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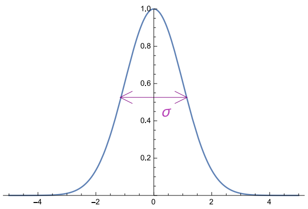
Lecture 11: Wavepackets and dispersion

1 Wave packets

The function

$$g(x) = e^{-\frac{1}{2}\left(\frac{x}{\sigma_x}\right)^2} = \tag{1}$$



is called a **Gaussian**. For a Gaussian, note that $g(\pm\sigma_x) = \frac{1}{\sqrt{e}} g(0) \approx 0.6\, g(0)$, so when $x = \pm\sigma_x$, the Gaussian has decreased to about 0.6 of its value at the top. Alternatively, the Gaussian is at half its maximal value at $x = \pm 1.1\sigma_x$. Either way, σ_x indicates the width of the Gaussian. The plot above has $\sigma_x = 1$. (You may recall that the power of a driven oscillator is given by a Lorentzian function $l(x) = \frac{\gamma}{x^2 + \gamma^2}$, which has roughly similar shape to a Gaussian and decays to half of its value at the top at $x = \pm\gamma$. Try not to get the functions confused.)

The Fourier transform of the Gaussian is

$$\tilde{g}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\, e^{-ikx} g(x) = \frac{\sigma_x}{\sqrt{2\pi}} e^{-\frac{1}{2}\sigma_x^2 k^2} = \frac{1}{\sqrt{2\pi}\sigma_k} e^{-\frac{1}{2}\left(\frac{k}{\sigma_k}\right)^2}, \quad \text{where} \quad \sigma_k = \frac{1}{\sigma_x} \tag{2}$$

This is also a Gaussian, but with width $\sigma_k = \frac{1}{\sigma_x}$. Thus, the narrower the Gaussian is in position space ($\sigma_x \to 0$), the broader its Fourier transform is ($\sigma_k \to \infty$), and vice versa

When $\sigma = \infty$, the Gaussian is infinitely wide: it takes the same value at all x. Then $\tilde{g}(k)$ becomes a δ-function at $k = 0$. That is, to construct a constant, one only needs the infinite wavelength mode (recall $\lambda = \frac{2\pi}{k}$). To construct something narrower than a constant, one needs more and more wavenumbers. To construct a very sharp Gaussian in x ($\sigma_x \to 0$) the Fourier transform flattens out: one needs an infinite number of wavenumbers to get infinitely sharp features.

As you know, if we shift the Gaussian $g(x + x_0)$, then the Fourier transform rotates by a phase. Conversely, if we shift the Fourier transform, the function rotates by a phase. Even with these extra phases, the Fourier transform of a Gaussian is still a Gaussian:

$$\boxed{f(x) = e^{-\frac{1}{2}\left(\frac{x - x_0}{\sigma_x}\right)^2} e^{ik_c x} \quad \Longleftrightarrow \quad \tilde{f}(k) = \frac{\sigma_x}{\sqrt{2\pi}} e^{-\frac{\sigma_x^2}{2}(k - k_c)^2} e^{-ix_0(k - k_c)}} \tag{3}$$

The Gaussian is called a **wavepacket** because of its Fourier transform: it is a packet of waves with frequencies/wavenumbers clustered around a single value k_c (the subscript "c" is for "carrier", as we explain below).

2 Amplitude modulation

One of the most important applications of wavepackets is in communication. How do we encode information in waves?

The simplest way is just to play a single note. For example, we produce a simple sin wave, $\sin(2\pi\nu_c t)$ for some ν_c, say 40 Hz for a low E. If this is all that ever happens, then no information is actually being transferred from one point to another. To transmit a signal, we can start and stop the note periodically. For example, suppose we modulate our note by turning it on and off once a second (think whole notes). Then we would have something like

$$A(t) = f(t)\sin(2\pi\nu_c t) \tag{4}$$

where $f(t)$ has a frequency of $\nu_m \sim 1$ Hz. So let's say $f(t) = \sin(2\pi\nu_m t)$. Since $\nu_m \ll \nu_c$, the curves will look like what we had for beats:

