A Goldilocks Theory of Fiscal Deficits*

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Abstract

This paper proposes a tractable framework to analyze fiscal space and the dynamics of government debt, with a possibly binding zero lower bound (ZLB) constraint. Without the ZLB, a greater primary deficit unambiguously raises debt. However, debt need not explode: When $R < G - \varphi$, where φ is the sensitivity of $R - G$ to debt, a modest permanent increase in the deficit can be sustained forever—a "free lunch" policy. With the ZLB, the relationship between deficit and debt can become non-monotone. Both high and low deficits can increase debt, as the latter weaken demand and reduce nominal growth at the ZLB. A rise in income inequality expands fiscal space outside the ZLB, but contracts it at the ZLB. Calibrating the model, we find little space for "free lunch" policies for the United States in 2019, but significant space for Japan.

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1 Introduction

Advanced economies in recent years have been characterized by very low interest rates, also known as "secular stagnation" (Summers 2014). This has led economists and policymakers to challenge the textbook view on the relationship between public debt and (primary) deficits; raising current deficits may no longer have to be offset by lowering future deficits and raising taxes. Instead, when the interest rate R lies below the growth rate G, there may be a "free lunch" (Blanchard 2019), according to which deficits can be increased permanently without causing explosive debt dynamics; and higher debt levels can be sustained without reduced deficits. In other words, when R lies below G, the fiscal cost of increased debt may be zero or even negative.

This paper systematically studies the fiscal cost of borrowing and the joint dynamics of public debt and primary deficits. Our starting point is a tractable model with two main ingredients. First, R can lie below G, and R increases in government debt. We model this simply by assuming government debt provides convenience benefits (e.g., Krishnamurthy and Vissing-Jorgensen 2012, Greenwood, Hanson and Stein 2015) but also show that our results carry over to other microfoundations. The second ingredient is a zero lower bound (ZLB) constraint on the nominal interest rate R, which allows for the possibility that weak demand reduces output and inflation, and thus also the nominal growth rate G of the economy.

Building on the model, our paper makes four contributions. First, we show that the correct condition for the existence of a free lunch policy is not $R < G$; instead, it is a tighter condition, $R < G - \varphi$, where φ is the sensitivity of $R - G$ to the logarithm of public debt to GDP. As a consequence, even for countries in which $R < G$, borrowing more may not be free. The intuition for why $R < G - \varphi$ is the free lunch condition is the following. Suppose that $R < G$. The government decides to borrow one additional percent of GDP and plans to roll it over forever. This fiscal choice will have two opposing effects on the government's budget constraint. On the one hand, the rolling over of the additional debt produces a positive cash flow for the government equal to $G - R$. On the other hand, however, the additional debt also tightens the budget constraint because of its impact on the interest rate on all infra-marginal outstanding units of debt. This latter effect is precisely captured by φ and combining the two effects gives us $R < G - \varphi$ as the free lunch condition.

Our second contribution is to characterize the dynamics of debt and deficits at or near the ZLB. This is important, as many economies with low R today are close to the ZLB. There, deficits are important instruments to increase aggregate demand (Blanchard and Tashiro 2019, Furman and Summers 2020). We show that this aggregate-demand channel may "invert" the textbook view on deficits and debt at the ZLB: Greater deficits may reduce, rather than increase, debt. This is because greater deficits raise aggregate demand and inflation; higher inflation translates into higher nominal growth rates; this pushes debt down, as it increases the speed at which debt is "inflated away". This indirect effect through the nominal growth rate can be sufficiently strong to overwhelm the direct effect of greater deficits on debt.

The third contribution of our paper is to study the role of inequality and tax progressivity. Inequality matters for government debt since as much as 69% of U.S. government debt held by U.S. households is directly or indirectly held by households in the top 10% of the U.S. wealth distribution (Mian, Straub and Sufi 2020). To evaluate the role of inequality in our framework, we allow for saver and spender households, as in Campbell and Mankiw (1989), Mankiw (2000), Galí, López-Salido and Vallés (2007), and Bilbiie (2008). Using these two types of households, we show that increased inequality, modeled as a greater share of income earned by savers, increases fiscal space and increases the availability of free lunch policies outside the ZLB. We believe that this finding is interesting as it points to a potential conflict between reducing inequality (e.g. via progressive taxation) on the one hand, and funding large deficits on the other.

The fourth contribution of our paper is the development of a quantitative version of our model. We calibrate our model first to the pre-Covid United States, and then Japan in order to illustrate how our model can be used in practice to evaluate fiscal space. Our results suggest that the United States was just inside the free lunch region before Covid, and could sustain a maximum permanent primary deficit of just over 2% of GDP at a stable debt-to-GDP ratio of about 120%. A naive $R < G$ rule might suggest that the U.S. had considerable additional fiscal space prior to Covid, but since empirical estimates of φ for the U.S. are around 2 pp, the true constraint faced by the United States is significantly tighter. Thus higher deficits beyond the free lunch limit of 120% have to be paid for either through higher future taxes or reduced spending.

In contrast to the United States, our calibration for Japan as of December 2019 shows significant room for free lunch policies. In fact, we find that Japan is in the "inverted" regime, in which a modest increase in deficits would reduce debt levels, precisely due to the effect on aggregate demand and inflation.We also provide direct empirical evidence to show that Japanese output and inflation respond sufficiently aggressively to government spending shocks during the Japanese ZLB episode in order to "invert" the relationship between fiscal deficit and government debt.

We make our four contributions by analyzing our model in an intuitive "deficit-debt diagram", in which a locus characterizes the feasible set of steady state combinations of the primary deficit and debt. The deficit-debt locus is hump-shaped: steady state deficits are zero both for zero debt and when debt is sufficiently large that $R = G$. In between, deficits are positive, consistent with the idea that $R < G$ allows an economy to permanently run positive deficits. The locus characterizes where a free lunch policy is available, namely exactly on the left branch of the locus, to the left of its peak. It also explains how, at the ZLB, the inverted relationship between deficits and debt levels occurs because of a "backward-bending" shape of the deficit-debt locus. While we focus on our tractable model for the most part in our paper, we plot the deficit-debt diagram also for a number of alternative models, to illustrate how our results generalize.

We provide two extensions to our basic framework in the main body of our paper, and several more in the appendix. The first extension is the introduction of aggregate risk into our model, building on the framework of Mehrotra and Sergeyev (2020). We prove that, even with aggregate risk, our free lunch condition remains informative. The second extension adds capital to the production function, and allows government debt to crowd out capital, as in Blanchard and Weil (2001). Interestingly, we show that greater crowding out of capital increases fiscal space and makes a free lunch policy more likely to exist as it reduces the sensitivity of the interest rate to the level of government debt.

Related Literature. This study is part of a growing body of theoretical work inspired by two key facts. First, the nominal interest rate on government debt is lower than the nominal growth rate on average, i.e. $R < G$.[1] Second, the demand curve for government debt slopes down empirically, i.e. $\varphi > 0$, as the interest rate on government debt rises when the government issues more debt (Engen and Hubbard 2004, Laubach 2009, Krishnamurthy and Vissing-Jorgensen 2012, Greenwood, Hanson and Stein 2015, Presbitero and Wiriadinata 2020).[2]

The literature has explored several ways to explain one or both of these facts. Bohn (1995) and Barro (2020) suggest that $R < G$ can naturally occur in complete markets economies with aggregate risk. Due to Ricardian equivalence (Barro 1974), the model suggests that government debt neither affects R, nor can the government run a permanent deficit in each state of the world. According to Jiang, Lustig, Van Nieuwerburgh and Xiaolan (2019), this approach cannot explain the valuation of U.S. government debt.

The perhaps largest literature on $R < G$ is based on OLG models, going back to Samuelson (1958) and Diamond (1965). One branch of this literature studies when $R < G$ is a sign of dynamic inefficiency (Abel, Mankiw, Summers and Zeckhauser 1989, Blanchard

[1]See Feldstein (1976), Bohn (1991), Ball, Elmendorf and Mankiw (1998), Blanchard (2019), Mehrotra and Sergeyev (2020) for more recent papers documenting the historical patterns of R vs. G.

[2]For recent work estimating the convenience yield of central bank reserves, see Lopez-Salido and Vissing-Jorgensen (2023).

and Weil 2001, Ball and Mankiw 2021); another branch evaluates the welfare implications of increased debt levels (Ball, Elmendorf and Mankiw 1998, Blanchard 2019, Brumm, Feng, Kotlikoff and Kubler 2021*a*,*b*, Brumm and Hußmann 2023); the most relevant branch of the OLG literature for us is the one concerned with the possibility of "free lunch" policies (Blanchard and Weil 2001, Blanchard 2019). These papers show that, when $R < G$, a debt-rollover policy is more likely to succeed when the economy is inefficient and production is linear in capital, but no general condition is developed.[3] Our paper develops a precise condition for a free lunch to exist in a deterministic model, $R < G - \varphi$, which is significantly stricter than $R < G$. We show that the condition still has bite with aggregate risk, and even holds in the Blanchard (2019) model itself. Interestingly, in recent work, Aguiar, Amador and Arellano (2021) find that a similar condition is indicative of the possibility of robust welfare improvements.

The above facts have also been approached using liquidity premia. Woodford (1990) illustrates how liquidity demand by producers or consumers can lead to $R < G$. Angeletos, Collard and Dellas (2020) microfound a convenience yield function based on liquidity needs to revisit the optimality of the Barro (1979) tax smoothing results.[4] Bayer, Born and Luetticke (2021) estimate the response of the liquidity premium to fiscal policy shocks empirically and model it with an estimated two-asset HANK model. Domeij and Ellingsen (2018) obtain $R < G$ in a Bewley-Aiyagari model. The closest paper to ours among this class of models is Reis (2021). The paper microfounds liquidity and safety premia of government debt and shows that a "bubble premium" emerges on public debt, which can be used to sustain permanent primary deficits. Different from Reis (2021), we focus on the dynamics of debt and deficits, both with and without the ZLB, and show when a free lunch exists.[5]

Angeletos, Lian and Wolf (2023) ask whether "deficits can finance themselves" without increases in tax *rates*. Their self-financing policies are typically financed by increased tax *revenue*, however, coming from a booming economy and an increased tax base. The free lunch policies we study are, instead, policies which do not rely on increased tax revenue (or reduced government spending) at any point in time.

Mehrotra and Sergeyev (2020) share with our paper the assumption of a convenience utility function $v(b)$ over government debt, which they employ in a model with aggregate risk and a specific deficit rule that yields a particularly tractable law of motion of debt-to-

[3] There is also a long literature on the private production of assets when the return on non-government assets is also below the growth rate, see, e.g., Tirole (1985), Kocherlakota (2009), Farhi and Tirole (2012), Hirano and Yanagawa (2016), and Martin and Ventura (2018).

[4] See also Canzoneri, Cumby and Diba (2016), Bhandari, Evans, Golosov and Sargent (2017), Azzimonti and Yared (2019).

[5] In Appendix D, we show that the economy in Reis (2021) can also be represented in a deficit-debt diagram.

GDP. They use it to show that slower trend growth and higher output risk can increase debt-to-GDP.[6] By comparison, our paper focuses on free lunch policies and the possibility of a binding ZLB. Kocherlakota (2021) microfounds a linear utility from holding bonds with a small probability disaster shock and shows how this allows the government to improve welfare by increasing debt. Michau (2020) presents a model with net wealth in the utility and a ZLB constraint and uses it to study fiscal policy plans that lead the economy away from the ZLB (see also Michau, Ono and Schlegl 2023). Guerrieri, Lorenzoni and Rognlie (2021) argue that a potentially binding ZLB constraint in the future can be a rationale for preserving fiscal space.

Our model is based on the assumption that monetary policy is active in stabilizing inflation whenever it is not constraint by the ZLB. A recent branch of the literature explores deviations from this assumption. Kaplan, Nikolakoudis and Violante (2023) study fiscal dominance in heterogeneous-agent and bonds-in-utility models.[7] Brunnermeier, Merkel and Sannikov (2020*a*,*b*) derive a Laffer curve for the rate of inflation in a model with liquidity needs among producers. Sims (2019) argues that fiscal policy should, in general, use this "inflation tax" to generate seignorage-like revenue and reduce distortionary taxes (different from Chari and Kehoe 1999). The deficit-debt schedule that we derive, and on which our phase diagram is based, may seem similar to the inflation Laffer curve, but is quite distinct (see Section 3.4 and Appendix C).

This study is also closely related to the burgeoning literature on the sources and implications of safe asset demand (e.g., Caballero, Farhi and Gourinchas 2008, Caballero and Farhi 2018*a*, and Farhi and Maggiori 2018). In their model of the international monetary system, Farhi and Maggiori (2018) explore an equilibrium in which there is large demand for debt issued by a hegemon government. When this is met by too much issuance, default risk emerges. When there is too little issuance, the ZLB may bind. This pattern resembles our deficit-debt diagram with a potentially binding ZLB, albeit it emerges in our case as the steady state locus of a dynamic model, rather than as a one-shot choice of the government as in Farhi and Maggiori (2018).

Finally, the notion of a free lunch formalizes an intuition that is often associated with "Modern Monetary Theory" (MMT). However, unlike common renditions of MMT (see Bisin 2020, Leeper 2022 for critical reviews), our model spells out the exact conditions under which a free lunch policy works or does not work. In line with intuition by Lerner (1943), we find that a free lunch policy always exists if an economy faces a persistent demand shortage at the ZLB.

[6]See also Kaldorf and Roettger (2023) for a model with convenience yield and default risk.

[7]See also the recent work by Bassetto and Cui (2018) and Bianchi and Melosi (2019). See Bassetto and Sargent (2020) for an excellent survey.

2 Model

We begin with a stylized model that we extend in later sections. The model runs in continuous time and is deterministic.[8] It consists of a government, a household side with savers and spenders, and a monetary authority. The government issues government debt, spends, and raises lump-sum taxes. Spenders and savers consume and savers draw convenience benefits from holding government debt. The monetary authority targets inflation.

Throughout, we denote by R_t the net nominal interest rate on government debt and by $G_t \equiv \gamma + \pi_t$ the net nominal growth rate, which is equal to real trend growth γ plus inflation π_t. $G^* \equiv \gamma + \pi^*$ corresponds to nominal trend growth, when inflation is at its target π^*.

To save on notation, we will conduct our analysis entirely in the context of a model that is de-trended with the nominal growth rate. Potential output y^* in the de-trended model is constant and we normalize it to one, $y^* \equiv 1$. Any quantities, such as the level of government debt b_t are to be understood as government debt relative to potential GDP. Moreover, we refer to $R_t - G_t$ as the "de-trended rate of return" on government debt, as it is the return R_t net of the re-investment that is necessary to keep a constant ratio of government debt to potential GDP. We abstract from capital in our baseline model, but discuss it at length in Sections 6 and 7.

Households. The economy is populated by a unit mass of savers and a unit mass of spenders, as in Campbell and Mankiw (1989) and Mankiw (2000). Savers choose paths of consumption c_t and government debt holdings b_t in order to maximize

$$\max_{\{c_t, b_t\}} \int_0^\infty e^{-\rho t} \{\log c_t + v(b_t)\} \, dt \tag{1}$$

subject to the consolidated budget constraint

$$c_t + \dot{b}_t \leq (R_t - G_t) b_t + (1 - \mu) w_t n_t - \tau_t. \tag{2}$$

The objective (1) involves flow utility from consumption $\log c_t$ and a utility $v(b_t)$ from holding government debt (relative to potential GDP). The latter captures safety and liquidity benefits that have been used extensively and are well documented in the literature (e.g. Krishnamurthy and Vissing-Jorgensen 2012). In line with this literature, we assume that the

[8] We separately study aggregate risk in Section 5.

utility over government debt is twice differentiable, increasing and concave, $v' \geq 0, v'' \leq 0$.[9] Flow utility is discounted at rate ρ.

Each saver has a labor endowment of $1-\mu$, where $\mu \in [0,1)$ captures the income share of spenders. Savers sell a fraction $n_t \leq 1$ of their endowments at real wage w_t each instant. n_t can lie strictly below 1 if there is rationing (see below). Savers pay lump-sum taxes τ_t.

Spenders are hand-to-mouth.[10] Each spender has a labor endowment of μ, and also sells a fraction n_t of it at real wage w_t. Spenders pay lump-sum taxes $\tilde{\tau}_t$. Thus, their consumption is equal to

$$\tilde{c}_t = \mu n_t w_t - \tilde{\tau}_t. \tag{3}$$

Representative firm. We assume that labor is used by a representative firm with linear production technology $y_t = n_t$. The firm sets flexible prices, pinning down the real wage to 1 at all times, $w_t = 1$. In contrast, we assume that nominal wages are downwardly rigid. Similar to Schmitt-Grohé and Uribe (2016), the path of nominal wages W_t satisfies

$$\frac{\dot{W}_t}{W_t} \geq \pi^* - \kappa(1-n_t). \tag{4}$$

This implies that, whenever labor demand is falling short of the labor endowments, wage inflation will fall short of π^*. The lower labor demand is, the lower wage inflation will be, just like in a standard Phillips curve. $\kappa > 0$ parameterizes the slope of the Phillips curve. Price inflation π_t in our de-trended model is equal to wage inflation and therefore determined by (4). Observe that potential output, with $n_t = 1$, is indeed equal to one, $y^* = 1$. The term $1-n_t$ in (4) is therefore simply equal to the output gap, $(y^* - y_t)/y^*$.

Government. The government sets fiscal and monetary policy. Fiscal policy consists of paths $\{x, b_t, \tau_t, \tilde{\tau}_t\}$ of government spending x, government debt b_t and taxes $\tau_t, \tilde{\tau}_t$, subject to the flow budget constraint

$$x + (R_t - G_t)\, b_t = \dot{b}_t + \tau_t + \tilde{\tau}_t. \tag{5}$$

The primary deficit is given by

$$z_t \equiv x - \tau_t - \tilde{\tau}_t. \tag{6}$$

[9] We also assume that v is defined over some domain $(\underline{b}, \infty)$ with $\underline{b} \leq 0$, with Inada conditions $\lim_{b \to \underline{b}} v'(b) = \infty$, $\lim_{b \to \infty} v'(b) = 0$. Moreover, we assume that $v'' < 0$ whenever $v' > 0$.

[10] One can easily microfound this behavior by assuming that spenders do not enjoy any convenience benefits from holding government bonds and are unable to borrow.

We assume taxes adjust to ensure that z_t follows a given fiscal rule $z_t = \mathcal{Z}(b_t)$. Our baseline assumption is that taxes on spenders are zero $\tilde{\tau} = 0$ and taxes on savers τ_t adjust. We consider the case where $\tilde{\tau} \neq 0$ in Section 3.5. Typically, $\mathcal{Z}(b)$ is downward-sloping in debt b, corresponding to a lower deficit, or greater surplus, when the level of debt is higher.

Government debt b_t is short-term and real in our baseline model. We study long-term debt in Appendix B.2 and a general asset market structure in Appendix B.3. Government spending $x \geq 0$ is assumed to be constant for now. Our analysis below is similar to one in which government spending is allowed to vary while taxes are kept fixed.

Monetary policy is "dominant" in our model, that is, it successfully implements the natural allocation whenever feasible. In particular, we denote by $\{R_t^*\}$ the path of the nominal natural interest rate, which is consistent with full employment, $n_t = 1$, at all dates t. We assume that the actual nominal interest rate then follows a Taylor rule with an infinite slope coefficient, subject to the ZLB:

$$R_t = 0 \text{ if } \pi_t < \pi^*, \qquad R_t \in [0, \infty) \text{ if } \pi_t = \pi^*, \qquad R_t = \infty \text{ if } \pi_t > \pi^* \tag{7}$$

This rule ensures that, whenever the interest rate is positive, it tracks the flexible-price (natural) interest rate R_t^*; in that case, the economy is at potential $y_t = n_t = 1$ and inflation is at its target $\pi_t = \pi^*$. When the natural rate is negative, however, R_t is constrained to be equal to zero by the ZLB. In that case, we will find that the economy falls below potential, $y_t = n_t < 1$. Labor endowments are rationed, equally across the two types of agents.[11]

Equilibrium. We define equilibrium in our model as follows.

Definition 1. Given an initial level of debt b_0 and a fiscal rule $\mathcal{Z}(\cdot)$, a (competitive) equilibrium consists of a tuple $\{c_t, \tilde{c}_t, y_t, n_t, b_t, R_t, G_t, \pi_t, \tau_t, \tilde{\tau}_t, z_t, w_t\}$, such that: (a) $\{c_t, b_t\}$ maximizes savers' objective (1) subject to (2), and $\tilde{c}_t$ satisfies (3); (b) the deficit $\{z_t\}$ follows the fiscal rule $\mathcal{Z}$ and taxes are in line with (6); (c) debt evolves in line with the flow budget constraint (5) and remains bounded; (d) monetary policy sets the nominal rate R_t in line with the rule (7) and the nominal rate R_t is always finite; (e) inflation π_t is determined by the Phillips curve (4); (f) output y_t is given by $y_t = n_t$ and the real wage is $w_t = 1$; (g) the goods market clears $c_t + \tilde{c}_t + x = y_t$. A *steady state equilibrium* is an equilibrium in which all quantities, real prices, and inflation are constant.

Interpretation of $v(b)$. There are two ways to interpret the convenience utility $v(b)$, either as coming from the asset supply or the asset demand side.

[11] This is similar to the rationing equilibria in Barro and Grossman (1971), Malinvaud (1977), and Benassy (1986).

$v(b)$ as coming from asset supply. According to this view, government debt offers asset-specific benefits, due to liquidity, safety, regulatory requirements, or international institutional demands. These benefits, or some subset of them, are often grouped together as "convenience benefits", and collectively explain why certain government bonds may have a particularly low yield relative to seemingly similar other assets (Krishnamurthy and Vissing-Jorgensen 2012, Caballero, Farhi and Gourinchas 2017, Jiang, Lustig, Van Nieuwerburgh and Xiaolan 2020*b*, Koijen and Yogo 2020, Mota 2020). In Appendix B.1 we offer a simple microfoundation for our convenience utility $v(b)$ based on a low-probability disaster shock (as in Barro 2020), after which a government may default on its debt.

$v(b)$ as coming from asset demand. According to this view, savers require higher yields in order to hold greater amounts of government debt. This can be microfounded with life-cycle (as in Diamond 1965, Blanchard 2019) or precautionary saving motives (as in Aiyagari and McGrattan 1998). These models will be part of our quantitative investigation in Section 7. We find that the results are similar to those we find with our $v(b)$ utility.

3 Fiscal space without the ZLB

In this section, we focus on the case without a ZLB constraint, so that $R_t = R_t^*$ in all periods, effectively implementing the full employment allocation. We study the role of the ZLB in Section 4. We begin our analysis by characterizing steady state equilibria.

3.1 Steady state equilibria

Our model admits a set of steady state equilibria, indexed by the level of steady state debt $b \geq 0$. For each b, one can find a primary deficit z such that $\dot{b} = 0$ and the economy remains steady at that level of debt b. The interest rate is equal to the natural rate, $R_t = R_t^*$, output and employment are at potential, $y_t = n_t = 1$, inflation is at its target, $\pi_t = \pi^*$, and the nominal growth rate is equal to nominal trend growth, $G_t = G^*$.

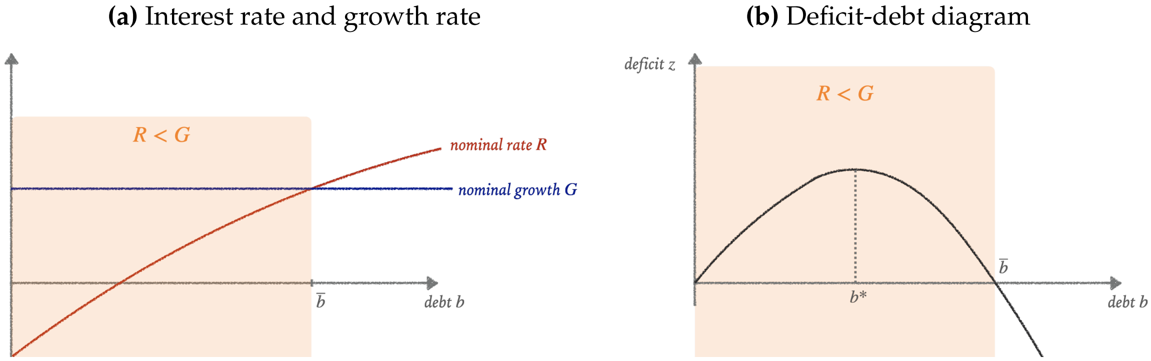
To see how the natural rate is determined, consider the savers' Euler equation

$$\frac{\dot{c}_t}{c_t} = R_t^* - G^* - \rho + v'(b_t)c_t. \tag{8}$$

Here, $v'(b_t)$ enters as it is the marginal convenience utility from saving one more unit in government bonds. It enters with the opposite sign as the discount rate ρ and therefore effectively makes the household more patient when saving in government bonds.

In a steady state, savers' consumption is constant and equal to $1 - x - \mu$ by goods market clearing, where x is government spending and μ consumption of spenders. This

Figure 1: Interest rate, growth rate, and deficits. Case without ZLB.



Note. Panel a plots the nominal interest rate R and nominal growth rate G as functions of real debt (relative to potential GDP). Panel b plots the steady-state primary deficit as a function of the steady-state debt level.

lets us solve (8) for the natural interest rate,

$$R^*(b) = \rho + G^* - \underbrace{v'(b) \cdot (1 - x - \mu)}_{\text{convenience yield}}. \tag{9}$$

This expression for the natural interest rate on government debt is intuitive. The natural rate is equal to $\rho + G^*$, which would be the steady state return on any non-convenience-bearing assets, minus the steady state convenience yield $v'(b) \cdot (1 - x - \mu)$. The expression already suggests how R^* moves with debt. As v is a concave utility function, R^* weakly increases in government debt b.

3.2 Steady state deficits and deficit-debt diagram

It is useful to represent $R^*(b)$ and G^* in a diagram, Figure 1a. The threshold for $R = G$ is determined by $v'(\bar{b})(1 - x - \mu) = \rho$. It is positive, $\bar{b} > 0$, if $v'(0)\,(1 - x - \mu) > \rho$.[12] For any given level of debt b, the primary deficit in (5) that keeps debt constant at b is

$$z(b) = (G^* - R^*(b))\, b. \tag{10}$$

We plot $z(b)$ in Figure 1b. We refer to this diagram as the *deficit-debt diagram* and we will use it extensively in this paper. Each point (b, z) on the locus corresponds to a steady state equilibrium with constant debt level b and constant primary deficit z. The locus is naturally hump-shaped. If $R^*(0)$ is finite, the steady state primary deficit is zero when debt is zero,

[12] Different from the discussion in Reis (2021), the bound $\bar{b}$ here can be unboundedly large relative to GDP as $\rho \to 0$.

as well as when $R = G$. Between 0 and $\bar{b}$, the primary deficit is positive.

The deficit-debt diagram is an intuitive description of an economy's "fiscal space": for any given initial level of debt, it exactly shows what the maximum primary deficit is for which debt does not increase.

To characterize the shape further, we define the semi-elasticity of the convenience yield

$$\varphi(b) \equiv \frac{\partial (R^*(b) - G^*)}{\partial \log b} - (1 - x - \mu)\frac{\partial v'(b)}{\partial \log b} = -(1 - x - \mu)v''(b)b.$$

φ is effectively the inverse (semi-)elasticity of savers' demand for government debt. With φ at hand, we then have the following result.

Proposition 1. *If $z(b)$ has a local maximum at some $b^* \in (0, \bar{b})$, we have that*

$$R^*(b^*) = G^* - \varphi(b^*). \tag{11}$$

If, in addition, $\varphi(b)$ is weakly increasing in b, then b^* is the unique local (and global) maximum, with primary deficit $z^* \equiv \varphi(b^*)b^*$.[13]

3.3 The free lunch condition $R < G - \varphi$

One idea that has garnered considerable attention in the literature surrounding $R < G$ (see, e.g., Blanchard 2019) is that the condition seemingly allows economies to run larger deficits temporarily, and then simply "grow out" of the resulting increased debt levels without a need to raise taxes. We refer to this idea as the "free lunch" property of higher deficits. Formally, a steady state with deficit z_0 and debt b_0 admits a free lunch if there exists an equilibrium (with bounded debt levels) in which deficits weakly dominate z_0 in all periods, $z_t \geq z_0$, with strict inequality on some time interval.

A free lunch can easily be derived from the government budget constraint (5), under the assumption of a constant interest rate R and a constant growth rate $G > R$. Then,

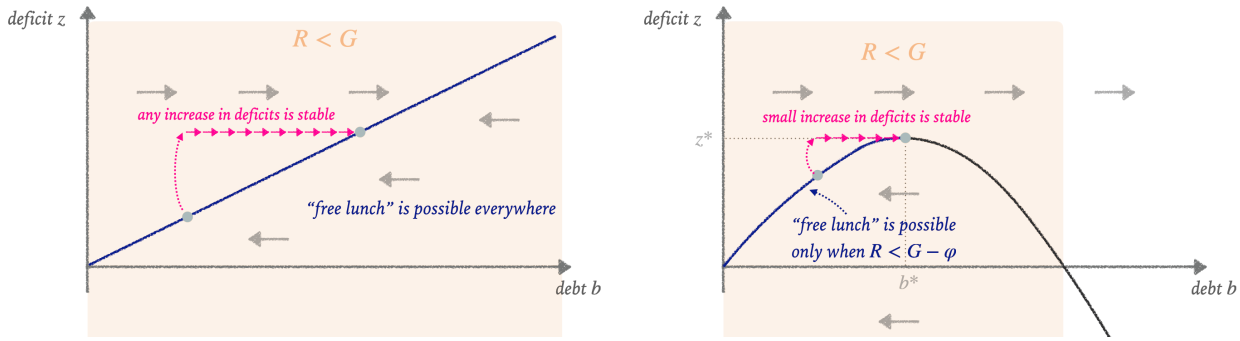
$$\dot{b}_t = -(G - R)\, b_t + z \tag{12}$$

describes a stable differential equation for debt b. This implies that temporary increases in deficits of arbitrary magnitude, leading to greater debt levels, can always be grown out of over time. Also, a permanent increase in deficits by some Δz simply raises steady state

[13]The fact that the maximum is in the interior of $[0, \bar{b}]$ echoes a similar finding in Bassetto and Sargent (2020) in an OLG setting.

Figure 2: Transitions when changing the deficit

(a) Model with exogenous $R < G$ **(b)** Model with endogenous $R^*(b)$



Note. Panel a shows a deficit-debt diagram with exogenous $R < G$. Any increase in deficits induces a stable trajectory of debt, and thus, represents a free lunch. Panel b shows a deficit-debt diagram like ours in which R is endogenous to the debt level. Here, a free lunch is only feasible left of the peak and only for modest increases in deficits.

debt levels by $\Delta z/(G - R)$, with no need for a reduction in deficits, i.e. an increase in taxes, at any point.

We next investigate when our model, in which R is endogenous to the debt level, admits a free lunch. Since savers' consumption remains constant at $1 - x - \mu$ along transitions, the natural rate $R^*(b_t)$ is still given by (9) in our model. Therefore, the dynamics of the debt level simply follow

$$\dot{b}_t = -\left(G^* - R^*(b_t)\right) b_t + z_t \tag{13}$$

for an exogenous path of deficits z_t.[14]

Representing transitions in the deficit-debt diagram. The effects of temporary or permanent changes in deficits can be studied in the deficit-debt diagram. In Figure 2 we indicate with arrows the direction the economy travels in when deficits are moved above or below the steady state locus.

