Math 320-1: Midterm 2 Solutions
Northwestern University, Fall 2015

1. Give an example of each of the following. You do not have to justify your answer.
(a) A function on $\mathbb{R}$ which is nowhere continuous.
(b) A function $f : \mathbb{R} \to \mathbb{R}$ which is uniformly continuous on $[2, 100]$ but not on all of $\mathbb{R}$.
(c) A function on $\mathbb{R}$ which is differentiable but not twice differentiable.
(d) A function $f : \mathbb{R} \to \mathbb{R}$ which is differentiable at 3 and nowhere else.

Solutions. (a) The function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

is nowhere continuous.

(b) The function $f(x) = x^2$ is uniformly continuous on any closed interval but not on all of $\mathbb{R}$.

(c) The function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is differentiable but not twice differentiable, as shown in class or on a homework assignment.

(d) The function defined by

$$f(x) = \begin{cases} (x-3)^2 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

works. This is differentiable at 3 since

$$\lim_{x \to 3} \frac{f(x) - f(3)}{x - 3} = 0$$

by considering the cases where $f(x) = (x-3)^2$ or $f(x) = 0$ in the numerator separately. However, f is not continuous at any $x \neq 3$, so it is not differentiable at such x either. $\square$

2. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function such that $\lim_{x \to 2} f(x) = L$ exists and

$$2 < L < 5.$$

Show that there exists $\delta > 0$ such that $2 < f(x) < 5$ for all $x \in (2 - \delta, 2 + \delta)$ except possibly $x = 2$.

Proof. Since $L - 2 > 0$ and $5 - L > 0$, $\min\{L - 2, 5 - L\} > 0$. Hence there exists $\delta > 0$ such that

$$|f(x) - L| < \min\{L - 2, 5 - L\} \text{ if } 0 < |x - 2| < \delta.$$

Thus for $x \in (2 - \delta, 2 + \delta)$ such that $x \neq 2$, we have

$$-(L - 2) \leq -\min\{L - 2, 5 - L\} < f(x) - L < \min\{L - 2, 5 - L\} \leq 5 - L,$$

which implies after adding L throughout that

$$2 < f(x) < 5$$

for such x, as was to be shown. $\square$

3. Show that the function $f : (0,4) \to \mathbb{R}$ defined by

$$f(x) = \frac{1}{x^2}$$

is continuous at $a = \frac{1}{3}$ and that it is not uniformly continuous on $(0,4)$. When showing continuity at $\frac{1}{3}$ you MUST verify the ϵ-δ definition directly and cannot simply quote the fact that quotients of continuous functions are continuous whenever the denominator is nonzero.

Proof. Let $\epsilon > 0$ and let $\delta = \min\{\frac{\epsilon}{324(4+\frac{1}{3})}, \frac{1}{6}\}$, which is also positive. Suppose that $|x - \frac{1}{3}| < \delta$. Then in particular

$$|x - \frac{1}{3}| < \frac{1}{6}, \text{ so } -\frac{1}{6} < x - \frac{1}{3} \text{ and hence } \frac{1}{6} < x.$$

Thus

$$\left|\frac{1}{x^2} - \frac{1}{1/3^2}\right| = \frac{|x^2 - (\frac{1}{3})^2|}{x^2/9} = \frac{|x - \frac{1}{3}||x + \frac{1}{3}|}{x^2/9} < \frac{\delta(4 + \frac{1}{3})}{1/(36 \cdot 9)} \leq \epsilon,$$

so we conclude that f is continuous at $\frac{1}{3}$.

Since f cannot be extended to a continuous function on $[0,4]$, it is not uniformly continuous on $(0,4)$. Another way to see this is to note that the sequence $\frac{1}{n}$ is Cauchy in $(0,4)$ but the sequence $f(\frac{1}{n}) = n^2$ is not, and uniformly continuous functions should send Cauchy sequences to Cauchy sequences. □

4. Suppose that $g : \mathbb{R} \to \mathbb{R}$ is continuously differentiable everywhere but not twice differentiable at 1. Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = (x-1)g(x)$$

is twice differentiable at 1. Hint: The product rule will say right away that f is differentiable everywhere, but it won't immediately say that f is twice differentiable.

Proof. Since g and $x - 1$ are differentiable, the product rule implies that f is differentiable and

$$f'(x) = g(x) + (x-1)g'(x)$$

for any $x \in \mathbb{R}$. In particular this gives $f'(1) = g(1)$. Now, we have

$$\frac{f'(x) - f'(1)}{x - 1} = \frac{g(x) + (x-1)g'(x) - g(1)}{x - 1} = \frac{g(x) - g(1)}{x - 1} + g'(x).$$

Since g is differentiable at 1 we have

$$\lim_{x \to 1} \frac{g(x) - g(1)}{x - 1} = g'(1)$$

and since g' is continuous we have

$$\lim_{x \to 1} g'(x) = g'(1).$$

Thus

$$\lim_{x \to 1} \frac{f'(x) - f'(1)}{x - 1} = g'(1) + g'(1)$$

exists, so f is twice differentiable at 1. □

5. Prove that $1 - \sin x \leq e^x$ for all $x \geq 0$. Hint: Find a good function to which you can apply the Mean Value Theorem.

Proof. Let $f(x) = e^x + \sin x$. First, $f(0) = e^0 + \sin 0 = 1$, so $1 - \sin 0 = e^0$ and the claimed inequality holds in this case. Now fix $x > 0$. Since f is differentiable, the Mean Value Theorem says that there exists c between x and 0 such that

$$f(x) - f(0) = f'(c)(x - 0) = (e^c + \sin c)x.$$

Since $c \geq 0$, $e^c \geq 1 \geq \sin c$, so $e^c + \sin c \geq 0$. Hence

$$f(x) - f(0) \geq 0,$$

so

$$e^x + \sin x - 1 \geq 0$$

and the desired inequality follows by moving $\sin x$ and -1 to the right-hand side. □