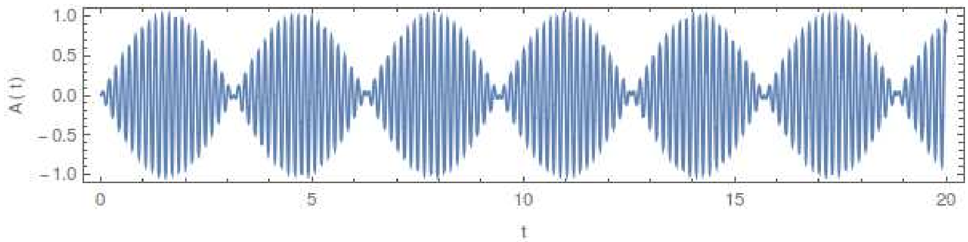


Figure 1. A note encoded with high frequency oscillations

We know that since these curves look like the beat curves, they are really the sum of Fourier modes with $\nu = \nu_c \pm \nu_m$. In other words $A(t) = \frac{1}{2}[\cos(39\text{ Hz} \times 2\pi t) - \cos(41\text{ Hz} \times 2\pi t)]$. There is still not any information carried in the signal. But by adding a few more frequencies, we can get something more interesting. For example, consider

$$A(t) = \cos(39t) - \cos(41t) + 0.5\cos(38t) + 2\cos(43t) - 2.5\cos(41.5t) \tag{5}$$

(I didn't write the 2πHz everywhere to avoid clutter). This looks like this

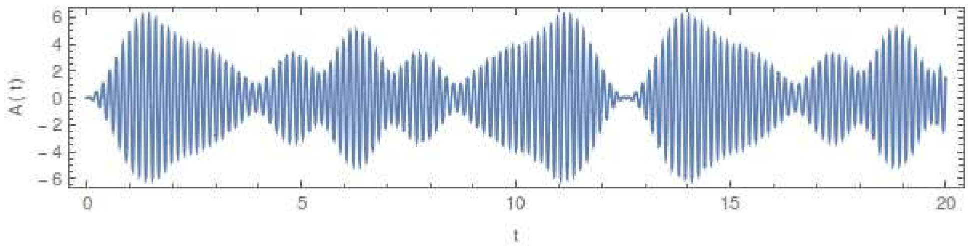


Figure 2. Combining frequencies close to the carrier frequency of 40 Hz we can encode information in the signal.

Note that this signal is constructed using only frequencies within 3 Hz of the carrier frequency of 40 Hz.

Rather than combining particular frequencies, it's somewhat easier to think about writing the amplitude as in Eq. (4) with $f(t) = F(t)\sin(2\pi\nu_m t)$ and the function $F(t)$ having a constant value which changes after each node of the modulated signal. For example, something like this

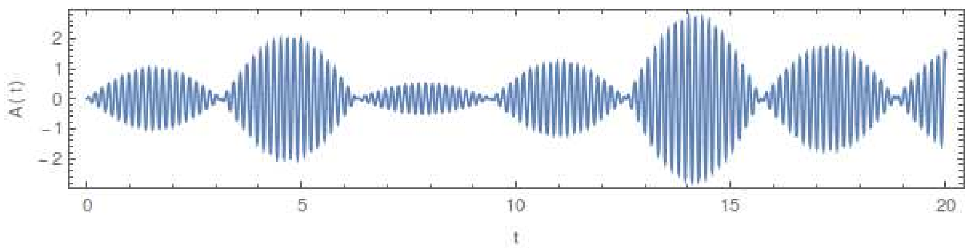
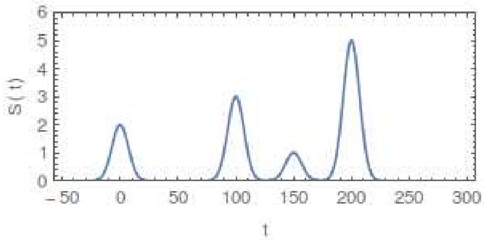


Figure 3. Varying a the amplitude at a frequency of $\nu_m = 1$ Hz using a $\nu_c = 40$ Hz carrier frequency. This is an amplitude-modulated signal.