As the figure shows, when deficits are raised above the steady state locus, debt grows, until either the steady state locus is hit, or until, at some point in the future, the deficit is reduced again down to the steady state locus. When deficits are reduced below the steady state locus, debt falls over time. Mathematically, this follows immediately from (13).

Figure 2a plots the evolution of debt in a model with exogenous R and G. As one can see, in this case, any increase in deficits is stable. A free lunch policy is always available. This

[14]If deficits followed a fiscal rule $z_t = \mathcal{Z}(b_t)$ instead, one would simply have to replace z_t in (13). The dynamics of debt are perfectly backward looking because savers' consumption is constant at $1 - x - \mu$ even along transitions.

would be the outcome of a model with a linear convenience utility $v(b)$, and is analyzed in Kocherlakota (2021), who microfounds a linear utility over bonds by allowing for a small probability disaster state with high and constant marginal utility of wealth. This is also why the debt rollover experiments in Blanchard and Weil (2001) and Blanchard (2019) are stable with linear technology.

The free lunch region in our model. By contrast, Figure 2b allows us to see the region of the state space in which the government can obtain a "free lunch" in our model, with endogenous R. Indeed, any steady state on the increasing part to the left of the peak at b^* allows for some form of a free lunch. For example, starting at any of these steady states, a permanent increase in the deficit to any value below or equal to z^* can be sustained indefinitely. If the deficit increase is temporary, it can exceed z^*, as long as it is reduced back to z^* or below in time. We show an example transition along these lines in Figure 2b.

While the diagram in Figure 2b illustrates how a "free lunch" policy is indeed possible, it also makes the limits of such a policy very clear. Even starting left of b^*, if deficits are increased by too much or for too long, a free lunch cannot be obtained. There is also no free lunch right of b^*, on the downward-sloping branch of the deficit-debt locus in Figure 2b.[15] In this case, any deficit increase must ultimately be met by reduced deficits. Crucially, this logic applies *even if* the economy displays $R < G$ throughout.

How is this possible? As the debt level increases, so does the interest cost on all (inframarginal) outstanding debt positions. This can undo the positive effect of a greater debt position on the government budget constraint when $R < G$ that we highlighted at the beginning of this section. In fact, as Figure 2b illustrates, this precisely happens for debt levels greater than b^*. In this region, the economics behind the financing of fiscal deficits are entirely conventional: greater debt must be repaid by raising taxes. Whether $R < G$ or $R > G$ is totally irrelevant for this question. As summarized in the following corollary, the correct threshold for R is not G, but $G - \varphi$.

Corollary 1. *Assume the deficit-debt diagram $z(b)$ is single-peaked. Then, there is a free-lunch policy available at a steady state with debt level $b_0 > 0$ if and only if $R^*(b_0) < G^* - \varphi(b_0)$.*

3.4 Discussion

Is a free lunch policy always Pareto-improving? We largely refrain from making welfare statements in this paper, partly because different microfoundations for the convenience

[15] Strictly speaking, there could be multiple local maxima of $z(b)$ in our model. The condition for the absence of a free lunch policy is that there can be no steady state with a greater debt level and a greater or equal deficit z.

utility $v(b)$ exist, and they carry different welfare implications.

If the model in Section 2 is taken literally, then a free lunch policy always constitutes a Pareto improvement. It is easy to see why: consumption of both agents remains unchanged in all periods, while debt increases. Since debt enters the utility of savers, welfare increases. In fact, by a similar logic, increases in the debt level even beyond the upper bound b^* of the free lunch region can be welfare improving.

This becomes a bit more nuanced if one assumes that both agents are paying taxes, e.g. for simplicity $\tau_t = \tilde{\tau}_t$ with both taxes adjusting in response to the policy. Now, a free lunch policy is still a Pareto-improvement since it is associated with tax reductions for both agents, and higher interest rates for savers. However, raising debt beyond b^* is no longer Pareto improving, echoing results in Aguiar, Amador and Arellano (2021).

Transversality condition. The transversality condition of the saver associated with utility maximization problem (1) is given by $e^{-\rho t} c_t^{-1} b_t \to 0$. This is clearly satisfied in the equilibria described above, as $c_t = 1 - x$ and b_t always converges to a finite value. The transversality condition rules out paths along which debt levels explode.

Present value vs. flow budget constraint. Our analysis illustrates the usefulness of working with the government's flow budget constraint. We have found the present value budget constraint of the government to be somewhat less practical. To see why, let us discount the flow budget constraint (5) at some arbitrary rate θ_t. We obtain

$$\int_0^T e^{-\int_0^t \theta_u du} z_t dt + b_0 = e^{-\int_0^T \theta_u du} b_T - \int_0^T e^{-\int_0^t \theta_u du} \left(R(b_t) - G^* - \theta_t\right) b_t dt \tag{14}$$

(14) is equivalent to the flow budget constraint (5). However, (14) is less useful than typical present value budget constraints. This is because in (5), the interest rate $R(b_t)$ is a function of the stock of debt b_t. Irrespective of how θ_t is chosen, the path of debt b_t cannot be eliminated from (14), defeating one of the main purposes of writing a present value constraint. If θ_t is chosen to be entirely unrelated to $R(b_t) - G^*$, e.g. equal to the household discount rate ρ, the dependence on b_t enters in the final term in (14); if, instead, θ_t is chosen to be equal to $R(b_t) - G^*$, the final term in (14) disappears but the dependence on b_t enters in (14) through θ_t. Moreover, if $\theta_t < 0$, one cannot take the limit $T \to \infty$ in (14). This is why we prefer to work with the flow budget constraint (5) instead.

If one had to work with (14), a natural choice for θ_t is the marginal cost of borrowing, $\theta_t = R(b_t) - G^* + \varphi(b_t)$, which includes $\varphi(b_t)$. Locally around a steady state with debt b_{ss}, interest rate $R_{ss} = R(b_{ss})$ and $\varphi = \varphi(b_{ss})$, we then find a present value constraint

$$\int_0^\infty e^{-(R_{ss}-G+\varphi)t} z_t dt + b_{ss} = \frac{\varphi}{R_{ss} - G + \varphi} b_{ss}. \tag{15}$$

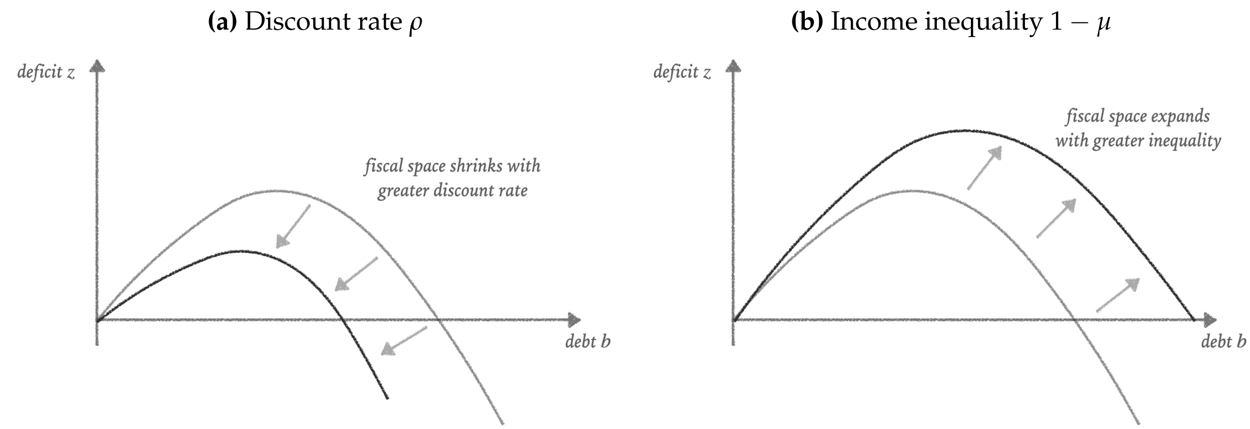
(15) is well-defined whenever there is no free lunch, that is, $R_{ss} > G - \varphi$. Relative to a standard present value condition, it includes an extra term (one may call this a "debt revenue term" as in Reis 2021) on the right hand side of (15), capturing additional fiscal space afforded by the convenience utility. The fact that discounting includes φ means that this condition is well defined even if $R_{ss} < G_{ss}$, so long as there is no free lunch, $R_{ss} > G_{ss} - \varphi$. If there is a free lunch, (15) is not well-defined as, locally, there are no constraints on deficits z_t around a free lunch steady state.

Comparison with a Sidrauski (1967) money-in-the-utility model. Our model is related to money-in-the-utility (MiU) models in that a real asset enters the utility function directly. As we show in Appendix C, a straightforward money-in-the-utility version of our model would give a steady state first order condition $\rho + G^* = v'(M/P) \cdot (1 - x - \mu)$ where M is money supply and P is the price level. Despite the similarities between this equation and our steady state condition (9), transitions differ significantly in the two models. In our model, as shown in Corollary 1, higher deficits raise debt levels, and a free lunch is available left of the peak, that is, when $R < G - \varphi$. In the baseline MiU model with perfectly flexible prices, one can always jump to the peak directly, without any transition, simply by choice of the optimal growth rate of money supply. In that sense, a free lunch is available left *and* right of the peak.

In Appendix C, we study MiU models with a small amount of nominal rigidity, preventing the price level from jumping. We show that this implies a free lunch is only available *right* of the peak—the opposite of our results. To get an intuition, observe that, in MiU models, the nominal interest rate on money is zero, $R_t = 0$, implying a real interest rate that falls with greater deficits. In our analysis above, by contrast, the real rate rises with greater deficits. Thus, MiU models with minimal amounts of nominal rigidity predict similar steady states but very different free lunch policies.[16]

[16] An alternative way to prevent the price level from jumping is to simply assume that the government cannot enact any policies that lead to sudden changes in the price level, known as a "honest government constraint" (Auernheimer 1974). Such a constraint can rule out transitions from right of the peak since those then require the government to reduce money supply by running surpluses in the short run. The "honest government constraint" is no longer a constraint in an economy with mild nominal rigidities, like the one we analyze in Appendix C.

Figure 3: What determines fiscal space?



Note. Panel a illustrates how an increase in the household discount rate ρ, capturing an increase in private demand, raises fiscal space. Panel b illustrates how greater inequality expands fiscal space.

3.5 What determines fiscal space?

What does the size and shape of the deficit-debt locus depend on? This section investigates the role of four factors: discount rates, trend growth, income inequality, and tax policy.

Discount rates. A greater discount rate, which can capture an increase in aggregate demand, makes savers more impatient, pushing up R^*. Figure 3a sketches the deficit-debt diagram, for two values of ρ. For higher ρ, we see that fiscal space shrinks, as R^* is increased and $G^* - R^*$ falls. We confirm this in the following result.

Corollary 2. *An increase in the discount rate ρ strictly reduces fiscal space.*

Trend growth. A reduction in nominal trend growth G^*—whether caused by a productivity growth slowdown, falling inflation expectations, or declining population growth—seems like it may tighten fiscal space by moving G^* closer to R. But this is not obvious as slower growth rates lead to a greater desire for saving by households, pushing R^* down alongside G^*. With log preferences over consumption as in (1), R^* falls one for one with G^*, as in (9), leaving $G^* - R^*$ unchanged.[17] This is why, without a ZLB in our model, growth does not affect steady state deficits. We revisit this comparative static in Sections 4.3 and 7 with a potentially binding ZLB.

[17] For an analysis of changing growth rates with intertemporal elasticities different from one, see Mehrotra and Sergeyev (2020).

Income inequality. Inequality is relevant for fiscal sustainability, as it is mainly richer households that, directly or indirectly, own government debt. The top 10% of the wealth distribution in the United States hold 69% of the outstanding government debt held by the U.S. household sector. The bottom 50% of the wealth distribution hold almost no government debt at all (Mian, Straub and Sufi 2020). The willingness and ability of richer households to save may thus be a primary factor in the determination of interest rates on government debt. We formalize this as a shift in the income share of spenders μ.

Corollary 3. *Absent a ZLB constraint, greater inequality, $\mu \downarrow$, expands fiscal space $z(b)$ in* (10).

Greater income inequality unambiguously expands fiscal space without ZLB. Savers in our model have a greater propensity to save out of an increase in permanent income compared to spenders, so that any increase in inequality reduces R^* and thus increases fiscal space as $z = (G^* - R^*)b$. Figure 3b illustrates these findings.

The model provides intuition behind the observation that rising income inequality has been accompanied by rising fiscal deficits and government debt levels in many advanced economies. Rising income inequality allows governments to borrow more cheaply from savers. Appendix H confirms this prediction empirically for the U.S., finding suggestive evidence that greater inequality reduced convenience yields in the regression used by Krishnamurthy and Vissing-Jorgensen (2012).

Tax policy. Similar to changes in the income distribution, tax policy also affects fiscal space. To see how, allow for nonzero taxes (or transfers) on spenders, $\tilde{\tau} \neq 0$, as well as consumption taxes τ^c paid by both types of agents and capital income taxes τ^b. The budget constraint of savers is then given by

$$(1+\tau^c)\, c_t + \dot{b}_t \leq \left(\left(1-\tau_t^b\right) R_t^{pre} - G_t\right) b_t + (1-\mu)\, w_t n_t - \tau_t$$

where we use R_t^{pre} as the pre-tax interest rate. We use R_t and R_t^* to denote post-tax interest rates. This changes the Euler equation of savers, leading to an updated equation for the (post-tax) natural interest rate,

$$R^*(b_t) = \rho + G^* - v'(b_t)\left((1+\tau^c)(1-x) - \mu + \tilde{\tau}\right). \tag{16}$$

The relationship between R^* and $\tau^c, \tilde{\tau}$ then gives us the following result.

Corollary 4. *Absent a ZLB constraint, increased regressive income taxes $\tilde{\tau}$ and consumption taxes τ^c expand fiscal space. Increased capital income taxes τ^b leave fiscal space unchanged.*

Increased taxes $\tilde{\tau}, \tau^c, \tau^b$ are, by construction, met by a reduction in lump-sum taxes on savers τ. Raising taxes $\tilde{\tau}$ on spenders is thus regressive, akin to an increase in income inequality, expanding fiscal space. Consumption taxes τ^c have a similar effect on fiscal space, even though both agents pay them. This is because savers trade off consuming and saving in their Euler equation, and their desire to save is partly due to convenience benefits. Increased capital income taxes τ^b are irrelevant here as the before-tax return on government debt immediately adjusts to keep the after-tax return $R^*(b)$ constant.[18]

These results have two important implications. First, they suggest a potential dilemma. Large redistributive programs may reduce fiscal space, potentially limiting the extent to which such programs can be deficit-financed. Second, regressive taxation is able to finance a greater level of government debt than progressive taxation, holding fixed the overall tax burden. Governments with sufficiently large debt levels and interest rates R near or above G may thus be forced to resort to such regressive taxation.

4 Fiscal space near the ZLB

We are now ready to re-introduce the ZLB constraint. As evidenced by the extended period of time many advanced economies have spent at the ZLB (or a similar effective lower bound) over the past decade or more, this is an important constraint that needs to be analyzed jointly with fiscal policy. Along the existing deficit-debt locus, $R^*(b)$ hits the ZLB precisely when $b = b^{ZLB}$ with $v'(b^{ZLB}) = \frac{\rho+G^*}{1-x-\mu}$ and deficit $z^{ZLB} = z(b^{ZLB})$.

4.1 Deficit-debt diagram with ZLB

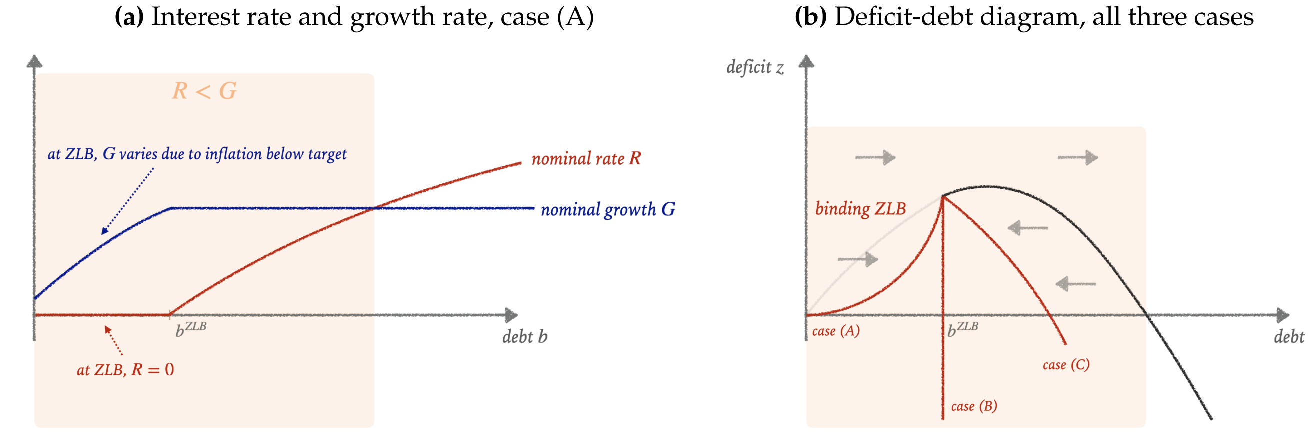
When the ZLB is binding, monetary policy is passive, as $R = 0$ no longer adjusts to achieve full employment. This means that output is determined by aggregate demand,

$$y_t = c_t + \tilde{c}_t + x \tag{17}$$

and inflation will fall short of its target. This reduces nominal growth G, thus raising the de-trended interest rate $R - G = -G^* + \kappa(1 - y_t)$. Savers respond by reducing their

[18]Two caveats: First, with longer-duration debt or large surprise taxes at date $t = 0$, there is some initial expropriation from capital income taxes, which can be used to reduce government debt. Second, if other types of capital income were present in the model, the capital income tax would become more similar to a tax on savers' income.

Figure 4: Fiscal space with ZLB constraint



Note. Panel a plots nominal interest rate R and nominal growth rate G as function of real debt (relative to potential GDP), for case (A). Panel b compares the three different shapes of the deficit-debt diagram.

consumption in line with their Euler equation

$$\frac{\dot{c}_t}{c_t} = R_t - G_t - \rho + v'(b_t)c_t. \tag{18}$$

This feedback between low demand on the one hand and increased $R - G$ on the other crucially depends on the slope of the Phillips curve, κ. A steeper Phillips curve implies a stronger feedback mechanism. As it turns out, an important threshold for κ is given by $\hat{\kappa} \equiv \frac{1-\mu}{1-x-\mu}(\rho + G^*)$. The following result characterizes ZLB steady states.

Proposition 2. *(b, z) is a steady state at the ZLB with positive primary deficit $z > 0$ if:*

(A) for $\kappa < \hat{\kappa}$, $b < b^{ZLB}$ and $z(b) = G(b)b$, where nominal growth is equal to

$$G(b) = G^* - \frac{\kappa}{v'(b)(1-\mu) - \kappa}(-R^*(b)). \tag{19}$$

(B) for $\kappa = \hat{\kappa}$, $b = b^{ZLB}$ and $z < z^{ZLB}$.

(C) for $\kappa > \hat{\kappa}$, $b > b^{ZLB}$ and $z(b) = G(b)b$, where $G(b)$ is as in (19).

To get an intuition for these results, consider case (A). In this case, for debt levels below b^{ZLB}, the aforementioned feedback loop kicks in, and nominal growth is reduced (Figure 4a). This squeezes the gap between G and R, reducing fiscal space at the ZLB (Figure 4b).

To understand case (B), it is useful to write the government budget constraint as

$$\dot{b}_t = \underbrace{z_t}_{\text{direct effect}} \underbrace{-\left(G_t - R_t\right) b_t}_{\text{indirect effect}}. \tag{20}$$

Imagine starting at point (b^{ZLB}, z^{ZLB}), just outside the ZLB, where $R = R^* = 0$. Any reduction in the primary deficit clearly has a negative direct effect on the evolution of government debt. In case (A), as well as in the absence of the ZLB, this direct effect is sufficient to reduce government debt, $\dot{b}_t < 0$.

In addition to this direct effect, there is also an indirect effect, operating through aggregate demand and growth rates. At the ZLB, a reduced primary deficit lowers aggregate demand and nominal growth rates. In case (B), when $\kappa = \hat{\kappa}$, this indirect effect is precisely strong enough to offset the direct effect. The deficit-debt locus becomes vertical in Figure 4b. When $\kappa > \hat{\kappa}$, in case (C), the indirect effect is, in fact, sufficiently strong for debt levels to rise with lower primary deficits. This leads to a "backward-bending" deficit-debt locus, shown in Figure 4b.

One way to tell in practice in which of the three cases an economy is in, is, in fact, to gauge whether the indirect effect of a fiscal policy change will dominate the direct effect. If a permanent deficit expansion Δz raises output by Δy and inflation by $\kappa \Delta y$, the indirect effect dominates precisely when

$$\kappa \frac{\Delta y}{\Delta z} b > 1.$$

This "sufficient statistic" condition is more likely to hold when κ, the multiplier $\frac{\Delta y}{\Delta z}$, and the initial debt level b are large. For realistic κ's in the range 0.1 to 0.3, and a debt level as large as Japan's (238%), this is already satisfied for multipliers $\frac{\Delta y}{\Delta z}$ the range $1.5 - 2$. In our quantitative model in Section 7, this condition is satisfied for Japan, but not for the U.S.

4.2 Free lunch at the ZLB

We next revisit the question of when free lunch policies exist. To do so, we study transitional dynamics. Different from the analysis in Section 3, consumption is now no longer constant along the transition. Instead, the economy is governed by a system of two differential equations, the Euler equation (18),

$$\frac{\dot{c}_t}{c_t} = \kappa \left(1 - \frac{x + c_t}{1 - \mu}\right) - G^* - \rho + v'(b_t) c_t \tag{21}$$

in addition to the government budget constraint

$$\dot{b}_t = \left(\kappa \left(1 - \frac{x + c_t}{1 - \mu} \right) - G^* \right) b_t + z_t. \tag{22}$$

It is hard to characterize dynamics and prove uniqueness in full generality, for arbitrary paths of z_t and functional forms for $v'(\cdot)$. Our next proposition shows that, under some assumptions on $v'(\cdot)$ and the fiscal rule, we can make progress.

Proposition 3. *Assume the fiscal rule $z_t = \xi b_t$, where $\xi \in (0, G^*)$, and assume that $v'(b)b$ is strictly increasing in b, with a well-behaved positive limit as $b \to 0$. Then, for any initial debt level $b_0 > 0$, there exists a unique equilibrium. The sign of $\dot{b}_t$ is exactly determined by the position of (b_t, z_t) in the deficit-diagram in Figure 4, at all time periods, that is: $\dot{b}_t < 0$ iff z_t lies strictly below the locus (cases (A), (B)), or if z_t lies strictly between $z(b)$ and the ZLB locus $G(b)b$ for $b_t > b^{ZLB}$ (case (C)).*

This proposition establishes that there are still well-behaved dynamics at the ZLB and the phase diagram in Figure 4b can still be used to study them. Importantly, the result implies that there is always a free lunch at the ZLB.[19]

Corollary 5. *Under the assumptions of Proposition 3, there exists a free lunch policy at the ZLB.*

To see this for case (A), notice that the ZLB region in Figure 4 is always on the left branch of the hump-shaped deficit-debt locus. An increase in the primary deficit, say, to the level z^{ZLB} right at the border of the ZLB region, ensures both a free lunch and an exit out of the liquidity trap. For cases (B) and (C), it is even simpler. A second non-ZLB steady state exists, with the same debt level but a greater deficit. With the fiscal rule described in Proposition 3, one can ensure a free lunch simply by setting $\xi = \frac{z^{ZLB}}{b^{ZLB}}$.

4.3 What determines fiscal space near the ZLB?

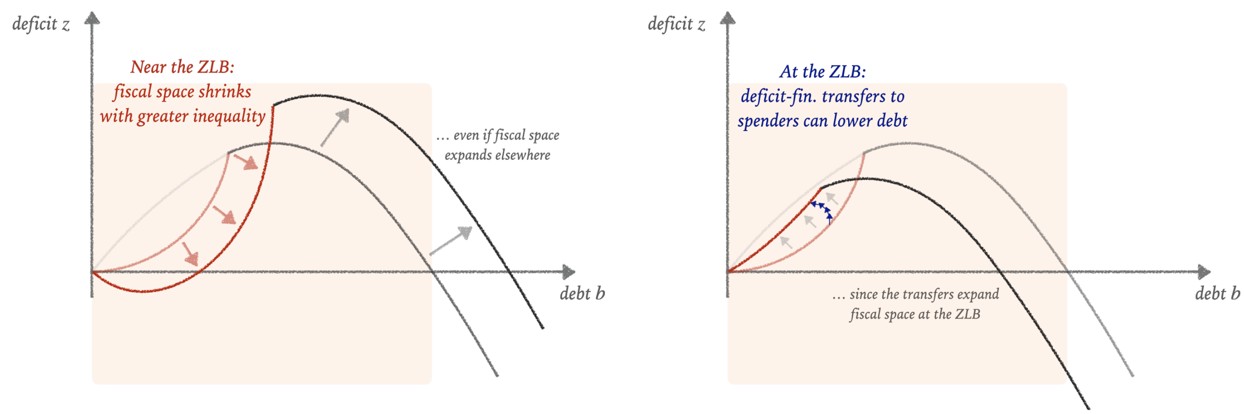
Next, we revisit the role of some of the drivers of fiscal space, only now allowing for a potentially binding ZLB constraint. For simplicity, we focus on the non-backward-bending case (A), that is, $\kappa < \hat{\kappa}$. The other cases behave similarly.

Growth slowdown. A slowdown in trend growth G^* does not have an effect on fiscal space outside the ZLB since the interest rate R shifts down one for one with G^*, leaving the

[19] If an economy is not literally in a steady state with a binding ZLB constraint, this result is to be understood as: The closer an economy is to the ZLB, and the more frequently it hits it, the more likely it becomes that a permanent deficit expansion is possible.

Figure 5: Drivers of fiscal space at the ZLB

(a) Increasing inequality can reduce fiscal space **(b)** Stimulus checks can reduce debt / GDP



Note. Panel a shows how greater inequality can reduce fiscal space at the ZLB. Panel b illustrates how stimulus checks can reduce debt to GDP at the ZLB.

gap $G^* - R^*$ unchanged. Yet, at the ZLB, R is stuck at zero, so that a growth slowdown reduces $G - R$ and fiscal space.

Corollary 6. *With a binding ZLB constraint, a reduction in trend growth G^* reduces fiscal space: $z(b) = (G(b) - R(b))b$ falls with lower G^*.*

Income inequality. Rising income inequality (falling μ) unambiguously increases fiscal space without the ZLB (Section 3.5). At the ZLB, we have the opposite result.

Proposition 4. *At the ZLB with debt level b, increased inequality locally reduces fiscal space $z(b)$.*

Income inequality can reduce fiscal space at the ZLB as it weighs down on aggregate demand, pulling down nominal growth G relative to a fixed interest rate $R = 0$. This reduces fiscal space.

Tax policy. An immediate implication of this result is that, at the ZLB, more progressive taxation (which here is identical to redistribution) increases fiscal space.

Corollary 7. *At the ZLB, redistribution raises fiscal space $z(b)$ if $\kappa > 0$.*

In particular, this result implies that greater redistribution at the ZLB reduces the debt level. It turns out that even deficit-financed stimulus checks may ultimately reduce the debt, by way of increased nominal growth. We spell this out in the next result.

Proposition 5. *Starting from a ZLB steady state, a permanent increase in transfers to spenders, $\tilde{\tau} < 0$, without change in taxes on savers τ, reduces the debt level in the long run if $\kappa >$*

$(1-\mu)\, v'(b)/\,(v'(b)b+1)$. *A necessary and sufficient condition for this to hold for some* $b < b^{ZLB}$ *is* $\kappa > \hat{\kappa}/\,\left(v'(b^{ZLB})b^{ZLB}+1\right)$.

This result is a close cousin of case (C) in Figure 4b. In that figure, the locus bent backwards because deficit-financed transfers to savers at the ZLB can ultimately lower the debt level. We showed that this happens when $\kappa > \hat{\kappa}$.

Proposition 5 highlights that deficit-financed transfers to spenders can also reduce the debt level. In fact, this happens under a looser condition on κ. Figure 5b reveals that this is because a transfer to spenders increases fiscal space at the ZLB and thus makes it more likely that the debt level falls, even if the economy is in case (A) where $\kappa < \hat{\kappa}$.

5 Fiscal space under aggregate risk

So far, we have analyzed a purely deterministic economy. We now introduce aggregate risk and study the implications for fiscal space and the viability of free lunch policies. To keep things tractable, we omit the ZLB in this section. Our model with aggregate risk builds on the representative-agent model of Mehrotra and Sergeyev (2020). We provide details on model derivations in Appendix A.12.

5.1 Introducing aggregate shocks

Instead of constant (de-trended) potential output, we now assume that potential output y_t^* is risky and follows a geometric Brownian motion,

$$d\log y_t^* = \gamma dt + \sigma dZ_t \tag{23}$$

where we explicitly allow for productivity growth γ and aggregate risk with volatility $\sigma > 0$. Z_t is a standard Brownian motion. We assume government spending is a fixed share of y_t^*, xy_t^*. Furthermore, we allow agents to have an intertemporal elasticity of substitution different from 1, denoted by $\nu^{-1} > 0$. Savers thus maximize expected utility

$$\max_{\{C_t,B_t\}} \mathbb{E}_0 \int_0^\infty e^{-\rho t} \left\{ \frac{C_t^{1-\nu}}{1-\nu} + (y_t^*)^{1-\nu}\, v\left(\frac{B_t}{y_t^*}\right) \right\} dt \tag{24}$$

Here, C_t denotes real consumption, and B_t denotes real, riskless debt. We continue to use $c_t = C_t/y_t^*$ and $b_t = B_t/y_t^*$ for normalized consumption and debt. While B_t is riskless, b_t is not. We define the growth-adjusted discount rate to be $\hat{\rho} \equiv \rho + (\nu-1)\,\gamma$ and assume it is positive. The utility function in (24) is set up to be scale invariant, as in Mehrotra and

Sergeyev (2020). The budget constraint is

$$dB_t = \left(\left(R_t - \pi^* \right) B_t + \left(1 - \mu \right) w_t n_t - T_t - C_t \right) dt \tag{25}$$

where $R_t - \pi^*$ is the riskless real interest rate earned on bonds B_t, and T_t are unnormalized taxes on savers. We note that $\mathbb{E}_t dB_t = dB_t$, so B_t is predictable.[20] We continue to use $\tau_t \equiv T_t / y_t^*$. Normalized consumption c_t in this economy follows the Euler equation

$$\nu \frac{dc_t}{c_t} = \left(R_t - G^* - \hat{\rho} + \frac{1}{2} \nu^2 \left(\sigma_{ct} + \sigma \right)^2 dt + \nu \frac{1}{2} \sigma_{ct}^2 dt + c_t^{\nu} v'(b_t) \right) dt + \sigma_c dZ_t \tag{26}$$

where σ_{ct} denotes the instantaneous volatility of $\frac{dc_t}{c_t}$. In equilibrium, $c_t = 1 - x - \mu$ as before, and $\sigma_{ct} = 0$, which determines the interest rate as[21]

$$R_t = R(b_t) = G^* + \hat{\rho} - \frac{1}{2} \nu^2 \sigma^2 - \left(1 - x - \mu \right)^{\nu} v'(b_t). \tag{27}$$

Aside from the adjusted discount rate $\hat{\rho}$, the new term in (27) relative to (9) is $-\frac{1}{2}\nu^2\sigma^2$. It captures the role of aggregate risk, which, all else equal, reduces the interest rate on government debt due to a precautionary motive. Since the new term is constant, $R(b_t)$ has the same functional form as (9).

