Question 4

Train A runs back and forth on an east-west section of railroad track. Train A's velocity, measured in meters per minute, is given by a differentiable function $v_A(t)$, where time t is measured in minutes. Selected values for $v_A(t)$ are given in the table above.

t (minutes)	0	2	5	8	12
$v_A(t)$ (meters/minute)	0	100	40	-120	-150

(a) Find the average acceleration of train A over the interval $2 \le t \le 8$.

(b) Do the data in the table support the conclusion that train A's velocity is -100 meters per minute at some time t with $5 < t < 8$? Give a reason for your answer.

(c) At time $t = 2$, train A's position is 300 meters east of the Origin Station, and the train is moving to the east. Write an expression involving an integral that gives the position of train A, in meters from the Origin Station, at time $t = 12$. Use a trapezoidal sum with three subintervals indicated by the table to approximate the position of the train at time $t = 12$.

(d) A second train, train B, travels north from the Origin Station. At time t the velocity of train B is given by $v_B(t) = -5t^2 + 60t + 25$, and at time $t = 2$ the train is 400 meters north of the station. Find the rate, in meters per minute, at which the distance between train A and train B is changing at time $t = 2$.

(a) average accel $= \dfrac{v_A(8) - v_A(2)}{8 - 2} = \dfrac{-120 - 100}{6} = -\dfrac{110}{3}$ m/min^2

1 : average acceleration

(b) v_A is differentiable $\Rightarrow$ v_A is continuous

$v_A(8) = -120 < -100 < 40 = v_A(5)$

Therefore, by the Intermediate Value Theorem, there is a time t, $5 < t < 8$, such that $v_A(t) = -100$.

$2 : \begin{cases} 1 : v_A(8) < -100 < v_A(5) \\ 1 : \text{conclusion, using IVT} \end{cases}$

(c) $s_A(12) = s_A(2) + \int_2^{12} v_A(t)\,dt = 300 + \int_2^{12} v_A(t)\,dt$

$$\int_2^{12} v_A(t)\,dt \approx 3 \cdot \frac{100 + 40}{2} + 3 \cdot \frac{40 - 120}{2} + 4 \cdot \frac{-120 - 150}{2} = -450$$

$s_A(12) \approx 300 - 450 = -150$

The position of Train A at time $t = 12$ minutes is approximately 150 meters west of Origin Station.

$3 : \begin{cases} 1 : \text{position expression} \\ 1 : \text{trapezoidal sum} \\ 1 : \text{position at time } t = 12 \end{cases}$

(d) Let x be train A's position, y train B's position, and z the distance between train A and train B.

$z^2 = x^2 + y^2 \Rightarrow 2z\dfrac{dz}{dt} = 2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt}$

$x = 300,\ y = 400 \Rightarrow z = 500$

$v_B(2) = -20 + 120 + 25 = 125$

$500\dfrac{dz}{dt} = (300)(100) + (400)(125)$

$\dfrac{dz}{dt} = \dfrac{80000}{500} = 160$ meters per minute

$3 : \begin{cases} 2 : \text{implicit differentiation of distance relationship} \\ 1 : \text{answer} \end{cases}$

NO CALCULATOR ALLOWED

t (minutes)	0	2	5	8	12
$v_A(t)$ (meters/minute)	0	100	40	−120	−150

4. Train A runs back and forth on an east-west section of railroad track. Train A's velocity, measured in meters per minute, is given by a differentiable function $v_A(t)$, where time t is measured in minutes. Selected values for $v_A(t)$ are given in the table above.

(a) Find the average acceleration of train A over the interval $2 \le t \le 8$.

$$\frac{v(8)-v(2)}{8-2} \rightarrow \frac{-120-100}{6} \rightarrow \frac{-220}{6} \rightarrow \boxed{\frac{-110}{3} \text{ m/min}^2}$$

(b) Do the data in the table support the conclusion that train A's velocity is -100 meters per minute at some time t with $5 < t < 8$? Give a reason for your answer.

Yes, because $v(8) = -120$ and $v(5) = 40$ and the function is differentiable and thus continuous, the train's velocity must be -100 m/min at some point between $5 < t < 8$ according to the intermediate value theorem.

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NO CALCULATOR ALLOWED

(c) At time $t = 2$, train A's position is 300 meters east of the Origin Station, and the train is moving to the east. Write an expression involving an integral that gives the position of train A, in meters from the Origin Station, at time $t = 12$. Use a trapezoidal sum with three subintervals indicated by the table to approximate the position of the train at time $t = 12$.

$$x(12) = \int_2^{12} V_A(t)\,dt + x(2) \rightarrow \boxed{x(12) = \int_2^{12} V_A(t)\,dt + 300}$$

$$x(12) \approx 3 \cdot \tfrac{1}{2} \cdot (140) + 3 \cdot \tfrac{1}{2} \cdot (-80) + 4 \cdot \tfrac{1}{2} \cdot (-270) + 300$$