Finally, we observe that the we can separate the pulses quite cleanly if we construct them with wavepackets, as long as the width of the packets is smaller than the distance between them. For example, we can add Gaussians with different widths and amplitudes:

$$S(t) = 2f(t) + 3f(t-100s) + f(t-150\,s) + 5f(t-200\,s) =$$



Now, we would like to construct these pulses with a carrier of frequency ν_c. Think of this as trying to draw little hills using a pen which wiggles up and down at a rate ν_c. The high-frequency pen changes each packet from $f(t)$ to

$$f_c(t) = e^{-\frac{1}{2}\left(\frac{t}{\sigma_t}\right)^2}\cos(2\pi\nu_c t) = \mathrm{Re}\left[e^{-\frac{1}{2}\left(\frac{t}{\sigma_t}\right)^2} e^{2\pi i \nu_c t}\right] \tag{6}$$

As long as the carrier frequency is larger than the width of the wavepacket, $\nu_c \gtrsim \frac{1}{\sigma_t}$, the wiggles in the carrier will be imperceptible and the packet will be faithfully reconstructed. For example, in $S(t)$ above, the pulses are separated by $100\,s$, so taking $\sigma_t = 10s$ should do. The corresponding width of the packets is then $\gamma = \frac{1}{\sigma} = 0.1\,\mathrm{Hz}$. The following plots shows the amplitude squared, centered around $\nu_c = 0.1\,\mathrm{Hz}$ and $\nu_c = 1\,\mathrm{Hz}$.

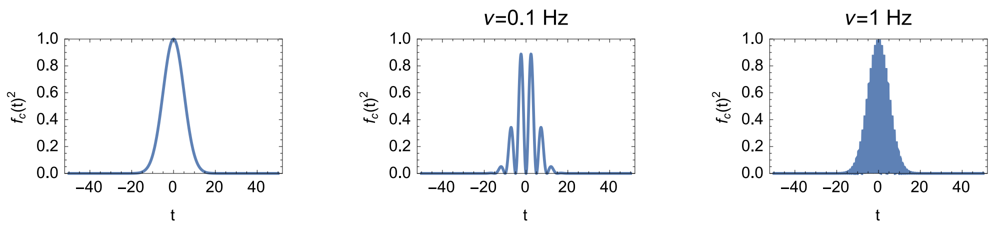


Figure 4. The Gaussian wave-packet (left) with $\gamma = \frac{1}{\sigma} = 0.1\,\mathrm{Hz}$ is well approximated by varying the amplitude of a $\nu_c = 1\,\mathrm{Hz}$ signal (right). Using $\nu_c = 0.1\,\mathrm{Hz}$ (middle) it's not that well constructed.

This example shows that information can be conveyed in $S(t)$ at the rate of $\nu_m = \frac{1}{100\,s} = 0.01\,\mathrm{Hz}$ using a carrier frequency of $\nu_c = 1\,\mathrm{Hz}$.

More generally, this is how **AM (Amplitude Modulated)** radio works. In radios, the information is conveyed at the **information rate** of $\nu_m \sim \mathrm{Hz}$ on the **carrier frequency** ν_c typically in the 100MHz range. For cell phones and wireless, GHz frequencies are used as carrier frequencies.

In terms of time and frequency, Eq. (3) becomes

$$f(t) = e^{-\frac{1}{2}\left(\frac{t-t_0}{\sigma_t}\right)^2} e^{i\omega_c t} \quad \Longleftrightarrow \quad \tilde{f}(\omega) = \frac{\sigma_t}{\sqrt{2\pi}} e^{-\frac{\sigma_t^2}{2}(\omega-\omega_c)^2} e^{-it_0(\omega-\omega_c)} \tag{7}$$

From this, we see that to construct a signal $f(t)$ with width σ_t, we can use frequencies within a range $\sigma_\omega = \frac{1}{\sigma_t}$ centered around *any* ω_c. The central frequency (the carrier frequency ω_c) can be anything. The key is that enough frequencies around ω_c be included. More precisely, we need a band of width $\sigma_\omega = \frac{1}{\sigma_t}$ to construct pulses of width σ_t. The pulses should be separated by, at minimum, σ_t. Thus the feature which limits how much information can be transmitted is the **bandwidth**. To send more information (smaller distance $\sim\sigma_t$ between pulses) a larger bandwidth is needed.