We denote the primary deficit to GDP ratio by $z_t = x - \tau_t$. The normalized government budget constraint is then

$$db_t = z_t dt + \left(R(b_t) - G^* + \frac{1}{2} \sigma^2 \right) b_t dt - b_t \sigma dZ_t. \tag{28}$$

Finally, the transversality condition for savers in this economy is $\mathbb{E}_0 \left[e^{-\rho t} C_t^{-\nu} B_t \right] \to 0$. A sufficient condition for it to hold is that there is an $\epsilon > 0$ such that for large t,

$$R(b_t) - G^* + \frac{z_t}{b_t} + \frac{1}{2} \nu^2 \sigma^2 < \hat{\rho} - \epsilon. \tag{29}$$

5.2 Deficit-debt diagram and free lunch

Just like before, we can plot the locus $z(b) \equiv \left(G^* - R(b) \right) b$. Given that $R(b_t)$ has the same functional form, this locus looks like that we plotted in Figure 1b. The interpretation is

[20] $\mathbb{E}_t dB_t = dB_t$ is the continuous-time version of the measurability constraint imposed by Aiyagari, Marcet, Sargent and Seppälä (2002) on the present value of future surpluses. For recent papers exploring the "asset pricing" approach on government debt, see e.g. Jiang, Lustig, Van Nieuwerburgh and Xiaolan (2019, 2020*a*). For reasons explained in Section 3.4, we cannot simply adopt this approach in our analysis.

[21] Here, the real growth rate γ of the economy not only enters via $G^* = \gamma + \pi^*$, but also via the adjusted discount rate $\hat{\rho} = \rho + (\nu - 1)\gamma$.

different, however. Before, sitting on the locus $z_t = z(b_t)$ ensured a steady state equilibrium. Here, $z_t = z(b_t)$ only ensures that log government debt remains unchanged *in expectation*, $\mathbb{E}_t [d \log b_t] = 0$. In that sense, the locus here corresponds to a "risky steady state" in the spirit of Coeurdacier, Rey and Winant (2011). Just like before, when the economy is above the locus, $z_t > z(b_t)$, log debt rises on average. As in Bohn (1998), a fiscal rule is necessary here to avoid violating the transversality condition (29). We write it as $z_t = \mathcal{Z}(b_t)$.

To study the analogue of a "free lunch" in this economy, we fix a fiscal rule that is consistent with (29) and leads to a well-defined stationary distribution of debt to GDP levels. Fix an initial debt level b_0 and denote the stochastic process of primary deficits implied by the fiscal rule by z_t. We construct the following counterfactual path of government debt b_t^Δ: It starts at an increased initial debt level $b_0^\Delta = b_0 + \Delta$, where $\Delta > 0$, but otherwise follows the exact same deficit path,

$$db_t^\Delta = z_t dt + \left(R(b_t^\Delta) - G^* + \frac{\sigma^2}{2} \right) b_t^\Delta dt - b_t^\Delta \sigma dZ_t.$$

In other words, b_t^Δ is the path of government debt that arises when the government runs a one-time deficit Δ at date 0, but otherwise keeps its deficit unchanged. We refer to the probability that the shifted path b_t^Δ converges back to the original debt level b_t, $P(b_t^\Delta \to b_t)$, as the *success probability of a free lunch policy.* While before, any free lunch had a success probability of 1, this is no longer the case with aggregate risk.

Despite these differences, our condition $R < G - \varphi$ is still relevant with aggregate risk. To show this in a formal result, we focus on a common linear parametrization of the convenience yield (e.g. Krishnamurthy and Vissing-Jorgensen 2012)

$$v'(b)\,(1 - \mu - x) = v'(b_0)\,(1 - \mu - x) - \varphi \frac{b - b_0}{b_0} \tag{30}$$

for some parameters $b_0, \varphi > 0$.[22] Our formal result is then as follows.

Proposition 6. *Denote by $\mathcal{F}(b)$ the cdf of the stationary distribution of debt to GDP in the model with aggregate risk. Assume the convenience yield is of the form (30). Denote by $\overline{R} \equiv \int R(b)\mathcal{F}(db)$ the average interest rate and by $\overline{\varphi}$ the average semi-elasticity $\overline{\varphi} \equiv \int \frac{\partial (R-G)}{\partial \log b} \mathcal{F}(db)$. The success probability of a free lunch policy of size Δ approaches 1 for small Δ, $\lim_{\Delta \to 0} P(b_t^\Delta \to b_t) = 1$, if*

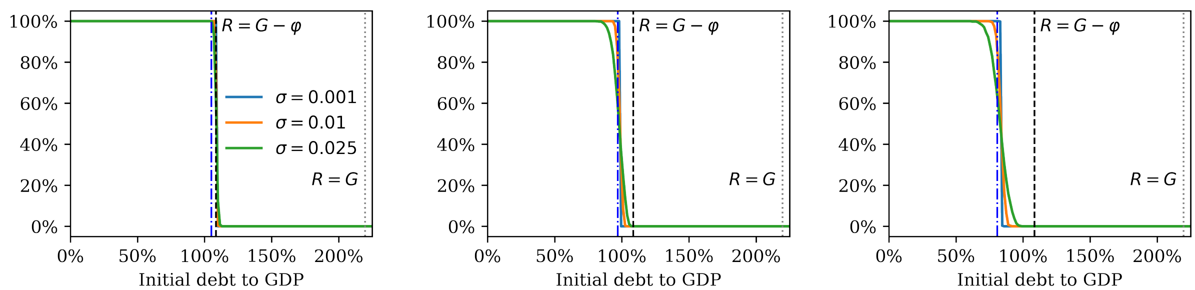
$$\overline{R} < G^* - \overline{\varphi}.$$

By contrast, $P(b_t^\Delta \to b_t) = 0$ for any Δ if $\overline{R} > G^* - \overline{\varphi}$.

[22]We set $v'(b) = 0$ for any b sufficiently large to cause the right hand side to move below zero.

Figure 6: Success probabilities of running a free lunch with aggregate risk

(a) Size $\Delta = 0.001\%$ of GDP **(b)** Size $\Delta = 20\%$ of GDP **(c)** Size $\Delta = 50\%$ of GDP



Note. The probabilities are computed by simulating 1,000 sample paths for each b_0. Convergence criterion: $|b_t^\Delta - b_t| < 0.01\%$ at any point $t < 10,000$. The vertical dash-dotted line is the threshold for b_0 under which the deterministic model allows for a free lunch of size Δ, see footnote 24.

The result generalizes our condition for a free lunch policy from Section 3. Instead of $R < G^* - \varphi$ being the relevant condition, evaluated at a given initial level of debt, it is the average of the condition that matters. Precisely when $\overline{R} < G^* - \overline{\varphi}$, a free lunch success probability arbitrarily close to 1 can be ensured by choosing a suitable Δ.

Going from current $R < G^* - \varphi$ to average $\overline{R} < G^* - \overline{\varphi}$ can make a meaningful difference. In practice, the latter condition requires an assessment of the long run average interest rate a government is paying on its debt, the long run average growth rate, as well as the long run average sensitivity φ. Especially during periods with unusually low interest rates, the average condition can be quite a bit tighter than the current one.

Proposition 6 can be expanded to allow for an upper bound $\overline{b}$ beyond which the government switches back to the original deficit rule $z_t = \mathcal{Z}(b_t)$. The result then holds in a slightly modified way, $\lim_{\Delta\to 0} \lim_{\overline{b}\to\infty} P(b_t^\Delta \to b_t) = 1$.

5.3 Evaluating the stochastic free lunch condition

We next evaluate Proposition 6 numerically. For the purposes of the simulation here, three calibration targets suffice: the initial debt to GDP ratio (assumed to be 100%), the initial interest rate growth rate differential $R_0 - G^*$ (assumed to be -2%) and the sensitivity of $R - G$ to debt (assumed to be 1.7%). All three choices are discussed in Section 7.2.[23]

Figure 6 illustrates the result in Proposition 6 for this parameterization. It plots free lunch success probabilities as a function of the initial debt level b_0. For each b_0, we choose a simple fiscal rule $\mathcal{Z}(b) = 0.1 \cdot (b_0 - b)$. We vary the standard deviation of aggregate risk σ

[23]The individual values of R_0, G^*, ν, ρ, μ, x are irrelevant for the simulation in this section. We have verified numerically that the transversality condition (29) holds for $\nu^{-1} = 1$ and values of ρ above 3%.

from a small value of 0.001 to 0.025, the standard deviation of post-WWII U.S. GDP growth.

The panels vary the size of the one-time deficit Δ. Panel a shows success probabilities for a very small value of Δ. As can be seen, success probabilities are essentially a step function: 100% to the left of the threshold $R < G - \varphi$, and 0% to the right. This is a numerical confirmation of Proposition 6. With greater Δ, there is no longer a clean step function. However, across σ, the success probabilities still line up closely with the vertical dash-dotted blue line, which is the deterministic free lunch threshold for that Δ.[24]

The applicability of the condition $\overline{R} < G^* - \overline{\varphi}$ is not limited to the model here. In Appendix D.3 (e.g. Figure A.8), we show that it works just as well in the Blanchard (2019) model. We conclude that, both theoretically and numerically, the condition $R < G - \varphi$ generalizes to an economy with aggregate risk.

6 Crowding out of capital

There is no privately issued asset in our baseline model. Next, we extend our model to include capital. To keep things simple, we assume there is no ZLB constraint, as in Section 3. We let (potential) output now be a Cobb-Douglas aggregate of capital k and labor n, which is still equal to $n = 1$ without the ZLB. Thus, $y_t = k_t^\alpha$ after de-trending, letting $\alpha \in [0,1]$ be the capital share. We let $\delta_k \geq 0$ denote the depreciation rate of capital and assume that government spending is a share x of potential output as y_t may now differ from 1. Following Ball and Mankiw (2021), we allow for an exogenous markup $m \geq 1$; pure profits are earned by savers.

Whether capital is affected by the debt level in our model is not obvious. If capital does not carry a convenience yield, it is entirely unaffected by the debt level.[25] In the literature, capital is often influenced by the debt level as both are treated as substitutable (see e.g. the alternative models in Section 7). In our model, this can be captured by including capital in the convenience utility,

$$\max_{\{c_t, b_t\}} \int_0^\infty e^{-\rho t} \left\{ \log c_t + v\left(\frac{b_t + k_t}{y_t}\right) \right\} dt. \tag{31}$$

We now also divide $b_t + k_t$ by (potential) output y_t explicitly. Before, this was unnecessary as potential output was equal to 1. The budget constraint of savers now includes capital

[24] This threshold is simply computed as the value of b_0 for which $b_0 + \Delta$ just converges back to b_0 holding the deficit constant at $z(b_0)$. In other words, b_0 satisfies $z(b_0) = z(b_0 + \Delta)$.

[25] This is for instance the case in the microfoundation proposed in Appendix B.1.

and pure profits,

$$c_t + \dot{b}_t + \dot{k}_t \leq (R_t - G_t)\, b_t + \left(r_t^k - \gamma\right) k_t + (1-\mu)\, w_t n_t - \tau_t + \left(1 - m^{-1}\right) y_t$$

where $r_t^k \equiv \alpha k_t^{\alpha-1} - \delta_k$ denotes the real net return on capital. By no arbitrage, $R_t = r_t^k + \pi^*$. Two first order conditions jointly pin down the capital stock k and interest rate R^* as a function of debt,[26]

$$R^* = \rho + G^* - (1 - x - \mu)\, v'\left(\frac{b+k}{y}\right) \qquad R^* - G^* = m^{-1}\alpha k^{\alpha-1} - \delta_k - \gamma. \tag{32}$$

Expanding both conditions to first order, we find the sensitivity of capital to debt

$$\frac{d\,(k/y)}{d\,(b/y)} = -\frac{\frac{\varphi^{\text{no K}}\left(\frac{b+k}{y}\right)}{r^k+\delta_k}}{1 + k/b\left(1 + \frac{\varphi^{\text{no K}}\left(\frac{b+k}{y}\right)}{r^k+\delta_k}\right)} \tag{33}$$

and the sensitivity of $R - G$ to debt

$$\frac{d\,(R-G)}{d\log\,(b/y)} = \varphi^{\text{no K}}\left(\frac{b+k}{y}\right) \cdot \frac{1}{1 + k/b\left(1 + \frac{\varphi^{\text{no K}}\left(\frac{b+k}{y}\right)}{R-G+\gamma+\delta_k}\right)}. \tag{34}$$

Here, $\varphi^{\text{no K}}$ denotes the sensitivity of $R - G$ to debt in an economy without capital, $\varphi^{\text{no K}}(a) = -(1-x-\mu)v''(a)a$.

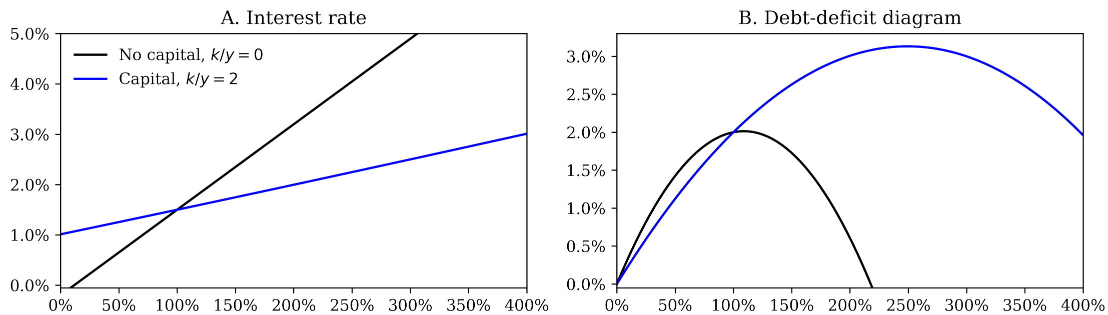
Equation (33) gives us the extent of "crowding out" of capital. Crowding out happens when (a) there is positive capital, $k/b > 0$, requiring that $\alpha > 0$; and (b) the interest rate is sensitive to wealth, $\varphi^{\text{no K}} > 0$. Equation (34) shows that the sensitivity of $R - G$ to b/y is smaller with positive capital. This implies an extended region in which a free lunch is available.

Proposition 7. *Crowding out of capital unambiguously increases the free lunch region. The condition for a free lunch is now given by*

$$R < G - \varphi^{no\ K}\left(\frac{b+k}{y}\right) \cdot \frac{1}{1 + k/b\left(1 + \frac{\varphi^{no\ K}\left(\frac{b+k}{y}\right)}{R-G+\gamma+\delta_k}\right)}. \tag{35}$$

[26] Observe that even if $R^* < G^*$, we may have dynamic efficiency here, that is, $\alpha k^{\alpha-1} > \delta_k + \gamma$, for m sufficiently above 1.

Figure 7: Fiscal space with crowding out of capital



Note. Black line = baseline model without capital. Blue line = calibration of model with capital such that $k/y = 2$ in the initial steady state, with $\delta_k = 0.08$ and productivity growth of $\gamma = 2\%$. See Section 5.3 for the other calibration targets needed to pin down these graphs.

This result may seem surprising at first: Isn't it the case that crowding out of capital increases the marginal product of capital, and hence the interest rate more quickly?

The answer is no. More crowding out due to a higher capital stock k/y, by definition, implies that household wealth $b/y + k/y$ increases *less* quickly with government debt. This, by (32), leads to a weaker interest rate response. This is why k/y reduces the elasticity of $R - G$ to debt in (34). We illustrate this in Figure 7 using the Section 5.3 parameterization.

Note also that the modified free lunch condition (35) is independent of the markup m. While m matters for welfare, as it determines whether the economy is dynamically inefficient or not when $R < G$, m is irrelevant for whether there exists a free lunch or not (conditional on k/b).

7 Quantitative exploration

So far, we have studied fiscal space in relatively stylized models, both with regard to the household side as well as the production side. Next, we explore the quantitative predictions in richer versions of our model. These versions feature one of four different household sides, a production function with capital subject to adjustment costs, as well as a forward-looking wage Phillips curve. We keep our model description brief here, and focus directly on the de-trended economy but provide all details in Appendix G.

7.1 Quantitative model

As before, the economy consists of a government, a household side with savers and spenders, and a monetary authority. Different from before, time is discrete. The models are solved in the sequence space (Auclert, Bardóczy, Rognlie and Straub, 2021).

Spenders. A unit mass of spenders are hand-to-mouth, just like before, consuming their after-tax income, $\tilde{c}_t = \mu (1 - \chi_t) w_t n_t$. Here, μ is the labor income share of spenders, χ_t is a proportional income tax, and w_t the pre-tax wage. As we describe below, labor supply n_t is chosen by unions, and identical across households.

Savers: Overview. A continuum $[0, 1]$ of savers consume c_{it} and save b_{it} in government bonds as well as $\tilde{k}_{it}$ in physical capital, via a capital accumulation firm. Government bonds pay a (de-trended) return of $(1 + R_{t-1}) / (1 + G_t)$ at date t; physical capital pays a return $1 + r_t$ to be specified below. Saver i earns post-tax labor income $(1 - \mu)(1 - \chi_t) w_t e_{it} n_t$ as well as per-capita profit income $d_t e_{it}$. Both are multiplied by e_{it}, an idiosyncratic productivity shifter.[27] e_{it} is equal to 1 except for our heterogeneous-agent savers below. Saver i's budget constraint is given by

$$c_{it} + b_{it} + \tilde{k}_{it} \leq \frac{1 + R_{t-1}}{1 + G_t} b_{i,t-1} + (1 + r_t) \tilde{k}_{i,t-1} + (1 - \mu)(1 - \chi_t) w_t e_{it} n_t + d_t e_{it}. \tag{36}$$

b_{it} and $\tilde{k}_{it}$ are restricted to be non-negative. Aggregate consumption is given by $c_t = \int c_{it} di$, aggregate bonds by $b_t = \int b_{it} di$, and capital investments by $\tilde{k}_t = \int \tilde{k}_{it} di$.

Savers solve one of three utility maximization problems. These three approaches will ultimately lead to four different calibrated household sides, as explained below.

Savers: Bonds-in-utility (BU). In this version of the model, all savers are identical, with $e_{it} = 1$, maximizing, subject to (36), the discrete-time analog of objective (1),

$$\max_{\{c_t, b_t, \tilde{k}_t\}} \sum_{t=0}^{\infty} e^{-\rho t} \left\{ \log c_t + v\left(b_t + \tilde{k}_t\right) - h(n_t) \right\}. \tag{37}$$

Savers: Overlapping generations (OLG). This version follows Blanchard (1985) and Yaari (1965). A mass $1 - \zeta > 0$ of savers is born each instant with zero assets and survives each period with probability ζ. A saver i, born at date t_0 without bonds or capital, solves

$$\max_{\{c_{it}, a_{it}\}} \sum_{t=t_0}^{\infty} \left(\zeta e^{-\rho}\right)^{t-t_0} \left\{ \log c_{it} - h(n_t) \right\} \tag{38}$$

[27] To avoid too strong a response of spending to dividends, we follow Debortoli and Galí (2017) in assuming that dividends are not paid out lump-sum and instead are proportional to other kinds of income or wealth.

subject to (36). Since agents have access to the standard annuities market (Blanchard, 1985), the returns on bonds and capital in this economy are $\zeta^{-1}\frac{1+R_{t-1}}{1+G_t}$ and $\zeta^{-1}(1+r_t)$ respectively. There are no idiosyncratic productivities, $e_{it}=1$.

Savers: Heterogeneous agents with idiosyncratic risk. In this version, savers are hit by idiosyncratic productivity shocks as in Aiyagari (1994) and Aiyagari and McGrattan (1998), with saver $i \in [0,1]$ solving

$$\max_{\{c_{it},a_{it}\}} \sum_{t=0}^{\infty} e^{-\rho t} \left\{ \frac{\left(c_{it}e^{-h(n_t)}\right)^{1-\nu}}{1-\nu} \right\} \tag{39}$$

subject to (36).[28] Here, e_{it} follows a Markov process that is iid across savers i. We allow for a non-unitary elasticity of intertemporal substitution ν^{-1} as in King, Plosser and Rebelo (1988). Since bonds and capital are perfect substitutes in the steady state of the model, we assume that they are held in equal proportions across savers i.[29]

Production. Final goods y_t are a CES aggregate over a continuum of symmetric intermediate goods. Intermediate goods producers operate a Cobb-Douglas technology in capital and labor and compete monopolistically. Since the math is standard, we relegate it to Appendix G. There, we show that production of final goods can be described as

$$y_t = k_{t-1}^{\alpha} n_t^{1-\alpha} - \overline{y} \tag{40}$$

where $\alpha \in (0,1)$ is the capital share, $\overline{y} \geq 0$ are fixed costs. In the de-trended economy introduced here, there is no productivity growth. We allow for productivity growth at some rate $\gamma > 0$ in Appendix G.

Capital k_{t-1} is rented in a spot market from investors (described below) at rental rate r_t^K. This gives the usual first order conditions,

$$m^{-1}(1-\alpha)\frac{y_t+\overline{y}}{n_t} = w_t \tag{41}$$

$$m^{-1}\alpha\frac{y_t+\overline{y}}{k_{t-1}} = r_t^k \tag{42}$$

where $m > 1$ is the monopoly markup charged by intermediate goods producers. Profits

[28] See Domeij and Ellingsen (2018) and Bayer, Born and Luetticke (2021) for more recent models of this sort.
[29] One way this can be implemented is via a mutual fund that holds all bonds and capital.

are given by

$$d_t = \left(1 - m^{-1}\right)(y_t + \overline{y}) - \overline{y}. \tag{43}$$

Investors. There is a representative investor that takes household funds $\tilde{k}_{t-1}$ at the end of period $t-1$ to purchase physical capital k_{t-1} at price q_{t-1}, that is, $q_{t-1}k_{t-1} = \tilde{k}_{t-1}$. It then earns return $r_t^k k_{t-1}$ in period t, invests i_t, and pays an adjustment cost. At the end of period t, the investor sells k_t units of capital at price q_t. Altogether, the investor solves

$$\max_{k_{t-1},k_t} \frac{1}{1+r_t}\left(q_t k_t - i_t - \Phi\left(k_t/k_{t-1}\right)k_{t-1} + r_t^k k_{t-1}\right) - q_{t-1}k_{t-1} \tag{44}$$

subject to the growth-adjusted law of motion of capital,

$$(1+\gamma)\,k_t = i_t + (1-\delta)\,k_{t-1}.$$

$\Phi(x) = \frac{1}{2\epsilon_I \delta}x^2$ is a standard quadratic adjustment cost function where $\epsilon_I > 0$ is the sensitivity of gross investment to Tobin's Q.

Nominal rigidity. We follow Erceg, Henderson and Levin (2000) and Auclert, Rognlie and Straub (2018, 2020) in assuming that a mass of labor unions exists in our economy, allocating a given amount of labor demand n_t equally among households. The disutility of labor that we assume is $h(n_t) = \tilde{h}\frac{1}{1+\phi^{-1}}n_t^{1+\phi^{-1}}$ with ϕ being the Frisch elasticity of labor supply. Unions set nominal wages subject to Rotemberg (1982) adjustment costs and index to trend inflation π^{w*}, giving rise to a Phillips curve for nominal wage inflation π_t^w,

$$\begin{aligned}
&\left(\pi_t^w - \pi^{w*}\right)\left(1 + \pi_t^w - \pi^{w*}\right) \\
&= \kappa \cdot \frac{1-\zeta e^{-\rho}}{\tilde{h}\left(1+\phi^{-1}\right)}\left(n_t h'(n_t) - \frac{(1-\chi_t)\,n_t w_t}{c_t + \tilde{c}_t}\right) + \zeta e^{-\rho}\left(\pi_{t+1}^w - \pi^{w*}\right)\left(1 + \pi_{t+1}^w - \pi^{w*}\right).
\end{aligned} \tag{45}$$

The first term on the right hand side scales with the gap in the first-order condition for labor of the average worker in the economy. If this gap is positive, the average worker is less willing to work, and unions negotiate nominal wage gains. The opposite happens if the gap is negative. The first term is scaled by $\frac{1-\zeta e^{-\rho}}{\tilde{h}\left(1+\phi^{-1}\right)}$ such that a 1% permanent increase in employment n_t generates κ percent inflation from increasing the disutility $h'(n_t)$. This makes κ roughly comparable to empirical estimates, e.g. those in Hazell, Herreno, Nakamura and Steinsson (2020). Goods inflation is given by

$$1+\pi_t = \frac{1+\pi_t^w}{(1+\gamma)\, w_t / w_{t-1}} \tag{46}$$

with trend inflation $1+\pi^* = \frac{1+\pi^{w*}}{1+\gamma}$. Nominal growth is $1+G_t = (1+\gamma)(1+\pi_t)$.

Government. The government chooses fiscal policy consisting of the paths $\{x_t, b_t, \chi_t\}$ subject to the de-trended flow budget constraint

$$x_t + \frac{1+R_{t-1}}{1+G_t} b_{t-1} = b_t + \chi_t w_t n_t \tag{47}$$

The primary deficit is given by

$$z_t = x_t - \chi_t w_t n_t.$$

Just like in (7), monetary policy targets inflation wherever feasible,

$$R_t = 0 \text{ if } \pi_t^w < \pi^{w*}, \qquad R_t \in [0,\infty) \text{ if } \pi_t^w = \pi^{w*}, \qquad R_t = \infty \text{ if } \pi_t^w > \pi^{w*}. \tag{48}$$

It is conceptually more natural to have monetary policy target wage inflation in this economy as that will be a better measure of economic slack than price inflation. Our results are similar if price inflation is targeted.

Equilibrium. We define equilibrium in this model in Definition 3 in Appendix G.2.

7.2 Calibration

We calibrate four versions of this model. The four versions only differ in their household sides; every other aspect is identical, e.g. the production function or fiscal policy. The *BU-quad* model has a bonds-in-utility household side (37) with quadratic convenience utility $v(a)$ and thus a linear convenience yield similar to (30), $v'(a) = v'(a_{ss}) - \tilde{\varphi}\frac{a-a_{ss}}{a_{ss}}$ for some $\tilde{\varphi} > 0$; a_{ss}denotes the steady state wealth-to-GDP ratio. The *BU-log* model also has a bonds-in-utility household side (37), but with logarithmic $v(a)$ so that $v'(a) = \tilde{\varphi} a_{ss}/a$ for some $\tilde{\varphi} > 0$. The *OLG* model has savers with overlapping generations (38). The *HA* model has heterogeneous-agent savers (39). We calibrate each of the four models to Japan and to the U.S. in 2019. One period equals one year.

Calibration to the U.S. For the U.S. calibration, we set trend inflation $\pi^* = 2\%$, target the nominal interest rate $R = 1.5\%$ (in line with nominal interest rates in December 2019)[30] and set trend productivity growth γ to 1.5%, equal to the average peak-to-peak growth rate from 2008 through 2019. We choose a (private) capital-output ratio k/y of 2.0 and a rate of depreciation of 8%. The capital adjustment cost parameter is set to $\epsilon_I = 4$, as in Auclert, Rognlie and Straub (2018). The markup is set equal to $m = 1.3$, well within recent markup estimates, with a fixed cost of $\overline{y} = 0.3$. We target initial government debt to be equal to 100% of GDP and government spending to be equal to 14% of GDP (in line with their values in 2019 Q4). Taxes are set as a residual to balance the government's budget. This gives a labor tax rate of $\chi = 12\%$. The Phillips curve slope κ is assumed to be 0.20, in the middle of the range of estimates in Hazell, Herreno, Nakamura and Steinsson (2020).[31]

U.S. savers are calibrated to achieve two targets. First, they need to be willing to hold all assets in equilibrium; second, an exogenous permanent expansion of government debt by some small $d \log b$ needs to increase $R^* - G$ by $\varphi d \log b$, with $\varphi = 0.017$. As we describe at great length in Appendix F, we view this number as a midpoint estimate of the effect of government debt on $R - G$.

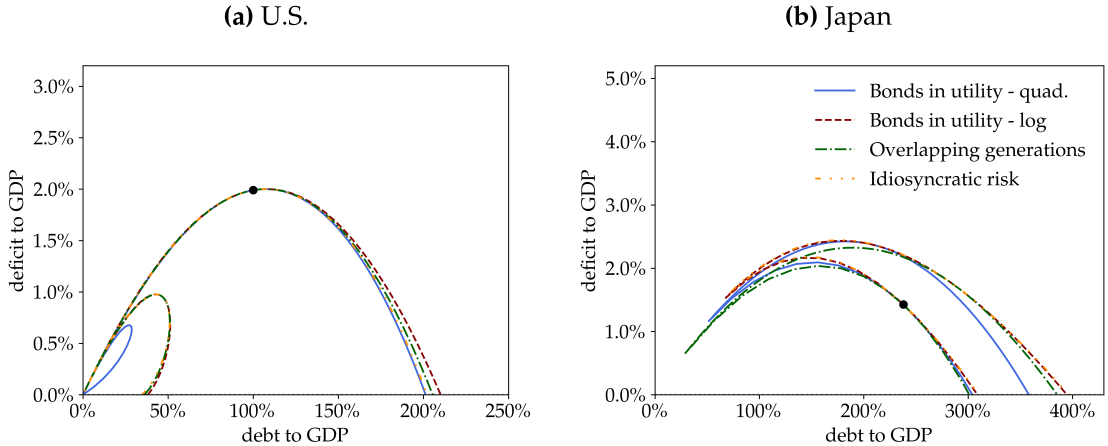
For BU-quad savers, there are three variables to calibrate: the discount rate ρ, $v'(a_{ss})$ and $\tilde{\varphi}$. We use two of the variables to hit the two targets, as well as the third to ensure that savers' consumption-wealth ratio is unaffected by changes in wages or profits. We view this as a baseline that makes the BU-quad model comparable to the other models, which all share this property. The BU-log model only has two degrees of freedom, ρ and $\tilde{\varphi}$, and we choose them to hit both targets. Similarly, the OLG model is calibrated using ρ and ζ; and the HA model is calibrated using ρ and ν^{-1}. We assume savers are the top 10% of the U.S. income distribution. We set their share of $1 - \mu$ to 45.7%, in line with evidence on the top 10% national income share from the world inequality database (Piketty, Saez and Zucman, 2018). The process for e_{it} is a standard one, an AR(1) with persistence $\rho_e = 0.90$ and cross-sectional standard deviation $\sigma_e = 0.92$. We assume a standard Frisch elasticity of $\phi = 0.2$. Finally, we choose $\tilde{h}$ to normalize $n = 1$ in the initial steady state. The parameters of our U.S. calibration are summarized in Table A.3.

Calibration to Japan. For the Japanese economy, the calibration strategy is similar. We also set trend inflation to $\pi^* = 2\%$. The economy is at the ZLB, $R = 0$, and had a peak-to-peak nominal growth rate of 0.6% from 2008 through 2019. During this time, inflation in Japan was 1.7% below the target of 2%. We therefore set real trend growth to $\gamma = 0.3\%$. We

[30]The effective federal funds rate was 1.55%, the 5-year Treasury yield was 1.68%, the 10-year yield just above that. The implied 5-year 5-year forward rate was 2.04%.

