The simplest polynomials are those of the *first degree*. Here we discuss some properties, and make the connection between the related concepts *polynomial*, *function* and *equation*.

Definition 1. Let $a, b \in \mathbb{R}$ be $a \neq 0$.

- A **first degree *polynomial* with one unknown** x is an expression of the form

$$ax + b$$

- With this polynomial we can associate a **first degree *function***:

$$x \mapsto ax + b$$

- A **first degree *equation*** (or ***linear*** **equation**) **with one unknown** x is an expression of the form

$$ax + b = 0$$

Remark 1.

- The word *linear* initially means 'like a *line*'. Because a line has an equation of the first degree, the use of the word *linear* has been extended to 'of the first degree'. So for equations, 'linear' and 'first degree' are synonyms. But for functions, the term linear in mathematics has a narrower and more technical meaning. A function is *linear* if for all numbers a and b it holds that $f(ax + by) = af(x) + bf(y)$. In this sense the function $x \mapsto 2x$ is linear, but the function $x \mapsto 2x+1$ is *not* linear. Therefore we avoid the use of the word *linear* for functions and always speak about *first degree functions*. However, some textbooks have different conventions.

- In the expression $ax+b$, x is considered to be a *letter* (having the role of *variable* or *unknown*), while a and b are considered to be *random numbers* (or *parameters*). Instead of x, the letters y, z or t are sometimes used, and for a and b, the letters $k, l, p, q, m, \alpha, \beta, \ldots$ are also common.

  So for functions you should always clearly indicate *which letter* has the role of variable or unknown, and which letters are used as parameters. The notation $x \mapsto ax + b$ says exactly that x is the variable, while z is the variable in $z \mapsto pz + q$, and a is the variable in $a \mapsto 2a + 3$.

Example 1.

- $7x + 9$ is a first degree polynomial (in x).
- $x \mapsto 7x + 9$ is a first degree function (with variable x).
- $7x + 9 = 0$ is a first-degree equation (with one unknown x).

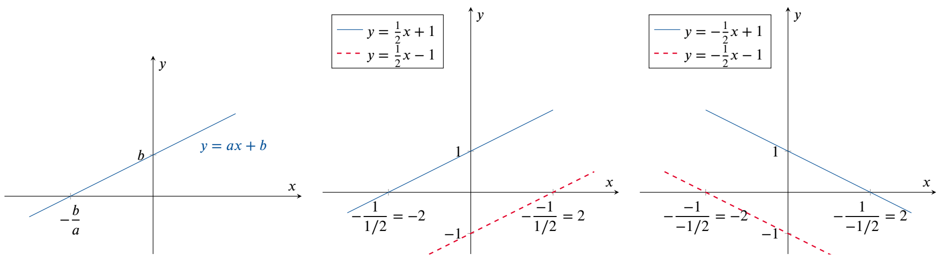
Proposition 1. Let $a, b \in \mathbb{R}$ be $a \neq 0$. Then for the first degree function $x \mapsto ax + b$:

- the graph of the function is a *straight line* with equation $y = ax + b$ and **slope** a.
- if a is positive, the function is strictly *increasing* and if a is negative the function is strictly *decreasing*.
- the (only) intersection point of the graph with the y axis is the point $(0, b)$.
- the (only) intersection point of the graph with the x axis is the point $(-\frac{b}{a}, 0)$. The number $-\frac{b}{a}$

is also the (only) *zero point* of the function, and the (only) solution of the equation $ax+b=0$.

We summarize this information in the following **graphs** (also *sign analysis table*) for the function $x \mapsto ax+b$

x		$-b/a$	
$ax+b$ where $a>0$	$-$	0	$+$
$ax+b$ where $a<0$	$+$	0	$-$



Remark 2.

- We usually speak of *zero points* of a function, *solutions* of an equation and *roots* of polynomials, but these terms mean the same thing.
- If $a=0$, the function reduces to $x \mapsto 0x+b=b$, which is not a first degree function but a *constant* function. The graph is then a *horizontal* line with equation $y=b$. This function has *no zero points* (and therefore no intersection points with the x-axis), unless $b=0$. In the latter case there are *infinitely many zero points* (and therefore intersection points), because then the function is the zero function $x \mapsto 0$.

Example 2.

1. Find out for which $x \in \mathbb{R}$, $8-4x<0$.

Elaboration: The zero point is 2. The coefficient $a=-4$ is negative, so the sign analysis table is

x		2	
$8-4x$	$+$	0	$-$

So the solution set is $V=]2,+\infty[$.

Definition 2. Let $a,b,c \in \mathbb{R}$ where $a \neq 0$ or $b \neq 0$ (so a and b cannot both be zero).
A **first degree equation** (or **linear equation**) **with two unknowns** x,y is an expression of the form

$$ax+by+c=0$$

Proposition 2.
The set of all points (x,y) in the plane that satisfy the equation $ax+by+c=0$ forms a line:

- if $b=0$, the equation becomes $ax+c=0$, from which it follows that $x=-\dfrac{c}{a}$. Note that there is no condition on y, y can take all possible values. Consequently, all points (x,y) in the plane that satisfy this equation form the vertical line $x=-\dfrac{c}{a}$.

