DUALITY AND STRUCTURE OF LOCALLY COMPACT ABELIAN GROUPS ...... FOR THE LAYMAN*

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1. Introduction

It is well-known that any finitely generated abelian group can be expressed as a direct product of cyclic groups. However, the equally attractive generalization of this to topological groups is known only to a small group of specialists. This is a real pity as the theory is not only elegant, but also a very pleasant mixture of algebra and topology.

Our aim is to describe the principal structure theorem for locally compact abelian groups, and to acquaint the reader with the Pontryagin - van Kampen duality theorem. The duality theorem is a deep result and the usual discussions assume a knowledge of measure theory and Banach algebras. In order to make the material accessible to as large an audience as possible we make no such assumption. Rather our presentation here will be uncluttered by analysis.

2. History

The concept of a topological group has its roots in the work of Felix Klein (1849-1925) and Marius Sophus Lie (1842-1899). In a visit to Paris in 1870, Klein met Lie who had become interested in mathematics only a short time before. The young men were much influenced by the

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work of the French mathematicians who included in their number Camille Jordan. Jordan had just written his treatise on substitution groups and Galois' theory of equations. Klein and Lie began to see the importance of group theory. In 1872 Klein became professor at Erlangen and declared in his inaugural address that one can classify geometries according to properties left invariant under groups of transformations. The study of any classical geometry such as Euclidean geometry, affine geometry, projective geometry, etc. may be regarded as an investigation of a particular transformation group.

With such transformation groups in mind Lie conceived the concept of continuous groups of transformations of manifolds. Of course Lie took differentiability for granted. So while Klein, as a rule concentrated on discontinuous transformation groups, Lie devoted his whole life to the systematic study of continuous transformation groups and their invariants. He demonstrated their central importance as a classifying principle in geometry, mechanics and ordinary and partial differential equations.

In 1900 David Hilbert presented to the International Congress of Mathematicians in Paris a series of twenty-three research projects. It is worth quoting a translation of Hilbert's fifth problem:

> "It is well-known that Lie with the aid of the concept of continuous groups of transformations, had set up a system of geometrical axioms and, from the standpoint of his theory of groups has proved that this system of axioms suffices for geometry. But since Lie assumes, in the very foundation of his theory, that the functions defining his group can be differentiated, it remains undecided in Lie's development, whether the assumption of the differentiability in connection with the question as to the axioms of geometry is actually unavoidable, or whether it may not appear rather as a

consequence of the group concept and the other geometrical axioms. This consideration, as well as certain other problems in connection with the arithmetical axioms bring before us the more general question:

How far Lie's concept of continuous groups of transformations is approachable in our investigations without the assumption of the differentiability of the functions."

Hilbert's question inspired many important works before it was answered by Deane Montgomery, Leo Zippin and Andrew Gleason in 1952.

Lie's notion of continuous group fell short of our current notion of topological group not only because he made a differentiability assumption but also because he did not consider abstract groups but rather groups of differentiable transformations of a manifold into itself. Until the 1880's groups were generally thought of in terms of permutations or substitutions or in connection with residues and number theory.

In 1854 Arthur Cayley began to publish articles explicitly devoted to the theory of abstract groups. In the first of these he gave a definition of a group: "A set of symbols 1, α, β, ... all of them different and such that the product of any two of them (no matter in what order), or the product of any one of them into itself, belongs to the set, is said to be a group. It follows that if the entire group is multiplied by any one of the symbols, either as a further or nearer factor the effect is simply to reproduce the group." In 1878 he wrote again on finite abstract groups and stressed that a group can be considered as a general concept and need not be limited to substitution groups, though he does point out that every (finite) group can be represented as a substitution group. With the work of Walther Dyck in 1882 and 1883 the three main roots of group theory - the theory of equations, number theory and infinite transformation groups - were all subsumed under the abstract group concept. Dyck, who was a student of Felix Klein, was influenced by Cayley. Dyck's definition of a group called for a set of elements and an operation that satisfy the closure property, the associative but not the commutative property, and the existence of an inverse element of each element.

General topology as it is understood today began with Felix Hausdorff who in 1914 introduced the notion of neighbourhood spaces. In 1922 Kazimierz Kuratowski defined the notion topological space by means of a closure operation. Alexandroff, Tychonoff, Urysohn and many others then developed a general theory of topological spaces.