[31]See Table I in their paper. Our κ is scaled to roughly correspond to their ψ estimates.

Figure 8: Deficit-debt diagrams in the U.S. and Japan



Note. Panel a shows the deficit-debt diagram implied by the U.S. calibration of our quantitative model. Panel b shows the diagram for the Japanese calibration. The loci shown correspond to possible steady states of the model.

match Japanese government debt to GDP of 238% (2019 value) and government spending of 20% of GDP (2019 value). Taxes are set as a residual to balance the government budget, giving $\chi = 18.6\%$. We keep parameters $\delta, \epsilon_I, m, \overline{y}$ as in the U.S. to facilitate the comparison. We calibrate the household sides using the exact same strategy as before. The top 10% income share $1 - \mu$ in Japan is set to 44.2% based on estimates of the top 10% national income share from the World Inequality Database.

The only remaining parameter to calibrate for Japan is the slope of the Phillips curve, κ. We pick κ so as to match evidence on the inflationary effects of Japanese government spending shocks at the ZLB in Miyamoto, Nguyen and Sergeyev (2018). We provide the details of this approach in Appendix G.4. We find a value of κ equal to 0.43, somewhat above the value we use for the U.S. economy. Table A.4 summarizes the Japanese calibration.

7.3 Deficit-debt diagrams

Figure 8 plots the levels of deficits and debt in units of our calibrated GDP across steady states of the model.

The left panel of Figure 8 shows the U.S. deficit-debt diagram across the four models. As is evident, the four models make relatively similar predictions for U.S. fiscal space before the pandemic. Locally around the calibrated steady state, this is by construction, as the slope φ is a calibration target. But even further away from the calibrated steady state, fiscal space looks similar across models. All models predict that the peak sustainable primary deficit is around 2% of GDP, obtained for a debt level around 120% of GDP. The models also imply that the interest rate R lies above the growth rate G for debt levels until around 200% of GDP. Finally, all models imply that a small ZLB region exists.

The right panel of Figure 8 shows the Japanese deficit-debt diagram across the models. Again, the models, despite their differences, make relatively similar predictions for Japanese fiscal space. First, all models imply that the calibrated steady state is at the ZLB, in the backward-bending part of the deficit-debt locus. This means that, according to all four models, a modest permanent fiscal expansion lowers debt levels in the long run (see section 7.4). The peak sustainable primary deficit is equal to around 2.5% of GDP, and obtained for debt levels around 180% of GDP.

A seemingly puzzling finding here is that Japan, with its very high debt level, seems to have access to greater fiscal space, including a greater peak sustainable deficit, than the U.S. For the most part, this is the result of a very low calibrated discount rate ρ (see Table A.4 in Appendix G.3), which is necessary to explain the very low interest rate in Japan before Covid despite the high debt levels.

Higher primary deficits are sustainable in Japan not despite the high debt level, but *because* of it. Higher deficits generate inflation, and inflation lowers real interest rates at the zero lower bound. As explained in Section 4.1, this effect, ceteris paribus, pushes for lower debt levels, especially with a high initial debt level. This explains why Japan, an economy with very high initial debt level that still finds itself at the zero lower bound, has such large fiscal space.

7.4 Free lunch in the quantitative model

We illustrate next how both economies, according to our calibration, were in the free lunch region before the Covid pandemic. To do so, we shock the primary deficit z_t,

$$z_t - z = \Delta z^{\text{perm}} + \Delta z^{\text{trans}} \cdot \rho_z^t, \qquad \text{where } \Delta z^{\text{perm}}, \Delta z^{\text{trans}} > 0 \tag{49}$$

in the U.S. and Japan. Δz^{perm} is a permanent deficit increase and Δz^{trans} a transitory one.

Panel a in Figure 9 shows the deficit, debt and output in the U.S. economy in response to a small positive deficit increase ($\Delta z^{\text{perm}} = 0.01\%$, $\Delta z^{\text{trans}} = 0.5\%$, $\rho_z = 0.9$). As shown, both the deficit and debt levels increase permanently, but debt does not diverge, just like explained in Section 3.3. All four models behave similarly in response to the shock. A similar deficit shock is fed into our model of the Japanese economy in Panel b. As Japan is in the backward bending part of its deficit-debt locus (see Figure 8), debt actually falls in response to the shock, because of greater inflation.

To show that a free lunch is contingent on the magnitude of the deficit increase, Panel c of Figure 9 simulates the deficit shock again, for the U.S. economy, but with larger Δz^{perm}, Δz^{trans}. Now, debt starts to explode which is why the deficit ultimately needs to be reduced. Such a large deficit shock is not a free lunch.

Finally, Panel d of Figure 9 simulates the response of a Sidrauski (1967) style economy with a nominal debt growth policy to a sudden increase in nominal debt growth (see Appendix C for details). On impact, the deficit increases and the debt level shrinks, akin to the dynamics in the Japanese economy at the zero lower bound (Panel b). However, here, as the debt level falls, deficits start falling below their original level. This is because, as illustrated in Figure A.5 in Appendix C, a free lunch policy in such an economy is only attainable to the right of the peak sustainable deficit, the opposite of our baseline setting.

7.5 Role of inequality and growth for fiscal space

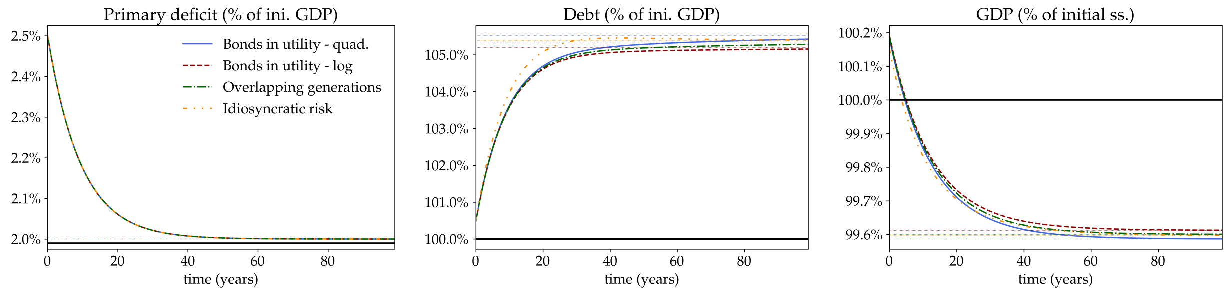
We end this section by revisiting two crucial comparative statics regarding the roles of inequality and growth in shaping fiscal space. For simplicity, we conduct these comparative statics in the BU-quad model.

Panel a of Figure 10 shows two deficit-debt diagrams, one for the U.S. in 1980, with a top 10% income share of $1 - \mu = 33.9\%$, and one for the U.S. in 2019, with a top 10% income share of $1 - \mu = 45.7\%$. On the one hand, the figure clearly shows that the increase in inequality significantly increased fiscal space. On the other, it also shows how some of that increased fiscal space was necessary to avoid getting stuck at the zero lower bound. In other words, had the U.S. government not expanded its deficits, not only would the zero lower bound have started to bind earlier than it did, but debt levels would have increased anyway, driven by inflation below target and higher real interest rates.

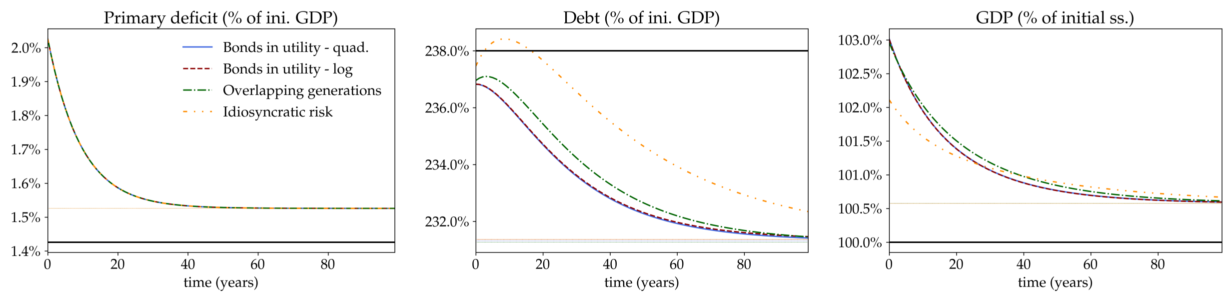
Panel b of Figure 10 shows two deficit-debt diagrams of the Japanese economy, one with a hypothetical real growth rate of $\gamma = 4.5\%$—its average real growth rate 1980–1990—and one with a growth rate of $\gamma = 0.3\%$—its average real growth rate 2008–2019. The growth decline did not expand fiscal space, despite lowering interest rates, as both R and G fell. However, it made the zero lower bound more likely to bind, which, just like for the U.S., forced the Japanese government to expand its debt and deficit.

Figure 9: Free lunch in the U.S. and Japan

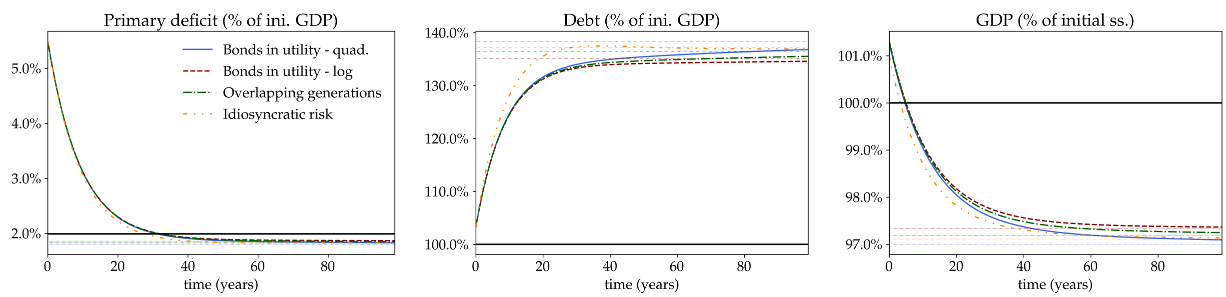
(a) Free lunch in the U.S.



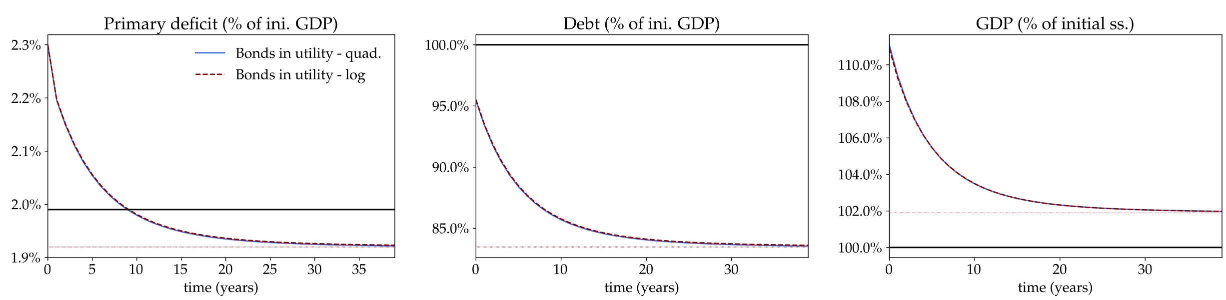
(b) Free lunch in Japan



(c) No free lunch for large shock in the U.S.



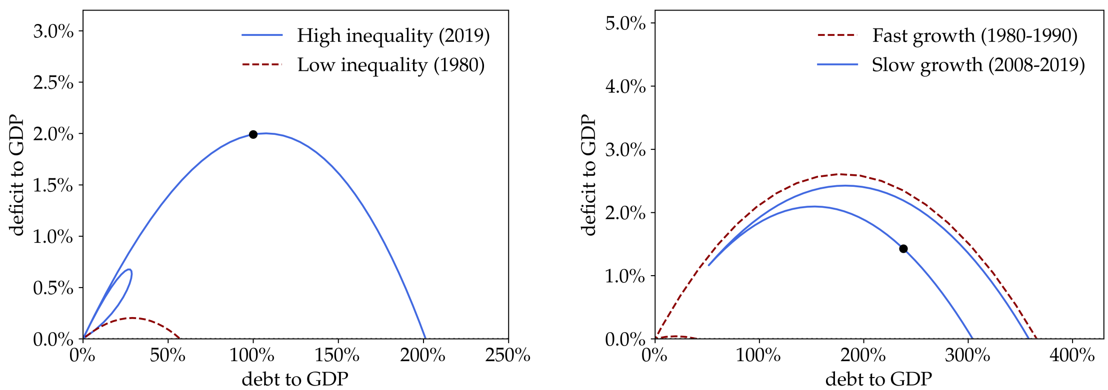
(d) No free lunch with nominal-debt-growth rule in the U.S.



Note. Panels a and b show free lunch policies for small deficit increases in the U.S. and Japan, respectively. A large deficit surge in the U.S. economy is no longer a free lunch (Panel c). Panel d illustrates what would happen with a nominal debt growth rule. The parameters used here are the following. Panel a: $\Delta z^{\text{perm}} = 0.01\%$, $\Delta z^{\text{trans}} = 0.5\%$, $\rho_z = 0.9$. Panel b: $\Delta z^{\text{perm}} = 0.1\%$, $\Delta z^{\text{trans}} = 0.5\%$, $\rho_z = 0.9$. Panel c: $\Delta z^{\text{perm}} = 0.5\%$, $\Delta z^{\text{trans}} = 3\%$, $\rho_z = 0.9$ with additional term $-0.018\,(b_t - b)$ in (49) to bring the deficit back down to a steady state. Panel d uses fiscal rule $z_t = \left(\frac{z}{b} - 0.023\right) b_t$ that induces constant nominal debt growth. Panel d assumes little price rigidity, $\kappa = 1$, in line with our discussion in Appendix C.

Figure 10: The making of fiscal space: Rising inequality and declining growth

(a) Effect of rising inequality in the U.S. **(b)** Effect of slowing growth in Japan



Note. Panel a compares the deficit-debt diagram of the U.S. economy in 2019, with high inequality, to the one in 1980, with low inequality, illustrating how much more fiscal space rising income inequality has caused. Panel b compares the deficit-debt diagram of the Japanese economy in the 2010s, with slow growth, to the one in the 1980s, with fast growth, illustrating how slowing growth made the ZLB region more binding, causing the backward-bending shape of the diagram.

8 Conclusion

The textbook view of debt and deficits is that raising deficits leads to an explosive path for government debt unless, at some point, deficits are reduced below their original level. In this paper, we argued that debt may not explode if $R < G - \varphi$ and the increase in deficits is modest ("free lunch"); and that debt may not even rise at all if the economy is at the ZLB and the nominal growth rate is sufficiently responsive to increased deficits. We further illustrated how inequality increases fiscal space outside the ZLB, but may reduce it at the ZLB. In the United States we found very little room for free lunch policies in 2019; but significant room for free lunch policies in Japan.

We have mostly focused on characterizing long-run dynamics in our paper. Our modeling approach, however, is very much amenable to being integrated in richer dynamic models of the short-run, including models with additional adjustment frictions, such as habits, inertial inflation, and other countries. We believe that such models can usefully connect short-run effects of fiscal deficits to the long-run effects we characterize here.

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A Goldilocks Theory of Fiscal Deficits
— Online Appendix —

Atif Mian Ludwig Straub Amir Sufi

A Additional proofs and model details

A.1 Proof of Proposition 1

From (10) we see that $z'(b) = G^* - R^*(b) - R^{*\prime}(b)b$. Substituting (9) into $z'(b)$ yields that $z'(b^*) = 0$ holds precisely when (11). If $\varphi(b) = R^{*\prime}(b)b$ is weakly increasing, $z'(b)$ is strictly decreasing and hence b^* is the unique global maximum.

A.2 Proof of Corollary 1

If $z(b)$ is single peaked, the unique global maximum b^* is also the unique local maximum. All points $b_0 < b^*$ are then necessarily characterized by $z'(b_0) > 0$, or equivalently, $R^*(b_0) < G^* - \varphi(b_0)$. For any such point, a permanent deficit increase by $\Delta z \equiv z(b_1) - z(b_0) > 0$, for some $b_1 \in (b_0, b^*)$ is a free lunch policy. Any point $b_0 \geq b^*$ does not allow for a free lunch as $z'(b_0) \leq 0$ there.

A.3 Proof of Corollary 2

The result follows directly by substituting out the interest rate (9) in the equation for the primary deficit (10),

$$z(b) = \left(v'(b)\left(1 - x - \mu\right) - \rho\right) b. \tag{A.1}$$

The locus $z(b)$ shifts down with higher ρ. In fact, if ρ rises above $v'(0)\left(1 - x - \mu\right)$, the government has to run a primary surplus at any positive level of debt.

A.4 Proof of Corollary 3

See (A.1) above.

A.5 Proof of Corollary 4

The post-tax interest rate $R^*(b)$ in (16) is the correct one to use in the government budget constraint. From (16), we see that for any b, $R^*(b)$ falls in τ^c, $\tilde{\tau}$, and is independent of τ^b. Substituting (16) into (10) then proves the statements.

A.6 Proof of Proposition 2

A steady state at the ZLB satisfies the following set of equations:

- the Euler equation of savers (18) at a steady state with $R_t = 0$

$$0 = \frac{\dot{c}_t}{c_t} = -G - \rho + v'(b)c \tag{A.2}$$

- the goods market clearing condition (17)

$$y = c + \tilde{c} + x \tag{A.3}$$

- spenders' budget constraint (3) with $\tilde{\tau}_t = 0$

$$\tilde{c} = \mu y \tag{A.4}$$

- the Phillips curve (4), expressed in terms of nominal growth rates,

$$G = G^* - \kappa(1 - y) \tag{A.5}$$

- the government budget constraint (5)

$$z + (0 - G)\, b = 0 \tag{A.6}$$

Combining (A.3), (A.4), (A.5), we obtain

$$G = G^* - \kappa \left(1 - \frac{c + x}{1 - \mu}\right) \tag{A.7}$$

Solving the Euler equation (A.2) for c and substituting it into (A.7), we ultimately find

$$(G^* - G)\left((1 - \mu)\, v'(b) - \kappa\right) = -\kappa R^*(b) \tag{A.8}$$

where the natural rate $R^*(b)$ is as in (9). We define $\hat{b}$ as the debt level which satisfies

$$v'(\hat{b}) = \frac{\kappa}{1 - \mu}.$$

Case (A). Now consider the case $\kappa < \hat{\kappa} = \frac{1-\mu}{1-x-\mu}(\rho + G^*)$ (case (A) of Proposition 2). In that case, for any $b < b^{ZLB}$, we have

$$v'(b) > v'(b^{ZLB}) = \frac{\rho + G^*}{1 - x - \mu} = \frac{\hat{\kappa}}{1-\mu} > \frac{\kappa}{1-\mu}.$$

This allows us to solve (A.8),

$$G(b) = G^* - \frac{\kappa}{v'(b)(1-\mu) - \kappa}(-R^*(b))$$

confirming (19). It only remains to be shown that there cannot be a steady state with a binding ZLB and $b \geq b^{ZLB}$. $b = b^{ZLB}$ cannot be a steady state, as there $R^*(b) = 0$ and thus, by (A.8), $G = G^*$ and $y = y^* = 1$. The ZLB is not binding there. If $b > b^{ZLB}$, $R^*(b) > 0$, so the only way for $G < G^*$ as part of a steady state is that $b > \hat{b}$ and $(1-\mu)v'(b) < \kappa$. In this case, we can write $G(b)$ as

$$G(b) = G^* - \frac{\frac{\rho+G^*}{1-x-\mu} - \frac{\kappa}{1-\mu}}{1 - \kappa^{-1}(1-\mu)v'(b)} - \frac{\kappa}{1-\mu} \tag{A.9}$$

which is clearly increasing in b for $b > \hat{b}$. The highest it gets is for $b \to \infty$, where $v'(b) \to 0$ and

$$G(b) \to G^* - \frac{\rho + G^*}{1 - x - \mu} < 0$$

This implies that the primary deficit z implied by (A.6) would have to be negative, which we ruled out in Proposition 2.

Case (B). If $\kappa = \hat{\kappa}$, then we can write steady state G as (A.9) whenever $\kappa^{-1}(1-\mu)v'(b) \neq 1$. In that case, (A.9) reads

$$G(b) = G^* - \frac{\kappa}{1-\mu} = G^* - \frac{\rho + G^*}{1 - x - \mu} < 0$$

and is ruled out for the same argument as above. If $\kappa^{-1}(1-\mu)v'(b) = 1$, then this implies $b = b^{ZLB}$ for $\kappa = \hat{\kappa}$. In that case, (A.8) is precisely satisfied since both sides of the equation are zero. The ZLB is binding precisely when $z < z^{ZLB}$ or else (A.6) would imply $G \geq G^*$.

Case (C). If $\kappa > \hat{\kappa}$, then for any $b > b^{ZLB}$,

$$v'(b)(1-\mu) < v'(b^{ZLB})(1-\mu) = \hat{\kappa} < \kappa$$

Thus, (19) is well defined there. It remains to be shown that no ZLB steady state exists with positive deficits and $b < b^{ZLB}$. In that range, $R^*(b) < 0$, so by virtue of (A.8) the only way for a ZLB steady state to exist is $(1-\mu)\, v'(b) > \kappa$, i.e. $b < \hat{b}$. Rearranging (A.9) as

$$G(b) = G^* - \frac{\frac{\kappa}{1-\mu} - \frac{\rho+G^*}{1-x-\mu}}{\kappa^{-1}(1-\mu)\, v'(b) - 1} - \frac{\kappa}{1-\mu}$$

the largest growth rate is obtained if $v'(b) \to \infty$, in which case

$$G(b) \to G^* - \frac{\kappa}{1-\mu} < G^* - \frac{\rho + G^*}{1-x-\mu} < 0$$

Thus, in this region, steady states at the ZLB must have a positive primary surplus, which we don't focus on here. This shows that with $\kappa > \hat{\kappa}$, the only feasible positive-deficit steady states are in set $\{(b,z) : b > b^{ZLB}, z(b) = G(b)b\}$.

A.7 Proof of Proposition 3

The dynamic equations characterizing equilibrium are (21) and (22). We first derive the associated phase diagram, showing that there are exactly two possible steady states for any given value of $\xi \in (0, G^*)$. We then argue that an equilibrium exists where $\dot{b}_t < 0$ in between the steady states, and $\dot{b}_t > 0$ left and right of the steady states. This will allow us to prove Proposition 3.

The dynamic equations can be written as

$$\frac{\dot{c}_t}{c_t} = -\rho - G^* + \kappa\left(1 - \frac{c_t + x}{1-\mu}\right) + v'(b_t)c_t$$

$$\dot{b}_t = \left(-G^* + \kappa\left(1 - \frac{c_t + x}{1-\mu}\right) + \xi\right) b_t$$

whenever $c_t < 1 - \mu - x$. When $c_t = 1 - \mu - x$, they are instead given by

$$\dot{c}_t = R_t - G^* - \rho + v'(b_t)c_t = 0$$

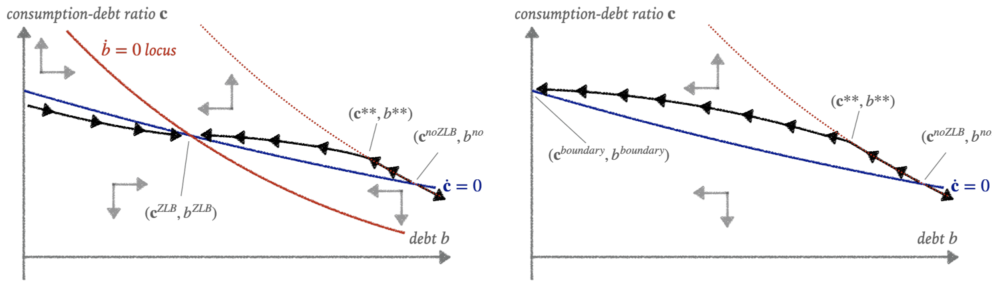
and

$$\dot{b}_t = (R_t - G^*)\, b_t + \xi b_t.$$

Variable transformation. In a first step, we transform the system of ODEs into new variables. The new variables are $\mathbf{c}_t \equiv c_t / b_t$ and b_t. $\mathbf{c}_t$ captures the ratio of consumption to

Figure A.1: Phase diagram with an occasionally binding ZLB

(a) Case A: $\kappa > \frac{1-\mu}{1-\mu-x}(G^* - \xi)$ **(b)** Case B: $\kappa \leq \frac{1-\mu}{1-\mu-x}(G^* - \xi)$



bonds and turns out to be a very useful object to study. It is straightforward to derive that

$$\frac{\dot{\mathbf{c}}_t}{\mathbf{c}_t} = v'(b_t)b_t\mathbf{c}_t - \xi - \rho$$

and

$$\frac{\dot{b}_t}{b_t} = \begin{cases} -G^* + \kappa\left(1 - \frac{b_t\mathbf{c}_t + x}{1-\mu}\right) + \xi & b_t\mathbf{c}_t < 1-\mu-x \\ \rho - v'(b_t)(1-\mu-x) + \xi & b_t\mathbf{c}_t = 1-\mu-x \end{cases}$$

Loci. The $\dot{\mathbf{c}} = 0$ locus is given by

$$\mathbf{c} = \frac{\xi + \rho}{v'(b)b}$$

and the $\dot{b} = 0$ locus is given by

$$\mathbf{c} = \frac{1}{b}(1-\mu-x)\left(1 - \frac{1-\mu}{1-\mu-x}\frac{G^* - \xi}{\kappa}\right).$$

We plot the two loci in Figure A.1. Under the assumption that $v'(b)b$ is increasing, the $\dot{\mathbf{c}} = 0$ locus is downward-sloping, starting from a finite and positive value at $b = 0$. We distinguish two cases.

Case A: $\kappa > \frac{1-\mu}{1-\mu-x}(G^* - \xi)$. In this case, there are points in the locus of $\dot{b} = 0$ in the first quadrant (with $c, b > 0$). This is shown in Figure A.1a. There exist two steady states, one at the ZLB and one outside. The ZLB steady state is given by

$$v'\left(b^{*ZLB}\right) = \frac{\xi + \rho}{1-\mu-x} \cdot \frac{\kappa}{\kappa - \frac{1-\mu}{1-\mu-x}(G^* - \xi)}. \tag{A.10}$$

and

$$\mathbf{c}^{*ZLB} = \frac{\xi + \rho}{v'\left(b^{*ZLB}\right) b^{*ZLB}}.$$

The steady state outside the ZLB is given by

$$v'\left(b^{*noZLB}\right) = \frac{\xi + \rho}{1 - \mu - x} \tag{A.11}$$

and

$$\mathbf{c}^{*noZLB} = \frac{1 - \mu - x}{b^{*ZLB}}.$$

In this case, as the phase diagram in Figure A.1a shows, there is a unique stable arm along which the economy converges to the ZLB steady state. On the right, the stable arm begins at one point $(\mathbf{c}^{**}, b^{**})$ on the line of full employment, where $b\mathbf{c} = 1 - \mu - x$. Further down along the line is the no-ZLB steady state. As long as the economy has not reached $(\mathbf{c}^{**}, b^{**})$, it must remain on the line of full employment. On that, the arrows point away from the no-ZLB steady state.

These arguments show that, in this case A, where $\kappa > \frac{1-\mu}{1-\mu-x}\left(G^* - \xi\right)$, we have that $\dot{b}_t < 0$ precisely in between the two steady states, and $\dot{b}_t \geq 0$ elsewhere. It is straightforward to see that the area between the two steady states precisely corresponds to the one described in Proposition 3.

Case B: $\kappa \leq \frac{1-\mu}{1-\mu-x}\left(G^* - \xi\right)$**.** In this case, there does not exist a ZLB steady state with $b, c > 0$. Instead, the economy converges to a boundary steady state, where

$$b^{boundary} = 0$$

and

$$\mathbf{c}^{boundary} = \frac{\xi + \rho}{\lim_{b \to 0} v'(b) b}.$$

There is still a unique stable arm that leads to this boundary steady state. The transversality condition holds along the trajectory, as the discounted marginal-utility weighted value of bonds converges to zero,

$$e^{-\rho t} b_t / c_t = e^{-\rho t} \frac{1}{\mathbf{c}_t} \to 0.$$

This follows directly from $\mathbf{c}_t \to \mathbf{c}^{boundary}$. Here, $\dot{b}_t < 0$ anywhere left of the no ZLB steady state, and $\dot{b}_t \geq 0$ right of it. It is straightforward to see that the area left of the steady state in this case precisely corresponds to the one described in Proposition 3.

A.8 Proof of Corollary 5

We prove that, starting from the ZLB steady state associated with some fiscal rule $z_t = \xi b_t$, $\xi = (0, G^*)$, we can always find a fiscal rule $z_t = \tilde{\xi} b_t$ for some $\tilde{\xi} > \xi$, that leads to an increase in primary deficits at all points in time.

We distinguish three cases, in line with the numbering introduced by Proposition 2.

Case (A): $\kappa < \hat{\kappa}$**.** In this case, we can choose $\tilde{\xi} = G^*$. With that fiscal rule, there is exactly a single steady state, as

$$\frac{\kappa}{\kappa - \frac{1-\mu}{1-\mu-x}\left(G^* - \xi\right)} = 1$$

so that the expressions in (A.10) and in (A.11) coincide. This steady state is exactly the steady state at the border of the ZLB region, whose deficit-debt ratio is given by $z^{ZLB}/b^{ZLB} = G^* = \xi$.

What happens to the deficit as we move from ξ to $\tilde{\xi}$? Certainly $\xi < \tilde{\xi}$ and debt b_t is slow-moving, so that primary deficits are bound to rise on impact. They also rise permanently as any ZLB steady state in this case has a debt level below b^{ZLB} (see Figure 4).

Case (B): $\kappa = \hat{\kappa}$**.** The argument in this case is identical, except that any ZLB steady state has a debt level equal to b^{ZLB} (see Figure 4).

Case (C): $\kappa > \hat{\kappa}$**.** In this case, for any ZLB steady state (z^{*ZLB}, b^{*ZLB}), there is a non-ZLB steady state at the same debt level and with greater deficits $z(b^{*ZLB}) > z^{*ZLB}$. Choosing $\tilde{\xi} = z(b^{*ZLB})/b^{*ZLB}$ ensures that deficits rise permanently.

A.9 Proof of Corollary 6

This follows from the fact that $z(b) = G(b)b$ at the ZLB. Differentiating (19) we find $\partial G(b)/\partial G^* = v'(b)\left(1-\mu\right)/\left(v'(b)\left(1-\mu\right)-\kappa\right) > 0$.

A.10 Proof of Proposition 4

Differentiating (19) with respect to μ, we see that

$$\frac{\partial G}{\partial \mu} = -\frac{\kappa v'(b)}{\left(v'(b)\left(1-\mu\right)-\kappa\right)^2}\left(-R^*(b)\right) + \frac{\kappa}{v'(b)\left(1-\mu\right)-\kappa} v'(b).$$

After some algebra, and using $\kappa > 0$, $\partial G/\partial\mu > 0$ is equivalent to $v'(b)x > \kappa - \rho - G^*$. For any $b < b^{ZLB}$, this is implied if $v'(b^{ZLB})x > \kappa - \rho - G^*$. Using $v'(b^{ZLB}) = \frac{\rho+G^*}{1-x-\mu}$, this is equivalent to $\kappa < \hat{\kappa}$, which we assume here. Thus, $\partial G/\partial\mu > 0$. From $z(b) = G(b)b$ we see that inequality reduces fiscal space.

