AP® CALCULUS AB
2015 SCORING GUIDELINES

Question 6

Consider the curve given by the equation $y^3 - xy = 2$. It can be shown that $\frac{dy}{dx} = \frac{y}{3y^2 - x}$.

(a) Write an equation for the line tangent to the curve at the point $(-1, 1)$.

(b) Find the coordinates of all points on the curve at which the line tangent to the curve at that point is vertical.

(c) Evaluate $\frac{d^2y}{dx^2}$ at the point on the curve where $x = -1$ and $y = 1$.

(a) $$\left.\frac{dy}{dx}\right|_{(x,\,y)=(-1,\,1)} = \frac{1}{3(1)^2 - (-1)} = \frac{1}{4}$$

An equation for the tangent line is $y = \frac{1}{4}(x + 1) + 1$.

$2 : \begin{cases} 1 : \text{slope} \\ 1 : \text{equation for tangent line} \end{cases}$

(b) $3y^2 - x = 0 \Rightarrow x = 3y^2$

So, $y^3 - xy = 2 \Rightarrow y^3 - (3y^2)(y) = 2 \Rightarrow y = -1$

$(-1)^3 - x(-1) = 2 \Rightarrow x = 3$

The tangent line to the curve is vertical at the point $(3, -1)$.

$3 : \begin{cases} 1 : \text{sets } 3y^2 - x = 0 \\ 1 : \text{equation in one variable} \\ 1 : \text{coordinates} \end{cases}$

(c) $$\frac{d^2y}{dx^2} = \frac{(3y^2 - x)\frac{dy}{dx} - y\left(6y\frac{dy}{dx} - 1\right)}{(3y^2 - x)^2}$$

$$\left.\frac{d^2y}{dx^2}\right|_{(x,\,y)=(-1,\,1)} = \frac{\left(3 \cdot 1^2 - (-1)\right) \cdot \frac{1}{4} - 1 \cdot \left(6 \cdot 1 \cdot \frac{1}{4} - 1\right)}{\left(3 \cdot 1^2 - (-1)\right)^2}$$

$$= \frac{1 - \frac{1}{2}}{16} = \frac{1}{32}$$

$4 : \begin{cases} 2 : \text{implicit differentiation} \\ 1 : \text{substitution for } \frac{dy}{dx} \\ 1 : \text{answer} \end{cases}$

NO CALCULATOR ALLOWED

6. Consider the curve given by the equation $y^3 - xy = 2$. It can be shown that $\frac{dy}{dx} = \frac{y}{3y^2 - x}$.

(a) Write an equation for the line tangent to the curve at the point $(-1, 1)$.

Slope $= \frac{dy}{dx}$ @ $(-1,1)$: $\frac{1}{3(1)^2-(-1)} = \frac{1}{4}$

Point $= (-1,1)$

$y - 1 = \frac{1}{4}(x+1)$

$y - 1 = \frac{x}{4} + \frac{1}{4}$

$\boxed{y = \frac{1}{4}x + \frac{5}{4}}$

(b) Find the coordinates of all points on the curve at which the line tangent to the curve at that point is vertical.

Vertical tangent line $\Rightarrow$ slope undefined

$\Rightarrow$ denominator $= 0$ $\Rightarrow$ $3y^2 - x = 0$

$3y^2 = x$

Substitute "x" for "$3y^2$" in the equation of the curve

$y^3 - [3y^2 \cdot y] = 2$

$y^3 - 3y^3 = 2$

$-2y^3 = 2$

$y^3 = -1$

$y = -1$

$(-1)^3 - x \cdot (-1) = 2$

$-1 + x = 2$

$x = 3$

$\boxed{\text{Vertical tangents at } (3,-1)}$

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(c) Evaluate $\frac{d^2y}{dx^2}$ at the point on the curve where $x = -1$ and $y = 1$. Point : $(-1, 1)$

$\frac{dy}{dx}$ @ $(-1,1) = \frac{1}{4}$

$3y^2 - x = 3 - (-1) = 4$

$$\frac{d^2y}{dx^2} = \frac{\left[y' \cdot (3y^2 - x)\right] - \left[(6y\overset{\frac{6}{4}}{y'} - \overset{\frac{4}{4}}{1}) \cdot (y)\right]}{(3y^2 - x)^2}$$

$$= \frac{\left(\frac{1}{\cancel{4}} \cdot \cancel{4}\right) - \left(\frac{1}{2} \cdot 1\right)}{16} = \frac{1 - \frac{1}{2}}{16} = \frac{\frac{1}{2}}{16} = \frac{1}{2} \cdot \frac{1}{16}$$

$$= \boxed{\frac{1}{32}}$$

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NO CALCULATOR ALLOWED

6. Consider the curve given by the equation $y^3 - xy = 2$. It can be shown that $\frac{dy}{dx} = \frac{y}{3y^2 - x}$.

(a) Write an equation for the line tangent to the curve at the point $(-1, 1)$.

