6.3000 Homework #8 Solutions

This homework assignment includes a lab. The lab check-in is due on Friday, October 28, at 5pm. Other parts of the lab are due with the remainder of homework 8 on Tuesday, November 1, at noon.

Solutions to the homework will be distributed shortly after it is due, and submissions will not be accepted after the solutions are posted.

Problems

1. Discrete Fourier Transforms

Using an analysis width $N = 32$, determine the DFTs of each of the following signals.

$$x_1[n] = \cos(\frac{\pi}{4}n)$$

$$\begin{aligned}
X_1[k] &= \frac{1}{32}\sum_{n=0}^{31}\cos\left(\frac{\pi}{4}n\right)e^{-j2\pi kn/32} = \frac{1}{32}\sum_{n=0}^{31}\frac{1}{2}\left(e^{j\pi n/4}+e^{-j\pi n/4}\right)e^{-j2\pi kn/32}\\
&= \frac{1}{64}\sum_{n=0}^{31}e^{-j2\pi(k-4)n/32} + \frac{1}{64}\sum_{n=0}^{31}e^{-j2\pi(k+4)n/32}\\
&= \frac{1}{64}\left(\frac{1-e^{-j2\pi(k-4)}}{1-e^{-j2\pi(k-4)/32}}\right) + \frac{1}{64}\left(\frac{1-e^{-j2\pi(k+4)}}{1-e^{-j2\pi(k+4)/32}}\right)
\end{aligned}$$

The numerator of the first term above is zero for all values of k because $2\pi(k-4)$ is an integer multiple of 2π for all integer values of k. The denominator of the first term is zero when $k-4$ is an integer multiple of 32. Therefore when $k = 4$, the first term is $0/0$, which we can evaluate using L'Hôpital's rule. Similar logic applies to the second term above when $k = 28$. The final answer is

$$X_1[k] = \frac{1}{2}\delta[k-4] + \frac{1}{2}\delta[k-28]$$

An alternative approach to solving this problem is to realize that the signal $x_1[n] = \cos(\pi n/4)$ is a fourth harmonic of the fundamental frequency $2\pi/32$. Therefore we can write $X_1[k]$ as $A\delta[k-4] + B\delta[k-28]$ where A and B are constants. Substituting this expression into the synthesis equation yields

$$\begin{aligned}
x_1[n] &= \sum_{k=0}^{31}X_1[k]e^{j2\pi kn/32} = \sum_{k=0}^{31}\Big(A\delta[k-4]+B\delta[k-28]\Big)e^{j2\pi kn/32}\\
&= Ae^{j2\pi 4n/32} + Be^{2\pi 28n/32} = Ae^{j2\pi 4n/32} + Be^{-2\pi 4n/32}\\
&= A\Big(\cos(\pi n/4) + j\sin(\pi n/4)\Big) + B\Big(\cos(\pi n/4) + j\sin(-\pi n/4)\Big)\\
&= (A+B)\cos(\pi n/4) + j(A-B)\sin(\pi n/4)
\end{aligned}$$

Therefore $A + B = 1$ and $A - B = 0$ yielding $A = B = 1/2$. It follows that

$$X_1[k] = \frac{1}{2}\delta[k-4] + \frac{1}{2}\delta[k-28]$$

$$x_2[n] = \sin(\frac{\pi}{4}n)$$

$$\begin{aligned}
X_2[k] &= \frac{1}{32}\sum_{n=0}^{31}\sin\left(\frac{\pi}{4}n\right)e^{-j2\pi kn/32} = \frac{1}{32}\sum_{n=0}^{31}\frac{1}{2j}\left(e^{j\pi n/4}-e^{-j\pi n/4}\right)e^{-j2\pi kn/32}\\
&= \frac{1}{64j}\sum_{n=0}^{31}e^{-j2\pi(k-4)n/32} - \frac{1}{64j}\sum_{n=0}^{31}e^{-j2\pi(k+4)n/32}\\
&= \frac{1}{64j}\left(\frac{1-e^{-j2\pi(k-4)}}{1-e^{-j2\pi(k-4)/32}}\right) - \frac{1}{64j}\left(\frac{1-e^{-j2\pi(k+4)}}{1-e^{-j2\pi(k+4)/32}}\right)\\
&= -\frac{j}{2}\delta[k-4] + \frac{j}{2}\delta[k-28]
\end{aligned}$$

$$x_3[n] = \cos(\frac{\pi}{8}n)$$

$x_3[n]$ is a cosine at half the frequency of $x_1[n]$:

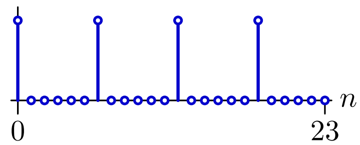
$$X_3[k] = \frac{1}{2}\delta[k-2] + \frac{1}{2}\delta[k-30]$$

$$x_4[n] = \left(\frac{1}{2}\right)^n u[n]$$

$$\begin{aligned}
X_4[k] &= \frac{1}{32}\sum_{n=0}^{31}x_4[n]e^{-j\frac{2\pi k}{32}n}\\
&= \frac{1}{32}\sum_{n=0}^{31}\left(\frac{1}{2}\right)^n e^{-j\frac{2\pi k}{32}n}\\
&= \frac{1}{32}\sum_{n=0}^{31}\left(\frac{1}{2}e^{-j\frac{2\pi k}{32}}\right)^n\\
&= \frac{1}{32}\left(\frac{1-\left(\frac{1}{2}e^{-j\frac{2\pi k}{32}}\right)^{32}}{1-\frac{1}{2}e^{-j\frac{2\pi k}{32}}}\right)\\
&= \frac{1}{32}\left(\frac{1-\left(\frac{1}{2}\right)^{32}}{1-\frac{1}{2}e^{-j\frac{2\pi k}{32}}}\right)
\end{aligned}$$

2. Discrete Fourier Transform Matching

Each of the following plots shows the first 24 samples of a discrete-time signal. Find the plot on the following page that corresponds to the 24-point Discrete Fourier Transform (DFT) for each of these signals. Enter the letter of the plot (A-N) in the box provided.

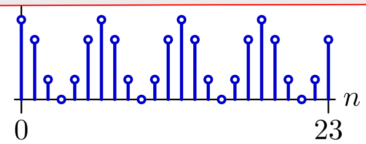


G

$x_1[n] = \delta[n] + \delta[n-6] + \delta[n-12] + \delta[n-18]$

$$X_1[k] = \frac{1}{24}\sum_{n=0}^{23} x[n]e^{-j2\pi kn/24} = 1 + e^{-j2\pi k/4} + e^{-j4\pi k/4} + e^{-j6\pi k/4} = 1 + (-j)^k + (-1)^k + j^k$$

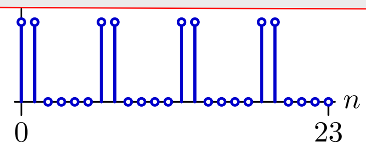
$$= \begin{cases} 1/6 & \text{if } k = 0, 4, 8, 12, 16, 20 \\ 0 & \text{otherwise} \end{cases}$$



I

$x_2[n] = 1 + \cos(2\pi n/6) = 1 + \frac{1}{2}e^{j2\pi n/6} + \frac{1}{2}e^{-j2\pi n/6}$

$$X_2[k] = \delta[k] + \frac{1}{2}\delta[k-4] + \frac{1}{2}\delta[k+4] = \delta[k] + \frac{1}{2}\delta[k-4] + \frac{1}{2}\delta[k-20]$$

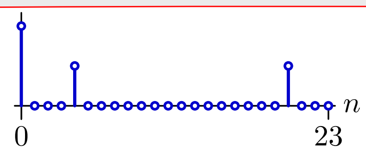


J

$x_3[n] = x_1[n] + x_1[n-1]$

$$X_3[k] = \left(1 + e^{-j2\pi k/24}\right) X_1[k] = 2e^{-j\pi k/24}\cos(\pi k/24)X_1[k]$$

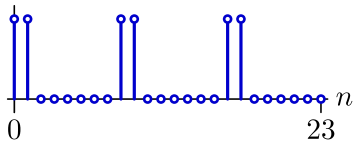
Similar to plot G except components near $k = 12$ are attenuated.



