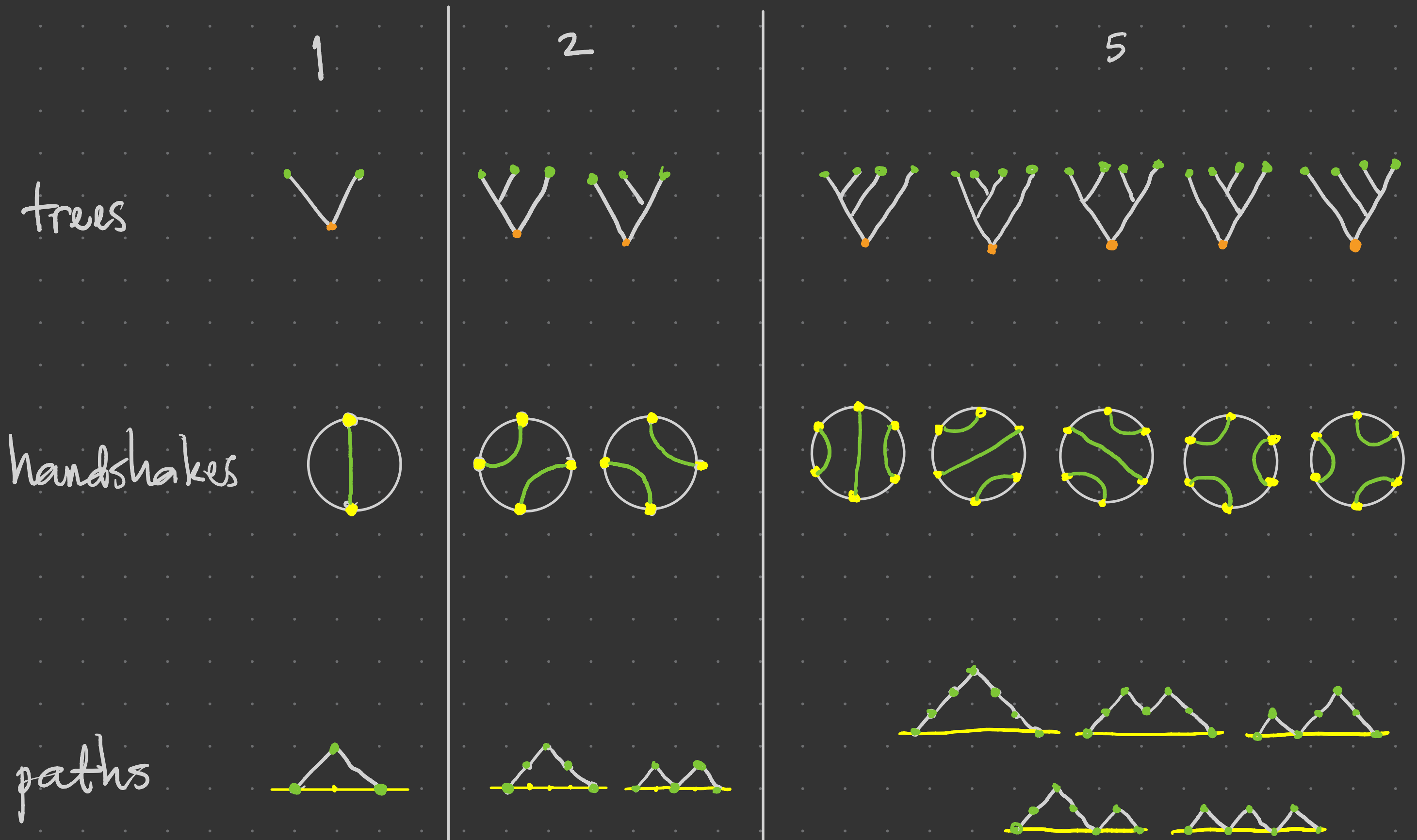
Counting in Catalan

handshakes, trees, & paths

UW Math Hour

2023 May 21

Kyle Ormsby


1
2
5
trees
handshakes
paths

And the pattern continues:

Trees, handshakes, and paths (and many other structures!) are counted by the Catalan numbers:

n	0	1	2	3	4	5	6	7	...
$\mathrm{Cat}(n)$	1	1	2	5	14	42	132	439	...