A.11 Proof of Proposition 5

A small transfer of $d\tilde{\tau} < 0$ with an associated increase in the primary deficit of $dz = -d\tilde{\tau} > 0$ leads to a reduction in debt if $d\dot{b} = -dz + bdG > 0$. Here, dG is the change in nominal growth, $dG = \kappa v'(b)/\left(v'(b)\left(1-\mu\right)-\kappa\right)dz$, which follows by substituting $R^*(b_t) = \rho + G^* - v'(b_t)\left(1 - x - \mu + \tilde{\tau}\right)$ (a special case of (16)) into (19). So, $d\dot{b} > 0$ iff $b \cdot dG/dz > 1$, which is equivalent to $\kappa > \frac{v'(b)}{v'(b)b+1}\left(1-\mu\right)$. This condition is loosest among all debt levels with a binding ZLB when $b = b^{ZLB}$, for which case we note that $v'(b^{ZLB})(1-\mu) = \hat{\kappa}$. This proves the results in the proposition.

A.12 Details on the model with aggregate risk

A.12.1 Derivation of (26)

From the geometric Brownian motion for y_t^*, (23), we see that y_t^* follows the process

$$dy_t^* = de^{\log y_t^*} = y_t^*\left(\gamma + \frac{1}{2}\sigma^2\right)dt + y_t^*\sigma dZ_t. \tag{A.12}$$

The saver solves (24) subject to (25). Denote by λ_t the costate of B_t. This implies a first order condition of

$$\lambda_t = C_t^{-\nu}$$

and a law of motion of λ_t of

$$\mathbb{E}_t\left[d\lambda_t\right] = \lambda_t\left(\rho + \pi^* - R_t\right)dt - \left(y_t^*\right)^{-\nu}v'(b_t)dt \tag{A.13}$$

Denoting by $\sigma_{\lambda t}$ the instantaneous volatility of $\lambda_t^{-1}d\lambda_t$, we move from $d\lambda_t$ to $d\log\lambda_t$

$$d\log\lambda_t = \frac{d\lambda_t}{\lambda_t} - \frac{1}{2}\sigma_{\lambda t}^2 dt. \tag{A.14}$$

Now define normalized consumption as $c_t \equiv C_t/y_t^*$ so that $\lambda_t = \left(c_t y_t^*\right)^{-\nu}$, or in logs,

$$\log\lambda_t = -\nu\log c_t - \nu\log y_t^*.$$

With σ_{ct} being the instantaneous volatility of $c_t^{-1} dc_t$, we have

$$d \log c_t = \frac{dc_t}{c_t} - \frac{1}{2}\sigma_{ct}^2 dt$$

so that

$$d \log \lambda_t = -\nu \frac{dc_t}{c_t} + \nu \frac{1}{2}\sigma_{ct}^2 dt - \nu\gamma dt - \nu\sigma dZ_t \tag{A.15}$$

which implies that $\sigma_{\lambda t} = -\nu\sigma_{ct} - \nu\sigma$. Substituting (A.13) and (A.15) into (A.14), we find

$$-\nu \mathbb{E}_t \frac{dc_t}{c_t} + \nu \frac{1}{2}\sigma_{ct}^2 dt - \nu\gamma dt = \mathbb{E}_t \frac{d\lambda_t}{\lambda_t} - \frac{1}{2}\sigma_{\lambda t}^2 dt$$

or simplified,

$$\nu \mathbb{E}\left[\frac{dc_t}{c_t}\right] = \left(R_t - \rho - \pi^* - \nu\gamma\right) dt + c_t^\nu v'(b_t) dt + \frac{1}{2}\nu^2 \left(\sigma_{ct} + \sigma\right)^2 dt + \nu \frac{1}{2}\sigma_{ct}^2 dt \tag{A.16}$$

Defining the growth-adjusted discount rate $\hat{\rho}$ by $\hat{\rho} \equiv \rho + (\nu - 1)\gamma$ and using $G^* = \gamma + \pi^*$ as before, we arrive at (26).

A.12.2 Derivation of $R(b_t)$ in (27) with aggregate risk

In equilibrium, $c_t = 1 - x - \mu$ is a constant by goods market clearing. This also means that $\sigma_{ct} = 0$. Substituting into (A.16), we find

$$R_t = R(b_t) = G^* + \underbrace{(\nu - 1)\gamma + \rho}_{=\hat{\rho}} - \frac{1}{2}\nu^2\sigma^2 - (1 - x - \mu)^\nu v'(b_t).$$

A.12.3 Derivation of normalized government budget constraint (28)

To derive (28), observe that the usual budget constraint still holds,

$$dB_t = \left(R(b_t) - \pi^*\right) B_t dt + \left(x y_t^* - T_t\right) dt \tag{A.17}$$

Therefore, using (A.12), the evolution of $b_t \equiv B_t / y_t^*$ is given by

$$db_t = z_t dt + \left(R(b_t) - G^* + \frac{1}{2}\sigma^2\right) b_t dt - b_t \sigma dZ_t$$

where the $\frac{1}{2}\sigma^2 = \sigma^2 - \frac{1}{2}\sigma^2$ term is the Ito correction σ^2 coming from the volatility of y_t^* minus $\frac{1}{2}\sigma^2$ from (A.12). Observe that the law of motion for $\log b_t$ does not have a correction,

$$d \log b_t = \frac{z_t}{b_t} dt + (R(b_t) - G^*)\, dt - \sigma dZ_t$$

confirming that $\mathbb{E}_t [d \log b_t] = 0$ if $z_t = (G^* - R(b_t))\, b_t$.

A.12.4 Derivation of sufficiency of (29) for the transversality condition

The transversality condition for the saver maximizing (24) is demanding that, at the optimum, the expected present shadow value of bonds at date t go to zero as $t \to \infty$,

$$\lim_{t\to\infty} e^{-\rho t} \mathbb{E}_0 [\lambda_t B_t] = 0.$$

To prove that this indeed holds under condition (29), we define $h_t = e^{-\rho t}\lambda_t B_t$. Building on (A.13) and (A.17), h_t evolves as

$$dh_t = -\rho h_t dt + h_t \frac{d\lambda_t}{\lambda_t} + h_t \frac{dB_t}{B_t},$$

or, substituting out $d\lambda_t$ and dB_t,

$$dh_t = -\rho h_t dt + h_t \frac{1}{2} \nu^2 \sigma^2 dt - \nu\sigma h_t dZ_t + (R(b_t) - \pi^* - \nu\gamma)\, h_t dt + \frac{z_t}{b_t} h_t dt.$$

If condition (29) holds, we can bound h_t above by process $\tilde{h}_t$, which evolves as

$$d\tilde{h}_t = -\epsilon \tilde{h}_t dt - \nu\sigma \tilde{h}_t dZ_t$$

at all times $t > T$ where T is chosen such that (29) holds thereafter. $\tilde{h}_t$ is a standard geometric Brownian motion whose expectation $\mathbb{E}_0 \tilde{h}_t$ converges to zero. Thus, $\mathbb{E}_0 h_t \le \mathbb{E}_0 \tilde{h}_t$ must converge to zero as well.

A.12.5 Proof of Proposition 6

The evolution of debt b_t without the increase in debt by Δ is given by

$$db_t = z_t dt + \left(R(b_t) - G^* + \frac{\sigma^2}{2} \right) b_t dt - b_t \sigma dZ_t.$$

The evolution of debt b_t^Δ after increasing debt by $\Delta > 0$ at date 0 is

$$db_t^\Delta = z_t dt + \left(R(b_t^\Delta) - G^* + \frac{\sigma^2}{2}\right) b_t^\Delta dt - b_t^\Delta \sigma dZ_t.$$

Given the convenience yield is affine-linear, as in (30), the interest rate schedule $R(b)$ has a constant slope $\phi \equiv R'(b)$. In the notation of (30), $\phi = \varphi / b_0$. Here, we use $\varphi(b)$ to denote the local semi-elasticity of R to debt around an arbitrary debt level b, $\varphi(b) = \frac{\partial R(b)}{\partial \log b} = \phi b$. Of course, around b_0, $\varphi(b_0)$ is exactly equal to the φ in (30).

We denote the difference between the two by $\Delta b_t \equiv b_t^\Delta - b_t$. It satisfies the SDE

$$d\left(\Delta b_t\right) = \left[R(b_t) - G^* + \varphi(b_t) + \frac{\sigma^2}{2} + \phi \Delta b_t\right] \Delta b_t dt - \Delta b_t \cdot \sigma dZ_t. \tag{A.18}$$

Our goal is to show that $\lim_{\Delta \to 0} P(\Delta b_t \to 0) = 1$. We do so by first analyzing a simpler process, $\widetilde{\Delta b_t}$, defined by

$$d\left(\widetilde{\Delta b_t}\right) = \left[R(b_t) - G^* + \varphi(b_t) + \frac{\sigma^2}{2}\right] \widetilde{\Delta b_t} dt - \widetilde{\Delta b_t} \cdot \sigma dZ_t \tag{A.19}$$

with same initial condition $\widetilde{\Delta b_0} = \Delta b_0 = \Delta$.

Characterizing the process $\widetilde{\Delta b_t}$. The SDE for $\log \widetilde{\Delta b_t}$ is given by

$$d \log \widetilde{\Delta b_t} = \left[R(b_t) - G^* + \varphi(b_t)\right] dt - \sigma dZ_t \tag{A.20}$$

We can integrate (A.20),

$$\log \widetilde{\Delta b_T} - \log \Delta = \int_0^T \left(R(b_t) - G^* + \varphi(b_t)\right) dt - \sigma Z_T$$

We note here that $\widetilde{\Delta b_T}$ scales with Δ.

Since b_t follows a stationary Markov process, the strong law of large numbers (Ergodic Theorem) holds,

$$\frac{1}{T} \int_0^T b_t dt \to \int b \mathcal{F}(db) \qquad \text{a.s.}$$

By linearity of R, φ, it follows that

$$\frac{1}{T} \int_0^T \left(R(b_t) - G^* + \varphi(b_t)\right) dt \to \overline{R} - G^* + \overline{\varphi} \qquad \text{a.s.} \tag{A.21}$$

with $\overline{R}, \overline{\varphi}$ as defined in the text of Proposition 6. Moreover, another application of the strong law of large numbers gives[A1]

$$\frac{1}{T} Z_T \to 0 \qquad \text{a.s.} \tag{A.22}$$

Together, (A.21) and (A.22) imply that

$$\frac{1}{T}\left(\log \widetilde{\Delta b_T} - \log \Delta\right) \to \overline{R} - G^* + \overline{\varphi} \qquad \text{a.s.}$$

We now distinguish two cases, depending on the sign of $\overline{R} - G^* + \overline{\varphi}$. Suppose first that $\overline{R} - G^* + \overline{\varphi} < 0$. Pick some $\delta > 0$ such that $\overline{R} - G^* + \overline{\varphi} + \delta < 0$. For almost any sample path of $\widetilde{\Delta b_T}$, we can find a time $\underline{T}$, such that for any $T > \underline{T}$,

$$\frac{1}{T}\left(\log \widetilde{\Delta b_T} - \log \Delta\right) \leq \overline{R} - G^* + \overline{\varphi} + \delta.$$

This implies

$$\widetilde{\Delta b_T} \leq \Delta \cdot \exp\left\{\left(\overline{R} - G^* + \overline{\varphi} + \delta\right) T\right\}.$$

Given $\overline{R} - G^* + \overline{\varphi} + \delta < 0$, this establishes that $\widetilde{\Delta b_T} \to 0$ along almost any sample path, and hence $\widetilde{\Delta b_T} \to 0$ almost surely. In addition, it establishes that $\widetilde{\Delta b_T}$ is integrable along almost any sample path, that is,

$$\int_0^\infty \widetilde{\Delta b_T} dT < \infty \qquad \text{a.s.}$$

Now consider the case $\overline{R} - G^* + \overline{\varphi} > 0$ and chose δ such that $\overline{R} - G^* + \overline{\varphi} - \delta > 0$. Then, for almost any sample path of $\widetilde{\Delta b_T}$, we can find a time $\underline{T}$, such that for any $T > \underline{T}$,

$$\frac{1}{T}\left(\log \widetilde{\Delta b_T} - \log \Delta\right) \geq \overline{R} - G^* + \overline{\varphi} - \delta$$

and therefore

$$\widetilde{\Delta b_T} \geq \Delta \cdot \exp\left\{\left(\overline{R} - G^* + \overline{\varphi} + \delta\right) T\right\}.$$

Given $\overline{R} - G^* + \overline{\varphi} - \delta > 0$, this establishes that in this case, $\widetilde{\Delta b_T} \to \infty$ along almost any sample path, and hence $\widetilde{\Delta b_T} \to \infty$ almost surely.

Having investigated the properties of $\widetilde{\Delta b_t}$, we now return to Δb_t.

[A1]See e.g. `https://www.stat.berkeley.edu/~pitman/s205s03/lecture15.pdf`, Example 15.6 for a proof.

Characterizing the process Δb_t. Δb_t differs from $\widetilde{\Delta b_t}$ as the former has an additional nonlinear term, $\phi\left(\Delta b_t\right)^2$, in its SDE (A.18). We therefore clearly have that $\Delta b_t \geq \widetilde{\Delta b_t}$. This already gives us our first result, namely that $\overline{R} - G^* + \overline{\varphi} > 0$ implies almost sure divergence of Δb_t, or in other words, $P(b_t^{\Delta} \to b_t) = 0$.

Accordingly, we focus on the case $\overline{R} - G^* + \overline{\varphi} < 0$ hereafter. We can formally study the difference between Δb_t and $\widetilde{\Delta b_t}$ by characterizing the SDE of $\widetilde{\Delta b_t}/\Delta b_t$ (which must lie between 0 and 1). After some algebra combining (A.18) and (A.19), we find

$$d\left(\frac{\widetilde{\Delta b_t}}{\Delta b_t}\right) = -\phi\widetilde{\Delta b_t}dt.$$

This SDE has the solution

$$\frac{\widetilde{\Delta b_T}}{\Delta b_T} - 1 = -\int_0^T \phi\widetilde{\Delta b_t}dt$$

or, equivalently,

$$\Delta b_T = \frac{\widetilde{\Delta b_T}}{1 - \int_0^\infty \phi\widetilde{\Delta b_t}dt}$$

which is well defined for any sample path with $\int_0^T \phi\widetilde{\Delta b_t}dt < 1$. Since we showed above that (a) for $T \to \infty$, $\int_0^T \widetilde{\Delta b_t}dt$ is finite almost surely and (b) $\widetilde{\Delta b_t}$ scales in Δ, it follows that for any Δ for which $\int_0^\infty \phi\widetilde{\Delta b_t}dt < 1$,

$$\lim_{T\to\infty} \Delta b_T = \frac{\lim_{T\to\infty}\widetilde{\Delta b_T}}{1 - \int_0^\infty \phi\widetilde{\Delta b_t}dt} = 0.$$

Thus,

$$P\left(\Delta b_T \to 0\right) \geq P\left(\int_0^\infty \phi\widetilde{\Delta b_t}dt < 1\right)$$

but the latter probability approaches 1 as we take $\Delta \to 0$, since $\widetilde{\Delta b_t}$ scales in Δ. Therefore,

$$\lim_{\Delta\to 0} P\left(\Delta b_T \to 0\right) = 1$$

which is what we set out to prove.

A.13 Details on the model with capital

We begin with the household optimization problem (31). The Euler equation for bonds is given by

$$\frac{\dot{c}_t}{c_t} = R_t - G_t - \rho + \frac{c_t}{y_t} v' \left(\frac{b_t + k_t}{y_t} \right). \tag{A.23}$$

The Euler equation for capital is given by

$$\frac{\dot{c}_t}{c_t} = r_t^k - \gamma - \rho + \frac{c_t}{y_t} v' \left(\frac{b_t + k_t}{y_t} \right). \tag{A.24}$$

We characterize the steady state. Then, subtracting (A.23) from (A.24), we find

$$R^* - G^* = r^k - \gamma = m^{-1} \alpha k^{\alpha - 1} - \delta_k - \gamma = m^{-1} \alpha \frac{y}{k} - \delta_k - \gamma.$$

Moreover, in a steady state, (A.23) looks as usual

$$R^* = \rho + G^* - (1 - x - \mu) \, v' \left(\frac{b+k}{y} \right).$$

This completes our derivation of (32). Linearizing the two equations in (32), we obtain

$$dR = \varphi \left(\frac{b+k}{y} \right) \left(d\frac{b}{y} + d\frac{k}{y} \right) \qquad \text{and} \qquad dR = -\left(k/y \right)^{-1} \frac{dR}{R - G^* + \delta_k + \gamma}$$

Combining these two equations, yields (33), (34), and Proposition 7.

B Additional model details and extensions

B.1 Microfoundation of the convenience utility

In the main body of the paper, we have taken the convenience yield $v(b)$ as given, deriving implications for fiscal space. We next propose a microfoundation for the convenience yield $v(b)$ that is fully consistent with our previous analysis.

Convenience benefits are typically thought of as either liquidity or safety premia. Many microfoundations exist for liquidity (e.g. Lagos and Wright 2005), and some have been shown to reduce to a $v(b)$ function (Angeletos, Collard and Dellas 2020). In this section, instead, we propose a model of safety premia, interpreting bonds as being safe if they are likely to pay out even after a big disaster.[A2]

[A2] We describe in Appendix D a number of alternative models and show that they numerically have similar

Consider an economy like the one in Section 2, with two changes. First, there is no ad-hoc convenience utility function v. Second, there is a flow probability $\lambda > 0$ with which a disaster occurs. Conditional on the disaster occurring, it reduces potential output y^* from 1, our normalized pre-disaster value, to $\delta \in (0, 1)$, with probability $f(\delta)$, where $\int_0^1 f(\delta)d\delta = 1$. The only friction that we assume in this model is that the government can only raise tax revenue τ_t from savers up to some fraction $\overline{\tau} + x$ of output.[A3] For simplicity, it cannot tax spenders, $\tilde{\tau} = 0$. If debt service requires greater taxes, we assume that the government defaults. We assume that default entails default costs (in the form of transfers to households, not resource costs) that are sufficiently large so that all bond wealth is lost.[A4]

We analyze this model in two steps. First, we focus on the economy after a disaster of size δ happened. Then, we study the economy before the disaster shock, and argue that it is isomorphic to our model in Section 2.[A5]

When a disaster of size δ materializes, the interest rate rises to $R = G^* + \rho$, as bonds lose their "specialness". This requires the economy to run a primary surplus of $\rho b/\delta$ relative to GDP. Given the upper bound on taxes of $\overline{\tau} + x$, default occurs when output after the shock δ falls below $\underline{\delta} \equiv \rho b/\overline{\tau}$. We denote by $\tilde{V}_t(b;\delta)$ the utility of an individual saver with bond position b after shock δ realizes.

Before the disaster occurs, savers now maximize utility

$$\rho V_t(b) \equiv \max_c \log c + \lambda \int_0^1 f(\delta) \left(\tilde{V}_t(b;\delta) - V_t(b)\right) d\delta + \dot{V}_t(b) + V_t'(b)\dot{b}_t \tag{A.25}$$

where $\dot{b}_t$ is given by the budget constraint (2). Combining the first order condition for c and the Envelope theorem for $V_t(b)$, this formulation can be shown to imply a natural rate before the disaster that depends on b and is given by

$$R^*(b) = \rho + G^* - \lambda F\left(\frac{\rho b}{\overline{\tau}}\right) \tag{A.26}$$

where we defined

$$F(\underline{\delta}) \equiv \int_{\underline{\delta}}^1 f(\delta)\delta^{-1}d\delta - 1. \tag{A.27}$$

$F(\underline{\delta})$ determines the insurance value of a bond that pays off whenever the shock is better

implications to our reduced-form convenience-yield model of Section 2.

[A3] We include the share of government spending x here so that the government can always finance its spending. This is equivalent to a cap on the primary surplus of $\overline{\tau}$.

[A4] This is equivalent to the government defaulting on all its debt. The case with partial default is very similar to the analysis below.

[A5] This is again similar to the "risky steady state" in Coeurdacier, Rey and Winant (2011), and to the Poisson shocks in Caballero and Farhi (2018*b*) and Caballero, Farhi and Gourinchas (2016).

than $\underline{\delta}$. If $F(\underline{\delta}) < 0$, this implies that the bond, on net, is risky, which will be the case for $\underline{\delta}$ close to 1. If $F(\underline{\delta}) > 0$, which will be the case for $\underline{\delta}$ closer to zero, the bond is, on net, safe. In that case, $\lambda F\left(\frac{\rho b}{\overline{\tau}}\right)$ corresponds to the convenience yield, analogous to $v'(b)(1-x)$ in (9). As before, the convenience yield falls in b.

The definition of F in (A.27) illustrates exactly the premium for "safety": bonds that pay out in very adverse states of the world, with low δ, carry a higher convenience yield. In the special case where the density is equal to $f(\delta) = 2\delta$ and $F(\underline{\delta}) = 1 - 2\underline{\delta}$, we find that the convenience yield is given by

$$\lambda F\left(\frac{\rho b}{\overline{\tau}}\right) = \lambda - 2\frac{\rho\lambda}{\overline{\tau}} b$$

microfounding our affine-linear specification (30).

Observe that this model here has the same deficit-debt locus as one in which savers have preferences (1) with convenience utility

$$v(b) = \frac{1}{1-x} \cdot \lambda \int_0^b F\left(\frac{\rho\tilde{b}}{\overline{\tau}}\right) d\tilde{b}.$$

B.2 Maturity structure and fiscal space

We begin by introducing long-term debt into the model. We denote by b_t^{ST} short-term debt and by b_t^{LT} long-term debt (relative to potential GDP). We assume that long-term debt also carriers convenience benefits for savers, albeit less than short -term debt. Thus, we assume a convenience utility of

$$v\left(b_t^{ST} + \alpha b_t^{LT}\right)$$

where $\alpha \in (0,1)$. This specification implies that the natural interest rate on short-term debt, which we continue to denote by R_t^*, is given by

$$R_t^* = \rho + G^* - v'(b_t^{ST} + \alpha b_t^{LT})\,(1 - x - \mu)\,.$$

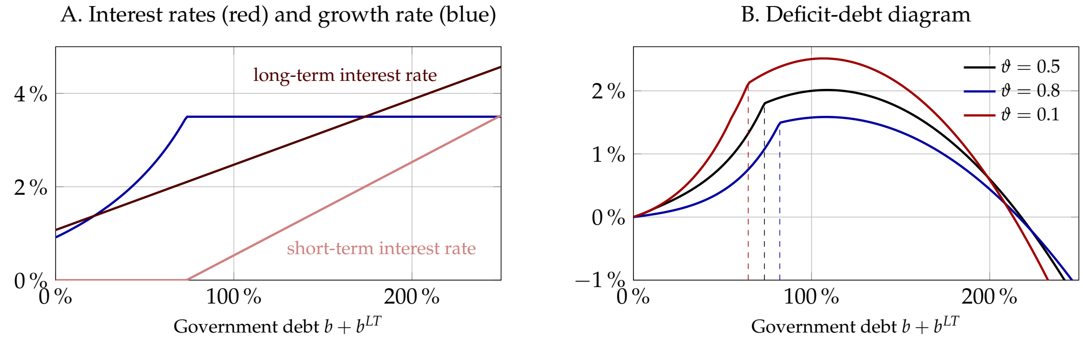
The ZLB binds whenever $R_t^* < 0$ so that $R_t = \max\{R_t^*, 0\}$. The interest rate on long-term debt, which we denote by R_t^{LT}, is then

$$R_t^{LT} = R_t + (1-\alpha)\,v'(b_t^{ST} + \alpha b_t^{LT})\,(1 - x - \mu)\,.$$

In particular, R_t^{LT} is strictly greater than R_t, and the spread between the two shrinks in $b_t^{ST} + \alpha b_t^{LT}$.

To see how this affects the deficit-debt diagram, we denote the share of LT debt issued

Figure A.2: Fiscal space with various shares ϑ of long-term debt



Note. Plot uses $\alpha = 0.7$. Left panel: $\vartheta = 0.5$. Right panel: $\vartheta \in \{0.1, 0.5, 0.8\}$. This figure is only illustrative. It uses a κ of $\kappa = 0.075$ and otherwise follows the parameterization in Section 5.3.

by the government by ϑ. The government budget constraint is then

$$\frac{d}{dt}\left(b_t^{ST} + b_t^{LT}\right) = \left(\overline{R}_t - G_t\right)\left(b_t^{ST} + b_t^{LT}\right) + z_t$$

where $\overline{R}_t = (1 - \vartheta) R_t + \vartheta R_t^{LT}$ and G_t is equal to G^* outside the ZLB and (19) at the ZLB, as before.

Figure A.2(A) plots R_t, R_t^{LT}, and G_t as function of total debt $b^{ST} + b^{LT}$, illustrating the positive spread between R_t and R_t^{LT}, which shrinks at higher debt levels. Figure A.2(B) plots the deficit-debt locus $z\left(b^{ST} + b^{LT}\right)$, as function of total debt $b^{ST} + b^{LT}$, for various shares of long-term debt ϑ. Two observations are noteworthy. First, with greater shares of long-term debt ϑ, there is less fiscal space at small debt levels; the ZLB region is greater; and the boundary of the free lunch region b^* generally shifts to the left. Second, with greater ϑ, there is generally *more* fiscal space at higher debt levels. This is a direct consequence of the fact that long-term debt has smaller convenience benefits, so both interest rates R_t and R_t^{LT} increase less rapidly in long-term debt.

A stylized way to think of large scale purchases of long-term government debt (one type of quantitative easing, QE) is that it changes the maturity composition of government liabilities towards short-term debt, effectively lowering ϑ. As Figure A.2(B) shows, this can help an economy escape the ZLB (as in Gertler and Karadi 2018, Caballero and Farhi 2018*b*, and Cui and Sterk forthcoming), and gives it greater fiscal space at low debt levels. However, it also highlights that QE may reduce fiscal space at higher debt levels.

B.3 Fiscal space with general asset market structure

We next show how the example of the previous subsection can be generalized to allow for a general asset market structure. For simplicity, we focus here on an economy without ZLB. We assume here that there are N distinct asset classes, labeled by $n = 1, \ldots, N$. Households have a general convenience utility

$$v(b_{1t}, \ldots, b_{Nt}). \tag{A.28}$$

If $b_{nt} > 0$, the government is indebted in asset class n to households; if $b_{nt} < 0$ the government is a saver in asset class n. Note that any constraints on assets (e.g. limits on borrowing or saving in a particular asset n) can be incorporated into v. The previous subsection is a special case of this setup where $N = 2$, $b_{1t} = b_t^{ST}$, $b_{2t} = b_t^{LT}$, and $v(b_1, b_2) = v(b_1 + \alpha b_2)$.

We denote the interest rate on asset n by R_{nt}. With the general convenience utility (A.28), the saver's Euler equation with respect to additional saving in asset class n implies

$$R_{nt} = R_n\left(b_{1t}, \ldots, b_{Nt}\right) = \rho + G^* - (1 - x - \mu)\frac{\partial v}{\partial b_n}\left(b_{1t}, \ldots, b_{Nt}\right).$$

The government budget constraint then reads

$$\sum_{n=1}^{N} \dot{b}_{nt} = \sum_{n=1}^{N}\left(R_{nt} - G^*\right) b_{nt} + z_t.$$

We define the asset specific sensitivity φ_n as

$$\varphi_n\left(b_1, \ldots, b_N\right) \equiv \sum_{m=1}^{N} b_m \frac{\partial R_m}{\partial b_n}(b_1, \ldots, b_N).$$

φ_n captures the impact on total borrowing cost of the government from the increase in interest rates caused by an additional unit of debt in asset class n.

We define the minimum weighted average interest rate by

$$\overline{R}\left(b\right) \equiv \min_{\sum b_n = b} \sum_{n=1}^{N} R_n\left(b_1, \ldots, b_N\right)\frac{b_n}{b}. \tag{A.29}$$

We assume that a minimum exists. $\overline{R}\left(b\right)$ is the interest rate the government pays if it chooses the optimal portfolio across asset classes, holding total debt fixed at b. We define total sensitivity as

$$\varphi(b) \equiv b \frac{\partial \overline{R}}{\partial b}(b).$$

At the minimum in (A.29), we have to have that the marginal cost of borrowing in asset class n, $R_n + \varphi_n$, is the same across asset classes; in that case, it is precisely equal to $\overline{R}(b) + \varphi(b)$.

We define the deficit-debt locus in this economy as

$$z(b) \equiv \left(G^* - \overline{R}(b)\right) b.$$

It is easy to see that an economy whose total debt is b, whose primary deficit is $z(b)$, and whose portfolio minimize the average interest rate as in (A.29) is in a steady state with $\dot{b}_{nt} = 0$ across asset classes.

We have the following result on free lunch policies in this environment.

Proposition 8. *Assume $z(b)$ is single-peaked. A free lunch policy exists in the economy with general asset classes with initial steady state debt b_0 and primary deficit z_0 if and only if either $z_0 < z(b_0)$ or*

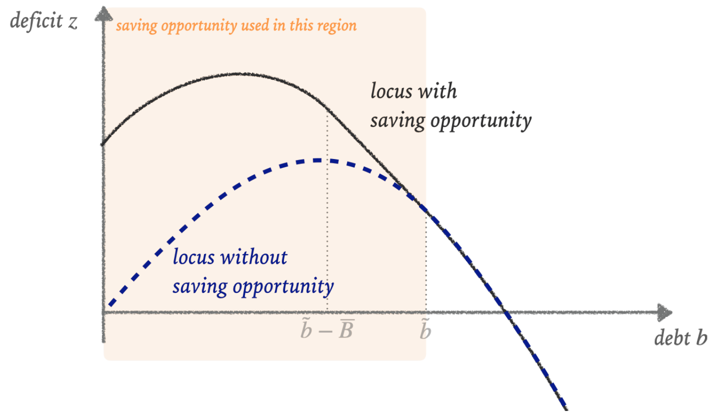
$$\overline{R}(b_0) < G - \varphi(b_0). \tag{A.30}$$

Proof. If $z_0 < z(b_0)$ then, by definition of $\overline{R}$ in (A.29), we can find a portfolio with the same initial total debt b_0 that allows for a greater primary deficit $z(b_0)$. Hence, there is a free lunch. The rest of the proof is exactly analogous to that of Corollary 1. If $z_0 = z(b_0)$ and (A.30), the argument exactly follows by raising deficits to some level between z_0 and the peak of $z(b)$. Vice versa, if $z_0 = z(b_0)$ and (A.30) does not hold, total debt cannot fall, and hence primary deficits cannot persistently rise, without first cutting deficits below z_0. □

Proposition 8 shows that there are now two ways in which a free lunch can materialize based on some initial steady state with total debt b_0 and deficit z_0. First, it can be that the steady state has a suboptimal portfolio, one that does not minimize the interest rate as in (A.29). In this case, the steady state will lie strictly below the deficit-debt locus. A free lunch here is simply given by moving up to the locus by way of adjusting the portfolio. Second, if the steady state lies on the deficit-debt locus already, a free lunch exists precisely under the same condition as before, $\overline{R} < G - \varphi$.