The modern concept of topological group was first introduced in 1927 by Frantischek Leja. Two years earlier Otto Schreier had given axioms for and studied groups that are "Frechet limit spaces" in which the group operations are continuous. Another axiomatic treatment inequivalent to Leja's was published by Reinhold Baer in 1929. In 1930 locally Euclidean groups were discussed per se by Elie Cartan. In 1931 David van Dantzig published his doctoral dissertation "Studien over topologische algebra" and in a series of papers investigated the properties of topological groups, rings and fields. Included in this work is the beautiful theorem proved independently by Lev Pontryagin in 1931 which states that up to topological isomorphism there are only two non-discrete locally compact fields - the field of real numbers and the field of complex numbers.

By the early 1930's many mathematicians were working with topological groups.

The work with which we are concerned is the creation of L.S. Pontryagin and E.R. van Kampen. In 1934 Pontryagin announced and proved the duality theorem for the case of compact abelian groups having a countable basis. In the same year J.W. Alexander dealt with the case of arbitrary discrete groups. In the following year E.R. van Kampen was able to complete the proof of the duality theorem for all locally compact abelian groups.

3. Definition of a Topological Group

A set G which is endowed with the structure of a group and a topological space is said to be a *topological group* if both of the maps

$$\begin{array}{ccc} G \longrightarrow G & \quad\text{and}\quad & G \times G \longrightarrow G \\ x \longrightarrow x^{-1} & & (x,y) \longrightarrow xy \end{array}$$

where $G \times G$ has the product topology, are continuous.

We mention three important examples:

(a) The additive group of real numbers with the usual topology is a topological group, denoted by R .

(b) The multiplicative group of all complex numbers of modulus one with the topology induced from the complex plane is called the circle group, and is denoted by T .

(c) If G is any group then by putting the discrete topology on G (i.e. every subset of G becomes an open set) we obtain a topological group. In the particular case that G is the group of integers, then G with the discrete topology is denoted by Z .

Let G be any topological group and g an arbitrary element of G . It is easily verified that a set U is a neighbourhood of g if and only if $g^{-1}U$ is a neighbourhood of the identity element, e , of G . Thus every question about the topology of G can be reduced to a question about the neighbourhoods of e .

4. LCA-groups

The most important class of topological groups is the class of locally compact Hausdorff groups.

A topological space X is said to be *locally compact* if each element of X has a compact neighbourhood. Clearly, a topological group is locally compact if and only if it has a compact neighbourhood of the identity.

A topological space X is said to be *Hausdorff* if each pair of distinct elements of X can be separated by open sets. In particular, a topological group is Hausdorff if and only if each point is a closed set. Thus a topological group is Hausdorff if and only if the set consisting of just the identity element is closed.

Each of the topological groups mentioned in 3 is locally compact and Hausdorff. For a compact neighbourhood of the identity in R we can choose the closed unit interval $[-1,1]$. In any discrete group the set $\{e\}$ is a compact neighbourhood of the identity element, e. The group T is, in fact, compact and so the set T is a compact neighbourhood of the identity.

We shall further restrict our attention to abelian groups. All groups will be written additively in future. For brevity, we introduce the shorthand:

LCA-group = *locally compact Hausdorff abelian topological group.*

5. The Dual Group of an Abelian Topological Group

We will denote by G^* the set of continuous homomorphisms from an abelian topological group G into the circle group T. Such continuous homomorphisms are called *characters* and G^* is called the *character group* or *dual group* of G.

The set G^* is an abelian group under the operation

$$(\phi + \psi)(g) = \phi(g) + \psi(g),\ g \in G,\ \phi \in G^*,\ \psi \in G^* .$$

We put a topology on G^* as follows:

Let K be a compact subset of G and U an open subset of T. Then the set $P(K,U) \subseteq G^*$ is defined to be

$$\{\phi : \phi \in G^* \text{ and } \phi(K) \subseteq U\} .$$

We define the sets $P(K,U)$ to be open subsets of G^* and, indeed, to be a subbasis for the topology of G^* ; that is, every open set is a union of sets of the form $P(K_1,U_1) \cap P(K_2,U_2) \cap \ldots \cap P(K_n,U_n)$, where each K_i is compact in G and each U_i is open in T. This is the well-known compact-open topology.

It is routine to verify that, with this topology, G^* is a Hausdorff topological group.

It is an interesting task to find the dual groups of R, T and Z. It is clear that for each $d \in R$ we can define a character γ_d of R by $\gamma_d(x) = exp(2\pi idx)$, for $x \in R$. It is not so easy to see that these are the only characters of R.