where

$$\mathrm{Cat}(n+1) = \mathrm{Cat}(0)\cdot\mathrm{Cat}(n) + \mathrm{Cat}(1)\cdot\mathrm{Cat}(n-1) + \mathrm{Cat}(2)\cdot\mathrm{Cat}(n-2) + \cdots$$
$$+ \mathrm{Cat}(n-1)\cdot\mathrm{Cat}(1) + \mathrm{Cat}(n)\cdot\mathrm{Cat}(0)$$
$$= \sum_{i=0}^{n} \mathrm{Cat}(i)\cdot\mathrm{Cat}(n-i) = \frac{1}{2n+1}\binom{2n+1}{n}.$$

(to be explained...)

Combinatorics — the mathematics of counting

Counting: it's easy!  6 pennies

Counting: it's hard! How many substitution codes ($A \to X$, $B \to F$, $C \to A$, ...) are there? What if no letter may substitute for itself?

A $26! = 26 \cdot 25 \cdot 24 \cdots 2 \cdot 1 \approx 4 \cdot 10^{26}$

or $26!\left(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \cdots + \frac{1}{26!}\right) \approx \frac{26!}{e} \approx 1.5 \cdot 10^{26}$

Euler's constant 2.71828...

What is counting?

Sequentially label with numbers so that each object gets exactly one label.

But...

What is a number?

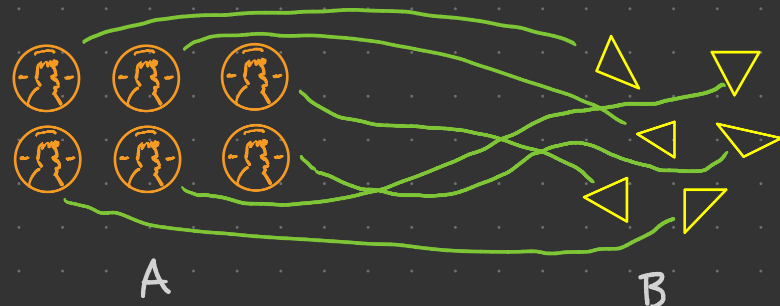
Potential Answer The number 6 is the collection of all collections that can be "registered" with $\{1,2,3,4,5,6\}$.

Foundational monsters nearby!

Bijection

aka "cardinality"

Two sets (collections of objects) have the same size when there is a one-to-one correspondence (aka bijection) between them:



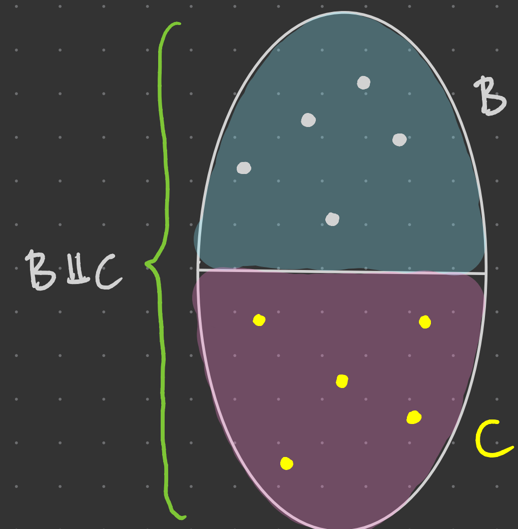
Write $|A| = |B|$ and call A and B equinumerous.

Note If we count A (say $|A| = n$) and A and B are equinumerous, then $|B| = n$ as well!

3 Counting Principles

1. Additive Counting

For a set A, write $A = B \amalg C$ if every element of A is in exactly one of B or C. This is a partition of A, which is the disjoint union of B, C.



E.g. $\{1,2,3,4,5,6\} = \{2,5\} \amalg \{1,3,4,6\}$

Theorem If $A = B \amalg C$, then $|A| = |B| + |C|$.