B

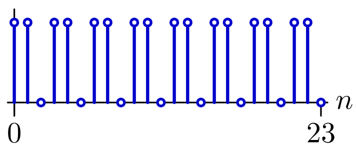
$x_4[n] = \delta[n] + \frac{1}{2}\delta[n-4] + \frac{1}{2}\delta[n-20]$

$$X_4[k] = \frac{1}{24} + \frac{1}{48}e^{-j2\pi k/6} + \frac{1}{48}e^{j2\pi k/6} = \frac{1}{24}\Big(1 + \cos(2\pi k/6)\Big)$$



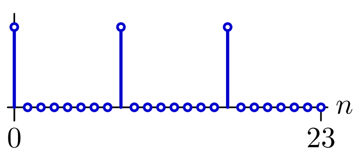
M

x_5 is imilar to x_3 except the period is 8 instead of 6. Therefore X_5 has non-zero components at $k = 0, 3, 6, ...$ and components near $k = 12$ are attenuated.



L

x_6 is similar to x_3 except the period is 3 instead of 6. Therefore X_6 has non-zero components at $k = 0, 8, 16$ and components near $k = 12$ are attenuated.



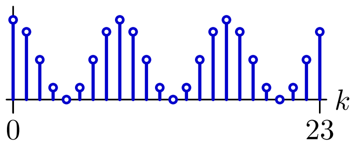
H

$$x_7[n] = \delta[n] + \delta[n-6] + \delta[n-12] + \delta[n-18]$$

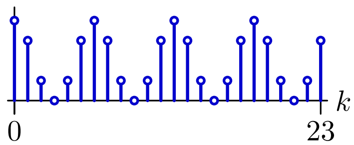
$$X_7[k] = \frac{1}{24}\sum_{n=0}^{23} x[n]e^{-j2\pi kn/24} = 1 + e^{-j2\pi k/3} + e^{-j4\pi k/3}$$
$$= \begin{cases} 1/6 & \text{if } k = 0, 4, 8, 12, 16, 20 \\ 0 & \text{otherwise} \end{cases}$$

Each of the following plots shows the magnitude of a DFT computed with an analysis window $N = 24$. The vertical scale for each plot is different: it has been normalized so that the peak value in each plot is 1.