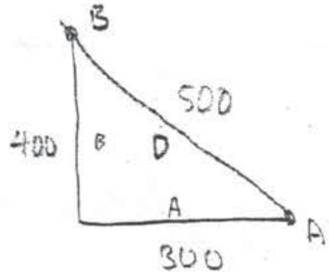
$$210 - 120 - 540 + 300$$

$$210 - 240 - 120$$

$$-30 - 120$$

$\boxed{-150}$ meaning it is 150 m W of origin station

(d) A second train, train B, travels north from the Origin Station. At time t the velocity of train B is given by $v_B(t) = -5t^2 + 60t + 25$, and at time $t = 2$ the train is 400 meters north of the station. Find the rate, in meters per minute, at which the distance between train A and train B is changing at time $t = 2$.



$$\frac{dB}{dt} = -5t^2 + 60t + 25 \rightarrow 125 \text{ m/min}$$

$$-20 + 120 + 25$$

$$120 + 5 \rightarrow 125$$

$$\frac{dA}{dt} = 100 \text{ m/min}$$

$$\begin{array}{r} 125 \\ 800 \\ \hline 100000 \end{array}$$

$$A^2 + B^2 = D^2$$

$$2A\frac{dA}{dt} + 2B\frac{dB}{dt} = 2D\frac{dD}{dt}$$

$$600(100) + 800(125) = 1000\frac{dD}{dt}$$

$$60000 + 100000 = 1000\frac{dD}{dt}$$

$$\frac{160{,}000}{1000} = \frac{dD}{dt} \rightarrow \boxed{\frac{dD}{dt} = 160 \text{ m/min}}$$

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 4B

NO CALCULATOR ALLOWED

t (minutes)	0	2	5	8	12
$v_A(t)$ (meters / minute)	0	100	40	−120	−150

4. Train A runs back and forth on an east-west section of railroad track. Train A's velocity, measured in meters per minute, is given by a differentiable function $v_A(t)$, where time t is measured in minutes. Selected values for $v_A(t)$ are given in the table above.

(a) Find the average acceleration of train A over the interval $2 \le t \le 8$.

$$\bar{a} = \frac{V_A(8) - V_A(2)}{8-2} = \frac{-120-100}{4} = \frac{-220}{4}$$

$$= \boxed{-55 \text{ meters/minute}^2}$$

(b) Do the data in the table support the conclusion that train A's velocity is -100 meters per minute at some time t with $5 < t < 8$? Give a reason for your answer.

Yes, since $V_A(t)$ is continuous and differentiable, The velocity ~~of~~ of train A must at some time t with $5<t<8$ equal -100 meters/minute because $V_A(5) = 40$ and $V_A(8) = -120$.

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NO CALCULATOR ALLOWED

(c) At time $t = 2$, train A's position is 300 meters east of the Origin Station, and the train is moving to the east. Write an expression involving an integral that gives the position of train A, in meters from the Origin Station, at time $t = 12$. Use a trapezoidal sum with three subintervals indicated by the table to approximate the position of the train at time $t = 12$.

$$X_A(12) = 300 + \int_2^{12} V_A(t)\,dt$$

$$= 300 + 3(100+40) + 3(40+(-120)) + 4(-120-150)$$

$$= 300 + 420 - 240 - 1080$$

$$= \boxed{-600 \text{ meters west of the Origin Station}}$$

$270 \times 4 = 1080$

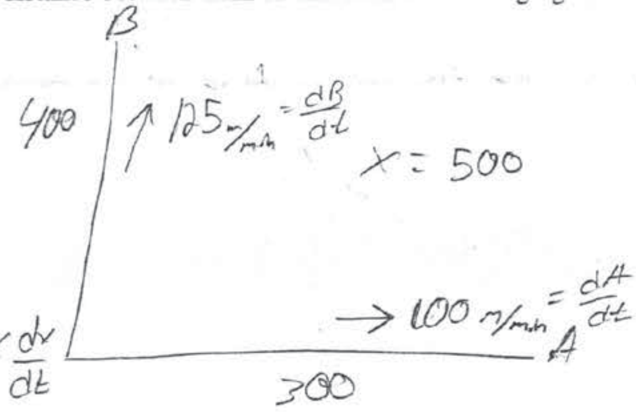
$1080 + 240 = 1320$

$-1320 + 720 = -600$

(d) A second train, train B, travels north from the Origin Station. At time t the velocity of train B is given by $v_B(t) = -5t^2 + 60t + 25$, and at time $t = 2$ the train is 400 meters north of the station. Find the rate, in meters per minute, at which the distance between train A and train B is changing at time $t = 2$.

$$V_B(2) = -5(2)^2 + 60(2) + 25$$

$$= 125 \text{ m/min}$$



$$A^2 + B^2 = \cancel{C}x^2$$

$$2A\frac{dA}{dt} + 2B\frac{dB}{dt} = 2x\frac{dx}{dt}$$

$$2(300)(100) + 2(400)(125) = 2(500)\frac{dx}{dt}$$

$$2(3)(100) + 2(4)(125) = 10\frac{dx}{dt}$$

$$600 + 1000 = 10\frac{dx}{dt}$$

$$\boxed{\frac{dx}{dt} = 160 \text{ m/min}}$$

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$v_A(t)$ (meters/minute)	0	100	40	−120	−150

4. Train A runs back and forth on an east-west section of railroad track. Train A's velocity, measured in meters per minute, is given by a differentiable function $v_A(t)$, where time t is measured in minutes. Selected values for $v_A(t)$ are given in the table above.