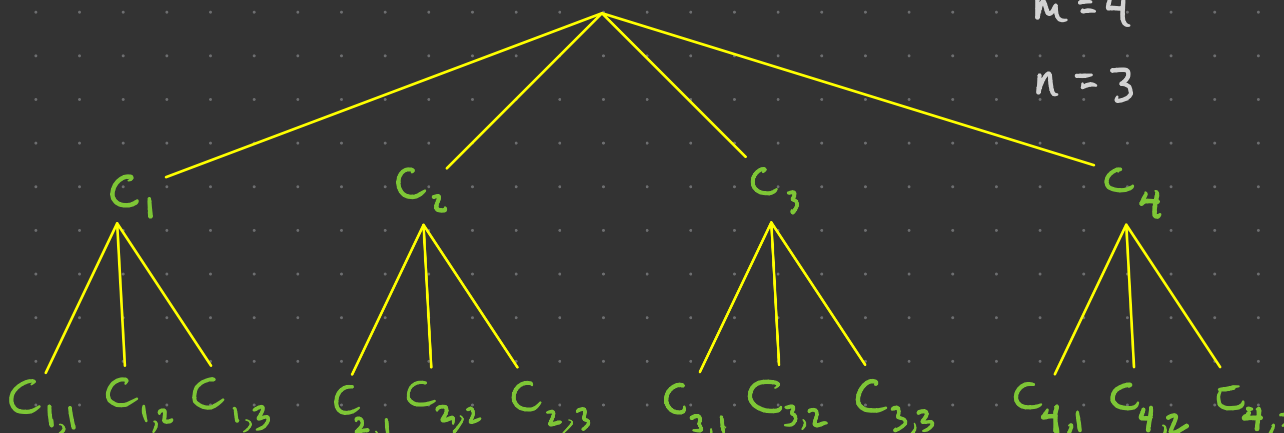
E.g. (ct'd) $6 = 2 + 4$

2. Multiplicative Counting

<u>Theorem</u> If we can enumerate (count) the elements of a set A by first making m choices and then making n choices, then $|A| = m \cdot n$.

$m = 4$

$n = 3$



Multiplicative Counting (ct'd)

Subsets

For sets A, B, suppose every element of A is also an element of B. We then call A a subset of B and write $A \subseteq B$.

Q If $|B| = n$, how many sets A are subsets of B?

(If there are 12 flowers and your friend says you can take whichever you like, how many different bouquets can you make?)

$$A \quad \underbrace{\text{in/out}}_{b_1} \quad \underbrace{\text{in/out}}_{b_2} \quad \underbrace{\text{in/out}}_{b_3} \quad \cdots \quad \underbrace{\text{in/out}}_{b_n}$$

Make 2 choices n consecutive times, so

$2 \cdot 2 \cdot 2 \cdots 2 = 2^n$ subsets.

Multiplicative Counting (ct'd)

Permutations

A permutation of $\{1,2,\dots,n\}$ is a reordering of the elements, or, equivalently, a bijection $\{1,2,\dots,n\} \to \{1,2,\dots,n\}$:

$3\ 1\ 6\ 4\ 2\ 5 \longleftrightarrow$

Theorem There are $n! := n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$ permutations of an n-element set.

n	1	2	3	4	5	6	7	8	9
$n!$	1	2	6	24	120	720	5040	40320	362880

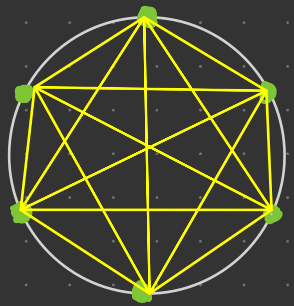
3 Counting Principles

3. Overcounting

Theorem If we count each element of a set A m times and our total count is N, then $|A| = \frac{N}{m}$.

Q Six colleagues at a business meeting each shake each others hands exactly once? How many handshakes occur in total?

A Each of the 6 people shake 5 hands, so $6 \cdot 5$ handshakes, but... We counted each handshake twice (Alice - Bob and (Bob - Alice) so $\frac{6 \cdot 5}{2} = 15$ handshakes.



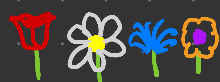
$$\frac{n(n-1)}{2} = (n-1) + (n-2) + (n-3) + \cdots + 1$$

Overcounting (ct'd)

Choosing k from n

Q How many different 4 flower bouquets can be made from 12 flowers?

First attempt $12 \cdot 11 \cdot 10 \cdot 9$

But  =  = ...

We've overcounted by a factor of $4!$, the number of permutations of each 4 flower bouquet.