- If $b \neq 0$, we can also write the equation as

$$y = -\frac{1}{b}(ax + c) \text{ or } y = -\frac{a}{b}x - \frac{c}{b}$$

We already noticed that the graph of a real function $x \mapsto mx + q$ is the same as the solution set of the equation $y = mx + q$.

So all points (x, y) in the plane that satisfy the equation $y = -\frac{a}{b}x - \frac{c}{b}$ form a line with gradient $-\frac{a}{b}$ and intersection with the y-axis $-\frac{c}{b}$.

Remark 3. We always looked at only *one* linear equation. Solving multiple equations *simultaneously* (with possibly more than two unknowns) is called *systems of linear equations* and belongs to the so-called *linear algebra*.

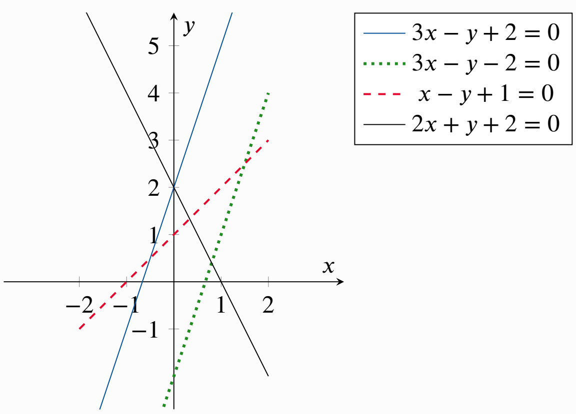
Excursus 1 (Equations, unknowns, functions and intersections)**.**
We can look at the relationship between a number of concepts and problems studied further in mathematics in this very simple case. In general, there is a strong relationship between:

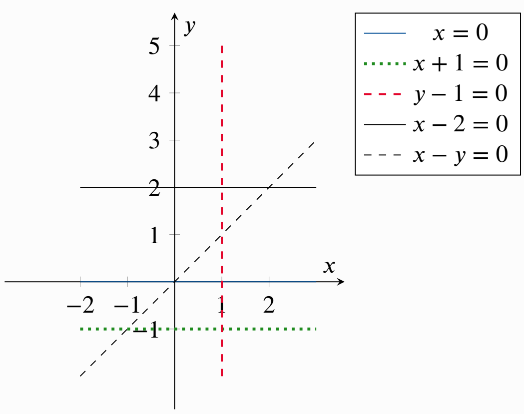
- finding the zero points of functions
- solving equations
- the intersection geometrical figures
- the *unknowns* in equations versus the *variables* in functions

Consider expressions of the form

$$ax + by + c = 0.$$

Such an expression can be regarded as an *equation with the two unknowns x and y*: it is then a condition which a pair of real numbers (x_0, y_0) can satisfy or not, and the *solution set* of the equation, i.e. the set of all pairs (x_0, y_0) so that $ax_0 + by_0 + c = 0$, form a *straight in the plane*. Examples are





We can make the following considerations about this equation $ax + by + c = 0$:

- $ax + by + c = 0$ is the general equation of a *line*, and we can find the **intersection points** of that line with the x-axis (the x-axis is the line with equation $y = 0$). Algebraically, that comes down to **solving the system of equations**

$$\begin{cases} ax + by + c = 0 \\ y = 0 \end{cases}$$

This system is equivalent to

$$\begin{cases} ax + c = 0 \\ y = 0 \end{cases}$$

If $a \neq 0$ then there is a unique solution, namely $x = -\frac{c}{a}$ and $y = 0$. If $a = 0$ and $c \neq 0$, then there are no solutions, and if $a = 0$ and also $c = 0$, then the system is equivalent to

$$\begin{cases} 0x + 1y = 0 \end{cases}$$

and that set has infinitely many solutions (the solution set is $\{(x, 0) \mid x \in \mathbb{R}\}$).

- If $b \neq 0$ then we can solve the equation for y , and write $y = -\frac{a}{b}x - \frac{c}{b}$. The solutions of this equation are the points of the **graph of the first degree function** (in one variable x):

$$x \mapsto -\frac{1}{b}(ax + c)$$

Thus we have rewritten the equation of these lines in the form

$$y = mx + q, \text{ where } m, q \in \mathbb{R}$$

- Solving a linear equation in one variable $ax + b = 0$ thus corresponds to finding the **zero points of the function** $x \mapsto ax + b$, and also to finding the **intersection points** of the line $y = ax + b$ with the x-axis.
- Every equation $y = mx + q$ can be written in the form $ax + by + c = 0$, but not every equation of the form $ax + by + c = 0$ can be written as $y = mx + q$. This is almost always possible, namely when $b \neq 0$. The case $b = 0$ corresponds exactly to the vertical lines, and these are also exactly the lines that are not the graph of a function.
- An equation of the form $ax + by + c = d$ is also linear, but yields nothing new, because it can be rewritten as $ax + by + (c - d) = 0$.
- Linear equations with three unknowns represent planes in space, and linear equations in n unknowns represent so-called *hyperplanes* in n-dimensional space. *Systems* of such equations are a central focus of study in *linear algebra*.