But once we know this and observe that $\gamma_{d_1} + \gamma_{d_2} = \gamma_{d_1+d_2}$ we have that the dual group of R is algebraically isomorphic to R itself under the isomorphism $d \longrightarrow \gamma_d$. Further examination reveals that R^* is indeed topologically isomorphic to R. (Topological groups G and H are said to be *topologically isomorphic* if there is a map $f : G \longrightarrow H$ which is both an algebraic isomorphism and a homeomorphism of G onto H).

Turning to T, we see that for each integer d a character is obtained by putting $\gamma(exp(2\pi ix)) = exp(2\pi idx)$. Once again these are the only characters, and so T^* is seen to be algebraically isomorphic to Z. It is not difficult to verify that T^* has the discrete topology and so T^* is topologically isomorphic to Z.

Considering the group Z, we see that any character γ is determined by its value on 1, as $\gamma(n) = n\gamma(1)$, $n \in Z$. (Note that

T is being written additively!) Of course $\gamma(1)$ can be any element of T, and for each $a \in T$ we obtain a character γ_a by putting $\gamma(1) = a$. Thus the mapping $a \to \gamma_a$ is an algebraic isomorphism of T onto Z^*. Further analysis shows that Z^* is topologically isomorphic to T.

It is also worth mentioning that if $A_1, A_2, \ldots, A_n$ are abelian topological groups then the dual group of the product topological group $A_1 \times A_2 \times \ldots \times A_n$ (that is, the product of the abstract groups $A_1, \ldots, A_n$ with the product topology) is the product of the dual groups. In other words, $(A_1 \times A_2 \times \ldots \times A_n)^*$ is topologically isomorphic to $A_1^* \times A_2^* \times \ldots \times A_n^*$.

We can infer from this that $(R^n)^*$, $(T^n)^*$ and $(Z^n)^*$ are topologically isomorphic to R^n, Z^n and T^n, respectively, for each positive integer n.

6. Dual Groups of Compact Groups and Discrete Groups

Those readers familiar with the compact-open topology will recall Ascoli's theorem which allows one to recognize sets which are compact in the compact-open topology. If we apply this theorem to dual groups we obtain the following key proposition.

For convenience, the identity elements of all groups will be denoted by 0.

Proposition 1. *Let G be an LCA-group, G^* its dual group and K any compact neighbourhood of 0 in G. If U is a "small" neighbourhood of 0 in T, more precisely if $U \subseteq \{exp(2\pi ix) : -\frac{1}{4} < x < \frac{1}{4}\}$, then $\overline{P(K,U)}$, the closure of the set $P(K,U)$, is a compact neighbourhood of 0 in G^*.*

Remark. Note the three crucial requirements of Proposition 1: (a) G be an LCA-group; (b) K be a compact neighbourhood of 0, rather than just a compact subset of G ; and (c) U be a "small" neighbourhood of 0 .

Corollary 1. *If G is an LCA-group, then G^* is an LCA-group.*

Proof. We have already commented that G^* is an abelian Hausdorff topological group. Proposition 1 says that G^* has a compact neighbourhood of 0 and consequently is also locally compact.

Corollary 2. *If G is an abelian discrete topological group, then G^* is a compact group.*

Proof. As G is discrete the set $K = \{0\}$ is a neighbourhood of 0 . Of course K is also compact, since it is finite. Let U be a "small" neighbourhood of 0 in T . Then by Proposition 1, the set $\overline{P(K,U)}$ is compact in G^* . But every homomorphism from G into T maps $K = \{0\}$ into U . Thus $P(K,U) = G^*$. Hence $G^* = \overline{P(K,U)}$, and so G^* is compact.

The converse of Corollary 2 is also true and easily proved.

Proposition 2. *If G is a compact Hausdorff abelian topological group, then G^* is discrete.*

Proof. As G is compact, the definition of the topology of G^* tells us that $P(G,U)$ is an open neighbourhood of 0 in G^* , for any neighbourhood U of 0 in T . If we choose U to be a "small' neighbourhood of 0 in T , then U contains no subgroup other than $\{0\}$. Thus any homomorphism which maps all of G into U , must map G onto 0 . Of course the only homomorphism which maps G onto 0 is the trivial homomorphism and so $P(G,U) = \{0\}$. So $\{0\}$ is an open subset of G^* . Hence G^* is discrete.

7. The Pontryagin - van Kampen Duality Theorem

We now state one of the most important theorems in mathematics. Naturally, we cannot include a proof here.