A $12 \cdot 11 \cdot 10 \cdot 9 / (4!) = 495$

Theorem There are $\dfrac{n \cdot (n-1) \cdot (n-2) \cdots (n-k+1)}{k!}$ k-element subsets of $\{1, 2, \ldots, n\}$.

The Binomial Triangle

- Write $\binom{n}{k}$ — read "n choose k" — for the # of k-element subsets of $\{1,2,\dots,n\}$, so $\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} = \frac{n!}{k!\,(n-k)!}$.
- Let $A = \{$such sets containing $n\}$, $B = \{$such sets not containing $n\}$ so that $\binom{n}{k} = |A| + |B|$ (by additive counting principle)
- To count A, choose $k-1$ elements of $\{1,2,\dots,n-1\}$ (then add n to the set); for B, choose k elements of $\{1,2,\dots,n-1\}$.

Theorem $$\boxed{\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}}$$ (Sometimes known as Pascal's identity)

The binomial triangle (ct'd)

$$\binom{0}{0}$$

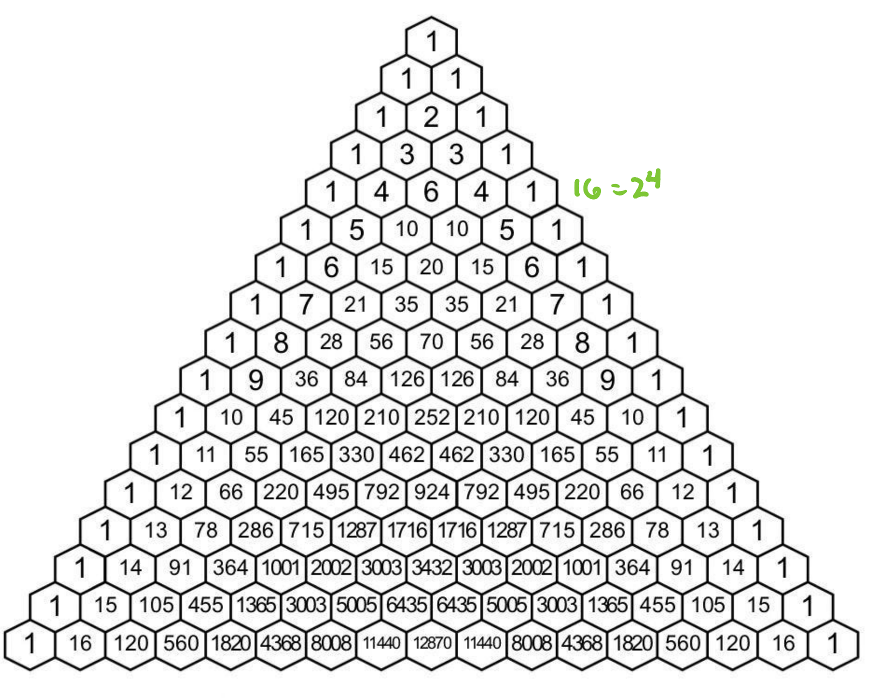
$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

The binomial triangle (ct'd)



$16 = 2^4$

Symmetry Theorem

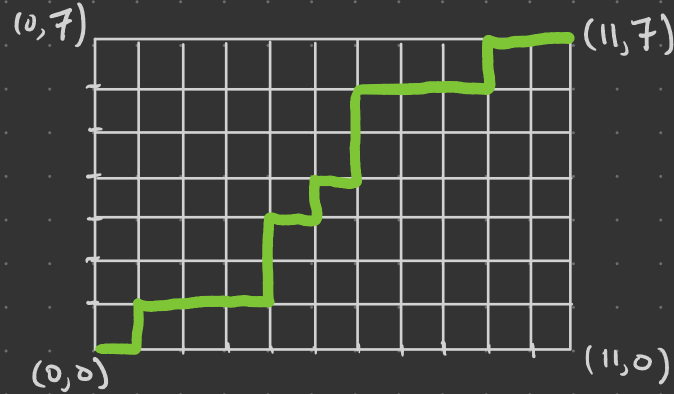
$$\binom{n}{k} = \binom{n}{n-k}$$

Row Sum Theorem

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$$

Binomials & Paths

Q How many "N/E lattice paths" on the grid from $(0,0)$ to (n,k)?



