Fubini's theorem

In mathematical analysis **Fubini's theorem**, named after Guido Fubini, is a result which gives conditions under which it is possible to compute a double integral using iterated integrals. As a consequence it allows the order of integration to be changed in iterated integrals.

Theorem statement

Suppose *A* and *B* are complete measure spaces. Suppose *f(x,y)* is $A \times B$ measurable. If

$$\int_{A\times B} |f(x,y)|\,\mathrm{d}(x,y) < \infty,$$

where the integral is taken with respect to a product measure on the space over $A \times B$, then

$$\int_A \left(\int_B f(x,y)\,\mathrm{d}y\right)\,\mathrm{d}x = \int_B \left(\int_A f(x,y)\,\mathrm{d}x\right)\,\mathrm{d}y = \int_{A\times B} f(x,y)\,\mathrm{d}(x,y),$$

the first two integrals being iterated integrals with respect to two measures, respectively, and the third being an integral with respect to a product of these two measures.

If the above integral of the absolute value is not finite, then the two iterated integrals may actually have different values. See below for an illustration of this possibility.

Corollary

If $f(x,y) = g(x)h(y)$ for some functions *g* and *h*, then

$$\int_A g(x)\,\mathrm{d}x \int_B h(y)\,\mathrm{d}y = \int_{A\times B} f(x,y)\,\mathrm{d}(x,y),$$

the integral on the right side being with respect to a product measure.

Alternate theorem statement

Another version of Fubini's theorem states that if *A* and *B* are σ-finite measure spaces, not necessarily complete, and if either

$$\int_A \left(\int_B |f(x,y)|\,\mathrm{d}y\right)\,\mathrm{d}x < \infty \quad \text{or} \quad \int_B \left(\int_A |f(x,y)|\,\mathrm{d}x\right)\,\mathrm{d}y < \infty$$

then

$$\int_{A\times B} |f(x,y)|\,\mathrm{d}(x,y) < \infty$$

and

$$\int_A \left(\int_B f(x,y)\,\mathrm{d}y\right)\,\mathrm{d}x = \int_B \left(\int_A f(x,y)\,\mathrm{d}x\right)\,\mathrm{d}y = \int_{A\times B} f(x,y)\,\mathrm{d}(x,y).$$

In this version the condition that the measures are σ-finite is necessary.

Tonelli's theorem

Tonelli's theorem (named after Leonida Tonelli) is a successor of Fubini's theorem. The conclusion of Tonelli's theorem is identical to that of Fubini's theorem, but the assumptions are different. Tonelli's theorem states that on the product of two σ-finite measure spaces, a product measure integral can be evaluated by way of an iterated integral for *nonnegative* measurable functions, regardless of whether they have finite integral.

In fact, the existence of the first integral above (the integral of the absolute value), can be guaranteed by Tonelli's theorem (see below).

A formal statement of Tonelli's theorem is identical to that of Fubini's theorem, except that the requirements are now that (X, A, μ) and (Y, B, ν) are σ-finite measure spaces, while f maps $X\times Y$ to $[0,\infty]$.

Kuratowski-Ulam theorem

The Kuratowski-Ulam theorem, named after Polish mathematicians Kazimierz Kuratowski and Stanisław Ulam, called also Fubini theorem for category, is a similar result for arbitrary second countable Baire spaces. Let X and Y be second countable Baire spaces (or, in particular, Polish spaces), and $A\subset X\times Y$. Then the following are equivalent if A has the Baire property:

1. A is meager (respectively comeager)
2. The set $\{x\in X: A_x \text{ is meagre (resp. comeagre) in Y}\}$ is comeagre in X, where $A_x=\pi_Y[A\cap\{x\}\times Y]$, where π_Y is the projection onto Y.

Even if A does not have the Baire property, 2. follows from 1.[1] Note that the theorem still holds (perhaps vacuously) for X - arbitrary Hausdorff space and Y - Hausdorff with countable π-base.

The theorem is analogous to regular Fubini theorem for the case where the considered function is a characteristic function of a set in a product space, with usual correspondences – meagre set with set of measure zero, comeagre set with one of full measure, a set with Baire property with a measurable set.

Applications

Gaussian integral

One application of Fubini's theorem is the evaluation of the Gaussian integral which is the basis for much of probability theory:

$$\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x=\sqrt{\pi}.$$

To see how Fubini's theorem is used to prove this, see Gaussian integral.

Rearranging a conditionally convergent iterated integral

Fubini's theorem tells us that if the integral of the absolute value is finite, then the order of integration does not matter; if we integrate first with respect to x and then with respect to y, we get the same result as if we integrate first with respect to y and then with respect to x. The assumption that the integral of the absolute value is finite is "Lebesgue integrability".