Theorem 1. (Pontryagin - van Kampen). *Let G be any LCA-group, G^* its dual group, and G^{**} the dual group of G^*. For fixed $g \in G$, let g' be the function : $G^* \longrightarrow T$ given by $g'(\gamma) = \gamma(g)$, for all $\gamma \in G^*$. If α is the mapping given by $\alpha(g) = g'$, then α is a topological isomorphism of G onto G^{**}.*

Roughly speaking the duality theorem says that every LCA-group is the dual group of its dual group. From this we deduce that every piece of information about G is stored as information about G^*. In the case of compact groups this is particularly interesting as G^* is discrete. So any compact Hausdorff abelian group can be completely described by the purely algebraic properties of its dual group; for example, if G is a compact Hausdorff abelian group then

(a) G is metrizable if and only if G^* is countable.

(b) G is connected if and only if G^* is torsion-free.

(c) The dimension of G as a topological space equals the torsion-free rank of G^* (that is, the number of elements in a maximal linearly independent subset of G^*).

We include a proof of result (a). However, the less patient reader can omit this proof without affecting his understanding of the later material.

Proposition 3. *Let G be a compact Hausdorff abelian topological group. Then G is metrizable if and only if G^* is countable.*

Proof. We will assume the standard result that a topological group is metrizable if and only if there is a countable base (or subbase) of neighbourhoods of 0.

Firstly, assume that G is metrizable. Then G has a countable base of compact neighbourhoods $K_1, K_2, \ldots, K_n, \ldots$ of 0. By Proposition 1, if U is a "small' neighbourhood of 0 in T, then the sets $\overline{P(K_i,U)}$ are compact neighbourhoods of 0 in G^*. As each $\gamma \in G^*$ is continuous, there exists a K_i such that $\gamma(K_i) \subsetneq U$. So $\gamma \in P(K_i,U)$.

Hence $G^* = \bigcup_{i=1}^{\infty} P(K_i,U)$ and so $G^* = \bigcup_{i=1}^{\infty} \overline{P(K_i,U)}$. But as G^* is discrete, each of the compact sets $\overline{P(K_i,U)}$ is finite. So G^* is a countable union of finite sets and so is countable.

Conversely, assume that G^* is countable.
For each positive integer n, let $U_n = \{exp(2\pi ix) \in T : -\frac{1}{n} < x < \frac{1}{n}\}$. Then each U_n is an open neighbourhood of 0 in T, and so the sets $P(K,U_n)$ are open in G^{**}, for any compact subset K of G^*. Indeed, it is easily verified that if we allow K to range over all compact subsets of G^* and n to range over all natural numbers then the sets $P(K,U_n)$ form a subbase of neighbourhoods of 0 in G^{**}. As G is compact, G^* is discrete and so each compact subset K of G^* is finite. Since G^* is countable, G^* has only a countable number of finite subsets. Thus there are only a countable number of sets $P(K,U_n)$ in the subbase of neighbourhoods of 0 in G^{**}.
Hence G^{**} is metrizable. The duality theorem tells us that G is topologically isomorphic to G^{**}, and so G too is metrizable.

8. Quotient Groups and Local Isomorphisms

Let N be a normal subgroup of a topological group G. If the quotient group G/N is given the quotient topology under the canonical homomorphism $p : G \longrightarrow G/N$ (that is, U is open in G/N if and only if $p^{-1}(U)$ is open in G), then G/N becomes a topological group - called the *quotient (topological) group* of G by N.

For example, we can form the quotient group R/Z. It is not difficult to verify that the topological group R/Z is topologically isomorphic to the circle group T.

Two topological groups G and H are said to be *locally isomorphic* if there exist neighbourhoods V of 0 in G and U of 0 in H, and a homeomorphism f of V onto U such that if x,y and $x + y$ are in V, then $f(x+y) = f(x) + f(y)$.

Example. Let $G = R$, $H = T$, $V = (-\frac{1}{4}, \frac{1}{4})$ and $U = \{exp(2\pi ix) : -\frac{1}{4} < x < \frac{1}{4}\}$. Then the map $f : V \longrightarrow U$ with $f(x) = exp(2\pi ix)$, $x \in V$ has the required property. So R and T are locally isomorphic.

With a little more effort one can prove

Proposition 4. *If D is a discrete normal subgroup of a topological group G, then the quotient group G/D is locally isomorphic to G.*

In passing, we mention that local isomorphisms play a central role in the theory of Lie groups, as two Lie groups have the same Lie algebra if and only if they are locally isomorphic. (Roughly speaking, a topological group is said to be a *Lie group* if the largest connected set containing the identity is an open set, and it has the additional structure of an analytic manifold with the operations $(x,y) \longrightarrow xy$ and $x \longrightarrow x^{-1}$ being analytic. Examples of Lie groups which we have met are R^n, T^n, for all $n \geq 1$, and discrete groups.)