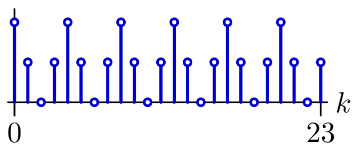
A



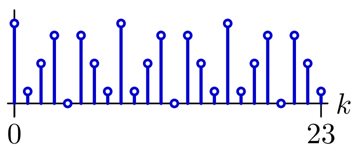
B



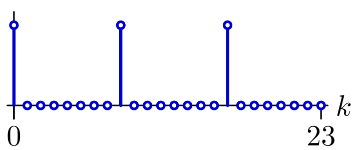
C



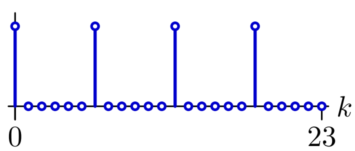
D



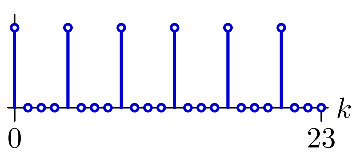
E



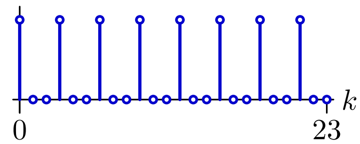
F



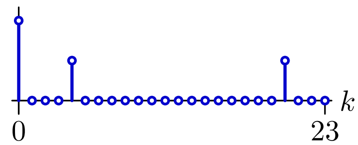
G



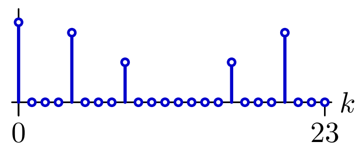
H



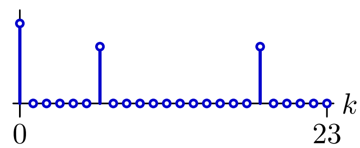
I



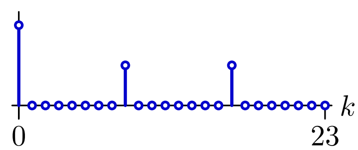
J



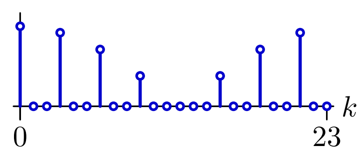
K



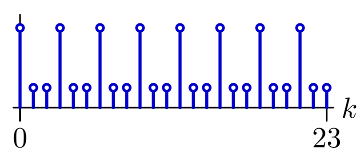
L



M

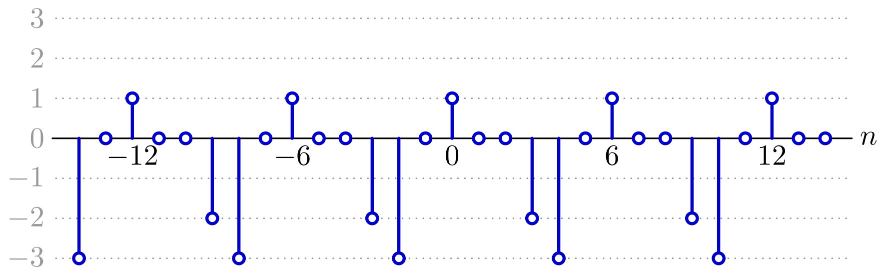


N



3. Signal Changes

Let $x[n]$ represent a periodic, discrete-time signal with a period of 6, a portion of which is shown below:



In this problem, we will explore alternative signals that we can make by changing a small number of samples per period in the signal above. Consider the related signals below (where $x_1[\cdot]$ through $x_5[\cdot]$ represent time-domain signals and $X_1[\cdot]$ through $X_5[\cdot]$ represent the associated Fourier series).

For each, we want to know whether it is possible to create a signal with the given properties by modifying **at most two samples per period** of $x[n]$. If it is possible, specify which value(s) you wish to change, and what you wish to change them to. If it is not possible, put an **X** in each box.

There may be multiple solutions to some of the following parts. You need only find one solution for full credit.

Part a. x_1

We would like to create $x_1[\cdot]$ by modifying **at most two** samples in $x[\cdot]$ so that $X_1[\cdot]$ is a symmetric function of k.

Which index should be changed?	2	What should the value be changed to?	-3
Which index should be changed?	**X**	What should the value be changed to?	**X**

If $X[k]$ is the DFT of $x[n]$, then $X[-k]$ will be the DFT of $x[-n]$. It follows that the antisymmetric part of $X[k]$ (which is $(X[k]-X[-k])/2$) is the DFT of the antisymmetric part of $x[n]$ (which is $(x[n]-x[-n])/2$). If the former is 0 for all k, then the latter will be zero for all n. Thus if we make $x_1[n]$ a symmetric function of n, $X_1[k]$ will be a symmetric function of k.

Part b. x_2

We would like to create $x_2[\cdot]$ by modifying **at most two** samples in $x[\cdot]$ so that $e^{j5\pi k/3}X_2[\cdot]$ is purely imaginary.

Which index should be changed?	0	What should the value be changed to?	3
Which index should be changed?	1	What should the value be changed to?	2

The DFT will be purely imaginary if the signal $x[n]$ is real and antisymmetric in n. The phase term $e^{j5\pi/3} = e^{-j2pik/6}$ corresponds to a delay of one sample. Therefore $X_2[k]$ will be purely imaginary if $x_2[n-1]$ is antisymmetric in n. antisymmetric.