A $n+k$ total steps, exactly k of which are N's, so

$$\binom{n+k}{k} = \binom{n+k}{n}.$$

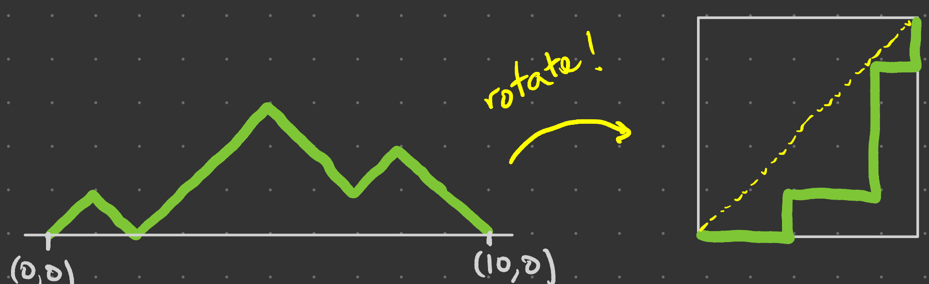
Catalan Structures

1. Mountain Ranges:

   NE/SE lattice paths from $(0,0)$ to $(2n,0)$ that never go below the x-axis

2. Dyck Paths:

   N/E lattice paths from $(0,0)$ to (n,n) that never go above the diagonal:



Theorem There are $\boxed{\frac{1}{n+1}\binom{2n}{n}} =: \mathrm{Cat}(n)$ Dyck paths from $(0,0)$ to (n,n).

Proof Idea: Let L_n be the collection of N/E lattice paths from $(0,0)$ to (n,n) so that $|L_n| = \binom{2n}{n}$. Now partition L_n as $E_0 \amalg E_1 \amalg \cdots \amalg E_n$ where $E_i \subseteq L_n$ consists of paths with i N steps above the diagonal.

(Note: $E_0 = \{\text{Dyck paths}\}$.) Then

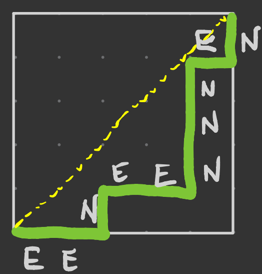
$$|L_n| = |E_0| + |E_1| + \cdots + |E_n|$$

by additive counting.

Claim: $|E_0| = |E_1| = \cdots = |E_n|$ (Try to prove it!)

Thus $\#\text{Dyck paths} = |E_0| = \dfrac{|L_n|}{n+1} = \dfrac{1}{n+1}\dbinom{2n}{n}$. $\square$

Parentheses & Handshakes

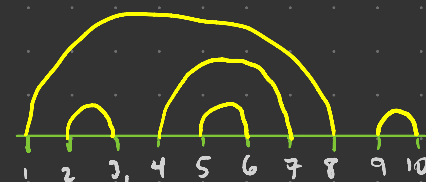


$\rightsquigarrow$ EENEENNNEN $\rightsquigarrow$ (()(()))()

Dyck word: n E's, n N's
#E's $\geq$ #N's reading left to right

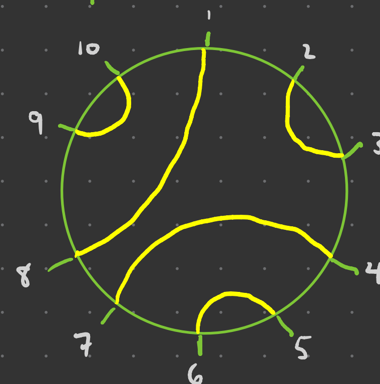
well-balanced parentheses

$\rightsquigarrow$



nested arcs joining $1, 2, \ldots, 2n$ in upper half plane

$\rightsquigarrow$

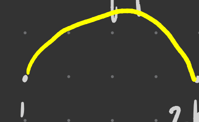


noncrossing handshakes

Theorem All of these structures are counted by $\frac{1}{n+1}\binom{2n}{n} = \mathrm{Cat}(n)$.