In developing the structure theory of locally compact abelian groups, we shall need to know precisely what topological groups are locally isomorphic to R^n.

Proposition 5. *Let G be a Hausdorff abelian topological group which is locally isomorphic to R^n. Then G is topologically isomorphic to $R^a \times T^b \times D$, where D is a discrete group and a and b are non-negative integers with $a + b = n$.*

9. The Principal Structure Theorem

We shall write $0 \to A \to G \to B \to 0$ to mean that A is a normal topological subgroup of G such that G/A is topologically isomorphic to B .

Proposition 6. *Corresponding to the sequence*

$$0 \longrightarrow A \longrightarrow G \longrightarrow B \longrightarrow 0$$

where A, G *and* B *are LCA-groups, there is a sequence*

$$0 \longleftarrow A^* \longleftarrow G^* \longleftarrow B^* \longleftarrow 0$$

The idea of the proof of Proposition 6 is quite simple.

Given $A \xrightarrow{f} G$ we want a map $f^* : G^* \longrightarrow A^*$. Let $\phi \in G^*$. So we have the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "f^*(\phi)"'] & G \arrow[d, "\phi"] \\ & T \end{tikzcd}$$

If we put $f^*(\phi) = \phi f$, we have the required map.

A topological group G is said to be *compactly generated* if it has a compact subset X such that X generates G algebraically.

R, Z and T are examples of compactly generated groups, while a discrete group is compactly generated if and only if it is finitely countable.

Observing that the subgroup of an LCA-group generated by any compact neighbourhood of the identity is open, we obtain

Proposition 7. *Every LCA-group has an open compactly generated subgroup* H . *(As every open subgroup is also closed,* H *is also an LCA-group).*

The trivial Proposition 7 allows us to focus our attention on compactly generated LCA-groups. The next proposition takes a little effort to prove.

Proposition 8. *If* G *is a compactly generated LCA-group, then there is a sequence*

$$0 \longrightarrow Z^n \longrightarrow G \longrightarrow K \longrightarrow 0$$

where K *is compact, and* n *is a non-negative integer.*

With this proposition we can prove the Principal Structure Theorem for LCA-groups.

Theorem 2. **(Principal Structure Theorem).** *If* G *is any LCA-group, then* G *has an open subgroup topologically isomorphic to* $R^a \times Z^b \times C$, *where* C *is a compact group and* a *and* b *are non-negative integers.*

Proof. By Proposition 7, G has an open compactly generated subgroup H , which is an LCA-group. By Proposition 8, there is a sequence

$$0 \longrightarrow Z^n \longrightarrow H \longrightarrow K \longrightarrow 0 \text{ , } K \text{ compact.}$$

Dualizing, Proposition 6 yields

$$0 \longleftarrow (Z^n)^* \longleftarrow H^* \longleftarrow K^* \longleftarrow 0$$

that is,

$$0 \longleftarrow T^n \longleftarrow H^* \longleftarrow K^* \longleftarrow 0$$

So T^n is topologically isomorphic to H^*/K^* , where K^* , being the dual of a compact group, is discrete. Proposition 4 then says that T^n is locally isomorphic to H^* . But T^n is locally isomorphic to R^n , so

H^* is locally isomorphic to R^n. Proposition 5 then says that H^* is topologically isomorphic to $R^a \times T^b \times D$, where D is discrete and a and b are non-negative integers. Thus H^{**} is topologically isomorphic to $R^a \times Z^b \times C$ where $C = D^*$ is compact. By the duality theorem, H is topologically isomorphic to H^{**}, so H is topologically isomorphic to $R^a \times Z^b \times C$, as required.

Corollary 1. *Any compactly generated LCA-group is topologically isomorphi to* $R^a \times Z^b \times C$, *where* C *is compact and* a *and* b *are non-negative integers.*

As every connected LCA-group is compactly generated we obtain

Corollary 2. *If* G *is a connected LCA-group,* G *is topologically isomorphic to* $R^a \times C$, *where* C *is compact and connected, and* a *is a non-negative integer.*

Corollary 3. *If* G *is any LCA-group, then* G *is topologically isomorphic to* $R^a \times H$, *where* H *is an LCA-group with a compact open subgroup, and* a *is a non-negative integer.*
Further, if G *is also topologically isomorphic to* $R^b \times H_1$, *where* H_1 *has a compact open subgroup, then* $a = b$.

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