Part c. x_3

We would like to create $x_3[\cdot]$ by modifying **at most two** samples in $x[\cdot]$ so that $\sum_{m=0}^{17} X_3[m] = 0$.

Which index should be changed?	0	What should the value be changed to?	0
Which index should be changed?	**X**	What should the value be changed to?	**X**

This sum is over three periods. Setting this sum to zero is the same as setting $x[0]$ to zero.

Part d. x_4

We would like to create $x_4[\cdot]$ by modifying **at most two** samples in $x[\cdot]$ so that $X_4[0] = 0$.

Which index should be changed?	0	What should the value be changed to?	5
Which index should be changed?	**X**	What should the value be changed to?	**X**

$X[0]$ will be zero if the average value of $x[n]$ is zero.

Part e. x_5

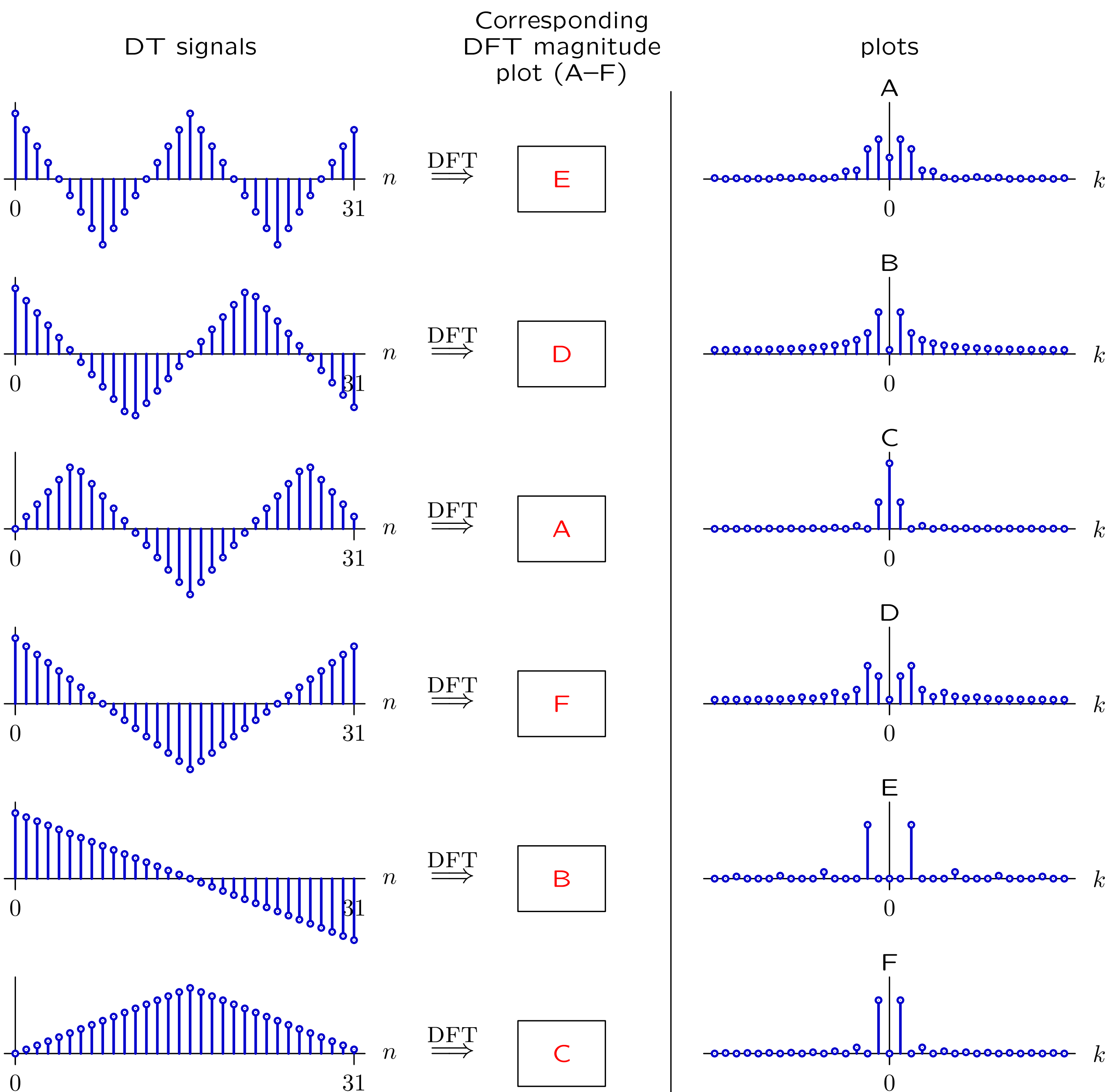
We would like to create $x_5[\cdot]$ by modifying **at most two** samples in $x[\cdot]$ so that $X_5[k] = -X_5[k+1]$ for all k.