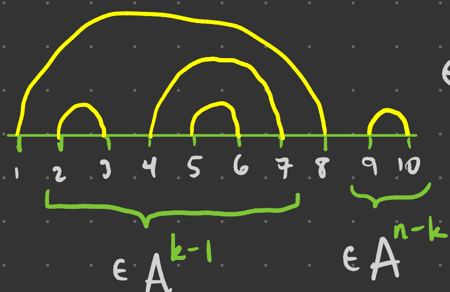
Catalan Recurrence

Let $A^n = \{$nested arcs joining $1, 2, \dots, 2n$ in upper half plane$\}$.

Partition A^n according to k such that 

so that

$$A^n = A_1 \sqcup A_2 \sqcup \cdots \sqcup A_n \quad (\star)$$



$\in A_8$ for $n=5$. By multiplicative counting,

$$|A_k| = |A^{k-1}| \cdot |A^{n-k}|$$
$$= \mathrm{Cat}(k-1) \cdot \mathrm{Cat}(n-k).$$

So by $(\star)$ + additive counting,

$$\mathrm{Cat}(n) = \mathrm{Cat}(0)\,\mathrm{Cat}(n-1) + \mathrm{Cat}(1)\,\mathrm{Cat}(n-2) + \cdots + \mathrm{Cat}(n-1)\,\mathrm{Cat}(0).$$

- $\mathrm{Cat}(0) = \boxed{1} = \frac{1}{1}\binom{0}{0}$
- $\mathrm{Cat}(1) = \mathrm{Cat}(0)\cdot\mathrm{Cat}(0) = \boxed{1} = \frac{1}{2}\binom{2}{1}$
- $\mathrm{Cat}(2) = \mathrm{Cat}(0)\,\mathrm{Cat}(1) + \mathrm{Cat}(1)\,\mathrm{Cat}(0) = 1\cdot 1 + 1\cdot 1 = \boxed{2} = \frac{1}{3}\binom{4}{2}$
- $\mathrm{Cat}(3) = \mathrm{Cat}(0)\,\mathrm{Cat}(2) + \mathrm{Cat}(1)\,\mathrm{Cat}(1) + \mathrm{Cat}(2)\,\mathrm{Cat}(0)$

  $= 1\cdot 2 + 1\cdot 1 + 2\cdot 1 = \boxed{5} = \frac{1}{4}\binom{6}{3}$
- $\mathrm{Cat}(4) = \mathrm{Cat}(0)\,\mathrm{Cat}(3) + \mathrm{Cat}(1)\,\mathrm{Cat}(2) + \mathrm{Cat}(2)\,\mathrm{Cat}(1) + \mathrm{Cat}(3)\,\mathrm{Cat}(0)$

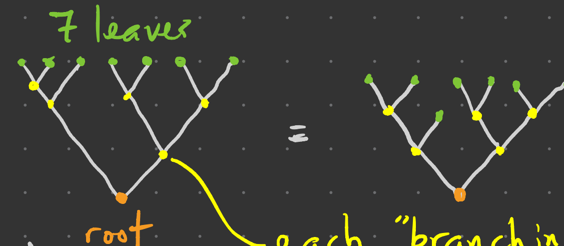
  $= 1\cdot 5 + 1\cdot 2 + 2\cdot 1 + 5\cdot 1 = \boxed{14} = \frac{1}{5}\binom{8}{4}$
- $\mathrm{Cat}(5) = \mathrm{Cat}(0)\,\mathrm{Cat}(4) + \mathrm{Cat}(1)\,\mathrm{Cat}(3) + \mathrm{Cat}(2)\,\mathrm{Cat}(2) + \mathrm{Cat}(3)\,\mathrm{Cat}(1)$

  $+ \mathrm{Cat}(4)\,\mathrm{Cat}(0)$

  $= 1\cdot 14 + 1\cdot 5 + 2\cdot 2 + 5\cdot 1 + 14\cdot 1 = \boxed{42} = \frac{1}{6}\binom{10}{5}$