3 Dispersion relations

An extremely important concept in the study of waves and wave propagation is *dispersion*. Recall the dispersion relation is defined as the relationship between the frequency and the wavenumber: $\omega(k)$. For non-dispersive systems, like most of what we've covered so far, $\omega(k)=vk$ is a linear relation between ω and k. An example of a dispersive system is a set of pendula coupled by springs (see Problem Set 3), where the wave equation is modified to

$$\frac{\partial^2 A(x,t)}{\partial t^2} - \frac{E}{\mu}\frac{\partial^2 A(x,t)}{\partial x^2} + \frac{g}{L}A(x,t)=0 \tag{8}$$

The dispersion relation can be derived by plugging in $A(x,t)=A_0 e^{i(kx+\omega t)}$, leading to the relation $\omega=\sqrt{\frac{E}{\mu}k^2+\frac{g}{L}}$, with $k=|\vec{k}|$.

Here is a quick summary of some physical systems and their dispersion relations

- Deep water waves, $\omega=\sqrt{gk}$, with $g=9.8\frac{m}{s^2}$ the acceleration due to gravity. Here, the phase and gorup velocity (see below) are $v_p=\sqrt{\frac{g\lambda}{2\pi}}$, $v_g=\frac{1}{2}v_p$ and the longer wavelength modes move faster. This regime applies if $\lambda \gg d$ with d the depth of the water
- Shallow water waves $\omega=\sqrt{gd}k$, where d is the depth of the water. This is a dispersionless system with $v_p=v_g=\sqrt{gd}$.
- Surface waves (capillary waves), like ripples in a pond: $\omega^2=k^3\sigma\rho$, with σ the surface tension and ρ density. Thus, $v_p=\sqrt{\frac{2\pi\sigma}{\rho\lambda}}$, $v_g=\frac{3}{2}v_p$ and shorter wavelength modes move faster. These involve surface tension so can be seen when the disturbance is small enough not to break the water's surface.
- Light propagation in a plasma: $\omega=\sqrt{\omega_p^2+ck^2}$, with ω_p the *plasma frequency* and c the speed of light. This is the same functional form as for the pendula/spring system above.
- Light in a glass $\omega=\frac{c}{n}k$. n is the index of refraction, which can be weakly dependent on wavenumber. In most glass, it is well described by $n^2=1+\frac{a}{k_0^2-k^2}$.

We'll talk about the water waves in Lecture 12 and light waves in later lectures.

4 Time evolution of modes: phase velocity

Now we will understand the importance of dispersion relations (and their name) by studying the time-evolution of propagating wavepackets.

To begin, let's think about how to solve the wave equation in a dispersive system with initial condition

$$A(x,t=0)=f(x) \tag{9}$$

Think about setting up a pulse of this form in a medium like a string and then sending down the string. For a non-dispersive wave, with $\omega(k)=vk$, the solution is easy

$$A(x,t)=f(x\pm vt) \tag{10}$$

with the sign determined by initial conditions.

Now say we want to solve the pendula/spring wave equation, Eq. (8) with $A(x,0)=f(x)$. So far, we have only solved Eq. (8) for solutions with fixed k.

$$A_k(x,t)=A_0 e^{i\left(kx-\sqrt{\frac{E}{\mu}k^2+\frac{g}{L}}\,t\right)} \tag{11}$$

This is indeed of the form $f(x-vt)$ for $v=\frac{\sqrt{\frac{E}{\mu}k^2+\frac{g}{L}}}{k}$. However, since v_p depends on k, this only works for if only one k is present in the Fourier transform. But if $A(x,0)$ is not of the form of a monochromatic (fixed frequency/wavenumber) plane wave, then this solution doesn't apply and we have to think a little harder.