$\frac{dy}{dx} = \frac{1}{3 \cdot 1^2 - (-1)} = -\frac{1}{4}$

$\boxed{y - 1 = \frac{1}{4}(x+1)}$

(b) Find the coordinates of all points on the curve at which the line tangent to the curve at that point is vertical.

$3y^2 - x = 0$ ← derrivative DNE when denominator $= 0$

$3y^2 - x + 2 = 2$

$y^3 - xy = 3y^2 - x + 2$ ← set equations equal to find x and y

$2 = y^3 - 3y^2 - xy + x$

$2 = y^2(y-3) - x(y+1)$

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(c) Evaluate $\frac{d^2y}{dx^2}$ at the point on the curve where $x = -1$ and $y = 1$.

$$\frac{d^2y}{dx^2} = \frac{\frac{dy}{dx}(3y^2-x) - y\left(6y\frac{dy}{dx}-1\right)}{(3y^2-x)^2}$$

$$= \frac{\left(\frac{y}{3y^2-x}\right)(3y^2-x) - y\left(6y\left(\frac{y}{3y^2-x}\right)-1\right)}{(3y^2-x)^2}$$

$$\frac{1 - 1\left(6\cdot 1\left(\frac{1}{3+1}\right)-1\right)}{(3+1)^2}$$

$$\frac{1 - \frac{3}{2} \cancel{1}}{16}$$

$$-\frac{3}{2}\cdot\frac{1}{16} = \boxed{\frac{-3}{32}}$$

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 6C

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6. Consider the curve given by the equation $y^3 - xy = 2$. It can be shown that $\frac{dy}{dx} = \frac{y}{3y^2 - x}$.

(a) Write an equation for the line tangent to the curve at the point $(-1, 1)$.

$m = \frac{dy}{dx} = \frac{1}{3(1)^2 - (-1)} = \frac{1}{5}$

$y - 1 = \frac{1}{5}(x + 1)$

$\frac{1}{5}x + \frac{1}{5}$

$+1$

$\boxed{y = \frac{1}{5}x + \frac{6}{5}}$

(b) Find the coordinates of all points on the curve at which the line tangent to the curve at that point is vertical.

$\frac{1}{5}x + \frac{6}{5} = \frac{y}{3y^2 - x}$

$(3y^2 - x)(\frac{1}{5}x + \frac{6}{5} = y$

$\frac{3}{5}yx^2 + \frac{18}{5}y^2 - \frac{1}{5}x^2 - 6/5\,x = y$

$(-1, 1)$?

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(c) Evaluate $\frac{d^2y}{dx^2}$ at the point on the curve where $x = -1$ and $y = 1$.

$$\frac{dy}{dx} = \frac{y}{3y^2 - x}$$

$$\frac{(y'(3y^2 - x)) - y(6yy' - 1)}{(3y^2 - x)^2} = \frac{d^2y}{dx^2}$$

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Question 6

Overview

In this problem students were given the equation of a curve, $y^3 - xy = 2$, with $\frac{dy}{dx} = \frac{y}{3y^2 - x}$. In part (a) students had to find an equation for the line tangent to the curve at the point $(-1, 1)$. Students were expected to use the given $\frac{dy}{dx}$ to find the slope of the curve at the point $(-1, 1)$. In part (b) students were asked to find the coordinates of all points on the curve at which there is a vertical tangent line. These are the points on the curve where $3y^2 - x = 0$, but $y \neq 0$. Students were expected to solve $y^3 - xy = 2$ with the condition that $3y^2 - x = 0$ and report only those pairs (x, y) where $y \neq 0$. In part (c) students were asked to evaluate $\frac{d^2y}{dx^2}$ at the point $(-1, 1)$ on the curve. Students had to use implicit differentiation with $\frac{dy}{dx}$ to find an expression for $\frac{d^2y}{dx^2}$, which required use of the chain rule and either the product rule or the quotient rule. The expression can be written in terms of x and y or can involve $\frac{dy}{dx}$. In either case, students needed to evaluate the expression for $\frac{d^2y}{dx^2}$ at $(-1, 1)$.

Sample: 6A
Score: 9

The response earned all 9 points.

Sample: 6B
Score: 6

The response earned 6 points: 2 points in part (a), 1 point in part (b), and 3 points in part (c). In part (a) the student's work is correct. In part (b) the student considers the equation $3y^2 - x = 0$, so the first point was earned. The student does not present an equation in one variable. In part (c) the student correctly differentiates and substitutes for $\frac{dy}{dx}$, so the first 3 points were earned. The student makes an error in computation, so the answer point was not earned.

Sample: 6C
Score: 3

The response earned 3 points: 1 point in part (a), no points in part (b), and 2 points in part (c). In part (a) the student makes an arithmetic error in computing the slope, so the first point was not earned. The student uses the slope to present a line that passes through $(-1, 1)$, so the second point was earned. In part (b) the student does not consider the equation $3y^2 - x = 0$. In part (c) the student correctly differentiates $\frac{dy}{dx}$, so the first 2 points were earned.