(a) Find the average acceleration of train A over the interval $2 \le t \le 8$.

$$\frac{-120-100}{8-2} = \boxed{\frac{-220}{6}}\ \text{m/min}^2$$

(b) Do the data in the table support the conclusion that train A's velocity is -100 meters per minute at some time t with $5 < t < 8$? Give a reason for your answer.

Yes, The velocity drops from 40 m/min to -120 m/min So at some point the velocity must have been at -100 m/min

Continue problem 4 on page 17.

NO CALCULATOR ALLOWED

(c) At time $t = 2$, train A's position is 300 meters east of the Origin Station, and the train is moving to the east. Write an expression involving an integral that gives the position of train A, in meters from the Origin Station, at time $t = 12$. Use a trapezoidal sum with three subintervals indicated by the table to approximate the position of the train at time $t = 12$.

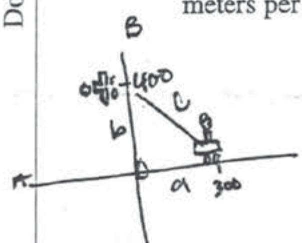
$$300 + \int_2^{12} v_A(t)\,dt \approx 300 + \frac{12-2}{6}\left(v_A(2) + 2v_A(5) + 2v_A(8) + v_A(12)\right)$$

$$= 300 + \frac{10}{6}(100 + 80 + -240 + -150)$$

$$= 300 + \frac{10}{6}(-210)$$

$$\boxed{= 300 - \frac{2100}{6}}$$ meters west of the origin Station

(d) A second train, train B, travels north from the Origin Station. At time t the velocity of train B is given by $v_B(t) = -5t^2 + 60t + 25$, and at time $t = 2$ the train is 400 meters north of the station. Find the rate, in meters per minute, at which the distance between train A and train B is changing at time $t = 2$.



$$a^2 + b^2 = c^2$$

$$300^2 + 400^2 = c^2$$

~~9600~~

$$900 + 1600 = c^2$$

$$2500 = c^2$$

$$500 = c$$

$\boxed{\text{500 meters per minute}}$

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Question 4

Overview

In this problem students were given a table of values of a differentiable function $v_A(t)$, the velocity of Train A, in meters per minute, for selected values of t in the interval $0 \le t \le 12$, where t is measured in minutes. In part (a) students were expected to know that the average acceleration of Train A over the interval $2 \le t \le 8$ is the average rate of change of $v_A(t)$ over that interval. The unit of the average acceleration is meters per minute per minute. In part (b) students were expected to state clearly that v_A is continuous because it is differentiable, and thus the Intermediate Value Theorem implies the existence of a time t between $t = 5$ and $t = 8$ at which $v_A(t) = -100.$ In part (c) students were expected to show that the change in position over a time interval is given by the definite integral of the velocity over that time interval. If $s_A(t)$ is the position of Train A, in meters, at time t minutes, then $s_A(12) - s_A(2) = \int_2^{12} v_A(t)\,dt$, which implies that $s_A(12) = 300 + \int_2^{12} v_A(t)\,dt$ is the position at $t = 12.$ Students approximated $\int_2^{12} v_A(t)\,dt$ using a trapezoidal approximation. In part (d) students had to determine the relationship between train A's position, train B's position, and the distance between the two trains. Students needed to put together several pieces of information from different parts of the problem and use implicit differentiation to determine the rate at which the distance between the two trains is changing at time $t = 2.$

Sample: 4A
Score: 9

The student earned all 9 points.

Sample: 4B
Score: 6

The student earned 6 points: no points in part (a), 2 points in part (b), 1 point in part (c), and 3 points in part (d). In part (a) the student makes an arithmetic mistake in computing the average acceleration. In part (b) the student's work is correct. In part (c) the student earned the point for the position expression, but the trapezoidal sum is incorrect. The student is not eligible for the answer point. In part (d) the student's work is correct.

Sample: 4C
Score: 3

The student earned 3 points: 1 point in part (a), 1 point in part (b), 1 point in part (c), and no points in part (d). In part (a) the student's work is correct. In part (b) the student encloses -100 within the required interval, but the student does not provide a reason. In part (c) the position expression is correct, but the trapezoidal sum is incorrect. The student is not eligible for the answer point. In part (d) the student's work did not earn any points.