Before thinking harder, we note that a fixed k solution is possible with any dispersion relation, not just this one. For a dispersion relation $\omega(k)$ the amplitude $A_0\exp\Big[ik\Big(x-\frac{\omega(k)}{k}t\Big)\Big]$ is a solution to the corresponding wave equation. We call the speed of this particular solution the phase velocity

$$\textbf{phase velocity}: \quad v_p(k)=\frac{\omega(k)}{k} \tag{12}$$

Thus $A(x,t)=A_0\exp[ik(x-v_p(k)t)]$ will always be a solution.

So what happens to $A(x,\,t)$ when $A(x,\,0)\neq e^{ikx}$ for some k? The easiest way to solve the wave equation is through Fourier analysis. We know we can write

$$A(x,t=0)=f(x)=\int dk e^{ikx}\tilde{f}(k) \tag{13}$$

where

$$\tilde{f}(k)=\frac{1}{2\pi}\int dx e^{-ikx}f(x)=\frac{1}{2\pi}\int dx e^{-ikx}A(x,0) \tag{14}$$

Eq. (13) writes the initial condition as a sum of plane wave (fixed k) modes. Then, since we know that each mode evolves by replacing $x\rightarrow x-v_p(k)t$, we have

$$\boxed{A(x,t)=\int dk e^{ik(x-v_p(k)t)}\tilde{f}(k)=\int dk e^{i\,(kx-\omega(k)t)}\tilde{f}(k)} \tag{15}$$

It's that simple. This is the exact solution to Eq. (8) with initial condition $A(x,0)=f(x)$.

Let us check that Eq. (15) satisfies Eq. (8) with $\omega(k)=\sqrt{\frac{E}{\mu}k^2+\frac{g}{L}}$. Plugging in we get

$$\left[\frac{\partial^2}{\partial t^2}-\frac{E}{\mu}\frac{\partial^2}{\partial x^2}+\frac{g}{L}\right]A(x,t)=\left[\frac{\partial^2}{\partial t^2}-\frac{E}{\mu}\frac{\partial^2}{\partial x^2}+\frac{g}{L}\right]\int dk e^{i\,(kx-\sqrt{\frac{E}{\mu}k^2+\frac{g}{L}}t)}\tilde{f}(k) \tag{16}$$

$$=\left[-\left(\frac{T}{\mu}k^2+\frac{g}{L}\right)+\frac{T}{\mu}k^2+\frac{g}{L}\right]\int dk e^{i\,(kx-\sqrt{\frac{E}{\mu}k^2+\frac{g}{L}}t)}\tilde{f}(k) \tag{17}$$

$$=0 \tag{18}$$

The boundary condition $A(x,0)=f(x)$ is also satisfied.

Another check is that in the special case of a dispersionless medium, where $\omega(k)=vk$ and so $v_p(k)=v$ constant, the solution is exactly what we expect:

$$A(x,t)=\int e^{ik(x-vt)}\tilde{f}(k)dk=f(x-vt) \tag{19}$$

which we already knew.

5 Time evolution of signals: group velocity

In this section, we take $\omega(k)$ to be arbitrary and take the initial signal shape to be our beautiful Gaussian wavepacket constricted with a carrier wave of wavenumber k_c. So

$$f(x)=e^{-\frac{1}{2}\left(\frac{x-x_0}{\sigma_x}\right)^2}e^{ik_cx} \tag{20}$$

where $k_0=k_c$ is the carrier wavenumber. Here, we let the signal be complex to efficiently encode phase information. One can always take the real part at the end, as we have done before. We again want to solve the general wave equation with dispersion relation $\omega(k)$ for $A(x,t)$ with initial condition $A(x,0)=f(x)$.