Which index should be changed?	0	What should the value be changed to?	0
Which index should be changed?	4	What should the value be changed to?	0

The DFT will alternate in sign if $x[n] = 0$ for $n \neq 3$.

4. Discrete Fourier Transforms

The left column below shows six discrete-time signals for $0 \le n \le 31$.
The right column shows plots of the magnitudes of six DFTs computed for $N = 32$.
For each discrete-time signal in the left column below, find the matching DFT magnitude (one of plots A–F) and enter its letter in the box provided.



The top signal shows two full cycles of a triangle wave. Therefore the fundamental frequency of the triangle wave falls at $k = 2$. There could also be harmonics of $k = 2$ (i.e., at $k = 4, 6, 8, \ldots$).

$\rightarrow$ plot E

The next two signals show 1.5 full cycles of a triangle wave. Therefore the magnitude will peak between $k = 1$ and $k = 2$. If the first of these is periodically extended, it will have a big discontinuity between periods. The second of these has a much smaller discontinuity. Also, the DC value of the second is much larger than the first.

→ the second signal corresponds to plot D

→ the third signal corresponds to plot A

The fourth signal shows 1 full cycle of a triangle wave. Therefore $k = 1$.

→ plot F

When periodically extended, the fifth signal will be a sawtooth with $k = 1$. There will also be a large discontinuity at the period boundaries, so that will generate contributions at nearby k's.

→ plot B

When periodically extended, the last signal will make a triangle wave at $k = 1$. Notice however that there is a large DC component.

→ plot C

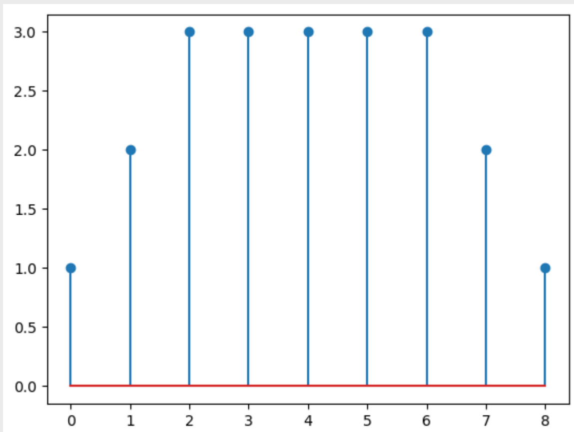
5. Convolution

Part a.

Write a Python program called `conv` to compute the convolution of two discrete-time signals: $x_1[n]$ and $x_2[n]$. Assume that the signal $x_1[n]$ is zero outside the range $0 \le n < N_1$ and that the signal $x_2[n]$ is zero outside the range $0 \le n < N_2$. Represent these signals as Python lists `x_1` and `x_2` where the first element in each list represents the value of the signal at $n = 0$ and the lengths of `x_1` and `x_2` are N_1 and N_2 respectively.

