Question 6

The Maclaurin series for a function f is given by $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{n} x^n = x - \frac{3}{2}x^2 + 3x^3 - \cdots + \frac{(-3)^{n-1}}{n} x^n + \cdots$ and converges to $f(x)$ for $|x| < R$, where R is the radius of convergence of the Maclaurin series.

(a) Use the ratio test to find R.

(b) Write the first four nonzero terms of the Maclaurin series for f', the derivative of f. Express f' as a rational function for $|x| < R$.

(c) Write the first four nonzero terms of the Maclaurin series for e^x. Use the Maclaurin series for e^x to write the third-degree Taylor polynomial for $g(x) = e^x f(x)$ about $x = 0$.

(a) Let a_n be the nth term of the Maclaurin series.

$$\frac{a_{n+1}}{a_n} = \frac{(-3)^n x^{n+1}}{n+1} \cdot \frac{n}{(-3)^{n-1} x^n} = \frac{-3n}{n+1} \cdot x$$

$$\lim_{n\to\infty} \left| \frac{-3n}{n+1} \cdot x \right| = 3|x|$$

$$3|x| < 1 \Rightarrow |x| < \frac{1}{3}$$

The radius of convergence is $R = \frac{1}{3}$.

$$3 : \begin{cases} 1 : \text{sets up ratio} \\ 1 : \text{computes limit of ratio} \\ 1 : \text{determines radius of convergence} \end{cases}$$

(b) The first four nonzero terms of the Maclaurin series for f' are $1 - 3x + 9x^2 - 27x^3$.

$$f'(x) = \frac{1}{1 - (-3x)} = \frac{1}{1 + 3x}$$

$$3 : \begin{cases} 2 : \text{first four nonzero terms} \\ 1 : \text{rational function} \end{cases}$$

(c) The first four nonzero terms of the Maclaurin series for e^x are $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$.

The product of the Maclaurin series for e^x and the Maclaurin series for f is

$$\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(x - \frac{3}{2}x^2 + 3x^3 - \cdots\right)$$
$$= x - \frac{1}{2}x^2 + 2x^3 + \cdots$$

The third-degree Taylor polynomial for $g(x) = e^x f(x)$ about $x = 0$ is $T_3(x) = x - \frac{1}{2}x^2 + 2x^3$.

$$3 : \begin{cases} 1 : \text{first four nonzero terms} \\ \quad\ \text{of the Maclaurin series for } e^x \\ 2 : \text{Taylor polynomial} \end{cases}$$

6A
1 of 2

NO CALCULATOR ALLOWED

6. The Maclaurin series for a function f is given by $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{n} x^n = x - \frac{3}{2}x^2 + 3x^3 - \cdots + \frac{(-3)^{n-1}}{n} x^n + \cdots$ and converges to $f(x)$ for $|x| < R$, where R is the radius of convergence of the Maclaurin series.

(a) Use the ratio test to find R.

$$\lim_{n\to\infty} \left| \frac{(-3)^n x^{n+1}}{n+1} \cdot \frac{n}{(-3)^{n-1} x^n} \right|$$

$$= \lim_{n\to\infty} |-3x| = \lim_{n\to\infty} |3x| < 1$$

$$|x| < \frac{1}{3}$$

$$R = \frac{1}{3}$$

Do not write beyond this border.

Continue problem 6 on page 21.

(b) Write the first four nonzero terms of the Maclaurin series for f', the derivative of f. Express f' as a rational function for $|x| < R$.

$f = x - \frac{3}{2}x^2 + 3x^3 - \frac{3^3}{4}x^4$

$f' = 1 - 3x + 9x^2 - 27x^3$

$a = 1 \quad r = -3x$

$f' = \frac{1}{1-(-3x)} = \frac{1}{1+3x}$ for $|x| < \frac{1}{3}$

(c) Write the first four nonzero terms of the Maclaurin series for e^x. Use the Maclaurin series for e^x to write the third-degree Taylor polynomial for $g(x) = e^x f(x)$ about $x = 0$.

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$g(x) = e^x f(x) \approx (1 + x + \frac{x^2}{2!} + \frac{x^3}{3!})(x - \frac{3}{2}x^2 + 3x^3)$

$= x - \frac{3}{2}x^2 + 3x^3 + x^2 - \frac{3}{2}x^3 + \cancel{3x^4} + \frac{x^3}{2!} - \cancel{\frac{3}{4}x^4} + \cancel{\frac{3}{2}x^5}$

$-\frac{3}{2} + \frac{2}{2} = -\frac{1}{2}$

$= x - \frac{3}{2}x^2 + x^2 + 3x^3 - \frac{3}{2}x^3 + \frac{1}{2}x^3$

$= x - \frac{1}{2}x^2 + 2x^3$

6. The Maclaurin series for a function f is given by $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{n} x^n = x - \frac{3}{2}x^2 + 3x^3 - \cdots + \frac{(-3)^{n-1}}{n}x^n + \cdots$ and converges to $f(x)$ for $|x| < R$, where R is the radius of convergence of the Maclaurin series.