Trees

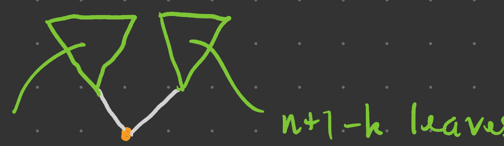
A full binary tree looks like



each "branching" has two offshoots, one to the left, the other to the right

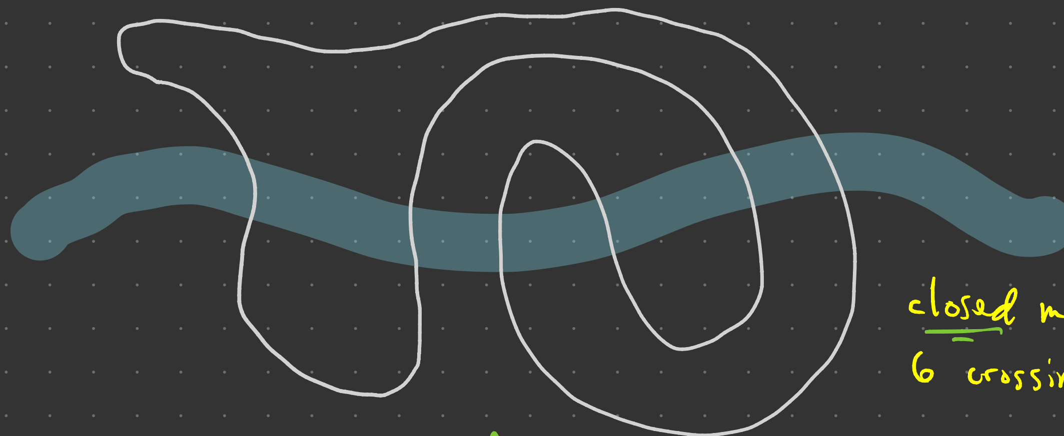
Theorem There are $\mathrm{Cat}(n) = \frac{1}{n+1}\binom{2n}{n}$ full binary trees with $n+1$ leaves.

Proof Idea Partition according to number of leaves on the left branch from the root:

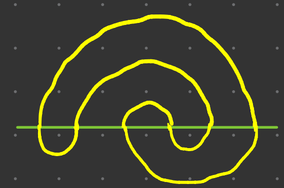


Use this to show that # full binary trees with $n+1$ leaves satisfies the Catalan recurrence. $\square$

Meanders (open problem!)



closed meander with 6 crossings



pair of nested arc systems joining 1,2,...,6

Meanders (ct'd)

There are $\mathrm{Cat}(n) \cdot \mathrm{Cat}(n) = \mathrm{Cat}(n)^2$ meandric systems with $2n$ crossings.

$n=3$:

Meanders (ct'd)

1 loop — closed meanders

2 loops

3 loops

There are 8 closed meanders with 6 crossings.

n	1	2	3	4	5	6	7	8	…
M_n	1	2	8	42	262	1828	13820	110 954	…

Open Problem

Let $M_n^{(k)} :=$ # meandric systems with $2n$ crossings and k loops

$M_n := M_n^{(1)} =$ # closed meanders with $2n$ crossings

Find a formular for M_n and $M_n^{(k)}$.

Henri Poincaré
1854 – 1912

Vladimir Arnol'd
1937 – 2010

Francesco - Golinelli - Guitter, 1995

Solution for $n-5 \leq k \leq n$:

$$M_n^{(n)} = \frac{(2n)!}{n!(n+1)!}$$

$$M_n^{(n-1)} = \frac{2(2n)!}{(n-2)!(n+2)!}$$

$$M_n^{(n-2)} = \frac{2(2n)!}{(n-3)!(n+4)!}(n^2+7n-2)$$

$$M_n^{(n-3)} = \frac{4(2n)!}{3(n-4)!(n+6)!}(n^4+20n^3+107n^2-107n+15)$$

$$M_n^{(n-4)} = \frac{2(2n)!}{3(n-5)!(n+8)!}(n^6+39n^5+547n^4+2565n^3-5474n^2+2382n-672)$$

$$\begin{aligned} M_n^{(n-5)} = \frac{4(2n)!}{15(n-6)!(n+10)!}&(n^8+64n^7+1646n^6 \\ &+20074n^5+83669n^4-323444n^3+257134n^2-155604n+45360) \end{aligned}$$

$k=1$: You, 2023?

Further Reading

https://kyleormsby.github.io/files/113full_text.pdf

Thank you to Dave and Reed's Math 113 students, and thank you!