Demonstrate the use of your program by convolving two rectangular pulses: one of length 3 and the other of length 7.

```
def conv(x,y):
    answer = []
    for n in range(len(x)+len(y)-1):
        answer.append(sum([x[m]*y[n-m] for m in range(max(0,n-len(y)+1),min(n+1,len(x)))]))
    return answer
```

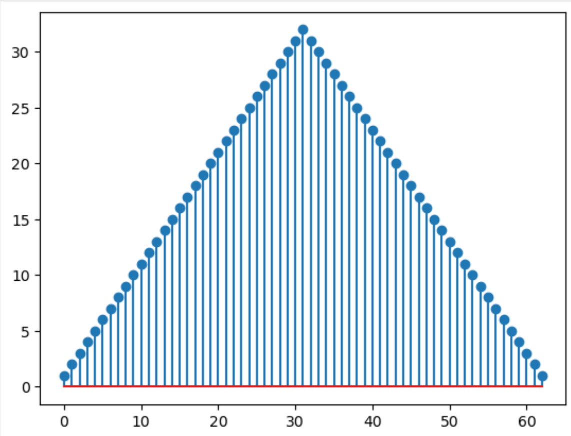
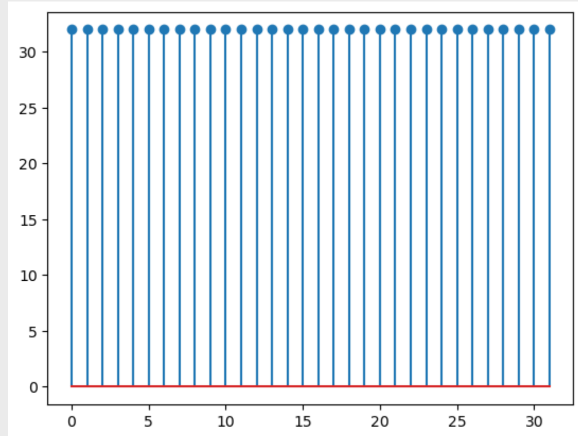


Part b.

Write a Python program called `circconv` to compute the circular convolution of two discrete-time signals: $x_1[n]$ and $x_2[n]$, as described in the previous part. Inputs to `circconv` should include the two input lists as well as the analysis width N of the circular convolution.

Demonstrate the use of your program by circularly convolving two rectangular pulses that are each of length 32 using an analysis width of $N = 32$. Compare the result of circular convolution with that of conventional convolution. Briefly explain the relation of these two results.

```
def circconv(x,y,N):
    answer = N*[0]
    z = conv(x,y)
    for n in range(len(z)):
        answer[n%N] += z[n]
    return answer
```

In this problem we convolved two rectangular signals with non-zero samples in the range $0 \le n < 32$ to get a result that had non-zero samples in the range $0 \le n < 64$.

However, the result of circular convolution is confined to the range $0 \le n < 32$. Samples outside that range are wrapped (i.e., aliased) back into the range, so that

$$(x_2 \circledast x_2)[n] = (x_2 * x_2)[n] + (x_2 * x_2)[n+32]$$

for $0 \le n < 32$. Because the inputs were rectangular, the convolution was triangular. Wrapping the triangle combined the portion of the convolution with positive slope with the portion with negative slope. The result was rectangular.