Example: Government can save in asset without convenience yield. We illustrate Proposition 8 with the following example with $N = 2$. Imagine asset class 1 is the usual debt with convenience yield coming from utility $v(b_1)$; asset class 2 instead allows the

Figure A.3: Deficit-debt diagram with additional asset class



Note. This figure compares our baseline deficit-debt diagram (Section 3) with one in which the government has the opportunity to save up to an amount $\overline{B}$ at interest rate $\rho + G^*$. This saving opportunity shifts up the deficit-debt locus to the solid line.

government to save at the interest rate $\rho + G^*$ up to some limit $\overline{B} > 0$, that is, $b_2 \in [-\overline{B}, 0]$. This interest rate is the one paid on investments that do not carry any convenience benefits.

Figure A.3 shows the deficit-debt diagram in this economy. In contrast to the deficit-debt diagram in Section A.3 without saving opportunity (dashed line), we now have a strictly higher deficit-debt locus (solid line). This makes sense as there now exists a region for total debt b to the left of some debt level $\tilde{b}$ where it is optimal for the government to take advantage of the higher-return saving opportunity while borrowing in a lower return asset class.

This automatically implies that any initial steady state left of $\tilde{b}$ in which the government does not use the opportunity, $b_2 = 0$, despite it being optimal to do so, now admits a free lunch policy. This can occur even if the initial steady state was to the right of the peak of the (dashed) deficit-debt locus without saving opportunity.

However, if the economy already starts with a steady state in which the saving opportunity is optimally used, then it is *less* likely that the economy admits a free lunch, as the peak of the deficit-debt locus shifts to the left with the saving opportunity.

C Comparison of our model with "money in utility" models

Next, we study a model that is inspired by recent models studying $R < G$ in a flexible price money demand model setting a la Sidrauski (1967). Models of this sort have recently been studied by Brunnermeier, Merkel and Sannikov (2020*a*,*b*) as well as Kaplan, Nikolakoudis and Violante (2023). As we explain in detail in this section, the key distinction between our work and these papers is that we assume that the central bank uses its nominal interest rate to target inflation whenever it can, whereas these papers allow the price level to jump.

C.1 A modified model

We take our model from Section 2 and make five modifications:

1. We assume $\mu = 0$, i.e. the economy is only populated by savers.

2. We assume away the zero lower bound.

3. We assume a symmetric (rather than only downward) nominal wage rigidity,

$$\pi_t = \frac{\dot{W}_t}{W_t} = \pi^* - \kappa \left(1 - n_t\right) \tag{A.31}$$

   with slope $\kappa > 0$. We denote the price level by P_t.

4. We assume the central bank targets inflation according to a Taylor rule

$$R_t = R^* + \phi \left(\pi_t - \pi^*\right) \tag{A.32}$$

   with coefficient $\phi \geq 0$.

5. We assume a general affine-linear fiscal rule,

$$z_t = z^* + \xi \left(b_t - b^*\right) \tag{A.33}$$

   where $\xi \in \mathbb{R}$ captures the responsiveness of the primary deficit to public debt.

This setup captures the essence of three different models:

1. Our model in Section 3 can be regarded as the limit $\phi \to \infty$ with a constant primary deficit, $\xi = 0$.

2. A money-in-utility (MiU) model à la Sidrauski (1967) is a model in which the government liability b_t represents real money balances, rather than the real value of bonds. This implies a constant nominal interest rate on money (typically assumed to be zero), and thus $\phi = 0$ in (A.32). Typical MiU models are analyzed with a constant money growth rule. Here, such a rule corresponds to constant growth in nominal debt B_t. We can nest this rule in (A.33) by assuming $z^* = b^* = 0$. In that case, the nominal

government budget constraint is

$$\dot{B}_t = P_t z_t + (R^* - \gamma) B_t = (\xi + R^* - \gamma) B_t.$$

Thus, by choice of ξ, any arbitrary growth rate of nominal bonds can be implemented. If prices in this economy are assumed to be flexible, the limit $\kappa \to \infty$ is to be taken.

3. A fiscal theory of the price level (FTPL) model as in Kaplan, Nikolakoudis and Violante (2023), with a constant primary deficit, $\xi = 0$, a passive monetary policy rule, $\phi = 0$, and flexible prices, $\kappa \to \infty$.

We focus on models 1 and 2 here, leaving a discussion of the FTPL model to Kaplan, Nikolakoudis and Violante (2023).

C.2 Laws of motion

This modified model can easily be seen to follow a system of two ordinary differential equations. The first equation, the Euler equation, is given by

$$\frac{\dot{c}_t}{c_t} = \underbrace{R^* - G^* + \kappa (\phi - 1) (x + c_t - 1)}_{R_t - G_t} - \rho + v'(b_t) c_t. \tag{A.34}$$

Here, we substituted the Taylor rule (A.32) in for R_t, and the Phillips curve (A.31) (with $x + c_t = n_t$) in for $G_t = \gamma + \pi_t$. Equation (A.34) describes the law of motion of consumption as a function of consumption c_t and real debt b_t.

The second equation is the law of motion of real debt b_t, which is given by

$$\dot{b}_t = z^* + \xi (b_t - b^*) + (R^* - G^* + \kappa (\phi - 1) (x + c_t - 1)) b_t. \tag{A.35}$$

We arrive at this expression by substituting the fiscal rule (A.33) and the expression for $R_t - G_t$ in (A.34) into the government budget constraint (5). Together, (A.34) and (A.35) are necessary equilibrium conditions.

C.3 Steady state result

Our first result concerns the steady states of this system of differential equations.

Proposition 9. *Assume $\phi \neq 1$. Then, in the limit of flexible prices, $\kappa \to \infty$, the primary deficit z and debt b in any steady state satisfy*

$$z = \left(v'(b) (1 - x) - \rho\right) b. \tag{A.36}$$

In particular, this relationship is independent of trend inflation π^*, the Taylor rule coefficient ϕ other than $\phi \neq 1$, and the parameters of the fiscal rule ξ, z^*, b^*. The same relationship holds when κ remains finite but $\phi \to \infty$.

Proof. The proposition is proved by imposing $\dot{b}_t = \dot{c}_t = 0$ in (A.34) and (A.35). Doing so, we find the following two equations:

$$R^* - G^* + (\phi - 1)\,\kappa\,(x + c - 1) - \rho + v'(b)c = 0 \tag{A.37}$$

and

$$z + (R^* - G^* + \kappa\,(\phi - 1)\,(x + c - 1))\,b = 0. \tag{A.38}$$

Substituting one into the other, we arrive at

$$z = \left(v'(b)c - \rho\right) b.$$

Finally, solving for c in (A.37) and applying $\kappa\,(\phi - 1) \to \infty$ or $-\infty$, we find $c = 1 - x$, establishing (A.36). □

Proposition 9 is useful as it shows that irrespective of monetary and fiscal policy rules, steady states always lie on the same steady state locus (A.36), equal to the one we solved for in Section 3.2. In particular, the three economies mentioned above—our Section 3 economy, the MiU economy and the FTPL economy—all share the same steady state locus.

In the following, we focus on the case where the steady state locus (A.36) is single-peaked, e.g., as depicted in Figure 1b. Just as before, steady states in the increasing part of the steady state locus are then characterized by $R - G < \varphi$, where φ is still defined as the derivative of steady state $R - G$—which by (A.37) is equal to $\rho - v'(b)\,(1 - x)$ in the limit $\kappa\,(\phi - 1) \to \infty$—with respect to $\log b$,

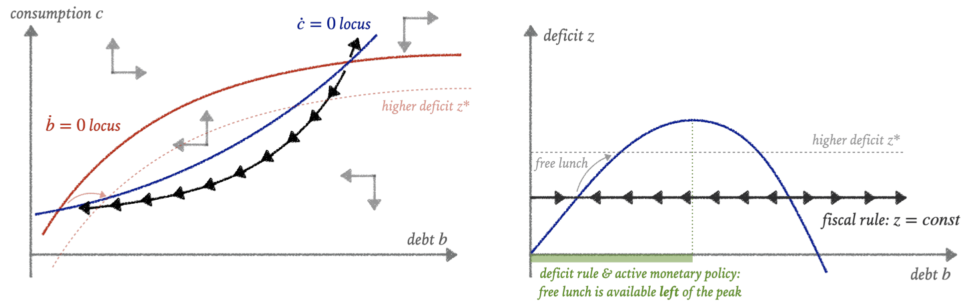
$$\varphi = -(1 - x)v''(b)b. \tag{A.39}$$

Vice versa, steady states in the decreasing part of the steady state locus are characterized by $R - G > \varphi$.

We next show that, different from the steady states, the *dynamics* of the model are, in fact, very much affected by monetary and fiscal policy rules, even in the limit of flexible prices $\kappa \to \infty$.

Figure A.4: Section 3 economy: Phase diagram and deficit-debt locus

(a) *c-b* phase diagram **(b)** Deficit-debt locus



C.4 Dynamics in the Section 3 economy

Figure A.4 shows the phase diagram and deficit-debt locus in the Section 3 economy, for $z^* > 0$ and for large $\kappa\,(\phi - 1)$. The $\dot{c} = 0$ locus is given by

$$c = \frac{G^* - R^* + \rho + (\phi - 1)\,\kappa\,(1 - x)}{v'(b) + (\phi - 1)\,\kappa}. \tag{A.40}$$

For large $(\phi - 1)\,\kappa$, this is an increasing function, which ultimately converges to $1 - x$ as $(\phi - 1)\,\kappa \to \infty$. The $\dot{b} = 0$ locus is given by

$$c = \frac{1}{(\phi - 1)\,\kappa}\left(G^* - R^* - \frac{z^*}{b}\right) + 1 - x.$$

This is also increasing for large $(\phi - 1)\,\kappa$, and also converges to $1 - x$ as $(\phi - 1)\,\kappa \to \infty$. The two loci intersect exactly twice for the two steady states with deficit z^*. As the saddle path shows in Figure A.4a, the left steady state is stable, while the right one is not.

When is there a free lunch in this economy? As shown with the dashed line, a modestly higher deficit z^* leads to a stable transition to a new steady state that still lies to the left of the peak in Figure A.4b, where $R < G - \varphi$, analogous to our result in Corollary 1. We formalize this in the following proposition.

Proposition 10. *Assume the Section 3 economy, with $\xi = 0$ and large $(\phi - 1)\,\kappa$. Any steady state with $R < G - \varphi$ has exactly one stable and one unstable root; any steady state with $R > G - \varphi$ as two unstable roots. A free lunch policy is available when $R < G - \varphi$.*

Proof. Begin with a steady state for c, b. We denote with a "hat" linear deviations from the steady state. The linearized laws of motion can be derived to be

$$\frac{\dot{\hat{c}}_t}{c} = \left((\phi - 1)\,\kappa + v'(b)\right)\hat{c}_t + v''(b)c\hat{b}_t$$

and

$$\dot{\hat{b}} = \kappa\left(\phi - 1\right) b\hat{c}_t - \frac{z^*}{b}\hat{b}_t,$$

or in matrix notation,

$$\begin{pmatrix} \dot{\hat{c}}_t \\ \dot{\hat{b}}_t \end{pmatrix} = \begin{pmatrix} \left(\phi - 1\right)\kappa + v'(b) & v''(b_t)c \\ \kappa\left(\phi - 1\right) b & -\frac{z^*}{b} \end{pmatrix} \begin{pmatrix} \hat{c}_t \\ \hat{b}_t \end{pmatrix}.$$

Any eigenvalue λ of this system is a zero of the characteristic polynomial

$$P(\lambda) = \lambda^2 - \left(\left(\phi - 1\right)\kappa + v'(b) - \frac{z^*}{b}\right)\lambda - \frac{z^*}{b}\left(\left(\phi - 1\right)\kappa + v'(b)\right) - v''(b)c\kappa\left(\phi - 1\right) b = 0.$$

We can rewrite the polynomial as

$$P(\lambda) = \lambda^2 - \kappa\left(\phi - 1\right)\left(\lambda + \left(\left(v''(b)b + v'(b)\right)(1 - x) - \rho\right)\right) - \left(v'(b) - \frac{z^*}{b}\right)\lambda - v'(b)\frac{z^*}{b}.$$

We see that, as $\kappa\left(\phi - 1\right) \to \infty$, one root converges to $+\infty$, while the other converges to

$$\lambda = \rho - \left(v''(b)b + v'(b)\right)(1 - x) = R - G - \varphi.$$

Exactly when $R < G - \varphi$, λ is negative and we have a saddle path stable equilibrium. When, instead, $R > G - \varphi$, λ is positive, and booth roots are explosive.

A free lunch in this economy exists precisely when a permanent increase in the deficit z^* is consistent with a stable evolution of the debt level. This is the case around the saddle-path stable steady state, where $R < G - \varphi$. This proves the result. □

C.5 Dynamics in the MiU economy

Figure A.5 shows the phase diagram and the deficit-debt locus for the MiU economy. Here, the $\dot{b} = 0$ locus is a horizontal line,

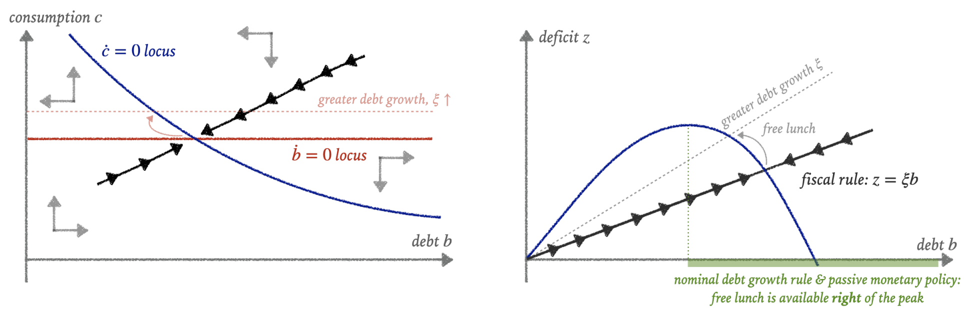
$$c = \frac{1}{\kappa}\left(\xi - G^* + R^*\right) + 1 - x.$$

The $\dot{c} = 0$ locus is again given by (A.40). There is a single steady state with positive debt for every choice of nominal debt growth, i.e. every choice of ξ. The steady state is stable.

When is there a free lunch in this economy? The dashed shifted line shows what happens when nominal debt growth, that is ξ, is shifted up. The deficit jumps up, then comes down. It settles below the original deficit precisely when $R < G - \varphi$, and above the

Figure A.5: MiU economy: Phase diagram and deficit-debt locus

(a) *c*-*b* phase diagram **(b)** Deficit-debt locus



original deficit when $R > G - \varphi$. This is exactly the flip-side of the result in the Section 3 economy. We formalize this as follows.

Proposition 11. *In the MiU economy, with $\phi = 0$ and $z^* = b^* = 0$, there is a unique steady state with positive debt. The steady state is saddle-path stable.*

Proof. Under the MiU assumptions, the steady state equations are then (A.36) together with the fiscal rule $z = \xi b$. This implies that any positive-debt steady state needs to satisfy

$$v'(b)c = \xi + \rho \tag{A.41}$$

Furthermore, (A.38), when substituting in the MiU fiscal rule, leads to

$$\xi + R^* - G^* + \kappa\,(-1)\,(x + c - 1) = 0$$

and thus $c = 1 - x + (\xi - (G^* - R^*))\,/\kappa$. The unique positive steady state level of debt b is therefore pinned down by

$$v'(b)\left(1 - x + \frac{\xi - (G^* - R^*)}{\kappa}\right) = \xi + \rho. \tag{A.42}$$

To show that the steady state is indeed saddle path stable, we linearize the system of ODEs again, noting that the linearized law of motion of debt (A.35) is now given by

$$\dot{\hat{b}}_t = -\kappa \hat{c}_t b$$

In matrix notation, we then have

$$\begin{pmatrix} \dot{\hat{c}}_t \\ \dot{\hat{b}}_t \end{pmatrix} = \begin{pmatrix} -\kappa + v'(b) & v''(b)c \\ -\kappa b & 0 \end{pmatrix} \begin{pmatrix} \hat{c}_t \\ \hat{b}_t \end{pmatrix}.$$

The characteristic polynomial is

$$P(\lambda) = \lambda^2 - \left(-\kappa + v'(b)\right)\lambda + v''(b)c\kappa b = 0$$

which, due to $P(0) < 0$ but $P''(\lambda) > 0$ always has exactly two roots, one negative and one positive and is thus saddle-path stable. □

Contrary to our discussion in the proof of Proposition 10, saddle-path stability here does *not* imply the availability of a free lunch policy. A free lunch policy is available if an increase in deficits can be sustained indefinitely. Fiscal policy is parameterized here with ξ, which shifts the rate of nominal debt growth. A free lunch exists when (a) one can increase ξ permanently, leading to higher deficits in the short run, and (b) this leads to higher debt levels in the long run. We next show that in the MiU economy, this is exactly flipped relative to the result in Proposition 10.

Corollary 8. *In the MiU economy, in the limit $\kappa \to \infty$, a free lunch exists when $R > G - \varphi$ and does not if $R < G - \varphi$.*

Proof. Starting at some steady state (c, b), contemplate a small increase in ξ, by some $\hat{\xi}$. This modifies the ODEs to be

$$\begin{pmatrix} \dot{\hat{c}}_t \\ \dot{\hat{b}}_t \end{pmatrix} = \begin{pmatrix} -\kappa + v'(b) & v''(b_t)c \\ -\kappa b & 0 \end{pmatrix} \begin{pmatrix} \hat{c}_t \\ \hat{b}_t \end{pmatrix} + \begin{pmatrix} 0 \\ \hat{\xi} b \end{pmatrix} \tag{A.43}$$

A free lunch exists if this leads to a permanently elevated path of deficits,

$$\hat{z}_t = \hat{\xi} b + \xi \hat{b}_t > 0. \tag{A.44}$$

A first necessary condition for a free lunch is that $\hat{z}_0 > 0$. As $\hat{b}_t$ is slow-moving, this is equivalent to $\hat{\xi} > 0$.

From (A.42) we can derive the long run effect of an increase in ξ on debt,

$$\hat{b}_\infty = \frac{1}{v''(b)c}\hat{\xi}\left(1 - \frac{v'(b)}{\kappa}\right). \tag{A.45}$$

A second necessary condition for a free lunch is that $\hat{z}_\infty > 0$, which, combining (A.44) and (A.45), is equivalent to

$$1 + \xi \frac{1}{v''(b)bc}\left(1 - \frac{v'(b)}{\kappa}\right) > 0.$$

As $\kappa \to \infty$, this condition is equivalent to

$$v''(b)bc + \xi > 0.$$

Based on (A.41), this can be rewritten as

$$v''(b)b + v'(b) - \rho > 0$$

which by (A.39) and the discussion around it, is precisely equivalent to $R > G - \varphi$.

This argument shows that $R > G - \varphi$ is necessary for a free lunch. It is also sufficient, as convergence of $\hat{b}_t$ to $\hat{b}_\infty$ must be monotone: The system of ODEs (A.43) is saddle-path stable, implying that there exists a unique trajectory to the steady state for any initial level of debt. □

This result may seem surprising: How come $R > G - \varphi$, a condition seemingly characterizing steady states with a high interest rate relative to the growth rate, and/or a high interest rate sensitivity to debt, is the right one for a free lunch in the MiU economy?

It comes down to the monetary policy rule assumed: In our Section 3 economy, monetary policy is active, $\phi > 1$, so that higher deficits lead to greater real interest rates, increasing the real debt of the economy. Vice versa, in the MiU economy, the nominal interest rate is fixed. Higher deficits then reduce real interest rates by raising inflation, thus lowering, not raising, debt levels. Precisely when interest rates are high and very sensitive to debt, permanently higher deficits leading to lower debt levels become a possibility.

The flip-side of this result may seem equally surprising: When $R < G - \varphi$ and (real) debt is low, how come we cannot simply push debt up in this economy and enjoy a free lunch like we did before? Can't we run very high deficits for a short instant, effectively amounting to an upwards jump in the debt level? This is infeasible, because, as mentioned, higher deficits do not increase real debt in this economy! By raising inflation they instead, inflate debt away. To see this mathematically, observe that $\hat{b}_t$ starts at zero at $t = 0$ and then falls to $\hat{b}_\infty < 0$ (see (A.45) for $\kappa \to \infty$). $\hat{b}_t$ is negative in between. The only way to raise real debt in this economy is, paradoxically, to lower deficits and inflation, and wait for debt to build up.

Our analysis of the ZLB constraint in Section 4 points towards a direction to see how "MiU-like" behavior can emerge in an economy at a binding ZLB. There, too, we saw that for a sufficiently flexible Phillips curve, greater deficits can potentially have a stronger negative effect on debt than positive effect. But within limits, as too high a deficit will pull the economy out of the ZLB, precipitating rising real interest rates and real debt levels.

D The deficit-debt diagram in other models

In this section, we derive the interest rate and growth rate schedules $R(b)$ and $G(b)$ in a variety of models, and compute the deficit-debt locus $z(b) \equiv (G(b) - R(b))\, b$.

D.1 Model inspired by Reis (2021)

Here, we sketch a version of the two-agent model in Reis (2021) and use it to derive the corresponding functions for $R(b)$ and $G(b)$.

There are two types of agents, entrepreneurs E and financiers F. Each instant t, an agent i is randomly allocated to be either E or F, with probabilities α and $1 - \alpha$ for E and F. Agents solve

$$\max \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \log c_t^i dt \right]$$

subject to the budget constraint

$$da_t^i = \left(R_t b_t^i + r_t^l l_t^i + r_t^i k_t^i - \tau_t - c_t^i \right) dt \tag{A.46}$$

where $a_t^i = b_t^i + l_t^i + k_t^i$ is agent i's total wealth, and subject to the constraint

$$b_t^i \geq 0, k_t^i \geq 0.$$

Here, b_t^i is agent i's holdings of bonds, l_t^i agent i's lending (or if negative, borrowing), and k_t^i agent i's holding of capital. τ_t is a lump-sum tax. Thus, each agent can invest in three different assets each instant: government bonds b_t^i paying rate R_t, loans l_t^i paying rate r_t^l and capital paying rate r^i.

The return on capital r^i crucially differs by type. If i is type E, then r_t^i is constant, and equal to $r_t^i = A - \delta \equiv m > 0$. If i is type F, r_t^i is subject to idiosyncratic investment risk and given by

$$r_t^i = \eta (A - \delta) - \sigma dz_t^i$$

where $\eta \in (0, 1)$ captures reduced capital quality in the hands of type F agents. We simplify the model here and set $\eta \to 0$. This essentially assumes that type F agents do not invest in capital.[A6]

To avoid too much investment on the side of type E agents, we also impose a borrowing constraint

$$-r_t^l l_t^i \leq \gamma r_t^i k_t^i$$

[A6] The case with $\eta > 0$ is similar, it just requires a case distinction.

for some $\gamma > 0$. For type F agents, the borrowing constraint is simply assumed to be $l_t^i \geq 0$. In equilibrium, aggregate bonds outstanding B_t have to equal the sum of all individual positions,

$$B_t = \int b_t^i di$$

and the market for loans has to clear,

$$0 = \int l_t^i di.$$

Our goal is to use this description of the household side to solve for both the steady state interest rate R and the steady state growth rate G as a function of the overall supply of steady state bonds B.

Given the iid type switching, we can split total wealth a_t into wealth held by E's, $a_t^E = \alpha a_t$ and wealth held by F's, $a_t^F = (1-\alpha) a_t$. E's always borrow to their maximum. Further, we assume that γ is sufficiently high so that E's do not hold any government bonds. Then, from (A.46) and the fact that agents always consume $c_t^i = \rho a_t^i$, E's wealth evolves as

$$\dot{a}_t^E = \frac{(1-\gamma)mr_t^l}{r_t^l - \gamma m} a_t^E - \rho a_t^E$$

with positions in capital and lending markets given by

$$a_t^E = k_t^E - \gamma \frac{mk^E}{R}.$$

Given capital k_t^E, output is simply

$$y_t = Ak_t^E. \tag{A.47}$$

F's hold all government bonds, and lend, so that $r_t^l = R_t$. Their wealth then evolves as

$$\dot{a}_t^F = (R_t - \rho) a_t^E$$

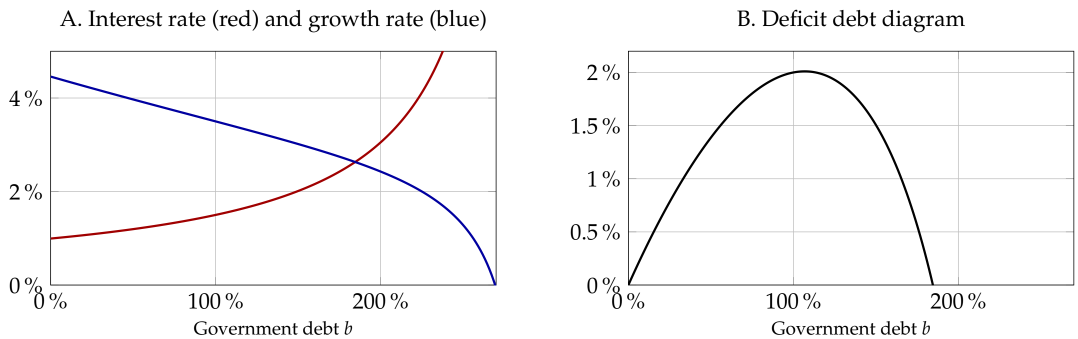
and is given by

$$a_t^F = \gamma \frac{mk_t^E}{R_t} + B_t. \tag{A.48}$$

In a steady state, total wealth evolves according to

$$\frac{\dot{a}_t}{a_t} = \alpha \frac{(1-\gamma)mR_t}{R_t - \gamma m} + (1-\alpha)R_t - \rho \tag{A.49}$$

Figure A.6: $R(b), G(b)$ and deficits in the Reis (2021) model



and is given by

$$a_t = k_t^E + B_t. \tag{A.50}$$

We denote by $b_t \equiv B_t / y_t$ government debt relative to GDP.

This gives us all the equations we need. Assuming that b is constant, we combine (A.47), (A.49) and (A.50) to find a steady state growth rate G of the economy of

$$G = \alpha \frac{(1-\gamma) m R}{R - \gamma m} + (1-\alpha) R - \rho. \tag{A.51}$$

The interest rate R is itself determined by the amount of lending in equilibrium, using (A.48), (A.50) and the fact that $a_t^F = (1-\alpha) a_t$,

$$\gamma \frac{m k^E}{R} + B = (1-\alpha) \left(k^E + B \right).$$

Solving for R we find

$$R(b) = \frac{\gamma m}{1 - \alpha - \alpha A b}. \tag{A.52}$$

Together with (A.51), we can solve for G as function of b as well,

$$G(b) = \frac{(1-\gamma) m}{1 + A b} + (1-\alpha) \frac{\gamma m}{1 - \alpha - \alpha A b} - \rho.$$

We sketch the two schedules in Figure A.6 and the implied deficit-debt diagram.[A7]

[A7] We calibrate the model exactly as above, matching $R_0 = 1.5\%$, $G_0 = 3.5\%$, $\varphi = 1.7\%$, $b_0 = 1$. This yields $\delta = 0.04$, $\rho = 0.03$, $\gamma = 0.033$, $A = 0.12$, $\alpha = 0.74$.

D.2 Model inspired by Diamond (1965)

We sketch the well-known Cobb-Douglas version of the Diamond (1965) model and show that it implies a simple closed-form deficit schedule $z(b)$, and derive the conditions under which there is a free lunch.

The model operates in discrete time and consists of two-period-lived overlapping generations. The generation born at date t has G^t members, where $G > 1$. Each maximizes preferences

$$(1-\beta)\log c_{yt} + \beta \log c_{ot+1}$$

over consumption when young c_{yt} and when old c_{ot+1}, subject to the budget constraints

$$c_{yt} + a_t \leq w_t (1-\tau_t) \qquad c_{ot+1} = R_{t+1} a_t.$$

We have $\beta \in (0,1)$, τ_t is an income tax. The policy function is then

$$a_t = \beta w_t (1-\tau_t). \tag{A.53}$$

The per capita saving a_t of generation t finances capital for $t+1$ and bonds maturing in $t+1$. Normalizing the latter two in terms of the population size at $t+1$, we have an asset market clearing condition

$$G^{-1} a_t = k_{t+1} + B_{t+1}. \tag{A.54}$$

Production in period t is neoclassical with aggregate output per capita

$$y_t = k_t^{\alpha} l_t^{1-\alpha}$$

where l_t is the labor endowment of each member of generation t, which we normalize to 1. Thus, the wage is $w_t = (1-\alpha) k_t^{\alpha}$ and, with a depreciation rate of 1, the return is $R_{t+1} = \alpha k_{t+1}^{\alpha-1} + 1 - \delta$. With (A.53) and (A.54), the law of motion for capital is then

$$k_{t+1} = G^{-1} \beta (1-\alpha)(1-\tau) k_t^{\alpha} - B_{t+1} \tag{A.55}$$

The government's budget constraint is simply given by

$$G B_{t+1} = R_t B_t - \tau_t w_t + X_t$$

where X_t denotes government spending per capita.

Next, we focus on steady states, at which all prices and per capita quantities are constant. Moreover, we normalize government debt and spending by output $y = k^{\alpha}$. We denote

$b \equiv B/y$ as before and $x = X/y$. Then, (A.55) becomes

$$k^{1-\alpha} = G^{-1}\beta(1-\alpha)(1-\tau) - b$$

and we can rearrange it to obtain an expression for the interest rate

$$R(b) = \frac{\alpha G}{\beta(1-\alpha)(1-\tau) - Gb}.$$

The normalize government budget constraint can be written as usual

$$z(b) = (G - R(b))\, b$$

where we defined the primary deficit relative to GDP as $z(b) \equiv x - \tau(1-\alpha)$. Different from our model in Section 2, it turns out that for this analysis, it is somewhat more tractable to fix the tax rate τ and instead vary government spending x if $z(b)$ changes.

We can analyze the deficit schedule $z(b)$ just like before. In particular, we can ask when higher debt levels allow for a greater primary deficit $z(b)$, which in this model is equivalent to dynamic inefficiency. The condition for this is

$$R(b) < G - b \cdot R'(b) \tag{A.56}$$

where $\varphi = b \cdot R'(b)$. Observe that the standard condition for dynamic inefficiency that is usually taught in this model is $R < G$, or in terms of primitives, $\frac{\alpha}{1-\alpha} < \beta(1-\tau)$. Yet, as (A.56) highlights this condition is only accurate for levels of government debt around zero, where $\varphi = 0$. When $b > 0$, $\varphi > 0$, and the relevant condition becomes $R < G - \varphi$. In terms of primitives, this corresponds to

$$b < \frac{\beta(1-\alpha)(1-\tau)}{G} - \frac{1}{G}\sqrt{\alpha \cdot \beta(1-\alpha)(1-\tau)} \equiv b^*$$

where b^* is, as before, the deficit-maximizing level of debt. The deficit associated with b^* is given by

$$z^* = \left(\sqrt{\beta(1-\alpha)(1-\tau)} - \sqrt{\alpha}\right)^2.$$

We thus find that OLG models based on Diamond (1965) admit a similar interest rate schedule as the one we derived in Section 3, and the relevant condition for a free lunch is given by $R < G - \varphi$, which only in the case without debt reduces to $R < G$.[A8]

[A8]Note that, in the Diamond (1965) model without markups, the dynamic inefficiency condition is still given by $R < G$.