The Fourier transform of this packet is

$$\tilde{f}(k)=\frac{\sigma_x}{2\sqrt{\pi}}e^{-\frac{\sigma_x^2}{2}(k-k_c)^2}e^{ix_0(k-k_c)} \tag{21}$$

as in Eq. (3). In Fourier space, the time evolution is easy to compute:

$$A(x,t) = \int dk e^{i(kx-\omega(k)t)} \tilde{f}(k) \tag{22}$$

As noted above, it is impossible to solve this in general. But since in our case $\tilde{f}(k)$ is exponentially suppressed away from $k = k_c$, we can Taylor expand the dispersion relation

$$\omega(k) = \omega(k_c) + (k - k_c)\omega'(k_c) + \cdots \tag{23}$$

$$= k_c v_p + (k - k_c) v_g + \cdots \tag{24}$$

$v_p = v_p(k_c)$ is the phase velocity at k_c and $v_g = v_g(k_c)$ is called the group velocity

$$\textbf{group velocity} \quad v_g(k) = \frac{d\omega(k)}{dk} \tag{25}$$

In general, both the phase and group velocities depend on k. Here, because of the Taylor expansion, we are only interested in the special value $v_g = v_g(k_c)$.

If we truncate the Taylor expansion to order $(k - k_c)$, then the solution for $A(x,t)$ is:

$$A(x,t) = \int dk e^{i(kx - k_c v_p t - (k-k_c) v_g t)} \tilde{f}(k) \tag{26}$$

$$= e^{-ik_c t(v_g - v_p)} \int dk e^{ik(x - v_g t)} \tilde{f}(k) \tag{27}$$

$$= e^{-ik_c t(v_g - v_p)} f(x - v_g t) \tag{28}$$

Thus we have found that the wave-packet moves at the velocity v_g.

Note that for a non-dispersive wave $v_p = v_g$ and we get back our original solution. Also note that in deriving this, we didn't need to use the exact form of the wavepacket, just that it was exponentially localized around k_c.

Stating our results in terms of time dependence and frequency, we have found that

- A pulse can be constructed with a group of wavenumbers in a band $k_c - \sigma_k < k < k_c + \sigma_k$ or equivalently with a group of frequencies in a band $\nu_c - \sigma_\nu < \nu < \nu_c + \sigma_\nu$.
- To send a pulse which lasts σ_t seconds using a carrier frequency ν_c, one needs frequencies within $\sigma_\nu = \frac{1}{\sigma_t}$ of ν_c.
- The pulse travels with the group velocity $v_g = \left.\frac{d\omega}{dk}\right|_{k=k_c}$ evaluated at the carrier wavenumber/frequency.

Note that because $\sigma_k \ll k_c$ ($\sigma_\nu \ll \nu_c$), the group velocity is roughly constant for all of the relevant wavenumbers, $k_c - \Delta k < k < k_c + \Delta k$. But it may be very different from the phase velocity. For example, if $\omega(k) = 5k^4$, then at $k_c = 100$, $v_p = 5 \times 10^8$ while $v_g = 20k^3 = 2 \times 10^7$. Again, for non-dispersive media, $v_g = v_p$. We will contrast group and phase velocity more in the next lecture when we have some concrete examples of dispersive systems.

6 Dispersion

Now we come to where dispersion relations got their name.

We just saw that to the first approximation, a wave-packet moves with velocity v_g. Of course, in the first order approximation in the Taylor expansion, the dispersion relation might as well be linear (non-dispersive). So let's add the second term to see the dispersion. Then

$$\omega(k) = \omega(k_c) + (k - k_c)\omega'(k_c) + \frac{1}{2}(k - k_c)^2 \omega''(k_c) + \cdots \tag{29}$$

$$= k_c v_p + (k - k_c) v_g + \frac{1}{2}(k - k_c)^2 \Gamma + \cdots \tag{30}$$

where $\Gamma = \omega''(k_c)$ is a new parameter. Note that if the wave is non-dispersive, so $\omega(k) = vk$, then $\omega_p = \omega_g$ and $\Gamma = 0$.