The iterated integral

$$\int_0^1\int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}\,\mathrm{d}y\,\mathrm{d}x$$

does not converge absolutely (i.e. the integral of the absolute value is not finite):

$$\int_0^1 \int_0^1 \left| \frac{x^2 - y^2}{(x^2 + y^2)^2} \right| \, \mathrm{d}y \, \mathrm{d}x = \infty.$$

That the assumption of Lebesgue integrability in Fubini's theorem cannot be dropped can be seen by examining this particular iterated integral. Putting "*dx dy*" in place of "*dy dx*" has the effect of multiplying the value of the integral by −1 because of the antisymmetry of the function being integrated. Therefore, unless the value of the integral is zero, putting "*dx dy*" in place of "*dy dx*" actually changes the value of the integral. That is indeed what happens in this case.

Proof

One way to do this without using Fubini's theorem is as follows:

$$\int_0^1 \int_0^1 \left| \frac{x^2 - y^2}{(x^2 + y^2)^2} \right| \, \mathrm{d}x \, \mathrm{d}y = \int_0^1 \left[\int_0^y \frac{y^2 - x^2}{(x^2 + y^2)^2} \, \mathrm{d}x + \int_y^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} \, \mathrm{d}x \right] \, \mathrm{d}y$$
$$= \int_0^1 \left(\frac{1}{2y} + \frac{1}{2y} - \frac{1}{y^2 + 1} \right) \, \mathrm{d}y = \int_0^1 \frac{1}{y} \, \mathrm{d}y - \int_0^1 \frac{1}{1 + y^2} \, \mathrm{d}y.$$

Evaluation

Firstly, we consider the "inside" integral.

$$\int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} \, \mathrm{d}y$$
$$= \int_0^1 \frac{x^2 + y^2 - 2y^2}{(x^2 + y^2)^2} \, \mathrm{d}y$$
$$= \int_0^1 \frac{1}{x^2 + y^2} \, \mathrm{d}y + \int_0^1 \frac{-2y^2}{(x^2 + y^2)^2} \, \mathrm{d}y$$
$$= \int_0^1 \frac{1}{x^2 + y^2} \, \mathrm{d}y + \int_0^1 y \frac{d}{\mathrm{d}y} \left(\frac{1}{x^2 + y^2} \right)$$
$$= \int_0^1 \frac{1}{x^2 + y^2} \, \mathrm{d}y + \left(\left[\frac{y}{x^2 + y^2} \right]_{y=0}^1 - \int_0^1 \frac{1}{x^2 + y^2} \, \mathrm{d}y \right) \quad \text{(by parts)}$$
$$= \frac{1}{1 + x^2}.$$

This takes care of the "inside" integral with respect to *y*; now we do the "outside" integral with respect to *x*:

$$\int_0^1 \frac{1}{1 + x^2} \, \mathrm{d}x = [\arctan(x)]_0^1 = \arctan(1) - \arctan(0) = \frac{\pi}{4}.$$

Thus we have

$$\int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} \, \mathrm{d}y \, \mathrm{d}x = \frac{\pi}{4}$$

and

$$\int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} \, \mathrm{d}x \, \mathrm{d}y = -\frac{\pi}{4}.$$

Fubini's theorem implies that since these two iterated integrals differ, the integral of the absolute value must be ∞.

Statement

When

$$\int_a^b \int_c^d |f(x,y)|\,\mathrm{d}y\,\mathrm{d}x = \infty$$

then the two iterated integrals

$$\int_a^b \int_c^d f(x,y)\,\mathrm{d}y\,\mathrm{d}x \text{ and } \int_c^d \int_a^b f(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

may have different finite values.

Strong versions

The existence of strengthenings of Fubini's theorem, where the function is no longer assumed to be measurable but merely that the two iterated integrals are well defined and exist, is independent of the standard Zermelo–Fraenkel axioms of set theory. Martin's axiom implies that there exists a function on the unit square whose iterated integrals are not equal, while a variant of Freiling's axiom of symmetry implies that in fact a strong Fubini-type theorem for [0, 1] does hold, and whenever the two iterated integrals exist they are equal.[2] See List of statements undecidable in ZFC.

References

[1] S. Srivastava *A course on Borel sets*. Springer, 1998, p. 112.

[2] Chris Freiling, *Axioms of symmetry: throwing darts at the real number line*, J. Symbolic Logic 51 (1986), no. 1, 190–200.

External links

- Kudryavtsev, L.D. (2001), "Fubini theorem" (http://www.encyclopediaofmath.org/index.php?title=F/f041870), in Hazewinkel, Michiel, *Encyclopedia of Mathematics*, Springer, ISBN 978-1556080104

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