D.3 The Blanchard model

Here, we compute the deficit-debt locus of the (Cobb-Douglas) model in Blanchard (2019). The model is a stochastic version of the model in the previous section, which we briefly recap here. The model operates in discrete time and consists of two-period-lived overlapping generations. Each period corresponds to $N = 25$ years. There is no population growth, $G = 0$, so all returns have to be considered de-trended. Households solve

$$\max \log c_{y,t} + \beta \frac{1}{1-\gamma} \log \mathbb{E}_t \left[c_{o,t+1}^{1-\gamma} \right]$$

over consumption when young c_{yt} and when old c_{ot+1}, subject to the budget constraints

$$c_{yt} + k_t + b_t \leq w_t (1 - \tau_t) \qquad c_{ot+1} = R_{t+1} k_t + R^f_{t+1} b_t.$$

Here, agents can choose between a risk-free bond, paying the risk free rate R^f_{t+1}, and risky capital, paying R_{t+1}. As before, production is Cobb-Douglas per head of generation t

$$y_t = A_t k_{t-1}^{\alpha} l_t^{1-\alpha}$$

where l_t is the labor endowment of each member of generation t, which we normalize to 1. Thus, the wage is $w_t = (1-\alpha) A_t k_{t-1}^{\alpha}$ and the return on capital is $R_{t+1} = \alpha A_{t+1} k_t^{\alpha-1}$. $\log A_t \sim \mathcal{N}(\mu, \sigma^2)$ is iid stochastic technology.

The government's budget constraint is still given by

$$b_t = R^f_t b_{t-1} - \tau_t w_t + x_t.$$

We look for a "risky steady state", characterizing the steady state of the path along which $\log A_t$ continues to realize at its mean μ. For this exercise, we set government spending to zero (as done by Blanchard 2019), $x_t = 0$. For a given (end of period) debt per capita b, the risky steady state is described by the following four equations: The two budget constraints

$$c_y = (1-\alpha) e^{\mu} k^{\alpha} + \left(1 - R^f\right) b \qquad c_o(A) = \alpha A k^{\alpha} + R^f b$$

and two Euler equations

$$\frac{1}{c_y} = \frac{\beta}{1-\gamma} \frac{1}{\mathbb{E}_A \left[c_o(A)^{1-\gamma} \right]} \mathbb{E}_A \left[c_o(A)^{-\gamma} \alpha A k^{\alpha-1} \right]$$

$$\mathbb{E}_A\left[c_o(A)^{-\gamma}\alpha A k^{\alpha-1}\right] = R^f \mathbb{E}_A\left[c_o(A)^{-\gamma}\right]$$

Together, the equations pin down $k, R^f, c_y, c_o(A)$ for any given b. By normalizing by output $y = e^{\mu}k^{\alpha}$ (all normalized variables are denoted with a hat), we can simplify this:

$$\hat{c}_y = 1 - \alpha + \left(1 - R^f\right)\hat{b} \qquad \hat{c}_o(A) = \alpha A e^{-\mu} + R^f \hat{b}$$

The risk free rate $R^f = R^f(\hat{b})$ then solves the risk-free Euler equation

$$\frac{1}{\hat{c}_y} = \beta \frac{1}{\mathbb{E}_A\left[\hat{c}_o(A)^{1-\gamma}\right]} R^f \mathbb{E}_A\left[\hat{c}_o(A)^{-\gamma}\right]$$

and the capital output ratio can then be computed from

$$\frac{k}{y} = \frac{\alpha}{R^f}\frac{\mathbb{E}_A\left[\hat{c}_o(A)^{-\gamma} A e^{-\mu}\right]}{\mathbb{E}_A\left[\hat{c}_o(A)^{-\gamma}\right]}$$

The expected return on capital is then equal to

$$\mathbb{E}R = \frac{\alpha}{k/y}\mathbb{E}_A\left[A e^{-\mu}\right]$$

We compare three different calibrations of the model. The first is a calibration inspired by the Cobb-Douglas calibration of Blanchard (2019). With zero initial debt, we choose $\alpha = 1/3$, a period length of 25 years, $\sigma = 0.2$, and calibrate γ and β to match jointly a riskless rate R^f equal to -1% annual (i.e. one percent below G^*) and an expected return on capital $\mathbb{E}R$ equal to $+2\%$ annual, (i.e. two percent above G^*). This yields $\gamma = 18.7$ and $\beta = 0.31$ (not annualized).

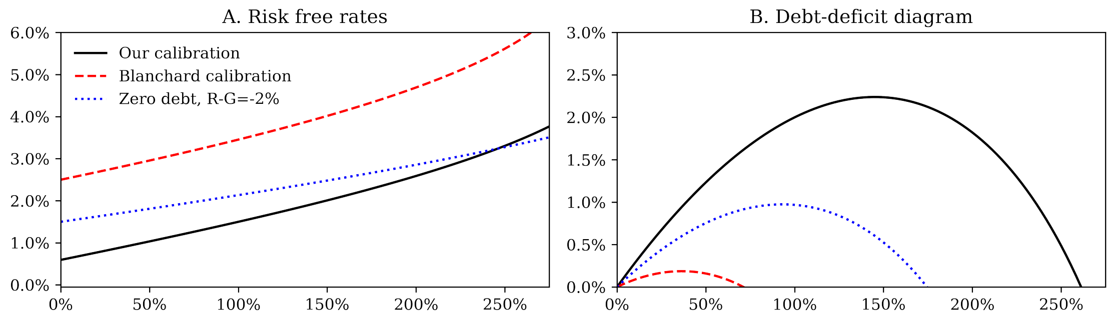
The second calibration, instead, targets a riskless rate R^f equal to -2% annual (i.e. two percent below G^*), and an expected return on capital $\mathbb{E}R = 1\%$, but keeps a zero initial debt position. This yields $\gamma = 18.8$ and $\beta = 0.40$.

The third calibration is like the second, except that we start the economy with an initial debt position of 100% of annual GDP. This yields $\gamma = 21.7$ and $\beta = 0.45$.

Figure A.7(A) shows the implied nominal annual risk-free rates across the three calibrations, once we add $G^* = 3.5\%$ of nominal trend growth to ease comparability with our own analysis in this paper. All three calibrations imply that interest rates increase with debt. Figure A.7(B) shows the annualized deficit-debt schedule implied by the calibrations, constructed as

$$z(\hat{b}) = \left(1 - \left(R^f(\hat{b})\right)^{1/N}\right) \cdot N\hat{b}.$$

Figure A.7: Interest rates and deficit-debt schedule in the risky steady state of the Blanchard (2019) model



Interestingly, the Blanchard (2019) implies a very small maximum sustainable deficit that can be run, while our calibration implies a much greater one. This difference is directly caused by the assumption of a zero initial debt level and a higher initial nominal interest rate in Blanchard (2019).

Simulating the model. To simulate the Blanchard (2019) model forward, we begin with initial values k_{-1}, b_{-1}, R^f_{-1}, a sequence of deficits z_t, and draw a random sequence of productivity shocks $\{A_t\}$.

At each step t, we compute output as

$$y_t = A_t k_{t-1}^{\alpha}$$

We evolve debt forward with

$$b_t = R^f_{t-1} b_{t-1} + z_t$$

We use this to write consumption of the currently young generation as

$$c_{yt} = (1-\alpha)\, y_t + z_t \qquad c_{ot+1}(A', R^f_t) = \alpha A' k_t^{\alpha} + R^f_t b_t$$

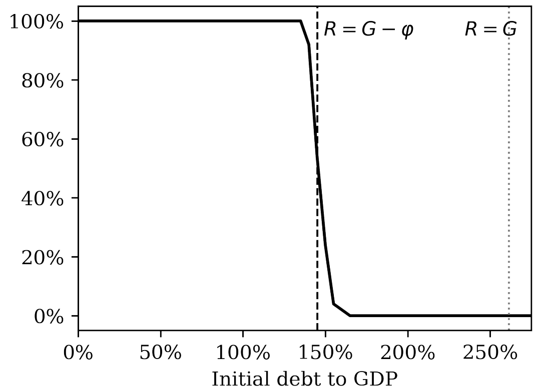
We solve for the unknown k_t and R^f_t by solving

$$\frac{1}{c_{yt}} = \beta \frac{1}{\mathbb{E}_{A'}\left[c_{ot+1}(A', R^f_t)^{1-\gamma}\right]} \mathbb{E}_{A'}\left[c_{ot+1}(A', R^f_t)^{-\gamma} \alpha A' {k_t}^{\alpha-1}\right]$$

$$\mathbb{E}_{A'}\left[c_{ot+1}(A', R^f_t)^{-\gamma} \alpha A' k'^{\alpha-1}\right] = R^f_t \mathbb{E}_{A'}\left[c_{ot+1}(A', R^f_t)^{-\gamma}\right]$$

To construct Figure A.8, we first simulate the economy for a steady state $k_{-1} = k_{ss}, b_{-1} = b_{ss}$ associated with a given initial debt-to-GDP level $\hat{b}_{ss}$. We assume a deficit rule that avoids

Figure A.8: Success probabilities of running a free lunch of 1% of GDP in the Blanchard (2019) model



Note. The probabilities are computed by simulating 50 sample paths for each b_0. Convergence criterion: $|b_t^\Delta - b_t| < 0.01\%$ at any point $t < 1,000$.

explosive debt levels,

$$z_t = z_{ss} - 0.08 \cdot N \cdot \left(R^f_{t-1} b_{t-1} - R^f_{ss} b_{ss} \right)$$

Then, for every $\hat{b}_{ss}$, we simulate the same economy again, for the same shocks $\{A_t\}$, except that (a) the initial debt level b_{-1} is now increased by 1% of initial GDP; and (b) the deficit path z_t is unchanged, taken directly from the economy *without* the initial 1% debt shift.

Free lunch in the Blanchard (2019) model. We next compute the probability of a free lunch succeeding in the Blanchard (2019) model. We use the recalibrated model with 100% initial debt and $R^f = -2\%$ as explained above. Figure A.8 plots the success probabilities of a free lunch policy that raises initial debt by 1% across different levels of initial debt. As in Section 5.3, we keep the path of deficits unchanged. The threshold $R = G - \varphi$ clearly matters for the viability of a free lunch policy in the Blanchard (2019) model.[A9]

D.4 Model with indebted demand and convenience yield

The model in Section 2 can easily be extended to allow for "indebted demand" as in Mian, Straub and Sufi (2021).[A10] To do so, we include a term in savers' preferences that captures

[A9]Here, $R = G - \varphi$ corresponds to the peak of the "risky steady state" locus $z(b)$ shown in Figure A.7(B).
[A10]We ignore the ZLB in this section.

their average saving motive, not specific to bonds,

$$\max_{\{c_t, b_t\}} \int_0^\infty e^{-\rho t} \left\{ \log c_t + v\left(b_t\right) + \hat{v}(b_t + d_t + h_t) \right\} dt \tag{A.57}$$

Here, $\hat{v}(b + d + h)$ is a utility over total wealth, bonds b as well as private debt d. For tractability, we include human capital h in total wealth. We denote the return on assets other than government bonds by $\hat{R}_t$. R_t continuous to denote the return on government debt.

Then, human wealth is equal to

$$\left(\hat{R}_t - G^*\right) h_t = (1-\mu)\, w_t n_t - \tau_t + \dot{h}_t$$

and the budget constraint can be written as

$$c_t + \dot{b}_t + \dot{d}_t \leq \left(R_t - G^*\right) b_t + \left(\hat{R}_t - G^*\right) d_t + (1-\mu)\, w_t n_t - \tau_t. \tag{A.58}$$

At the steady state, the first order conditions for bonds and other assets pin down R and $\hat{R}$,

$$\hat{R} = G^* + \frac{\rho}{1 + \hat{v}'(b+d+h)\,(b+d+h)}$$
$$R = \hat{R} - v'(b)\left(1 - x - \mu + \left(\hat{R} - G^*\right) d\right)$$

The first equation is like the one in Mian, Straub and Sufi (2021): Increased total wealth of savers means a reduced return on wealth $\hat{R}$. Despite this, interest rates on government debt behave in a more nuanced way. R unambiguously falls when other wealth (e.g. d) increases, since that increases savers incomes and their desired saving. But R can rise when savers' holdings of government debt increase, as it diminishes the convenience yield $\hat{R} - R$ on government debt, in line with the analysis in this paper.

E General free lunch result

For the most part, our paper considers a two-agent model in which government debt enters the the savers' utility function directly. We chose this setup as it gave us a tractable way to have an upward-sloping household demand curve for government debt. It allowed us, among other things, to derive our free lunch result in Section 3, namely that the government can run a free lunch precisely when $R < G - \varphi$, where φ was the response of $R - G$ to a 1% expansion in government debt. In this short section, we sketch out an argument that suggests that our condition is much more general than the specific model we consider. To do this, we build on recent advances in macroeconomics in the sequence space, see Auclert, Rognlie and Straub (2018); Auclert, Bardóczy, Rognlie and Straub (2021); Auclert, Rognlie and Straub (2023). We leave a formalization of the arguments presented here for future research.

Consider the following abstract model in discrete time. Households, in aggregate, demand government debt as a function of the paths of de-trended interest rates $r_t \equiv \frac{1+R_{t-1}}{1+G_t} - 1$, aggregate income, and taxes. We can write their demand of bonds at date t as

$$\mathcal{A}_t \left(\{ r_s, y_s, \tau_s \} \right).$$

As shown in Auclert, Rognlie and Straub (2018), many models of household consumption and saving behavior, even those with rich household heterogeneity, can be aggregated in such a form. Market clearing then becomes, period by period,

$$b_t = \mathcal{A}_t \left(\{ r_s, y_s, \tau_s \} \right). \tag{A.59}$$

We write the government budget constraint in discrete time as in (47),

$$x_t + (1 + r_t) \, b_{t-1} = b_t + \tau_t \tag{A.60}$$

where we use r_t for the de-trended interest rate again, to simplify notation. For this more general model, a single interest rate sensitivity φ is no longer appropriate. Instead, we define φ_j as the response of interest rates in period $t + j$ to a 1% expansion of debt in period t, for a far-out period $t \to \infty$. We introduce φ_j mathematically below.

Now consider the following thought experiment. Imagine the government chooses to raise deficits $z_t = x_t - \tau_t$ by some small amount $dz_t \geq 0$ in period t. For simplicity, we assume in this section that the increased deficits are used to financed government consumption, $dx_t = dz_t$ (the case of lower taxes is similar). We ask: Under what assumptions on household behavior, encapsulated by $\mathcal{A}$, does the implied evolution of debt db_t not explode and instead remain finite? Whenever db_t remains finite, we have found a free lunch; whenever db_t does not remain finite, the deficit expansion violates equilibrium—a free lunch was not found.

To analyze this question, we collect all derivatives of household asset demand in an infinite matrix **A**—known as a sequence-space Jacobian (Auclert, Bardóczy, Rognlie and Straub, 2021)—with elements

$$A_{t,s} \equiv \frac{\partial \mathcal{A}_t}{\partial r_s}.$$

As Auclert, Rognlie and Straub (2018) argue, **A** commonly has a very specific structure, namely that of a quasi-Toeplitz matrix. Quasi-Toeplitz matrices have entries that eventually start repeating as one goes down along the diagonal, that is,

$$a_j \equiv \lim_{t\to\infty} A_{t+j,t}$$

is well-defined. As explained in Auclert, Rognlie and Straub (2023), quasi-Toeplitz matrices are generically invertible whenever the graph of the complex function $z \mapsto \sum_{j\in\mathbb{Z}} a_j z^j$ has a winding number of 0. $\mathbf{A}$ typically has a winding number of zero and is thus generically invertible.

From (A.59), we see that the evolution of interest rates dr_t in response to the deficit expansion needs to satisfy

$$d\mathbf{b} = \mathbf{A}d\mathbf{r}.$$

Inverting $\mathbf{A}$, this becomes

$$d\mathbf{r} = \mathbf{A}^{-1}d\mathbf{b}. \tag{A.61}$$

The interest rate sensitivities φ_j mentioned above are then defined as the asymptotic behavior of $\mathbf{A}^{-1}$ along the diagonal,

$$\varphi_j \equiv b \lim_{t\to\infty} \left[\mathbf{A}^{-1}\right]_{t+j,t}.$$

Finally, linearizing (A.60), we find

$$d\mathbf{b} = d\mathbf{z} + bd\mathbf{r} + (1+r)\,\mathbf{L}d\mathbf{b}$$

where $\mathbf{L}$ denotes the lag matrix, $L_{t,s} = 1$ if $t = s+1$, and 0 else. Substituting in (A.61), we find

$$\left(\mathbf{I} - b\mathbf{A}^{-1} - (1+r)\,\mathbf{L}\right) d\mathbf{b} = d\mathbf{z}. \tag{A.62}$$

Equation (A.62) precisely determines the path of debt $d\mathbf{b}$. If (A.62) can be solved for a finite path $d\mathbf{b}$, there is a free lunch. If (A.62) cannot be solved for $d\mathbf{b}$, because $\mathbf{I} - b\mathbf{A}^{-1} - (1+r)\,\mathbf{L}$ is not invertible, there is no free lunch. Using the tools spelled out in Auclert, Rognlie and Straub (2023), we can see that $\mathbf{I} - b\mathbf{A}^{-1} - (1+r)\,\mathbf{L}$ is generically invertible precisely when the complex function

$$z \mapsto 1 - \sum_{j\in\mathbb{Z}} \varphi_j z^j - (1+r)\,z$$

has a winding number of zero. A sufficient condition for this is that

$$1 + r + \sum_{j\in\mathbb{Z}} |\varphi_j| < 1$$

which can be rewritten as

$$R < G - \sum_{j \in \mathbb{Z}} |\varphi_j|. \tag{A.63}$$

We thus find that a very similar condition to our $R < G - \varphi$ condition from Section 3 still determines the availability of a free lunch policy. The key difference is, however, that here it is the entire time profile φ_j of interest rate sensitivities to a one-time debt expansion that matters, not just the contemporaneous response of interest rates.

F Details on the calibration of φ

There are different ways to estimate the elasticity φ that are equivalent within the context of the model. Observe that the intercept $v'(b_0)(1-x-\mu)$ in the linear specification (30) is determined by the initial steady state, for which the Euler equation pins down the convenience yield $v'(b_0)(1-x-\mu)$ as

$$v'(b_0)(1-x-\mu) = \rho + G^* - R_0. \tag{A.64}$$

Thus, the convenience yield is nothing other than $\rho + G - R$. Since ρ is independent of b, we can write

$$\varphi = -\frac{\partial(\rho + G - R)}{\partial \log b} = -b_0 \frac{\partial(\rho + G - R)}{\partial b} = -\frac{\partial(G - R)}{\partial \log b} = -b_0 \frac{\partial(G - R)}{\partial b}. \tag{A.65}$$

As all expressions in (A.65) are valid ways to obtain φ, we will compare estimates across these specifications. Note, however, that the latter two specifications in (A.65) are slightly more robust, as they do not hinge on a convenience-yield interpretation of $R^*(b)$ and apply equally well to the alternative models in Appendix D.

The derivatives in equation (A.65) have been estimated in the literature, and we summarize these estimates in Table A.1.[A11] For equation (A.65), Krishnamurthy and Vissing-Jorgensen (2012) focus on estimates of $\frac{\partial(\rho+G-R)}{\partial \log b}$. This derivative measures how the difference between the rate of return on government debt R and the return on other assets $\rho + G$ varies with a change in the log government debt to GDP ratio. Krishnamurthy and Vissing-Jorgensen (2012) use the yield spread difference between corporate bonds rated Baa and 10-year Treasuries as the measure of $\rho + G - R$, and they show a semi-elasticity of -1.3% to -1.7%, depending on the sample. This implies that a 10 percent increase in debt

[A11] A detailed explanation of the exact specifications used from the existing literature to construct Table A.1 is below. We thank Sam Hanson, Andrea Presbitero, Quentin Vandeweyer, and Ursula Wiriadinata for helpful discussions.

Table A.1: How does government debt to GDP affect convenience yield and $G - R$?

Study	Countries	Sample	Object	Estimated φ
Convenience yield: $\rho + G - R$				
Krishnamurthy and Vissing-Jorgensen (2012)	USA	1926-2008	$b_0 \frac{\partial(\rho+G-R)}{\partial b}$	-0.011
Krishnamurthy and Vissing-Jorgensen (2012)	USA	1969-2008	$b_0 \frac{\partial(\rho+G-R)}{\partial b}$	-0.018
Krishnamurthy and Vissing-Jorgensen (2012)	USA	1926-2008	$\frac{\partial(\rho+G-R)}{\partial \log b}$	-0.013
Krishnamurthy and Vissing-Jorgensen (2012)	USA	1969-2008	$\frac{\partial(\rho+G-R)}{\partial \log b}$	-0.017
Greenwood, Hanson and Stein (2015)	USA	1983-2007	$b_0 \frac{\partial(\rho+G-R)}{\partial b}$	-0.014
Vandeweyer (2019) (natural experiment)	USA	2014-2016	$\frac{\partial(\rho+G-R)}{\partial \log b}$	-0.009
Jiang, Lustig, Van Nieuwerburgh and Xiaolan (2020*b*)	Eurozone	2002-2020	$b_0 \frac{\partial(\rho+G-R)}{\partial b}$	-0.008
Growth minus Interest Rate: $G - R$				
Presbitero and Wiriadinata (2020)	17 AEs	1950-2019	$b_0 \frac{\partial(G-R)}{\partial b}$	-0.014
Presbitero and Wiriadinata (2020)	31 AEs & 25 EMs	1950-2019	$b_0 \frac{\partial(G-R)}{\partial b}$	-0.013
This paper	17 AEs	1950-2019	$\frac{\partial(G-R)}{\partial \log b}$	-0.015 to -0.031
This paper	G7	1950-2019	$\frac{\partial(G-R)}{\partial \log b}$	-0.020 to -0.028
This paper	USA, Senate	Jan 2021	$\frac{\partial(G-R)}{\partial \log b}$	-0.022
Negative Real Interest Rate: $-R$				
Laubach (2009)	USA	1976-2006	$b_0 \frac{\partial(\pi-R)}{\partial b}$	-0.015 to -0.022*
Engen and Hubbard (2004)	USA	1976-2003	$b_0 \frac{\partial(\pi-R)}{\partial b}$	-0.015*

Notes. This table summarizes estimates from the existing literature of the effect of government debt to GDP ratios on convenience yields (upper panel) and $G - R$ (lower panel). All of the details on the exact specifications used are in the appendix. Further details on the country-year panel regressions done in this study and the evaluation of the Georgia Senate election results of January 2021 are also in the appendix.

* Estimates in Laubach (2009) and Engen and Hubbard (2004) are stated in terms of $\frac{\partial(-R)}{\partial b}$. To obtain $b_0 \frac{\partial(-R)}{\partial b}$, estimates were multiplied by $b_0 = 0.5$, the average level of total federal debt to GDP over the sample period.

to GDP leads to a 13 to 17 basis point decline in the convenience yield. Alternatively, one can use the Krishnamurthy and Vissing-Jorgensen (2012) estimates to measure $b_0 \frac{\partial(\rho+G-R)}{\partial b}$, which gives estimates between -1.1% and -1.8% when using the average debt to GDP ratio over the relevant sample period for b_0.[A12] Finally, Jiang, Lustig, Van Nieuwerburgh and Xiaolan (2020*b*) provide estimates of the effect of government debt to GDP ratios on convenience yields for Eurozone countries from 2002 to 2020. The implied estimate of $b_0 \frac{\partial(\rho+G-R)}{\partial b}$ from their main specification is -0.8%.

There is also a literature estimating $b_0 \frac{\partial(G-R)}{\partial b}$. In particular, the recent study by Presbitero and Wiriadinata (2020) estimate this derivative in a sample of 56 countries from 1950 to 2019. They provide estimates of $\frac{\partial(G-R)}{\partial b}$ for 17 advanced economies and for the full sample. After multiplying these estimates by b_0, which is the average debt to GDP ratio in each of the respective samples, the implied estimates of $b_0 \frac{\partial(G-R)}{\partial b}$ are -1.4%. For this study, we replicated the Presbitero and Wiriadinata (2020) results for the 17 advanced economies and also for the Group of 7 (G7) countries, and the coefficient estimate ranges are also reported in Table A.1. We show the full results from the regressions in Appendix F.2 below. The estimates of interest are robust to the inclusion of both time and country fixed effects. Overall, most of the estimates across the different samples and the two different objects fit between -1.0% and -2.5%.

An alternative technique to estimate $\frac{\partial(G-R)}{\partial \log b}$ is an analysis of the 2021 Georgia Senate run-off elections that took place on January 5th in the United States. Ex-ante, there was about an even probability of the two Democrat candidates winning their elections as there was that at least one of the two winning candidates was Republican. In the event of a Democrat win, Democrats would obtain the Senate majority, and would likely pass the \$1.9 trillion deficit-financed stimulus package already proposed by President-elect Biden. This was unlikely to happen otherwise. As shown in Figure A.9 in Appendix F.3 below, the wins by both Democrats in Georgia led to a significant persistent increase in real 10 year Treasury yields of about 8 basis points. The effect is concentrated right after the election. It is unlikely that the election was associated with a change in long-term growth prospects; as a result, we interpret the evidence as suggesting that the prospect of the \$1.9 trillion stimulus package, approximately corresponding to 7.4% of outstanding debt, led to a persistent 8 basis point reduction in $G-R$. As this the Democrat win was anticipated with 50% likelihood, this gives $\frac{\partial(G-R)}{\partial \log b} = -2.2\%$. The natural experiment yields an effect of government debt on $G-R$ that is in the same range as the estimates from the existing

[A12] Two other studies in the literature use short-term T-bills and more high frequency data. Greenwood, Hanson and Stein (2015) find estimates for $b_0 \frac{\partial(\rho+G-R)}{\partial b}$ in this range, around -1.4%, whereas Vandeweyer (2019) finds an estimate of -0.4%. The estimates in these two studies should be regarded as a lower bound as they are based on a local estimate of the demand for T-bills as opposed to demand for all government debt.

literature. We provide details on this calculation in Appendix F.3.

Finally, Laubach (2009) and Engen and Hubbard (2004) estimate the effect of government debt to GDP on real interest rates, finding effects in the range 3% to 4.4%. The average level of government debt (total federal debt) to GDP over their sample period was about 50%. Together, this gives an estimate of φ of $b_0 \frac{\partial(G-R)}{\partial b} \approx -1.5\%$ to -2.2% under the assumption that the real growth rate is unaffected by government debt.

Overall, while there is some variation, most of the implied elasticity estimates φ lie in the range 1.1% – 2.5%. We pick the average estimate $\varphi = 1.7\%$ for both countries as our baseline parameter but explore robustness to $\varphi = 1.2\%$ and $\varphi = 2.2\%$.

F.1 Further discussion of estimates from the literature

The estimates of $\frac{\partial(\rho+G-R)}{\partial \log b}$ from Krishnamurthy and Vissing-Jorgensen (2012) reported in Table A.1 above come from their Table 1, columns 4 and 5. The measure of the spread is the Baa corporate yield minus the Treasury bond yield, which they prefer because "Aaa bonds offer some convenience services of Treasuries and thus the Baa-Treasury spread is more appropriate for capturing the full effect of Treasury supply on the Treasury convenience yield." For the estimates of $b_0 \frac{\partial(\rho+G-R)}{\partial b}$, we collected the same data as in Krishnamurthy and Vissing-Jorgensen (2012) and regressed the Baa minus Treasury spread on the level of the debt to GDP ratio. We multiply this coefficient $\frac{\partial(\rho+G-R)}{\partial b}$ (which is -0.027 and -0.048 for the long and short time periods, respectively) by the average level of the debt to GDP ratio b_0 (which is 0.42 and 0.36 for the long and short timer periods, respectively) to get the estimate.

The Greenwood, Hanson and Stein (2015) estimate is from column 1 of Panel B of their Table 1. The measure of the spread is the difference between the actual yield on an 2-week Treasury bill and the 2-week fitted yield, based on the fitted Treasury yield curve in Gürkaynak, Sack and Swanson (2007). The derivative is with respect to the amount of Treasury bills outstanding scaled by GDP. The implied estimate of $\frac{\partial(\rho+G-R)}{\partial b}$ is -0.167, which we then multiply by the average Treasury bill to GDP ratio b_0 (which is 0.084) to get the estimate. We use the estimate from Panel B which goes only through 2007 because of the endogeneity issues discussed by Greenwood, Hanson and Stein (2015) surrounding the Great Recession and financial crisis (see the last full paragraph on page 1689 of their article). The Vandeweyer (2019) regression estimate comes from column 2 of Table 4 of his study. The measure of the spread is the 3-month T-bill rate minus the 3-month General Collateral Repo rate, and this is regressed on the ratio of outstanding T-bills to GDP. The implied estimate of $\frac{\partial(\rho+G-R)}{\partial b}$ is -0.040, which we then multiply by the average Treasury bill to GDP ratio b_0 (which is 0.010) to get the estimate. We use column 2 of Table 4, as this regression controls for the Federal Funds rate as suggested by Nagel (2016). The Vandeweyer (2019) natural experiment involves the 2016 money market reform which led to a large rise in demand for T-bills by money market funds. Money market funds increased their holdings of T-bills by $400 billion, which was about 20% of the stock outstanding. Vandeweyer (2019) uses a model-based counter-factual to show that this shock led to an 18 basis point reduction in yields on government debt, which gives $\frac{\partial(\rho+G-R)}{\partial \log b} = 0.009$. The estimate from Takaoka (2018) comes from Table 4, and the estimate from Jiang, Lustig, Van Nieuwerburgh and Xiaolan (2020*b*) comes from Table 5, panel A, column 2. For the Jiang, Lustig, Van Nieuwerburgh and Xiaolan (2020*b*) estimate of -0.01, we multiply by the average government debt to GDP ratio in their sample to get the final estimate of -0.008.

The estimates of $b_0 \frac{\partial(G-R)}{\partial b}$ come from Presbitero and Wiriadinata (2020), Table A3, column 1. The coefficients $\frac{\partial(G-R)}{\partial b}$come from that table (-0.027 for advanced economies, -0.024 for the full sample), and then these are multiplied by the average government debt to GDP ratiob_0 for the respective samples, which are 0.53 and 0.56 for the advanced economies and the full sample, respectively.

F.2 Regressions based on Presbitero and Wiriadinata (2020)

The other estimates from Table A.1 come from our own data analysis using a data set constructed exactly as the one used by Presbitero and Wiriadinata (2020). The associated regression table is Table A.2.

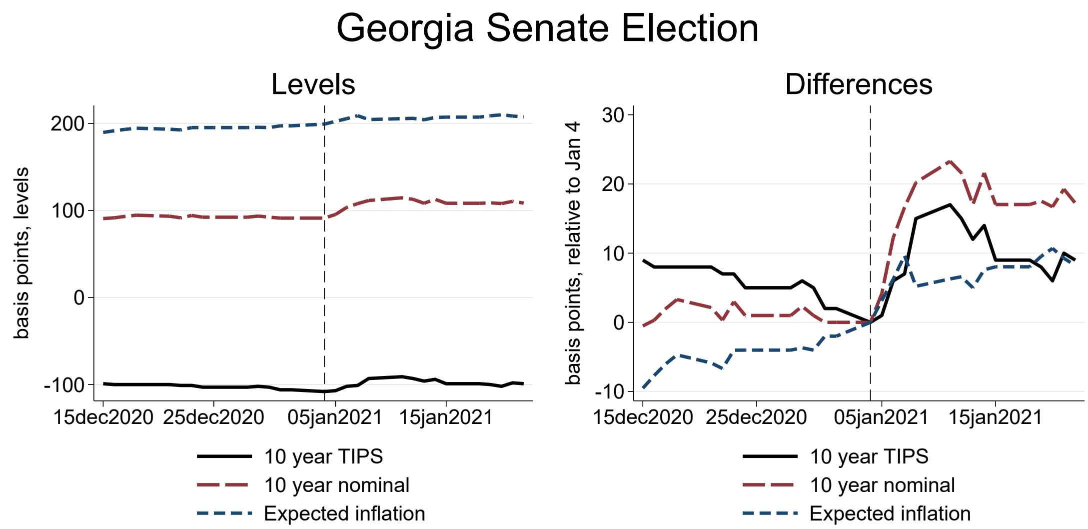
Table A.2: Results from regressions on Presbitero and Wiriadinata (2020) data

	Left hand side: G - R					
	(1)	(2)	(3)	(4)	(5)	(6)
Log(Gov Debt/GDP)	-0.024***	-0.031***	-0.015**	-0.025**	-0.028**	-0.020**
	(0.006)	(0.005)	(0.004)	(0.007)	(0.006)	(0.003)
Observations	1184	1184	1184	490	490	490
R^2	0.103	0.179	0.553	0.162	0.209	0.698
FE		Country	Country and Year		Country	Country and Year
Sample						

* $p < 0.1$, ** $p < 0.05$, *** $p < 0.01$.
Standard errors are heteroskedasticity-robust, clustered by country.