With this expansion, let's go back to our Gaussian. We start with

$$A(x,t=0)=f(x)=e^{-\frac{1}{2\sigma^2}(x-x_0)^2}e^{ik_c x} \tag{31}$$

Then,

$$\tilde{f}(k)=\frac{\sigma}{2\sqrt{\pi}}e^{e^{-\frac{\sigma_x^2}{2}(k-k_c)^2}}e^{ix_0(k-k_c)} \tag{32}$$

So

$$A(x,t)=\frac{\sigma}{2\sqrt{\pi}}\int e^{i\left(kx-\left[k_c v_p+(k-k_c)v_g+\frac{1}{2}(k-k_c)^2\Gamma\right]t\right)}e^{-\frac{\sigma_x^2}{2}(k-k_c)^2}e^{ix_0(k-k_c)}dk \tag{33}$$

If you stare at the exponent, you will see that it is still quadratic in k – still a Gaussian – so in this special case we can actually perform the inverse Fourier transform. And of course, we will get a new Gaussian. The result is

$$A(x,t)=\exp\left[-\frac{1}{2}\left(\frac{x-(x_0+v_g t)}{\sqrt{\sigma_x^2-i\Gamma t}}\right)^2\right]e^{ik_c x}e^{-ik_c t(v_g-v_p)} \tag{34}$$

This is the exact solution for the time dependence if $\omega(k)=k_c v_p+(k-k_c)v_g+\frac{1}{2}(k-k_c)^2\Gamma$ exactly. It is helpful to also pull the i out of the denominator, writing the solution

$$A(x,t)=\exp\left[-\frac{1}{2}\left(\frac{x-(x_0+v_g t)}{\sigma(t)}\right)^2\right]e^{i\phi(x,t)} \tag{35}$$

where

$$\sigma(t)=\sigma_x\sqrt{1+\frac{\Gamma^2}{\sigma_x^4}t^2} \tag{36}$$

and

$$\phi(x,t)=k_c x-k_c t(v_g-v_p)-\frac{t\Gamma}{t^2\Gamma^2+\sigma_x^4} \tag{37}$$

How do we interpret this solution? It has a magnitude and a phase. The phase just causes the real part to oscillate between -1 and 1, which is not that interesting. So let's concentrate on the magnitude. The magnitude of a Gaussian only has three parameters, its overall normalization, its center and its width. We have written Eq. (35) in a way so that it is easy to read of that at time t the packet is centered at $(x_0+v_g t)$. This is consistent with what we found above: the center of the Gaussian moves with the group velocity. We can also read off from Eq. (35) that the the width at time t is given by the functino $\sigma(t)$ in Eq. (36). Notice that the **width is increasing with time**. That is, the wave-packet is broadening. This is why we call it a dispersion relation. Recall that a non-dispersive wave has $\Gamma=0$, so with non-dispersive dispersion relations, wavepackets don't disperse.

Here's a comparison of a nondispersive pulse, with $v=1$ to one with dispersion relation $\omega(k)=\sqrt{k^2+50^2}$. We construct a wavepacket of width $\sigma_x=0.5$ with a carrier wavenumber of $k_c=30$.

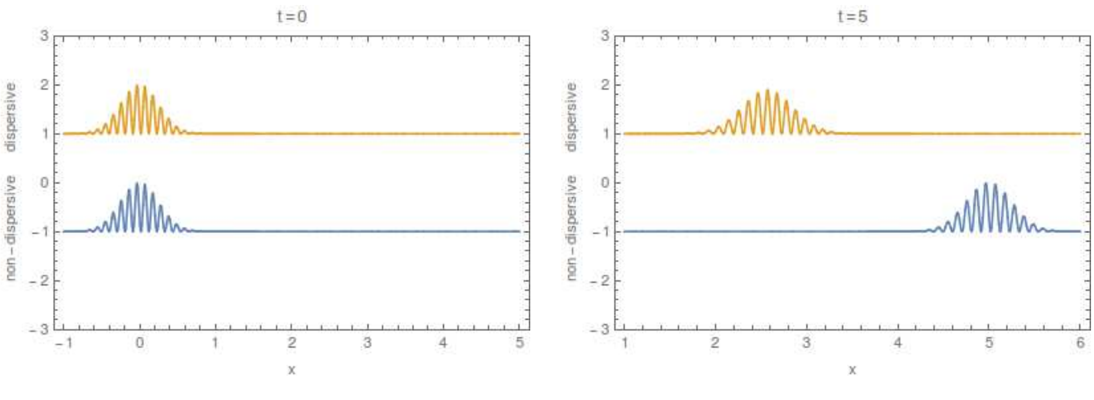


Figure 5. Pulses at t=0 and t=10. Dispersive packet is on top. Note that the dispersive one is moving at $v=0.5$ and the non-dispersive one at $v=1$.

