$