Note. This table presents coefficient estimates of $G - R$ on government debt to GDP ratios. The sample for the first three columns are the 17 advanced economies covered by the JST Macrohistory data base. The sample for columns 4 through 6 is G7 countries (Canada, France, Germany, Italy, Japan, United Kingdom, United States). The time period covered is 1950 to 2019. Please see Presbitero and Wiriadinata (2020) for more details.

Figure A.9: The change in real interest rates around the January 5th, 2021 Georgia run-off election.



F.3 Georgia Senate election

The Georgia Senate election of January 5, 2021 offers a unique opportunity to assess how markets perceive a sudden rise in expected government debt. On the eve of the election, trading at Electionbettingodds.com implied a 50.8% probability of the Republicans controlling the Senate, and a 49.1% probability of the Democrats controlling the Senate. It was widely reported in the press that President-Elect Biden's administration would propose a \$1.9 trillion "American Rescue Plan" once the President-Elect took office. Our assumption in the calculation below is that the win by the two Democrats in the Georgia Senate election of January 5, 2021 increased the expected government debt by \$2 trillion, which at the time was about 7.4% of total debt outstanding.

Figure A.9 shows the effect on the 10 year nominal interest rate, the 10 year TIPS interest rate, and expected inflation. As it shows the victory by the Democrats in the Georgia Senate election led to a 15 basis point immediate reaction which then declined to an 8 basis point reaction after a week. Taken together, these numbers imply that a 3.7% rise in total government debt outstanding relative to prior expectations led to an 8 basis point decline in $G - R$, which gives an estimate of $\frac{\partial(G-R)}{\partial \log b}$ of -0.022. The data for these calculations come from Bloomberg.

G Details on the quantitative model

In this section, we write down the quantitative model with trend, define equilibrium for it, and then de-trend it. We do so for the bonds-in-utility household side; the other household sides are analogous. All trending variables are denoted with uppercase letters. Any variable that was introduced in Section 7 will not be re-introduced here.

G.1 Model

Spenders. Spenders are hand-to-mouth, consuming their after-tax income, $\tilde{C}_t = \mu(1-\chi_t) W_t \tilde{n}_t$, with W_t being the real pre-tax wage.

Savers: Bonds-in-utility (BU). Savers maximize

$$\max_{\{C_t, n_t, A_t\}} \sum_{t=0}^{\infty} e^{-\rho t} \left\{ \log C_t + v\left(\frac{B_t + \tilde{K}_t}{\Theta_t}\right) - h(n_t) \right\} \tag{A.66}$$

subject to the real consolidated budget constraint

$$C_t + B_t + \tilde{K}_t \leq \frac{1+R_{t-1}}{1+\pi_t} B_{t-1} + (1+\tilde{r}_t)\tilde{K}_{t-1} + (1-\mu)(1-\chi_t) W_t n_t + D_t.$$

Here, B_t and $\tilde{K}_t$ are the savers' end of period holdings of government bonds and capital. $\tilde{r}_t$ denotes the real return on capital. Θ_t denotes productivity and ensures that the preferences are compatible with balanced growth. D_t are dividends earned by savers.

Production. Final goods are used for private and public consumption, investment (together denoted by Y_t), as well as for fixed costs (denoted by $\overline{Y}_t \geq 0$). Together, they are a CES aggregate over a continuum of intermediate good varieties Y_{jt}, labeled by $j \in [0,1]$,

$$Y_t + \overline{Y}_t = \left(\int_0^1 Y_{jt}^{\frac{\zeta-1}{\zeta}} dj \right)^{\frac{\zeta}{\zeta-1}}$$

Each intermediate good is produced using a Cobb-Douglas production function,

$$Y_{jt} = \Theta_t^{1-\alpha} K_{jt-1}^{\alpha} n_{jt}^{1-\alpha} \tag{A.67}$$

where TFP Θ_t is common across varieties and grows like

$$\Theta_t = \Theta_{t-1}(1+\gamma).$$

The fixed cost of $\overline{Y}_t \geq 0$ needs to be paid by each intermediate goods producer. Fixed costs also grow at rate γ,

$$\overline{Y}_{t+1} = \overline{Y}_t \left(1+\gamma\right).$$

Intermediate goods producers are monopolistically competitive. Capital K_{jt-1} is rented in a competitive spot market for capital, at rental rate r_t^K.

The price of variety j is denoted by p_{jt} with the price index given by

$$P_t \equiv \left(\int_0^1 p_{jt}^{1-\zeta} dj\right)^{\frac{1}{1-\zeta}}$$

such that demand for variety j is

$$Y_{jt} = Y_t \left(\frac{p_{jt}}{P_t}\right)^{-\zeta}. \tag{A.68}$$

We denote by

$$m \equiv \frac{\zeta}{\zeta - 1} > 1$$

the markup firms will charge in this environment. Since all firms are symmetric, they behave identically in equilibrium. We henceforth drop the subscripts j. Firm behavior is then characterized by the following optimality conditions:

$$Y_t = \Theta_t^{1-\alpha} K_{t-1}^{\alpha} n_t^{1-\alpha} - \overline{Y}_t \tag{A.69}$$

$$m^{-1}\left(1-\alpha\right)\frac{Y_t + \overline{Y}_t}{n_t} = W_t \tag{A.70}$$

$$m^{-1}\alpha\frac{Y_t + \overline{Y}_t}{K_{t-1}} = r_t^K \tag{A.71}$$

Firms' pure profits are given by

$$D_t^Y = \left(1 - m^{-1}\right)\left(Y_t + \overline{Y}_t\right) - \overline{Y}_t. \tag{A.72}$$

Investors. There is a representative investor that takes household funds $\tilde{K}_{t-1}$ at the end of period $t-1$ to purchase physical capital K_{t-1} at price Q_{t-1}, that is, $Q_{t-1}K_{t-1} = \tilde{K}_{t-1}$. It then earns return $r_t^k K_{t-1}$ in period t, invests I_t, and pays an adjustment cost. At the end of period t, the investor sells K_t units of capital at price Q_t. Altogether, the investor solves

$$D_t^I = \max_{K_{t-1},K_t} \frac{1}{1+\tilde{r}_t}\left(Q_t K_t - I_t - \tilde{\Phi}\left(K_t/K_{t-1}\right)K_{t-1} + r_t^k K_{t-1}\right) - Q_{t-1}K_{t-1}. \tag{A.73}$$

Here, $\tilde{\Phi}$ represents a standard quadratic adjustment cost function,

$$\tilde{\Phi}(K_{t+1}/K_t)K_t = \frac{1}{(1+\gamma)^2}\frac{1}{2\epsilon_I\delta}\left(K_{t+1}/K_t - 1 - \gamma\right)^2 K_t. \tag{A.74}$$

ϵ_I parametrizes the strength of the adjustment cost. The adjustment cost (A.74) does not matter along a balanced growth path. The γ terms in this expression are there to ensure balanced growth. The adjustment cost function is necessary to obtain a reasonable response of investment to fiscal policy surprises. We denote total profits by

$$D_t = D_t^I + D_t^Y \tag{A.75}$$

although we note that D_t^I will be equal to zero in equilibrium as the investor's technology has constant returns to scale. Aggregate investment is

$$I_t = K_t - (1-\delta)\, K_{t-1} \tag{A.76}$$

The investor's first order conditions for capital K_{t-1} and K_t are, respectively, for $t \geq 1$

$$r_t^K - \tilde{\Phi}\left(\frac{K_t}{K_{t-1}}\right) + \tilde{\Phi}'\left(\frac{K_t}{K_{t-1}}\right)\frac{K_t}{K_{t-1}} + (1-\delta) = Q_{t-1}$$

and

$$\frac{Q_t}{1+\tilde{r}_t} = \tilde{\Phi}'\left(\frac{K_t}{K_{t-1}}\right) - 1.$$

Nominal rigidity. We follow Erceg, Henderson and Levin (2000) and Auclert, Rognlie and Straub (2018, 2020) and assume that a mass of labor unions exists in our economy which allocates a given amount of labor demand n_t equally among households, such that $n_{it} = \tilde{n}_t = n_{t|t_0} = n_t$. Unions negotiate nominal wages, and index to trend inflation π^{w*}, giving rise to a Phillips curve for nominal wage inflation π_t^w

$$\left(\pi_t^w - \pi^{w*}\right)\left(1 + \pi_t^w - \pi^{w*}\right) = \tilde{\kappa}\cdot n_t\left(h'(n_t) - m^w\left(1-\chi_t\right)\frac{w_t}{c_t}\right) + \zeta e^{-\rho}\left(\pi_{t+1}^w - \pi^{w*}\right)\left(1 + \pi_{t+1}^w - \pi^{w*}\right) \tag{A.77}$$

where m^w is a wage markup. The first term on the right hand side scales with the gap in the first-order condition for labor of the average worker in the economy. If this gap is positive, the average worker is less willing to work, and unions negotiate nominal wage gains. The opposite happens if the gap is negative.

The disutility of labor that we assume is $h(n_t) = \tilde{h}\frac{1}{1+\phi^{-1}}n_t^{1+\phi^{-1}}$ and we write the slope parameter $\tilde{\kappa}$ to make it more interpretable as in (45), that is,

$$\tilde{\kappa} = \kappa \cdot \frac{1 - \zeta e^{-\rho}}{\tilde{h}\left(1 + \phi^{-1}\right)}.$$

We take the limit $m^w \to 1$ to simplify expressions in the main body of the paper. This is without loss for our results as it is equivalent to a simple rescaling $\tilde{h}$ for the exercises in our paper.

Goods inflation is given by

$$\frac{P_t}{P_{t-1}} = 1 + \pi_t = \frac{1 + \pi_t^w}{W_t / W_{t-1}} \tag{A.78}$$

with goods trend inflation defined as

$$1 + \pi^* = \frac{1 + \pi^{w*}}{1 + \gamma}.$$

Government. The government chooses fiscal policy consisting of the paths $\{X_t, B_t, \chi_t\}$ subject to the flow budget constraint

$$X_t + \frac{1 + R_{t-1}}{1 + \pi_t} B_{t-1} = B_t + \chi_t W_t n_t. \tag{A.79}$$

We assume that government spending is exogenous and grows at rate γ, $X_{t+1} = (1 + \gamma) X_t$. The primary deficit is given by

$$Z_t = X_t - \chi_t W_t n_t.$$

Monetary policy targets inflation wherever feasible,

$$R_t \begin{cases} = 0 & \text{if } \pi_t^w < \pi^{w*} \\ \in [0, \infty) & \text{if } \pi_t^w = \pi^{w*} \\ = \infty & \text{if } \pi_t^w > \pi^{w*} \end{cases} \tag{A.80}$$

It is conceptually more natural to have monetary policy target wage inflation in this economy as that will be a better measure of economic slack than price inflation. Our results are similar if price inflation is targeted.

Equilibrium. We define equilibrium in this model as follows.

Definition 2. Given initial levels of debt B_{-1} and capital $\tilde{K}_{-1}, K_{-1}$, a *(competitive) equilibrium* consists of a tuple $\{C_t, \tilde{C}_t, B_t, \tilde{K}_t, Y_t, D_t, Q_t, K_t, R_t, \tilde{r}_t, \pi_t^w, \pi_t, \chi_t, Z_t, W_t, r_t^K\}$, such that: (a)

$\{C_t, \tilde{C}_t, B_t, \tilde{K}_t\}$ maximizes the respective utility maximization problems of savers (A.66) and spenders; (b) $\{B_t, \chi_t\}$ satisfy the government budget constraint (A.79) with bounded debt-to-GDP B_t/Y_t given a path of government spending $\{X_t\}$; (c) wage and price inflation $\{\pi_t, \pi_t^w\}$ follow (A.77) and (A.78); (d) monetary policy follows (A.80) and implies a finite nominal rate R_t; (e) output Y_t is produced using (A.69), with dividends D_t given by (A.72) and (A.75); (f) N_t is consistent with (A.70); (g) the return on capital r_t^K is given by (A.71); (h) K_t is optimally chosen by investors according to (A.73) given r_t^K, $\tilde{r}_t$, Q_t; (i) the goods market clears at all times,

$$\tilde{C}_t + C_t + I_t + X_t = Y_t$$

and the capital market clears,

$$\tilde{K}_{t-1} = q_{t-1} k_{t-1}.$$

A *balanced growth path (BGP) equilibrium* is an equilibrium in which C_t, K_t, Y_t, B_t all grow at the same, constant rate.

G.2 De-trending

We next de-trend this economy. In the following, we go from uppercase to lowercase letters, as in $x_t \equiv \frac{X_t}{\Theta_t}$ and $y_t \equiv \frac{Y_t}{\Theta_t}$, and, normalize Θ such that $\Theta_0 = 1$. The only variable that is de-trended slightly differently is capital, where we define $k_t \equiv \frac{K_t}{\Theta_{t+1}}$ and $\tilde{k}_t \equiv \frac{\tilde{K}_t}{\Theta_{t+1}}$ as capital invested in period t is not used in production until period $t+1$.

De-trending households. The de-trended utility function (A.66) is

$$\log c_t + v\left(b_t + \tilde{k}_t\right) - h(n_t)$$

identical to (37). The de-trended budget constraint (37) is

$$c_t + b_t + \tilde{k}_t \leq \frac{1 + R_{t-1}}{(1+\gamma)(1+\pi_t)} b_{t-1} + \frac{1+\tilde{r}_t}{1+\gamma} \tilde{k}_{t-1} + (1-\mu)(1-\chi_t) w_t n_t + d_t.$$

This is identical to (36) upon defining nominal growth as

$$1 + G_t \equiv (1+\gamma)(1+\pi_t) \tag{A.81}$$

and the de-trended real interest rate as

$$1 + r_t \equiv \frac{1+\tilde{r}_t}{1+\gamma}.$$

De-trending production. We naturally have that de-trended production follows

$$y_t = k_{t-1}^{\alpha} n_t^{1-\alpha} - \overline{y}$$

as in (40) where $\overline{y}$ is constant as $\overline{Y}_t$ was assumed to grow at the same rate as productivity Θ_t. The first order conditions (41) and (42) follow naturally as de-trended versions of (A.70) and (A.71). Dividends d_t in (43) are the de-trended combination of (A.72) and (A.75).

De-trending investors. The investor's problem (A.73) can be written as,

$$D_t^I = \max_{K_{t-1},K_t} \frac{1}{1+\tilde{r}_t} \left(Q_t \Theta_{t+1} k_t - \Theta_t i_t - \Theta_t \tilde{\Phi} \left(\left(1+\gamma\right) k_t / k_{t-1} \right) k_{t-1} + \Theta_t r_t^k k_{t-1} \right) - \Theta_t Q_{t-1} k_{t-1}$$

which simplifies to

$$\left(1+\gamma\right) d_t^I = \max_{K_{t-1},K_t} \frac{1+\gamma}{1+\tilde{r}_t} \left(q_t k_t - i_t - \Phi \left(k_t / k_{t-1} \right) k_{t-1} + r_t^k k_{t-1} \right) - q_{t-1} k_{t-1}$$

where $\Phi(x) \equiv \tilde{\Phi} \left(\left(1+\gamma\right) x \right) = \frac{1}{2\epsilon_I \delta} x^2$ and we defined $q_t \equiv Q_t \left(1+\gamma\right)$. This maximization problem is identical to (44). De-trended investment (A.76) is

$$i_t = \left(1+\gamma\right) k_t - \left(1-\delta\right) k_{t-1}$$

exactly as in (A.76).

De-trending goods inflation. De-trending goods inflation (A.78), we obtain

$$1 + \pi_t = \frac{1 + \pi_t^w}{\left(1+\gamma\right) w_t / w_{t-1}}$$

exactly as in (46).

De-trending the government. The government budget constraint (A.79), de-trended, becomes

$$x_t + \frac{1 + R_{t-1}}{\left(1+\pi_t\right)\left(1+\gamma\right)} b_{t-1} = b_t + \chi_t w_t n_t$$

which is exactly equal to (47) given the definition of G_t in (A.81). The de-trended primary deficit is easily seen to be equal to (A.81).

Definition of equilibrium. We define equilibrium in the de-trended economy as follows.

Definition 3. Given initial levels of debt $\{b_{i,-1}\}$ and capital $\{\tilde{k}_{i,-1}, k_{-1}\}$, a *(competitive) equilibrium* consists of a tuple $\{c_{it}, \tilde{c}_t, b_{it}, \tilde{k}_{it}, y_t, d_t, q_t, k_t, R_t, r_t, \pi_t^w, \pi_t, \chi_t, z_t, w_t, r_t^K\}$, such that: (a) $\{c_{it}, \tilde{c}_t, b_{it}, \tilde{k}_{it}\}$ maximizes the respective utility maximization problems of savers and spenders; (b) $\{b_t, \chi_t\}$ satisfy the government budget constraint (47) with bounded debt b_t given a path of government spending $\{x_t\}$; (c) wage and price inflation $\{\pi_t, \pi_t^w\}$ follow (45) and (46); (d) monetary policy follows (48); (e) output y_t is produced using (40), with dividends d_t given by (43); (f) n_t is consistent with (41); (g) the return on capital r_t^K is given by (42); (h) k_t is optimally chosen by investors according to (44) given r_t^K, r_t, q_t; (i) the goods market clears,

$$\tilde{c}_t + c_t + i_t + x_t = y_t$$

where $c_t = \int c_{it} di$, and the capital market clears,

$$\tilde{k}_{t-1} = \int \tilde{k}_{it-1} di = q_{t-1} k_{t-1}.$$

A *steady state equilibrium* is an equilibrium in which c_t, k_t, y_t, b_t are all constant.

For each equilibrium satisfying Definition 2 there is an equilibrium satisfying Definition 3.

Summing up. These steps show that the model with trend we set up here in Section G.1 exactly turns into the de-trended model introduced in Section 7. An equilibrium in one corresponds to an equilibrium in the other. A balanced growth path in Section G.1 corresponds to a steady state in Section 7.

G.3 Calibration details

The U.S. calibration is summarized in Table A.3. Discount rates in the BU models are higher than one would expect, but these must be regarded in conjunction with the convenience utility, as households' true rate of time preference is a combination of the discount rate and the marginal convenience utility. Similarly, discount rates in the OLG economy must be regarded in conjunction with the survival probability. Together, $\zeta e^{-\rho} = 0.95$. As explained in Farhi and Werning (2019), among other papers, the survival probability in this model should not be interpreted literally; instead, it should be thought of as also capturing other reasons why households might have a limited planning horizon, such as occasionally binding borrowing constraints.

The Japanese calibration is summarized in Table A.4.

Table A.3: Calibration to the pre-Covid U.S. economy

Parameter name	Symbol	BU-quad	BU-log	OLG	HA
Gov. spending to GDP	X/Y	— 14% for all —			
Gov. debt to GDP	B/Y	— 100% for all —			
Labor tax rate	χ	— 12% for all —			
Nominal rate	R	— 1.5% for all —			
Trend inflation	π^*	— 2.0% for all —			
Labor inc. share savers	$1-\mu$	— 36% for all —			
Phillips curve slope	κ	— 0.20 for all —			
Rate of depreciation	δ	— 0.08 for all —			
TFP growth	γ	— 1.5% for all —			
Capital adjustment cost	ϵ_I	— 4 for all —			
Capital to GDP	K/Y	— 2.0 for all —			
Capital share	α	— 0.15 for all —			
Markup	m	— 1.3 for all —			
Fixed cost	$\overline{y}$	— 0.3 for all —			
Frisch elasticity	ϕ	— 1.0 for all —			
Effective disutility weight	$\tilde{h}$	— 1.09 for all —			
Discount rate	ρ	0.156	0.156	-0.196	0.059
Intertemporal elast.	ν^{-1}	1.000	1.000	1.000	0.219
Convenience utility level	$v'(a_{ss})$	0.286	---	---	---
Convenience utility slope	$\tilde{\varphi}$	0.286	0.286	---	---
OLG survival probability	ζ	---	---	0.783	---
Persistence e_{it}	ρ_e	---	---	---	0.900
Cross-sectional std. dev. e_{it}	σ_e	---	---	---	0.920

Table A.4: Calibration to the pre-Covid Japanese economy

Parameter name	Symbol	BU-quad	BU-log	OLG	HA
Gov. spending to GDP	X/Y	— 20% for all —			
Gov. debt to GDP	B/Y	— 238% for all —			
Labor tax rate	χ	— 18.6% for all —			
Nominal rate	R	— 0.0% for all —			
Trend inflation	π^*	— 2.0% for all —			
Labor inc. share savers	$1-\mu$	— 34% for all —			
Phillips curve slope	κ	— 0.432 for all —			
Rate of depreciation	δ	— 0.08 for all —			
TFP growth	γ	— 0.3% for all —			
Capital adjustment cost	ϵ_I	— 4 for all —			
Capital to GDP	K/Y	— 2.0 for all —			
Capital share	α	— 0.15 for all —			
Markup	m	— 1.3 for all —			
Fixed cost	$\bar{y}$	— 0.3 for all —			
Frisch elasticity	ϕ	— 1.0 for all —			
Effective disutility weight	$\tilde{h}$	— 0.920 for all —			
Discount rate	ρ	0.042	0.042	-0.054	0.018
Intertemporal elast.	ν^{-1}	1.000	1.000	1.000	0.299
Convenience utility level	$v'(a_{ss})$	0.080	---	---	---
Convenience utility slope	$\tilde{\varphi}$	0.080	0.080	---	---
OLG survival probability	ζ	---	---	0.923	---
Persistence e_{it}	ρ_e	---	---	---	0.900
Cross-sectional std. dev. e_{it}	σ_e	---	---	---	0.920

G.4 Calibration of κ in the Japanese calibration

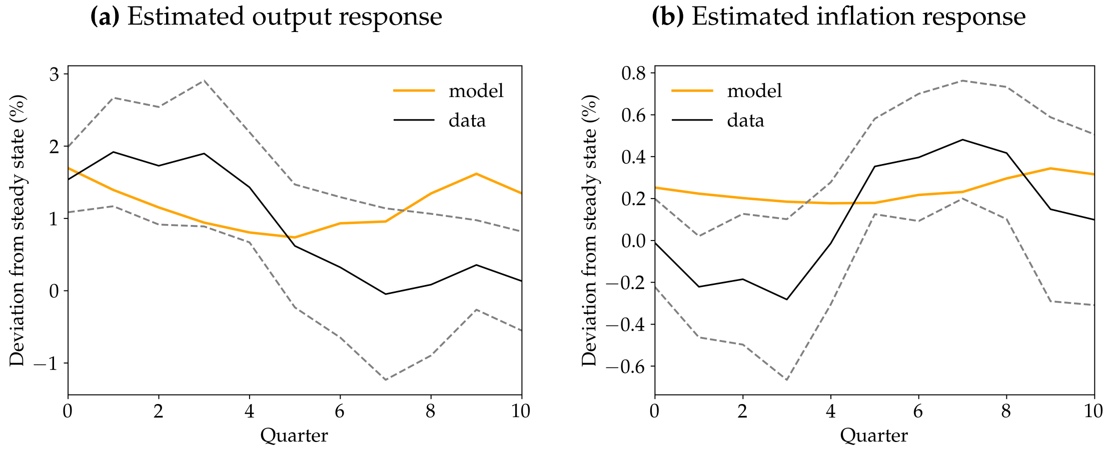
We base our calibration on the evidence of the effects of government spending shock in Japan in Miyamoto, Nguyen and Sergeyev (2018) (MNS). MNS use surprises in government spending relative to forecasted government spending a period ago as instruments for government spending shocks. Denote their shocks by $\tilde{\epsilon}_t$. We run simple local projections (as in equation (2) in MNS) of changes in the price level and changes in GDP on the shock $\tilde{\epsilon}_t$,

$$OUT_{t+h} = \alpha_h^o + \beta_h^o \cdot \tilde{\epsilon}_t + \psi_h^o(L)y_{t-1} + \epsilon_{t+h}^o. \tag{A.82}$$

where OUT_{t+h} is an outcome; $\psi_h^o(L)y_{t-1}$ are MNS's controls, and β_h^o is the IRF of the outcome of interest as a function of the horizon $h = 0, \ldots, H$. The three outcomes that are most relevant for us are the change in government spending, $\frac{G_{t+h}-G_{t-1}}{Y_{t-1}}$, which is the shock itself; the change in GDP, $\frac{Y_{t+h}-Y_{t-1}}{Y_{t-1}}$; and the change in the price level, $\frac{P_{t+h}-P_{t-1}}{P_{t-1}}$, where P_t is the GDP deflator in the data (just like in MNS). Just like in MNS, we estimate the shock over a total horizon $H = 15$, and the impulse responses over a horizon of $H = 10$.

We feed the shock β_h^G into our model as an exogenous, 16-period-long government spending shock. We then pick the slope of the Phillips curve κ to minimize the distance between the model generated IRFs for GDP and the price level and the corresponding empirical IRFs. Figure A.10 plots the impulse responses for GDP and the price level. The model reproduces the correct magnitudes, but is clearly too simple to get the precise timing in the IRFs right. This procedure yields $\kappa = 0.272$, slightly higher than the value we use for the U.S. economy.

Figure A.10: Output and inflation responses to government spending shock in Japan



Note. Impulse responses estimated as in Miyamoto, Nguyen and Sergeyev (2018) using equation (A.82). The slope of the Phillips curve κ was chosen in the model to minimize the sum of squared distances to both impulse responses.

H Rising inequality and convenience yields

Section 3.5 showed that rising inequality raises fiscal space by raising total saving available to finance government debt, and putting downward pressure on government borrowing cost. It is very difficult to generate completely exogenous variation in inequality at the macroeconomic level. Nonetheless time series evidence from the U.S. is consistent with these predictions.

Krishnamurthy and Vissing-Jorgensen (2012) (KVJ) show that convenience yield on U.S. debt—proxied by the Aaa-Treasury spread—is strongly negatively correlated with the amount of debt issued by the government as a share of GDP. This long-run negative relationship represents the traditional tradeoff between cost of government debt and supply of government debt, *holding all else equal*. Column (1) of Table A.5 replicates the key KVJ result by regressing the Aaa-Treasury spread (or convenience yield) on log of government debt to GDP over their full sample that goes from 1919-2008. The coefficient on log debt to GDP is -0.74. Column (2) extends the sample to 2022, the most recent available year. What is striking is that the magnitude of the coefficient drops to -0.45. Why does the convenience yield - debt relationship appear to weaken in recent years? As we show next, a possible reason is the sharp rise in wealth inequality since 1980. Once we control for that, we recover the original KVJ result even in the most recent sample.

We repeat the KVJ regression in the more recent period of 1962-2022 in column (3). This is the period with a consistent public tax return data, where Mian, Straub and Sufi (2020) show that rising inequality contributed to an expansion in the supply of savings,

Table A.5: Role of rising inequality in the Krishnamurthy and Vissing-Jorgensen (2012) regression

	1919-2008	1919-2022	1962-2022	1962-2008	1980-2022	1962-2022
	(1)	(2)	(3)	(4)	(5)	(6)
ln(Debt/GDP)	-0.746***	-0.459***	-0.241	-1.032***	-0.0469	-0.710**
	(0.0704)	(0.0795)	(0.123)	(0.221)	(0.147)	(0.248)
Share Top 1% Wealth						5.043*
						(2.340)
N	90	104	61	47	42	60
R-sq	0.449	0.218	0.049	0.283	0.002	0.121

Standard errors in parentheses

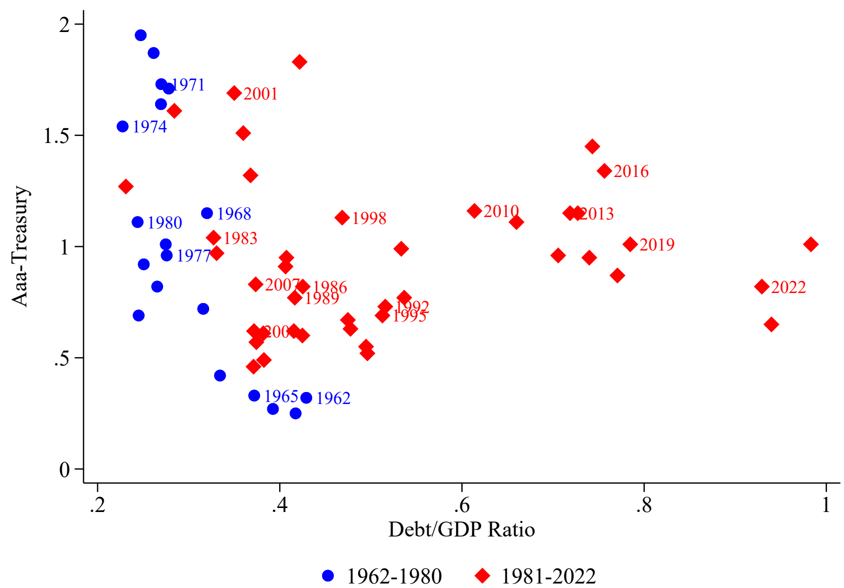
* $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$

and the top 1% use the additional savings to hold significantly more government debt. The coefficient on log debt to GDP is only -0.24, about one-third of the column (1) estimate. Columns (4) and (5) show that the decline in KVJ coefficient is entirely driven by post-1980 sample, when wealth inequality starts to rise appreciably. Importantly, once wealth inequality (measured as the share of wealth going to the top 1%) is included as a control variable in the 1962-2022 sample, the original KVJ coefficient is recovered once again.

This simple analysis suggests that the apparent breakdown of convenience yield - debt relationship is driven by a third variable, namely wealth inequality, rising in the post-1980 period. This is inline with our model which suggests that the rise in inequality should expand fiscal capacity, enabling the government to raise government debt without putting as much downward pressure on convenience yield as before.

The outward shift in fiscal space (and hence convenience yield) post-1980 can also be seen in Figure A.11 that plots the relationship between convenience yield and government debt to GDP. There is a clear upward shift in convenience yield relative to the earlier downward relationship between convenience yield and debt to GDP, post-1980 (red dots) as inequality starts to rise. Overall the post-1980 experience of the U.S. with rising inequality is consistent with the predictions of our model that fiscal space increases with inequality.

Figure A.11: Replication of Krishnamurthy and Vissing-Jorgensen (2012) regression



Note. This figure plots the Aaa-Treasury spread in % on the y axis against the debt-to-GDP ratio on the x axis.