(a) Use the ratio test to find R.

$$\lim_{n\to\infty} \left| \frac{\frac{(-3)^{(n+1)-1}}{n+1} x^{n+1}}{\frac{(-3)^{n-1}}{n} x^n} \right| = \lim_{n\to\infty} \left| \frac{n}{n+1} \cdot \frac{(-3)^n}{(-3)^{n-1}} \cdot \frac{x^{n+1}}{x^n} \right|$$

$$= \lim_{n\to\infty} \left| -3x \right| = |-3x| < 1$$

$$-\frac{1}{3} < x < \frac{1}{3}$$

$$\boxed{R = \frac{1}{3}}$$

Continue problem 6 on page 21.

6B

2 of 2

(b) Write the first four nonzero terms of the Maclaurin series for f', the derivative of f. Express f' as a rational function for $|x| < R$.

$$f'(x) = \frac{d}{dx} \sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{n} x^n = \sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{n} \cdot n x^{n-1} = \boxed{\sum_{n=1}^{\infty} (-3)^{n-1} x^{n-1}} = f'(x)$$

$$f'(x) \approx \boxed{1 + (-3)x + (-3)^2 x^2 + (-3)^3 x^3}$$

(c) Write the first four nonzero terms of the Maclaurin series for e^x. Use the Maclaurin series for e^x to write the third-degree Taylor polynomial for $g(x) = e^x f(x)$ about $x = 0$.

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \approx \boxed{1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3}$$

$$g(x) = e^x f(x)$$

$$g(0) = e^0 f(0) = f(0) = 0 \Rightarrow \boxed{g(0) = 0}$$

$$\sum_{n=0}^{\infty} \frac{g^n(c)}{n!}(x-c)^n$$

GO ON TO THE NEXT PAGE.

6. The Maclaurin series for a function f is given by $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{n} x^n = x - \frac{3}{2}x^2 + 3x^3 - \cdots + \frac{(-3)^{n-1}}{n} x^n + \cdots$ and converges to $f(x)$ for $|x| < R$, where R is the radius of convergence of the Maclaurin series.

(a) Use the ratio test to find R.

$$\frac{(-3)^n x^{n+1}}{n+1} \cdot \frac{n}{(-3)^{n-1} x^n} = (-3)(1)x$$

$-3x < 1$

$0 < x < -\frac{1}{3}$

$x > 0$
$x < -\frac{1}{3}$

$x < \frac{1}{3}$

R is infinite?

Do not write beyond this border.

Continue problem 6 on page 21.

(b) Write the first four nonzero terms of the Maclaurin series for f', the derivative of f. Express f' as a rational function for $|x| < R$.

$$1 - 3x + 9x^2 - \frac{27x^3}{4}$$

(c) Write the first four nonzero terms of the Maclaurin series for e^x. Use the Maclaurin series for e^x to write the third-degree Taylor polynomial for $g(x) = e^x f(x)$ about $x = 0$.

$$1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3$$

$$x - \frac{3}{2}x^3$$

Question 6

Overview

In this problem students were presented with the Maclaurin series $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{n} x^n = x - \frac{3}{2}x^2 + 3x^3 - \cdots + \frac{(-3)^{n-1}}{n}x^n + \cdots$ for a function *f*. The Maclaurin series converges to $f(x)$ for $|x| < R$, where R is the radius of convergence of the Maclaurin series. In part (a) students were asked to use the ratio test to find R. Students were expected to evaluate $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$ and use this limit to find R. Students were expected to show that $|x| < \frac{1}{3}$, and thus the radius of convergence is $R = \frac{1}{3}$. In part (b) students were asked to write the first four nonzero terms of the Maclaurin series for f', then express f' as a rational function for $|x| < R$. By using term-by-term differentiation, the first four nonzero terms are $1 - 3x + 9x^2 - 27x^3$. Because this series is geometric with a common ratio of $-3x$, the rational function is $f'(x) = \frac{1}{1+3x}$. In part (c) students needed to write the first four nonzero terms of the Maclaurin series for e^x and use this series to write a third-degree Taylor polynomial for $g(x) = e^x f(x)$ about $x = 0$. After showing that the first four nonzero terms of the Maclaurin series for e^x are $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$, students were expected to multiply to determine that the third-degree Taylor polynomial desired is $T_3(x) = x - \frac{1}{2}x^2 + 2x^3$.

Sample: 6A
Score: 9

The response earned all 9 points.

Sample: 6B
Score: 6

The response earned 6 points: 3 points in part (a), 2 points in part (b), and 1 point in part (c). In part (a) the student's work is correct. In part (b) the student writes the correct first four nonzero terms of f', so the first 2 points were earned. There is no rational function presented. In part (c) the student writes the correct first four nonzero terms of the Maclaurin series for e^x, so the first point was earned. The student does not present the correct third-degree Taylor polynomial for g.

Sample: 6C
Score: 3

The response earned 3 points: 1 point in part (a), 1 point in part (b), and 1 point in part (c). In part (a) the student writes the correct setup, so the first point was earned. The student does not indicate a limit, so the second point was not earned. The student does not determine a radius of convergence, so the third point was not earned. In part (b) the student writes three of the correct first four nonzero terms of the Maclaurin series for f', so 1 of the first 2 points was earned. There is no rational function presented. In part (c) the student writes the correct first four nonzero terms of the Maclaurin series for e^x, so the first point was earned. The student does not present the correct third-degree Taylor polynomial for g.