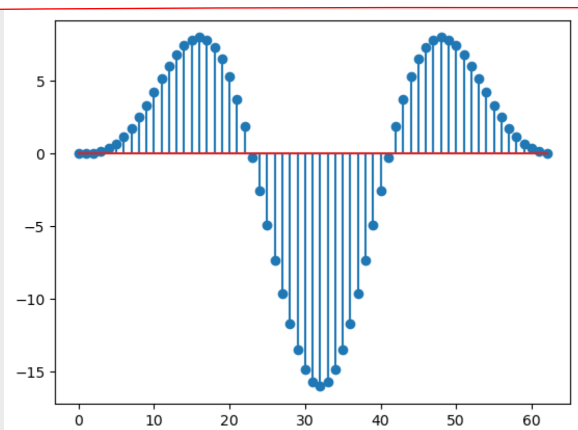
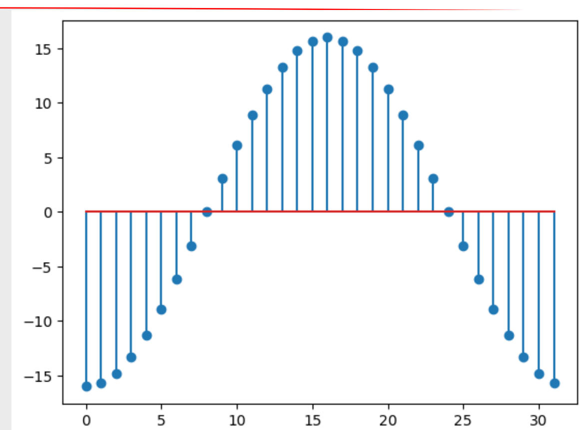
Part c.

Let $x_3[n]$ represent the following signal:

$$x_3[n] = \begin{cases} \sin(2\pi n/32) & \text{if } 0 \le n < 32 \\ 0 & \text{otherwise} \end{cases}$$

Compute the conventional convolution of $x_3[\cdot]$ with itself.

Compare the result with an analogous circular convolution, where the analysis window is $N = 32$. Briefly explain how the two results differ.

In this problem, one cycle of a sine wave with length 32 was convolved with itself. The result was greatest when the positive half cycles of the signals overlapped ($n = 16$) and smallest when the positive and negative halves overlapped ($n = 32$).

As in the previous part, circular convolution is confined to the range $0 \le n < 32$. Samples outside that range are wrapped (i.e., aliased) back into the range, so that

$$(x_2 \circledast x_2)[n] = (x_2 * x_2)[n] + (x_2 * x_2)[n+32]$$

for $0 \le n < 32$.

Interestingly, The result is a negative cosine wave. This result makes sense in the frequency domain. The Fourier series representation of the original sine wave is $X_4[k]$:

$$X_4[k] = -\frac{j}{2}\delta[k-1] + \frac{j}{2}\delta[k+1]$$

Convolution in time is equivalent to multiplication in frequency. If

$$x_5[n] = (x_4 * x_4)[n]$$

then

$$X_5[k] = X_4[k] \times X_4[k] = -\frac{1}{2}\delta[k-1] - \frac{1}{2}\delta[k+1]$$

which is a negative cosine signal.

Lab: Separating Harmonies

The file `am_resynth.wav` in this week's distribution **(here)**, is a synthesized version of a short tune using 3-part harmony. At each pont in time, there are three *voices:* bass, melody, and harmony. each playing a single, sinusoidal tone whose frequency changes with time. The frequencies for the three voices fall within the following ranges:

- Voice 1 (bass): 40-170Hz
- Voice 2 (melody): 170-370Hz
- Voice 3 (harmony): 370-750Hz

Our goal is to produce three new signals:

- **bass.wav** containing the bass part,
- **melody.wav** containing the melody, and
- **harmony.wav** containing the harmony.

Design a system to generate these .wav files.

Check-In.

Discuss the details of your plan for separating the components from each other, including how your implementation guarantees that the resulting signals will be purely real.

Generate the bass, melody, and harmony signals.

Upload a single '.zip' file containing your three tunes, as well as a file that contains the Python code that you used to generate them.

Were you able to perfectly separate the three voices? Explain any artifacts that remain.

More Realistic Example

The file `am.wav` contains a version of the same tune, but played on a guitar. In this version of the song, each of the voices has a slightly different range from the synthesized version:

- Voice 1 (bass): 65-208Hz
- Voice 2 (melody): 208-350Hz
- Voice 3 (harmony): 350-960Hz

Try your same method from above on this version of the tune.

You likely found that your approach does not work as well on `am.wav`, compared to `am_resynth.wav`, and that some of the three voices may have been easier or harder to isolate. Explain why this is the case.