For the non-dispersive pulse, the phase and group velocity are $v_p = v_g = 1$. Thus, after 5 seconds at is at $x = 5$, consistent with the figure. For the dispersive pulse, the phase and group velocities are

$$v_p = \frac{\omega(k_c)}{k_c} = 1.94, \quad v_g = \omega'(k_c) = 0.51 \tag{38}$$

One can see from the figure that at $t = 5$, the dispersive packet has gone half as far as the non-dispersive one, which is consistent with it traveling at the group velocity of $v_g = 0.51$.

At longer times, you can really see the pulse flatten out.

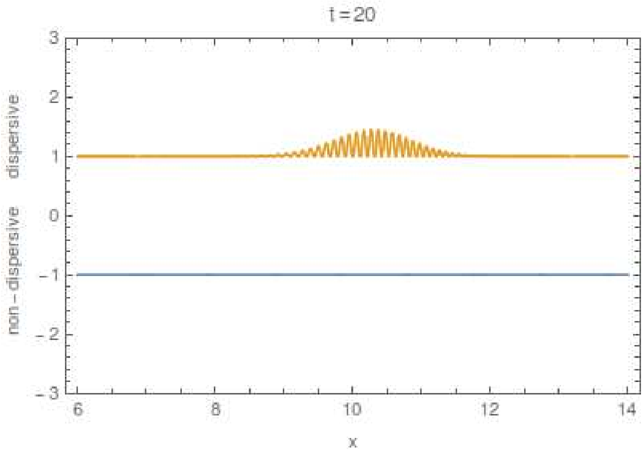


Figure 6. At t=20, the dispersive packet is significant broadened. The non-dispersive pulses is off the plot, near $x = 20$. It has the same shape it did originally.

Dispersion in optical media are critical to modern optics and to telecommunications. For example, high speed internet and long distance telephone communication are now done through fiber optic cables. Fiber optic cable contains a glass core surrounded by a lower index-of-refraction cladding. This allows light to be transported along the cable via total internal reflection. A key figure in telecommunication is the rate at which data can be communicated. In fiber optic telecommunications, information is transmitted via optical wavepackets in a glass fiber. Due to dispersion in the glass, pulses too close together begin to overlap, destroying the information. This sets a fundamental limit to the speed internet communications. Luckily, silica-based glass has very low dispersion (and absorption) in the near IR region (1.3-1.5 micron). This frequency band, now known as the telecomm band, has seen extensive technological development in the last 20 years due to its use for fiber optic communication. Fiber optics will be revisited when we discuss light.

By the way, in quantum mechanics, an electron is often effectively treated as a wavepacket. We will see that in Lecture 20 that non-relativistic dispersion relation for an electron is $\omega(k) = \frac{\hbar}{2m}\vec{k}^2$. So $v_g = 2v_p = \hbar\frac{k}{m} = \frac{p}{m}$ with $p = \hbar k$ the momentum and $\Gamma = \frac{\hbar}{m}$ the width. Thus the width becomes $\sigma(t) = \sigma_x\sqrt{1 + \left(\frac{\lambda_c c t}{2\pi\sigma_x^2}\right)^2}$ with $\lambda_c = \frac{h}{mc} = 2.42 \times 10^{-12}m$ called the Compton wavelength of the electron. The phase becomes $\phi(x,t) = k_c\left(x - \frac{p}{m}\right) + \frac{\lambda_c c t}{\lambda_c^2 c^2 t^2 + \sigma_x^4}$. At late times $\sigma(t) = \frac{\lambda_c}{2\pi\sigma_x}ct$ and $\phi(x,t) = k_c\left(x - \frac{p}{m}\right)$. Thus the center moves with the velocity $\frac{p}{m}$, as expected, and the width grows very rapidly: at the speed of light for an electron localized to within its Compton wavelength. The more you try to pin down an electron (smaller σ_x), the faster the wavepacket grows!